# FRACTIONAL BRAUER CONFIGURATION ALGEBRAS III: FRACTIONAL BRAUER GRAPH ALGEBRAS IN TYPE MS

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#### Abstract

In previous papers, we defined fractional Brauer graphs and studied their covering theory. In this paper, we develop a covering theory for the Brauer G-sets and use it to generalize the representation type results and the AR-components results on Brauer graph algebras to the scope of fractional Brauer graph algebras in type MS.

## 1. INTRODUCTION

In [\[7\]](#page-38-0), we introduced a class of locally bounded quiver algebras called fractional Brauer configuration algebras (abbr. f-BCAs). It was shown that f-BCAs in type S (abbr.  $f_s$ -BCAs) are locally bounded Frobenius algebras, and over an algebraically closed field, the representation-finite  $f_s$ -BCAs coincide with standard representation-finite basic self-injective algebras. In the present paper we concentrate on a subclass of  $f_s$ -BCAs, which are called fractional Brauer graph algebras in type MS (abbr.  $f_{ms}$ -BGAs). As a natural generalization of Brauer graph algebras (abbr. BGAs),  $f_{ms}$ -BGAs are self-injective special biserial and have tame representation type.

It is well-known that over an algebraically closed field, Brauer graph algebras coincide with symmetric special biserial algebras and whose representation types are classified in terms of the defining Brauer graphs (abbr. BGs) as follows (here we view a BG as a f-BC, see [\[7,](#page-38-0) Section 3]).

<span id="page-0-0"></span>**Theorem 1.1.** (cf. [\[1\]](#page-38-1) and [\[11\]](#page-38-2)) Suppose that the field k is algebraically closed. Let E be a finite connected Brauer graph and  $A_E$  be the corresponding BGA. Then

- (1)  $A_E$  is representation-finite if and only if E is a Brauer tree (abbr. BT).
- (2)  $A_E$  is 1-domestic if and only if one of the following holds
	- The underlying graph of E is a tree, with two vertices f-degree 2 and others f-degree 1.
- E is f-degree trivial and the underlying graph of  $E$  has a unique cycle of odd length.
- (3)  $A_E$  is 2-domestic if and only if E is f-degree trivial and the underlying graph of E has a unique cycle of even length.
- (4)  $A_E$  cannot be n-domestic for  $n \geq 3$ .

Moreover, recently Duffield determined explicitly in [\[2\]](#page-38-3) the Auslander-Reiten components (abbr. AR-components) of BGAs in terms of the defining BGs. Note that about thirty years before Erdmann and Skowronski had obtained a general description in  $[4]$  on the representation types and the AR-components of self-injective special biserial algebras. Since  $f_{ms}$ -BGAs (which are defined by fractional Brauer graphs in type MS, or shortly by  $f_{ms}$ -BGs) are self-injective special biserial algebras, it is natural to ask how to describe explicitly the representation types and ARcomponents of  $f_{ms}$ -BGAs in terms of defining  $f_{ms}$ -BGs.

To achieve this, our original idea is to use a covering theory for  $f_{ms}$ -BGs developed in [\[8\]](#page-38-5). However, since a quotient of a  $f_{ms}$ -BG by a group of automorphisms may not again be a  $f_{ms}$ -BG, we need a variation of the notion  $f_{ms}$ -BG. Therefore we define Brauer G-sets, which are generalizations of  $f_{ms}$ -BGs, and they are closed under quotients. Then we study the covering theory for Brauer G-sets, and compute the fundamental groups of a special class of Brauer G-sets, namely the modified Brauer graphs (abbr. modified BGs).

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For a  $f_{ms}$ -BG E, we first consider the modified BG  $E/\langle \sigma \rangle$ , which is the quotient of E by the group of automorphisms of E generated by the Nakayama automorphism  $\sigma$ . Using  $E/\langle \sigma \rangle$  we defined the reduced form  $R_E$  of E, which is a BG. We show that the  $f_{ms}$ -BGA  $A_E$  is representationfinite (resp. domestic) if and only if the BGA  $A_{R_E}$  is representation-finite (resp. domestic). Moreover, using the covering  $E \to E/\langle \sigma \rangle$  of Brauer G-sets, we calculate the fundamental group of E when  $A_E$  is representation-finite or domestic.

Finally, for a  $f_{ms}$ -BG E with  $A_E$  representation-finite or domestic, we construct E via the modified BG  $E/\langle \sigma \rangle$  using covering theory for Brauer G-sets, and determine the AR-components of  $A_E$ .

This paper is organized as follows. In Section 2 we introduce Brauer G-sets and study their covering theory; we define lines and bands for a Brauer G-set, and discuss the relations between the bands of a  $f_{ms}$ -BG and the bands of associated  $f_{ms}$ -BGA; we show that a  $f_{ms}$ -BGA  $A_E$ is representation-finite (resp. domestic) if and only if the BGA  $A_{R_E}$  of the reduced form  $R_E$ of  $E$  is representation-finite (resp. domestic) (see Theorem [2.29\)](#page-8-0). As a byproduct, we obtain some unexpected example of weakly symmetric  $f_{ms}$ -BGA which is not a BGA (Example [2.31\)](#page-9-0). In Section 3 we first calculate the fundamental groups of modified BGs using an analogy of Van Kampen theorem, and then together with covering theory for Brauer G-sets we calculate the fundamental groups of  $f_{ms}$ -BGs E with  $A_E$  representation-finite or domestic. In Section 4 we describe the Auslander-Reiten quivers (abbr. AR-quivers) of representation-finite  $f_{ms}$ -BGAs in terms of defining  $f_{ms}$ -BGs and show that these algebras coincide with basic representation-finite self-injective algebras of class  $A_n$ . In Section 5 we construction the defining  $f_{ms}$ -BGs of domestic  $f_{ms}$ -BGAs and describe their stable AR-components.

## DATA AVAILABILITY

The datasets generated during the current study are available from the corresponding author on reasonable request.

## 2. Brauer G-sets and covering theory

Throughout this paper we assume that k is a field.

## 2.1. Review on fractional Brauer graph of type MS.

**Definition 2.1.** (cf. [\[7,](#page-38-0) Section 3]) Let  $G = \langle g \rangle$  be an infinite cyclic group. A fractional Brauer configuration of type MS (abbr.  $f_{ms}$ -BC) is a quadruple  $E = (E, P, L, d)$ , where E is a G-set, P is a partition of E such that each class of P is a finite set, L is the partition of E given by  $L(e) = \{e\}$  for any  $e \in E$ , and  $d : E \to \mathbb{Z}_+$  is a function, such that

- if  $e_1$ ,  $e_2$  belong to same  $\langle g \rangle$ -orbit, then  $d(e_1) = d(e_2)$ ;
- $P(e_1) = P(e_2)$  if and only if  $P(g^{d(e_1)} \cdot e_1) = P(g^{d(e_2)} \cdot e_2)$ .

Moreover, if each class of P contains exactly two elements, then E is called a fractional Brauer graph of type MS (abbr.  $f_{ms}$ -BG).

Follows from [\[7,](#page-38-0) Remark 3.4], the elements of E are called angles, the  $\langle g \rangle$ -orbits of E are called vertices, the subsets of E of the form  $P(e)$  are called polygons (if  $P(e)$  contains two elements, we call  $P(e)$  an edge, and call an angle in  $P(e)$  a half-edge), and the function  $d: E \to \mathbb{Z}_+$  is called the degree function. If v is a vertex of E which is a finite set, define the f-degree  $d_f(v)$  of v as  $d(v)$  $\frac{d(v)}{|v|}$ ; E is called f-degree trivial if  $d_f(v) \equiv 1$ . The permutation  $\sigma: E \to E$ ,  $e \to g^{d(e)} \cdot e$  on E is called the Nakayama automorphism of E.

**Definition 2.2.** ([\[7,](#page-38-0) Definition 4.1 and Definition 4.4]) For a  $f_{ms}$ -BC  $E = (E, P, L, d)$ , the fractional Brauer configuration category in type MS (abbr.  $f_{ms}$ -BCC) associated with E is a kcategory  $\Lambda_E = kQ_E/I_E$ , where  $Q_E$  is a quiver defined as follows:  $(Q_E)_0 = \{P(e) \mid e \in E\}$ ,  $(Q_E)_1 = \{L(e) \mid e \in E\}$  with  $s(L(e)) = P(e)$  and  $t(L(e)) = P(g \cdot e)$ , and  $I_E$  is the ideal of path category  $kQ_E$  generated by the following relations:

- $L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e)L(e) L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h)L(h)$ , where  $P(e) = P(h)$ ;
- Paths of the form  $L(e_2)L(e_1)$  with  $g \cdot e_1 \neq e_2$ ;
- Paths of the form  $L(g^{n-1} \cdot e) \cdots L(g \cdot e) L(e)$  for  $n > d(e)$ .

Moreover, if E is a finite  $f_{ms}$ -BC, then we define  $A_E = \bigoplus_{x,y \in (Q_E)_0} \Lambda_E(x,y)$  (which is a finite dimensional k-algebra) and call  $A_E$  a fractional Brauer configuration algebra in type MS (abbr.  $f_{ms}$ -BCA).

According to [\[7,](#page-38-0) Proposition 6.5], if E is a  $f_{ms}$ -BC, then  $\Lambda_E$  is a locally bounded special multiserial Frobenius category, and if E is a  $f_{ms}$ -BG, then  $\Lambda_E$  is a locally bounded special biserial Frobenius category. Thus  $f_{ms}$ -BCAs and  $f_{ms}$ -BGAs are generalization of BCAs and BGAs respectively.

For the definitions of morphisms (coverings), walks and fundamental groups (groupoids) of  $f_{ms}$ -BCs, we refer to [\[8,](#page-38-5) Section 2].

2.2. Brauer G-set and fundamental group. In this subsection, we define Brauer G-sets and their fundamental groups.

Let  $E = (E, P, L, d)$  be a  $f_{ms}$ -BG. Since each edge  $P(e)$  of E contains 2 half-edges, we can define an involution  $\tau$  on E such that  $P(e) = \{e, \tau(e)\}\$ for every  $e \in E$ . Therefore a  $f_{ms}$ -BG can be considered as a triple  $E = (E, \tau, d)$ , where E is a G-set  $(G = \langle g \rangle \cong \mathbb{Z})$ ,  $\tau$  is an involution on E without fixed points, and  $d : E \to \mathbb{Z}_+$  is a function on E, such that

- if  $e_1, e_2 \in E$  belong to the same G-orbit, then  $d(e_1) = d(e_2)$ ;
- $g^{d(\tau(e))} \cdot \tau(e) = \tau(g^{d(e)} \cdot e)$  for every  $e \in E$ .

Moreover, if  $E = (E, P, L, d)$  is a  $f_{ms}$ -BC such that each polygon of E contains at most 2 elements, then we may regarded E as a quadruple  $(E, U, \tau, d)$ , where  $U = \{e \in E \mid |P(e)| = 2\}$  is a subset of E, and  $\tau$  is an involution on U without fixed points such that  $P(e) = \{e, \tau(e)\}\$ for every  $e \in U$ . The above discussion motivates us to introduce the following notion.

<span id="page-2-0"></span>**Definition 2.3.** A Brauer G-set is a quadruple  $E = (E, U, \tau, d)$ , where E is a G-set  $(G = \langle g \rangle \cong \langle g \rangle \cong \langle g \rangle$  $\mathbb{Z}$ ), U is a subset of E,  $\tau$  is an involution on U ( $\tau$  may have fixed points), and  $d: E \to \mathbb{Z}_+$  is a function on E, such that

 $(mf1) d(e_1) = d(e_2)$  if  $e_1, e_2$  belong to the same G-orbit;  $(mf2) \ \sigma(U) = U \ and \ \tau\sigma(e) = \sigma\tau(e) \ for \ every \ e \in U, \ where \ \sigma: E \to E, \ e \mapsto g^{d(e)} \cdot e.$ 

The elements of E are called half-edges of E; an element  $e \in U$  with  $\tau(e) = e$  is called a double half-edge of E; if  $e \in U$  and  $e \neq \tau(e)$ , then the subset  $\{e, \tau(e)\}\$  of E is called an edge; the function d is called degree function; the permutation  $\sigma: E \to E$ ,  $e \mapsto g^{d(e)} \cdot e$  on E is called Nakayama automorphism of E. Similar to fractional Brauer configuration, we have the concepts of vertex and f-degree for a Brauer G-set.

<span id="page-2-1"></span>**Example 2.4.** Let  $E = \{e, e'\}$  be a G-set with  $g \cdot e = e'$ ,  $g \cdot e' = e$ . Let  $U = E$ ,  $\tau = id_U$  be an involution on U, and  $d : E \to \mathbb{Z}_+$  be the function given by  $d(e) = d(e') = 2$ . Then  $(E, U, \tau, d)$  is a Brauer G-set, which is given by the diagram

$$
\frac{e}{\bullet} \bullet \frac{e'}{e'}.
$$

<span id="page-2-2"></span>Example 2.5. Let  $E = \{e, e', e_1, e_2\}$  be a G-set with  $g \cdot e = e_1, g \cdot e_1 = e', g \cdot e' = e_2, g \cdot e_2 = e$ . Let  $U = \{e, e', e_1\}, \tau$  be the involution on U given by  $\tau(e) = e'$ ,  $\tau(e') = e$ ,  $\tau(e_1) = e_1$ , and  $d : E \to \mathbb{Z}_+$ be the function given by  $d(e) = d(e') = d(e_1) = d(e_2) = 4$ . Then  $(E, U, \tau, d)$  is a Brauer G-set, which is given by the diagram



<span id="page-3-0"></span>**Remark 2.6.** According to the remarks before Definition [2.3,](#page-2-0) a  $f_{ms}$ -BG is identified with a Brauer G-set  $E = (E, U, \tau, d)$  such that  $U = E$  and  $\tau$  has no fixed points, and a  $f_{ms}$ -BC such that each polygon of it contains at most two elements is identified with a Brauer G-set  $E = (E, U, \tau, d)$  such that  $\tau$  has no fixed points. We shall frequently using these identifications.

**Definition 2.7.** A Brauer G-set  $E = (E, U, \tau, d)$  of integral f-degree with  $U = E$  is said to be a modified Brauer graph (abbr. modified BG).

Let  $E = (E, U, \tau, d)$  be a Brauer G-set. A walk of E is a sequence of the form

$$
w = e_n \frac{\delta_n}{e_{n-1}} e_{n-1} \frac{\delta_{n-1}}{\cdots} \frac{\delta_3}{e_2} e_1 \frac{\delta_2}{e_1} e_0,
$$

where  $e_i \in E$  for  $0 \le i \le n$  and  $\delta_j \in \{g, g^{-1}, \tau\}$  for  $1 \le j \le n$ , such that  $e_{i-1}, e_i \in U$  if  $\delta_i = \tau$  and

$$
e_i = \begin{cases} g \cdot e_{i-1}, & \text{if } \delta_i = g; \\ g^{-1} \cdot e_{i-1}, & \text{if } \delta_i = g^{-1}; \\ \tau(e_{i-1}), & \text{if } \delta_i = \tau. \end{cases}
$$

We may write  $w = (e_n|\delta_n \cdots \delta_1|e_0)$  or  $w = \delta_n \cdots \delta_1$  if there is no confusion. A Brauer G-set E is said to be connected if every two half-edges  $e_1, e_2$  of it can be connected by a walk.

**Remark 2.8.** For a  $f_{ms}$ -BC E, we have already defined the walks and the special walks in E in [\[8,](#page-38-5) Section 2]. Suppose that E is  $f_{ms}$ -BC such that each polygon of E contains at most 2 angles, by Remark [2.6,](#page-3-0) we can also view E as a Brauer G-set. Then every walk of the Brauer G-set E is a walk of the  $f_{ms}$ -BC E, and every special walk of the  $f_{ms}$ -BC E is a walk of E as a Brauer G-set. In particular, E is connected as a f-BC if and only if E is connected as a Brauer G-set.

**Definition 2.9.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set. Define the homotopy relation  $\approx$  on the set of walks of E as the equivalence relation generated by

- $(mh1)$   $(e|g^{-1}g|e) \approx (e|gg^{-1}|e) \approx (e|\tau^2|e) \approx (e||e)$  for every  $e \in E$ .
- $(mh2)$   $(g^{d(\tau(e))}\cdot \tau(e)|\tau|g^{d(e)}\cdot e)(g^{d(e)}\cdot e|g^{d(e)}|e) \approx (g^{d(\tau(e))}\cdot \tau(e)|g^{d(\tau(e))}|\tau(e))(\tau(e)|\tau|e)$  for every  $e \in U$ .

 $(mh3)$  If  $w_1 \approx w_2$ , then  $uw_1 \approx uw_2$  and  $w_1v \approx w_2v$  whenever the compositions make sense.

For a walk w of a Brauer G-set E, define  $[w] := \{\text{walks } v \text{ such that } v \approx w\}.$ 

In [\[8,](#page-38-5) Subsection 2.4] we have already define the homotopy relation ∼ on the set of walks of a f-BC E. Note that if E is a  $f_{ms}$ -BC E such that each polygon of E contains at most 2 angles, then for walks  $w_1, w_2$  of E as a Brauer G-set,  $w_1 \approx w_2$  implies  $w_1 \sim w_2$ .

Similar to f-BC, we can define the fundamental group  $\Pi_m(E, e)$  (resp. fundamental groupoid  $\Pi_m(E, A)$  of a Brauer G-set E at  $e \in E$  (resp. on a subset  $A \subseteq E$ ), using the homotopy relation ≈.

**Example 2.10.** Let  $E = (E, U, \tau, d)$  be the Brauer G-set in Example [2.4,](#page-2-1) and let  $A = \{e, e'\} = E$ . Then  $\Pi_m(E, A)$  is isomorphic to  $\mathscr{F}/\langle c^2 = 1_x, d^2 = 1_y, bac = cba, abd = dab\rangle$ , where  $\mathscr{F}$  is the fundamental groupoid of the quiver



**Example 2.11.** Let  $E = (E, U, \tau, d)$  be the Brauer G-set in Example [2.5.](#page-2-2) Then by Lemma [3.1](#page-10-0) and Lemma [3.3,](#page-12-0)  $\Pi_m(E, e) \cong F\langle x, y, z \rangle / \langle xy = yx, xz = zx, z^2 = 1 \rangle.$ 

In  $[8,$  Subsection 2.2] we have already define the fundamental group (groupoid) of a f-BC E. If E is a  $f_{ms}$ -BC such that each polygon of E contains at most 2 angles, then by Remark [2.6](#page-3-0) we can view it as a Brauer G-set, and we also have the notion of fundamental group (groupoid). The following Lemma shows that these two fundamental groups are isomorphic.

<span id="page-4-1"></span>**Lemma 2.12.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set such that  $\tau$  has no fixed points

(equivalently, E is a  $f_{ms}$ -BC such that each polygon of E contains at most 2 angles), and let A be a subset of E. Denote by  $\Pi_m(E, A)$  and  $\Pi(E, A)$  the fundamental groupoid of E on A as a Brauer G-set and as a  $f_{ms}$ -BC respectively. Then  $\Pi_m(E, A)$  is isomorphic to  $\Pi(E, A)$ .

*Proof.* In the following, when we say a walk of  $E$ , we always view  $E$  as a Brauer  $G$ -set unless otherwise stated. First we need to show the following fact: If  $w_1, w_2$  are two walks of E with  $w_1 \sim w_2$ , then  $w_1 \approx w_2$ . It can be shown that for each walk w of E, there exists some special walk v and some integer n such that  $w \approx (t(w)|g^{nd(t(w))}|t(v))v$ . For two walks  $w_1, w_2$  of E with  $w_1 \sim w_2$ , let  $w_i \approx (t(w_i)|g_i^{n_i d(t(w_i))}|t(v_i))v_i$   $(i = 1, 2)$ , where  $v_i$  is a special walk of E and  $n_i$  is an integer. Then  $(t(w_1)|g^{n_1d(t(w_1))}|t(v_1))v_1 \sim (t(w_2)|g^{n_2d(t(w_2))}|t(v_2))v_2$ , and by [\[8,](#page-38-5) Proposition 2.49] we have  $(v_1, n_1) = (v_2, n_2)$ . Then  $w_1 \approx (t(w_1)|g^{n_1 d(t(w_1))}|t(v_1))v_1 = (t(w_2)|g^{n_2 d(t(w_2))}|t(v_2))v_2 \approx w_2$ .

For each walk w of E as a  $f_{ms}$ -BC, choose a walk w' of E such that  $w \sim w'$  (by [\[8,](#page-38-5) Proposition 2.49] this can be done). For every  $a, b \in A$ , define a map  $F : \Pi(E, A)(a, b) \to \Pi_m(E, A)(a, b)$ ,  $\overline{w} \mapsto [w']$ . If  $w_1, w_2$  are two walks of E (as a  $f_{ms}$ -BC) with  $w_1 \sim w_2$ , then  $w'_1 \sim w'_2$  and  $w'_1 \approx w'_2$ . So  $F$  is well defined. Same argument shows that  $F$  does not depend on the choice of the walk  $w'$ for each walk w of E as a  $f_{ms}$ -BC.

For  $\overline{w_1} \in \Pi(E, A)(a, b)$  and  $\overline{w_2} \in \Pi(E, A)(b, c)$ , since  $w_2'w_1'$  is a walk of E such that  $w_2w_1 \sim$  $w_2'w_1'$ ,  $F(\overline{w_2w_1}) = [w_2'w_1'] = F(\overline{w_2})F(\overline{w_1})$ . Therefore F becomes a functor from  $\Pi(E, A)$  to  $\Pi_m(E, A)$ . If  $F(\overline{w_1}) = F(\overline{w_2})$ , then  $w'_1 \approx w'_2$ . Therefore  $w_1 \sim w_2$  and  $\overline{w_1} = \overline{w_2}$ , so F is faithful. For every morphism  $[w] \in \Pi_m(E, A)(a, b)$ , since w and w' are walks with  $w \sim w'$ , we have  $w \approx w'$ . Therefore  $F(\overline{w}) = [w'] = [w]$  and F is dense. Since F induces identity map on objects, F is an isomorphism.

2.3. Covering theory for Brauer G-sets. In this subsection, we define morphisms and coverings between Brauer G-sets; we compare different coverings between Brauer G-sets using their fundamental groups; we define special walks on Brauer G-sets and construct the universal cover of Brauer G-sets.

**Definition 2.13.** Let  $E = (E, U, \tau, d)$  and  $E' = (E', U', \tau', d')$  be Brauer G-sets. A morphism (resp. covering)  $f : E \to E'$  of Brauer G-sets is a morphism of  $\langle q \rangle$ -sets satisfying the following conditions  $(1), (2)$  and  $(3)$  (resp.  $(1'), (2)$  and  $(3)$ ) below:

- (1) For every  $e \in U$ , we have  $f(e) \in U'$ ;
- (1') For every  $e \in E$ ,  $e \in U$  if and only if  $f(e) \in U'$ ;
- (2)  $f(\tau(e)) = \tau'(f(e))$  for every  $e \in U$ ;
- (3)  $d'(f(e)) = d(e)$  for every  $e \in E$ .
- **Remark 2.14.** (1) Let E and E' as above. If E and E' are  $f_{ms}$ -BCs such that each polygon contain at most 2 angles (that is,  $\tau$  and  $\tau'$  has no fixed points), then for any map  $f : E \to$ E', f is a covering of Brauer G-sets if and only if f is a covering of f-BCs.
	- (2) A morphism  $f: E \to E'$  of Brauer G-sets maps each walk of E to a walk of E'. Moreover, if  $f: E \to E'$  is a covering of Brauer G-sets, then for every walk  $w' = (h'|\delta_n \cdots \delta_1|e')$  of  $E'$  and for every  $e \in E$  (resp.  $h \in E$ ) which lies over  $e'$  (resp.  $h'$ ), there exists a unique walk  $w$  of  $E$  which lies over  $w'$  whose source (resp. terminal) is e (resp. h).

<span id="page-4-0"></span>For Brauer G-set we have following proposition, which is an analogy to f-BC case.

**Proposition 2.15.** Let  $f : E \to E'$  be a covering of Brauer G-sets, u, v be two walks of E with  $s(u) = s(v)$  or  $t(u) = t(v)$ . Then  $u \approx v$  if and only if  $f(u) \approx f(v)$ .

Therefore each covering  $f : E \to E'$  of Brauer G-sets induces an injective map  $f_* : \Pi_m(E, e) \to$  $\Pi_m(E', f(e))$  of associated fundamental groups for each  $e \in E$ .

<span id="page-4-2"></span>Remark 2.16. Proposition [2.15](#page-4-0) suggests a general method to calculate the fundamental group of given Brauer G-set. Let  $f : E \to E'$  be a covering of Brauer G-sets and  $e' \in E'$ . Then  $f^{-1}(e')$ 

becomes a  $\Pi_m(E', e')$ -set: for any  $e \in f^{-1}(e')$  and  $[w'] \in \Pi_m(E', e')$ ,  $[w'] \cdot e$  is defined as the terminal of w, where w is the walk of E lying over w' with  $s(w) = e$  (Proposition [2.15](#page-4-0) ensures that this group action is well-defined). Then for each  $e \in f^{-1}(e')$ , the stabilizer subgroup of e in  $\Pi_m(E',e')$  equals to the image of  $f_*:\Pi_m(E,e)\to \Pi_m(E',e')$ , which is isomorphic to  $\Pi_m(E,e)$ .

<span id="page-5-1"></span>By analogy with [\[8,](#page-38-5) Proposition 2.28], we have

**Proposition 2.17.** Let E,  $E_1$ ,  $E_2$  be Brauer G-sets with  $E_1$  connected, and let  $f_1 : E_1 \rightarrow E$  and  $f_2: E_2 \to E$  be coverings of Brauer G-sets. For  $e_i \in E_i$   $(i = 1, 2)$  with  $f_1(e_1) = f_2(e_2)$ , there exists a covering  $\phi: E_1 \to E_2$  of Brauer G-sets such that  $f_1 = f_2 \phi$  and  $\phi(e_1) = e_2$  if and only if  $f_{1*}(\Pi_m(E_1, e_1)) \subseteq f_{2*}(\Pi_m(E_2, e_2))$ . Moreover, if such  $\phi$  exists, then it is unique.

Let  $E = (E, U, \tau, d)$  be a Brauer G-set and  $\Pi$  be a group of automorphism of E. Similar to the case of f-BC, we may define a quadruple  $E/\Pi = (E/\Pi, U', \tau', d')$  as follows:  $E/\Pi$  is the G-set of  $\Pi$ -orbit of E;  $U' := \{ [e] \in E/\Pi \mid e \in U \}$  is a subset of  $E/\Pi$ ; τ' is an involution of U' given by  $\tau'([e]) = [\tau(e)]$  for every  $[e] \in U'; d' : E/\Pi \to \mathbb{Z}_+$  is a function on  $E/\Pi$  given by  $d'([e]) = d(e)$  (we denote [e] the  $\Pi$ -orbit of  $e \in E$ ).

The following Lemma is an analogy of [\[8,](#page-38-5) Lemma 2.39] and Lemma [\[8,](#page-38-5) Lemma 2.40], and its proof is straightforward. Note that here the condition that  $\Pi$  acts admissibly on E is not needed.

<span id="page-5-0"></span>**Lemma 2.18.**  $E/\Pi = (E/\Pi, U', \tau', d')$  is a Brauer G-set, and the natural projection  $p : E \to E/\Pi$ is a covering of Brauer G-sets.

By analogy with [\[8,](#page-38-5) Subsection 2.4], we construct the universal cover of a Brauer G-set.

Let  $E = (E, U, \tau, d)$  be a Brauer G-set such that  $\tau$  has a fixed point. Define a Brauer G-set  $\widehat{E} = (\widehat{E}, \widehat{U}, \widehat{\tau}, \widehat{d})$  as follows:  $\widehat{E} = E_1 \sqcup E_2$  as a  $\langle g \rangle$ -set with  $E_1 = E_2 = E$ , and denote the element  $e \in E$  by  $e_i$  if we consider it as an element of  $E_i$ . Define  $U = U_1 \sqcup U_2$  as a subset of E, where  $U_i$ denotes the subset U of  $E_i$  for  $i = 1, 2$ , and for each  $e \in U$  and  $i = 1, 2$ , define

$$
\widehat{\tau}(e_i) = \begin{cases} \tau(e)_i, & \text{if } \tau(e) \neq e; \\ e_{3-i}, & \text{if } \tau(e) = e. \end{cases}
$$

Define  $\hat{d}(e_i) = d(e)$ . Note that  $\hat{\tau}$  has no fixed points, that is,  $\hat{E}$  is a  $f_{ms}$ -BC such that each polygon of it contains at most 2 angles.

Let  $\phi : E \to E$  be the map which sends  $e_i$  to  $e_{3-i}$  for any  $e \in E$  and  $i = 1, 2$ . It is straightforward to show that  $\phi$  is a morphism of Brauer G-sets. Since  $\phi^2 = id$ ,  $\phi$  is an automorphism of  $\widehat{E}$ . Moreover,  $\widehat{E}/\langle \phi \rangle \cong E$  as Brauer G-sets. By Lemma [2.18,](#page-5-0) we have

<span id="page-5-2"></span>**Lemma 2.19.** The map  $\pi : E \to E$ ,  $e_i \mapsto e$  is a covering of Brauer G-sets.

Let  $E = (E, U, \tau, d)$  be a Brauer G-set, a walk w of E is called special if it is of the form

$$
(g^{i_k} \cdot e_k | g^{i_k} | e_k) (e_k | \tau | g^{i_{k-1}} \cdot e_{k-1}) (g^{i_{k-1}} \cdot e_{k-1} | g^{i_{k-1}} | e_{k-1}) (e_{k-1} | \tau | g^{i_{k-2}} \cdot e_{k-2}) \cdots
$$
  

$$
(e_2 | \tau | g^{i_1} \cdot e_1) (g^{i_1} \cdot e_1 | g^{i_1} | e_1) (e_1 | \tau | g^{i_0} \cdot e_0) (g^{i_0} \cdot e_0 | g^{i_0} | e_0),
$$

where  $0 \leq i_0 < d(e_0)$ ,  $0 \leq i_k < d(e_k)$ , and  $0 < i_l < d(e_l)$  for all  $1 \leq l \leq k-1$ . Note that if  $\tau$  has no fixed points (that is,  $E$  is a  $f_{ms}$ -BC such that each polygon of it contains at most 2 angles), then a special walk of E is just a special walk of E as a  $f_{ms}$ -BC.

For a Brauer G-set  $E = (E, U, \tau, d)$  and for  $e \in E$ , similar to the case of  $f_{ms}$ -BC, we can define a connected Brauer G-set  $B_{(E,e)} = (B_{(E,e)}, U_e, \tau_e, d_e)$  such that  $\tau_e$  has no fixed points (that is,  $B_{(E,e)}$  is a  $f_{ms}$ -BC such that each polygon of  $B_{(E,e)}$  contains at most two angles) as follows:  $B_{(E,e)} = \{\text{special walks of } E \text{ starting at } e\}.$  The action of  $\langle g \rangle$  on  $B_{(E,e)}$  is given by

$$
g \cdot w = \begin{cases} g^{i_k+1} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0}, & \text{if } w = g^{i_k} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0} \text{ with } i_k < d(t(w)) - 1; \\ \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0}, & \text{if } w = g^{i_k} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0} \text{ with } i_k = d(t(w)) - 1. \end{cases}
$$

The subset  $U_e$  of  $B_{(E,e)}$  is given by  $U_e = \{w \in B_e \mid t(w) \in U\}$ , the involution  $\tau_e$  of  $U_e$  is given by

$$
\tau_e(w) = \begin{cases}\n(\tau(t(w)) | \tau | t(w)) w, & \text{if } w = g^{i_k} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0} \text{ with } i_k > 0; \\
g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0}, & \text{if } w = g^{i_k} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0} \text{ with } i_k = 0,\n\end{cases}
$$

and the degree function  $d_e$  is given by  $d_e(w) = d(t(w))$ . Note that  $B_{(E,e)}$  is f-degree trivial.

The Brauer G-set  $\mathbb{Z}B_{(E,e)}$  can also be defined similarly as in [\[8,](#page-38-5) Subsection 2.4], whose involution also has no fixed points. Note that when the involution  $\tau$  of E has no fixed points (that is, E is a  $f_{ms}$ -BC such that each polygon of E contains at most two angles), the  $f_{ms}$ -BC  $B_{(E,e)}$  (resp.  $\mathbb{Z}B_{(E,e)}$  defined as above is just the  $f_{ms}$ -BC  $B_{(E,e)}$  (resp.  $\mathbb{Z}B_{(E,e)}$ ) defined in [\[8,](#page-38-5) Subsection 2.4].

<span id="page-6-0"></span>**Proposition 2.20.** There exists a covering of Brauer G-sets  $q : \mathbb{Z}B_{(E,e)} \to E$ ,  $(w, n) \mapsto \sigma^n(t(w))$ , which is universal in the sense of [\[8,](#page-38-5) Corollary 2.30].

*Proof.* It is straightforward to show that q is a covering of Brauer G-sets. By Proposition [2.17,](#page-5-1) it suffices to show that  $\Pi_m(\mathbb{Z}B_{(E,e)}) = \{1\}$ . When the involution  $\tau$  of E has no fixed points, it follows from Lemma [2.12](#page-4-1) and [\[8,](#page-38-5) Proposition 2.47] that  $\Pi_m(\mathbb{Z}B_{(E,e)}) = \{1\}.$ 

When the involution  $\tau$  of E has a fixed point, let  $\pi : \widehat{E} \to E$  be the covering of Brauer G-sets in Lemma [2.19.](#page-5-2) Choose some  $x \in \widehat{E}$  with  $\pi(x) = e$ . Since  $\pi$  is a covering of Brauer G-sets,  $\pi$ maps special walks of  $\widehat{E}$  starting at x bijectively onto special walks of E starting at e. Therefore  $\pi$  induces an isomorphism between  $B_{(\widehat{E},x)}$  and  $B_{(E,e)}$ . Since E is a Brauer G-set whose involution has no fixed points, we have  $\Pi_m(\mathbb{Z}B_{(\widehat{E},x)}) = \{1\}$ . Therefore  $\Pi_m(\mathbb{Z}B_{(E,e)}) = \{1\}$ .

In general  $B_{(E,e)}$  is a finite or infinite tree for any  $f_{ms}$ -BG E, here is an example:

**Example 2.21.** Let E be the Brauer G-set in Example [2.4.](#page-2-1) Then  $B_{(E,e)}$  is an infinite tree with trivial f-degree, which is given by the diagram



<span id="page-6-1"></span>The following proposition is an analogy of [\[8,](#page-38-5) Proposition 2.49].

**Proposition 2.22.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set, w be a walk of E. Then there exists a unique special walk v and a unique integer n such that  $w \approx (t(w)|g^{nd(t(w))}|t(v))v$ .

*Proof.* It is straightforward to show that each walk of  $E$  is homotopic to a special walk. The rest of the proof is similar to that of Proposition [\[8,](#page-38-5) Proposition 2.49], using the covering  $q : \mathbb{Z}B_{(E,e)} \to E$ ,  $(u, n) \mapsto \sigma^n(t(u))$  of Brauer G-sets in Proposition [2.20,](#page-6-0) where  $e = s(w)$ .

2.4. Lines and bands. In this subsection, we define lines and bands on Brauer G-sets, and compare the numbers of equivalence classes of bands via a covering between Brauer G-sets. Moreover, we define the reduced form  $R_E$  (which is a BG) of a finite connected  $f_{ms}$ -BG E, and show that the  $f_{ms}$ -BGA  $A_E$  is representation-finite (resp. domestic) if and only if the BGA  $A_{R_E}$  is representation-finite (resp. domestic).

Let  $E = (E, P, L, d)$  be a finite  $f_{ms}$ -BG such that each edge of E contains a half-edge e with  $d(e) > 1$ . According to [\[7,](#page-38-0) Section 6],  $A_E \cong kQ'_E/I'_E$  with  $I'_E$  admissible, where  $Q_E$  is the subquiver of  $Q_E$  given by  $(Q'_E)_0 = (Q_E)_0$  and  $(Q'_E)_1 = \{L(e) | e \in E \text{ with } d(e) > 1\}$ , and  $I'_E$  is generated by the following three types of relations:

 $(fR1') L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e) L(e) - L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h) L(h)$ , where  $e, h \in E$  and  $d(e), d(h) > 1$ ;  $(fR2') L(e_1)L(e_2)$ , where  $e_1, e_2 \in E$ ,  $d(e_1), d(e_2) > 1$ , and  $e_1 \neq g \cdot e_2$ ; (fR3')  $L(g^{d(e)} \cdot e) \cdots L(g \cdot e) L(e)$ , where  $e \in E$  and  $d(e) > 1$ .

Therefore the string algebra  $A_E/\text{soc}(A_E)$  is given by the quiver  $Q'_E$  and the admissible ideal  $I''_E$ % of  $kQ_E'$  generated by the following two types of relations

- (a)  $L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e) L(e)$ , where  $e \in E$  and  $d(e) > 1$ ;
- (b)  $L(e_1)L(e_2)$ , where  $e_1, e_2 \in E$ ,  $d(e), d(h) > 1$ , and  $e_1 \neq g \cdot e_2$ .

We call b a band of  $A_E$  if b is a band of the string algebra  $A_E/\text{soc}(A_E) \cong kQ'_E/I''_E$  (for the definition of bands of a string algebra, see [\[3,](#page-38-6) II.2]).

**Definition 2.23.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set. A line of E is an infinite sequence

$$
l = \cdots \frac{\delta_3}{2} e_2 \frac{\delta_2}{2} e_1 \frac{\delta_1}{2} e_0 \frac{\delta_0}{2} e_{-1} \frac{\delta_{-1}}{2} e_{-2} \frac{\delta_{-2}}{2} \cdots,
$$

where  $e_i \in E$  and  $\delta_i \in \{g, g^{-1}, \tau\}$  for every  $i \in \mathbb{Z}$ , such that

- (a)  $e_{i-1}, e_i \in U$  if  $\delta_i = \tau$ ;
- (b)  $e_{i+1} = \delta_{i+1}(e_i)$  for every  $i \in \mathbb{Z}$ .

We will also write a line l as a family  $\{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$ . For such a line l and for any integer n, denote  $l[n]$  the line  $\{(e'_i, \delta'_i)\}_{i \in \mathbb{Z}}$ , where  $e'_i = e_{i+n}$  and  $\delta'_i = \delta_{i+n}$ . We call  $l[n]$  the *n*-th translate of l. We may consider a line  $l = \{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$  as a walk in E of infinite length. Moreover, define  $l^{-1}$ as the line  $\{(e_i'', \delta_i'')\}_{i \in \mathbb{Z}}$ , where  $e_i'' = e_{-i}$  and

$$
\delta_i'' = \begin{cases} g, \text{ if } \delta_{1-i} = g^{-1}; \\ g^{-1}, \text{ if } \delta_{1-i} = g; \\ \tau, \text{ if } \delta_{1-i} = \tau. \end{cases}
$$

Call the line  $l^{-1}$  the inverse of l.

**Definition 2.24.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set and l be a line of E such that  $l[n] = l$ for some positive integer n. Then l is called a band of  $E$  if it is of the form (consider l as a walk of infinite length)

$$
\cdots (e_2|\tau|g^{-l_1} \cdot h_1)(g^{-l_1} \cdot h_1|g^{-l_1}|h_1)(h_1|\tau|g^{k_1} \cdot e_1)(g^{k_1} \cdot e_1|g^{k_1}|e_1)
$$
  

$$
(e_1|\tau|g^{-l_0} \cdot h_0)(g^{-l_0} \cdot h_0|g^{-l_0}|h_0)(h_0|\tau|g^{k_0} \cdot e_0)(g^{k_0} \cdot e_0|g^{k_0}|e_0) \cdots,
$$

where  $0 < k_i < d(e_i)$  and  $0 < l_i < d(h_i)$  for all  $i \in \mathbb{Z}$ .

**Definition 2.25.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set. Define an equivalence relation ∼ on the set of all bands of  $E: ~ ∼$  is generated by

(a) 
$$
l \sim l[i]
$$
 for any integer *i*;  
(b)  $l \sim l^{-1}$ .

<span id="page-7-0"></span>For every band l of E, denote [l] the equivalence class of bands of E containing l.

**Lemma 2.26.** If  $E = (E, P, L, d)$  is a finite  $f_{ms}$ -BG such that each edge of E contains a half-edge e with  $d(e) > 1$ , then there exists a bijection between the set of equivalence classes of bands of E and the set of equivalence classes of bands of  $A_E$ .

*Proof.* For any band  $l = \{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$  of E, let n be the smallest positive integer such that  $l = l[n]$ . Then  $w = (e_n|\delta_n \cdots \delta_1|e_0)$  is a closed walk of E, which induces a closed walk  $\phi(l)$  of the quiver  $Q'_{E}$ . It can be shown that  $\phi(l)$  is a band of  $A_{E}$ . Moreover, if a band l' of E is obtained from the band l of E by a translation (resp. by taking inverse), then the band  $\phi(l')$  of  $A_E$  is obtained form the band  $\phi(l)$  of  $A_E$  by a rotation (resp. by taking inverse), so  $\phi$  induces a map  $\phi$  from the set of equivalence classes of bands of E to the set of equivalence classes of bands of  $A_E$ .

If  $l_1, l_2$  are two bands of E such that  $\phi(l_1), \phi(l_2)$  are equivalent, then there exists a sequence of bands  $b_0 = \phi(l_1), b_1, \dots, b_{k-1}, b_k = \phi(l_2)$  of  $A_E$  such that for each  $1 \leq i \leq k, b_i$  is obtained from  $b_{i-1}$  by a rotation or by taking inverse. Therefore there exists a band  $l'_1$  of E which is equivalent to  $l_1$  such that  $\phi(l'_1) = \phi(l_2)$ . It can be shown that  $l'_1 = l_2$ . So  $l_1, l_2$  are equivalent and therefore  $\phi$  is injective.

If b is a band of  $A_E$ , then up to equivalence we may assume that b is of the form

$$
\alpha_{2r,1}^{-1}\cdots\alpha_{2r,n_{2r}}^{-1}\alpha_{2r-1,n_{2r-1}}\cdots\alpha_{2r-1,1}\cdots\alpha_{4,1}^{-1}\cdots\alpha_{4,n_4}^{-1}\alpha_{3,n_3}\cdots\alpha_{3,1}\alpha_{2,1}^{-1}\cdots\alpha_{2,n_2}^{-1}\alpha_{1,n_1}\cdots\alpha_{1,1},
$$

where each  $\alpha_{ij}$  is an arrow of  $Q'_E$ . Assume that  $\alpha_{i,1} = L(e_i)$  for each  $1 \leq i \leq 2r$ . Then b induces a closed walk

$$
w = (e_1|\tau|e_{2r})(e_{2r}|g^{-n_{2r}}|g^{n_{2r}} \cdot e_{2r})(g^{n_{2r}} \cdot e_{2r}|\tau|g^{n_{2r-1}} \cdot e_{2r-1})(g^{n_{2r-1}} \cdot e_{2r-1}|g^{n_{2r-1}}|e_{2r-1}) \cdots
$$
  

$$
(e_3|\tau|e_2)(e_2|g^{-n_2}|g^{n_2} \cdot e_2)(g^{n_2} \cdot e_2|\tau|g^{n_1} \cdot e_1)(g^{n_1} \cdot e_1|g^{n_1}|e_1)
$$

of E. Denote  $n = \sum_{i=1}^{2r} n_i + 2r$ , and let  $l = \{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$  be the band of E such that  $l[n] = l$  and  $(e_n|\delta_n\cdots\delta_1|e_0) = w$ . Then  $b = \phi(l)$ , which implies that  $\phi$  is also surjective.

For any Brauer G-set  $E$  such that the number of equivalence classes of bands of  $E$  is finite, denote  $N_E$  the number of equivalence classes of bands of  $E$ .

<span id="page-8-1"></span>**Proposition 2.27.** Let  $E = (E, U, \tau, d)$  be a Brauer G-set and let  $\Pi$  be a finite group of automorphisms of E of order n. Then the number of equivalence classes of bands of E is finite if and only if the number of equivalence classes of bands of  $E/\Pi$  is finite. In this case we have  $N_{E/\Pi} \leq N_E \leq n N_{E/\Pi}$ .

Proof. Let  $E/\Pi = (E/\Pi, U', \tau', d')$ . For every band  $l = \{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$  of E, denote  $\overline{l}$  the band  $\{([e_i], \delta'_i)\}_{i\in\mathbb{Z}}$  of  $E/\Pi$ , where  $[e_i]$  is the  $\Pi$ -orbit of  $e_i$  and

$$
\delta_i' = \begin{cases} \delta_i, & \text{if } \delta_i = g \text{ or } \delta_i = g^{-1}; \\ \tau', & \text{if } \delta_i = \tau \end{cases}
$$

for each  $i \in \mathbb{Z}$ . Since  $\overline{l^{-1}} = \overline{l}^{-1}$  and  $\overline{l[i]} = \overline{l}[i]$  for any integer i, the map  $l \mapsto \overline{l}$  defines a map f from the set of equivalence classes of bands of E to the set of equivalence classes of bands of  $E/\Pi$ . For every band  $l' = \{(e'_i, \delta'_i)\}_{i \in \mathbb{Z}}$  of  $E/\Pi$  and for every  $e_0 \in E$  with  $[e_0] = e'_0$ , since the natural projection  $E \to E/\Pi$  is a covering, there exists a unique band  $l = \{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$  of E such that  $\overline{l} = l'$ . Therefore f is surjective.

For every band  $l' = \{(e'_i, \delta'_i)\}_{i \in \mathbb{Z}}$  of  $E/\Pi$ , choose a band  $l = \{(e_i, \delta_i)\}_{i \in \mathbb{Z}}$  of E such that  $\overline{l} = l'$ . Suppose that  $f([b]) = [l']$  for some band b of E. Then  $\overline{b} \sim l'$ . Since for every band v of E, the operation  $v \mapsto \overline{v}$  commutes with translation and taking inverse, we imply that there exists a band c of E such that  $b \sim c$  and  $\overline{c} = l'$ . Let  $c = \{(h_i, \epsilon_i)\}_{i \in \mathbb{Z}}$ . Since  $[h_0] = e'_0 = [e_0]$ , there exists some  $\pi \in \Pi$  such that  $\pi(e_0) = h_0$ . Since  $\pi(l) = \{(\pi(e_i), \delta_i)\}_{i \in \mathbb{Z}}$  and  $c = \{(h_i, \epsilon_i)\}_{i \in \mathbb{Z}}$  are bands of E which satisfy  $\overline{\pi(l)} = l' = \overline{c}$  and  $\pi(e_0) = h_0$ , we have  $\pi(l) = c$ . Therefore  $[b] = [c] = [\pi(l)]$  and  $f^{-1}([l']) = \{[\mu(l)] \mid \mu \in \Pi\}.$  Then we have  $1 \leq |f^{-1}([l'])| \leq n$  for every band l' of  $E/\Pi$ , and the conclusion holds.

Let  $E = (E, E, \tau, d)$  be a  $f_{ms}$ -BG and  $\sigma : E \to E$ ,  $e \mapsto g^{d(e)} \cdot e$  be the Nakayama automorphism of E. Denote  $\langle \sigma \rangle$  the group of automorphisms of E generated by  $\sigma$ . The following lemma is straightforward.

**Lemma 2.28.** Let E and  $\sigma$  be as above. If  $\langle \sigma \rangle$  is admissible, that is, each  $\langle \sigma \rangle$ -orbit of E meets each edge of E in at most one half-edge, then  $E/\langle \sigma \rangle$  is a Brauer graph; if  $\langle \sigma \rangle$  is not admissible, then  $E/\langle \sigma \rangle$  is a modified BG which contains a double half-edge.

If  $\langle \sigma \rangle$  is admissible, then  $E/\langle \sigma \rangle$  is a Brauer graph, and we define the reduced form  $R_E$  of E to be  $E/\langle \sigma \rangle$ . If  $\langle \sigma \rangle$  is not admissible, then  $E/\langle \sigma \rangle$  is a modified BG which contains a double half-edge, and  $E/\langle \sigma \rangle$  is defined, which is a Brauer graph (see the paragraph after Lemma [2.18\)](#page-5-0). In this case, define the reduced form  $R_E$  of E to be  $E/\langle \sigma \rangle$ . (For an example of this construction, see Example [2.31](#page-9-0) below.) Note that if E is finite (resp. connected), then so does  $R_E$ .

<span id="page-8-0"></span>**Theorem 2.29.** Suppose that the field k is algebraically closed. Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG. Then  $A_E$  is representation-finite (resp. domestic) if and only if  $A_{R_E}$  is representation-finite (resp. domestic).

Proof. If E contains a half-edge  $\{e, \tau(e)\}$  with  $d(e) = d(\tau(e)) = 1$ , then it can be shown that  $E = \{e, g \cdot e, \dots, g^{n-1} \cdot e, \tau(e), g \cdot \tau(e), \dots, g^{n-1} \cdot \tau(e)\}\$  for some positive integer n, where  $g^n \cdot e = e$ ,  $g^n \cdot \tau(e) = \tau(e), \tau(g^i \cdot e) = g^i \cdot \tau(e)$  for  $0 \le i \le n-1$ , and the degree of each half-edge of E is equal to 1. Moreover,  $R_E$  is a Brauer tree with trivial f-degree given by the diagram

Both  $A_E$  and  $A_{R_E}$  are Nakayama algebras of Loewy length 2, so they are both representationfinite. Therefore we may assume that each edge of E contains a half-edge e with  $d(e) > 1$ .

.

For a self-injective special biserial algebra  $A$ , it is well known that  $A$  is representation-finite if and only if A has no bands. Moreover, according to [\[4,](#page-38-4) Theorem 2.1], A is domestic if and only if the number of equivalence classes of bands of A is a positive integer.

Since E is finite, the group  $\langle \sigma \rangle$  of automorphisms of E is finite. If  $\langle \sigma \rangle$  is admissible, then  $R_E = E/\langle \sigma \rangle$ ; if  $\langle \sigma \rangle$  is not admissible, then we have  $R_E = \widehat{E}/\langle \sigma \rangle$  and  $E/\langle \sigma \rangle \cong R_E/\langle \phi \rangle$ , where φ is the automorphism of  $R_E = (E/\langle \sigma \rangle) \sqcup (E/\langle \sigma \rangle)$  given by  $\phi(h_i) = h_{3-i}$  for every  $h \in E/\langle \sigma \rangle$ (see the paragraph before Lemma [2.19\)](#page-5-2). By Proposition [2.27,](#page-8-1) the number of equivalence classes of bands of E is finite (resp. zero) if and only if the number of equivalence classes of bands of  $R_E$ is finite (resp. zero). Since each edge of  $E$  (resp.  $R_E$ ) contains a half-edge whose degree is larger than 1, by Lemma [2.26,](#page-7-0) we imply that the number of equivalence classes of bands of  $A_E$  is finite (resp. zero) if and only if the number of equivalence classes of bands of  $A_{R_E}$  is finite (resp. zero). Therefore  $A_E$  is representation-finite (resp. domestic) if and only if  $A_{R_E}$  is representation-finite (resp. domestic).

Remark 2.30. Theorem [2.29](#page-8-0) gives an effective way to determine the representation type of a  $f_{ms}$ -BGA in terms of the reduced form of its defining  $f_{ms}$ -BG.

<span id="page-9-0"></span>**Example 2.31.** Let E be the  $f_{ms}$ -BG given by the diagram

where the f-degree of the unique vertex of E is  $\frac{1}{2}$  and the G-action is induced from the clockwise order on the half-edges around this vertex. Then  $\langle \sigma \rangle$  is not admissible and  $E/\langle \sigma \rangle$  is a Brauer G-set given in Example [2.4.](#page-2-1) Therefore  $R_E$  is a BG with trivial f-degree given by the diagram

 $\left( \begin{array}{c} \end{array} \right)$ 

According to Theorem [1.1,](#page-0-0)  $A_{R_E}$  is domestic, therefore  $A_E$  is also domestic. One of particularly interests is that  $A_{R_E}$  is symmetric but  $A_E$  is weakly symmetric with nonidentity Nakayama automorphism. Indeed,  $A_E = kQ_E/I_E$ , where  $Q_E$  is the quiver



and I<sub>E</sub> is generated by  $\alpha_4\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_4, \alpha_2\alpha_1 - \alpha_4\alpha_3, \alpha_3\alpha_2 - \alpha_1\alpha_4$ . The structure of indecomposable projective modules are as follows:





The Nakayama automorphism of  $A_E$  is induced by  $e_i \mapsto e_i$  for  $i = 1, 2, \alpha_i \mapsto \alpha_{i+2}$  for  $i \in \mathbb{Z}/4\mathbb{Z} =$  $\{1, 2, 3, 4\}$ . However,  $A_{R_E}$  has the same quiver with the following structure of indecomposable projective modules:



3. Fundamental groups of fractional Brauer graphs of type MS

In this section, we calculate the fundamental group of a finite connected  $f_{ms}$ -BG E with  $A_E$ representation-finite or domestic. Our method is as follows: we first calculate the fundamental groups of modified BGs, then we use the covering  $E \to E/(\sigma)$  of Brauer G-sets (here  $E/(\sigma)$  is a modified  $BG$ ) to reduce the calculation of the fundamental group of E to the calculation of the fundamental group of  $E/\langle \sigma \rangle$  by a method mentioned at the end of [\[8\]](#page-38-5).

## 3.1. The fundamental groups of modified BGs.

In this subsection we calculate the fundamental groups of modified BGs using an analogy of Van Kampen theorem for Brauer G-sets.

We denote  $F\langle x_1, x_2, \cdots, x_n \rangle$  the free group on the set  $\{x_1, x_2, \cdots, x_n\}$ . The following Lemma should be compared with [\[8,](#page-38-5) Lemma 5.6].

<span id="page-10-0"></span>**Lemma 3.1.** Let  $E = (E, E, \tau, d)$  be the modified BG given by the diagram



where E contains  $n = a + b$  double half-edges  $e_1, \dots, e_{a+b}$ , and the f-degree of the unique vertex of E is m. Then  $\Pi_m(E, e) \cong F\langle x, y, z_1, \cdots, z_n \rangle / \langle x^m y = y x^m, x^m z_i = z_i x^m (1 \leq i \leq n), z_i^2 = 1 (1 \leq i \leq n)$  $i \leq n$ ).

Proof. Define a group homomorphism  $f' : F\langle x, y, z_1, \dots, z_n \rangle \to \Pi_m(E, e)$  as follows:  $f'(x) =$  $[(e|g^{n+2}|e)], f'(y) = [(e|\tau g^{a+1}|e)],$  and

$$
f'(z_i) = \begin{cases} [(e|g^{-i}\tau g^i|e)], & \text{if } 1 \le i \le a; \\ [(e|g^{-i-1}\tau g^{i+1}|e)], & \text{if } a+1 \le i \le n. \end{cases}
$$

By imitating the proof of [\[8,](#page-38-5) Lemma 5.6], it can be shown that for every closed special walk  $w = (e|g^{i_k} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0} | e)$  of E at e, [w] belongs to the image of f' (by induction on the number of times that  $\tau$  appears in w). Then according to Proposition [2.22,](#page-6-1)  $f'$  is surjective. Moreover, it is straightforward to show that the kernel of  $f'$  contains the normal subgroup of

 $F\langle x,y,z_1,\dotsb,z_n\rangle$  generated by the relations  $x^m y = yx^m, x^m z_i = z_i x^m (1 \leq i \leq n), z_i^2 = 1(1 \leq i \leq n)$  $i \leq n$ . Therefore f' induces a surjective group homomorphism  $f : F\langle x, y, z_1, \dots, z_n \rangle / \langle x^m y \rangle =$  $yx^m, x^m z_i = z_i x^m (1 \leq i \leq n), z_i^2 = 1(1 \leq i \leq n) \rangle \rightarrow \Pi_m(E, e).$ 

We need to show that f is also injective. Note that each element of  $F\langle x, y, z_1, \dots, z_n \rangle / \langle x^m y =$  $yx^m, x^m z_i = z_i x^m (1 \leq i \leq n), z_i^2 = 1(1 \leq i \leq n)$  is of the form  $x^{lm} \delta_k^{l_k} \cdots \delta_1^{l_1}$ , where  $l \in \mathbb{Z}$ ,  $l_1, \dots, l_k \in \mathbb{Z} - \{0\}, \, \delta_i \in \{x, y, z_1, \dots, z_n\} \text{ for } 1 \le i \le k, \text{ such that }$ (1)  $\delta_{i-1} \neq \delta_i$  for  $1 < i \leq n$ ; (2) if  $\delta_i = x$ , then  $0 < l_i < m$ ;

(3) if  $\delta_i = z_r$  for some  $1 \le r \le n$ , then  $l_i = 1$ . If  $x^{lm}\delta_k^{l_k}\cdots\delta_1^{l_1} \in \ker(f)$ , according to Proposition [2.22,](#page-6-1) it is straightforward to show that  $k=l=0$ , therefore  $x^{lm}\delta_k^{l_k}\cdots\delta_1^{l_1}=1$ .

The following Lemma should be compared with [\[8,](#page-38-5) Lemma 5.8]. For the definition of the fundamental groupoid of a quiver, we refer to [\[8,](#page-38-5) Section 3].

<span id="page-11-0"></span>**Lemma 3.2.** Let  $E = (E, E, \tau, d)$  be the modified BG given by the diagram



where E contains  $a+b$  double half-edges  $e_1, \dots, e_a, e_{a+1}, \dots, e_{a+b}$ , and the f-degree of the vertex on the left (resp. right) is m (resp. n). Let  $A = \{e, h\}$  be a subset of E. Then the fundamental groupoid  $\Pi_m(E, A)$  is isomorphic to

$$
\mathscr{F}/\langle tx^m = u^n t, x^m z_i = z_i x^m, z_i^2 = 1_p (1 \le i \le a), u^n v_j = v_j u^n, v_j^2 = 1_q (1 \le j \le b)\rangle,
$$

$$
\sim \left(\bigvee_{p \in \bigvee_{i=1}^{\infty} a_i} \bigvee_{p \in \bigvee_{i=1}^{\infty} a_i} \bigvee_{q \in \bigvee_{i=1}^{\infty} a_i} \bigvee_{p \in \bigvee_{i=1}^{\infty} a_i} \
$$

where  $\mathscr F$  is the fundamental groupoid of the quiver

*Proof.* Define a morphism of groupoids  $F': \mathscr{F} \to \Pi_m(E, A)$  by setting  $F(p) = e$ ,  $F(q) = h$ ,  $F(x) = [(e|g^{a+1}|e)], F(u) = [(h|g^{b+1}|h)], F(t) = [(h|\tau|e)], F(z_i) = [(e|g^{-i}\tau g^i|e)] (1 \le i \le a),$  $F(v_j) = [(h|g^{-j}\tau g^j|h)] (1 \le j \le b)$ . We first need to show that  $F'$  is full. According to Proposition [2.22,](#page-6-1) it suffices to show that for every special walk w with  $s(w)$ ,  $t(w) \in A$ , [w] belongs to the image of  $F'$ .

 $\widetilde{z}_1$   $\widetilde{v}_1$ 

.

Let  $w = g^{i_k} \tau g^{i_{k-1}} \tau \cdots \tau g^{i_1} \tau g^{i_0}$  be a special walk of E with  $s(w), t(w) \in A$ . We will show that [w] belongs to the image of F' by induction on k, that is, the number of times that  $\tau$  appears in w. If  $k = 0$ , then  $w = (e|g^{r(a+1)}|e)$  or  $w = (h|g^{s(b+1)}|h)$ , where  $r, s \in \mathbb{Z}$ , so [w] belongs to the image of F'. Now suppose that  $k > 0$ . We may assume that w contains no subwalks of the form  $(h|\tau|e)$  or  $(e|\tau|h)$ , otherwise w can be factored as a composition of special subwalks whose sources and terminals belong to A with k smaller. Therefore we may assume that  $w$  is of the form

$$
(e|g^{i_k}|c_k)(c_k|\tau|c_k)(c_k|g^{i_{k-1}}|c_{k-1})(c_{k-1}|\tau|c_{k-1})\cdots (c_2|\tau|c_2)(c_2|g^{i_1}|c_1)(c_1|\tau|c_1)(c_1|g^{i_0}|e),
$$

where  $c_1, \dots, c_k \in \{e_1, \dots, e_a\}$  and  $0 < i_0, i_1, \dots, i_k < m(a+1)$ . It is straightforward to show that  $[w]$  belongs to the image of  $F'$ .

Let  $y = t^{-1}ut$  and  $z_{a+i} = t^{-1}v_it$  for  $1 \le i \le b$ . Then  $\mathscr{F}(p,p)$  is a free group generated by  $x, y, z_1, \dots z_{a+b}$ . Let  $f' : \mathscr{F}(p, p) \to \Pi_m(E, e)$  be the group homomorphism induced by F'. Since *F'* is full, *f'* is surjective. We have  $f'(x) = [(e|g^{a+1}|e)], f'(y) = [(e|\tau g^{b+1}\tau |e)],$  and

$$
f'(z_i) = \begin{cases} [(e|g^{-i}\tau g^i|e)], & \text{if } 1 \le i \le a; \\ [(e|\tau g^{-i+a}\tau g^{i-a}\tau|e)], & \text{if } a+1 \le i \le a+b. \end{cases}
$$

It is straightforward to show that the normal subgroup of

 $\mathscr{F}(p, p) = F\langle x, y, z_1, \cdots, z_{a+b} \rangle$  generated by relations  $x^m = y^n, x^m z_i = z_i x^m, z_i^2 = 1 (1 \le i \le a+b)$ is contained in the kernel of  $f'$ , therefore  $f'$  induces a surjective group homomorphism

$$
f: \mathscr{F}(p,p)/\langle x^m=y^n, x^m z_i=z_i x^m, z_i^2=1(1\leq i\leq a+b)\rangle \to \Pi_m(E,e).
$$

It can be shown directly that each element of  $\mathscr{F}(p,p)/\langle x^m=y^n,x^m z_i=z_i x^m, z_i^2=1(1\leq i\leq n)$  $a + b$ ) is of the form  $x^{lm} \delta_k^{l_k} \cdots \delta_1^{l_1}$ , where  $l \in \mathbb{Z}, l_1, \dots, l_k \in \mathbb{Z} - \{0\}$ ,  $\delta_i \in \{x, y, z_1, \cdots, z_{a+b}\}\$  for  $1 \leq i \leq k$ , such that

(1)  $\delta_{i-1} \neq \delta_i$  for  $1 < i \leq n$ ; (2) if  $\delta_i = x$ , then  $0 < l_i < m$ ; (3) if  $\delta_i = y$ , then  $0 < l_i < n$ ; (4) if  $\delta_i = z_r$  for some  $1 \le r \le a + b$ , then  $l_i = 1$ . If  $x^{lm}\delta_k^{l_k}\cdots\delta_1^{l_1}\in\ker(f)$ , according to Proposition [2.22,](#page-6-1) it is straightforward to show that  $k=l=0$ ,

therefore  $x^{lm}\delta_k^{l_k}\cdots\delta_1^{l_1}=1$ . Thus f is also injective.

Let  $\mathscr G$  be the groupoid

$$
\mathscr{F}/\langle tx^m = u^n t, x^m z_i = z_i x^m, z_i^2 = 1_p (1 \le i \le a), u^n v_j = v_j u^n, v_j^2 = 1_q (1 \le j \le b) \rangle.
$$

Then it is straightforward to show that F' induces a morphism of groupoids  $F: \mathscr{G} \to \Pi_m(E, A)$ . Since  $\mathscr{G}(p,p) = \mathscr{F}(p,p)/\langle x^m = y^n, x^m z_i = z_i x^m, z_i^2 = 1(1 \leq i \leq a+b)\rangle$ , and since F induces a group isomorphism  $f : \mathscr{G}(p, p) \to \Pi_m(E, e)$ , by [\[8,](#page-38-5) Lemma 5.7], F is an isomorphism of groupoids.  $\Box$ 

<span id="page-12-0"></span>The following Lemma is an analogy of [\[8,](#page-38-5) Lemma 5.3]. We omit the proof of it.

**Lemma 3.3.** Let  $E = (E, U, \tau, d)$  be a connected Brauer G-set. Let C be a subset of  $E - U$  such that for each  $e \in E$ ,  $e \in C$  if and only if  $g^{d(e)}(e) \in C$  (We denote  $g^n(h)$  the action of  $g^n$  on h for every  $h \in E$  and  $n \in \mathbb{Z}$ ). Let  $E' = E - C$  and assume that  $E' \neq \emptyset$ . Define a Brauer G-set structure  $(E', U, \tau, d')$  on  $E'$  as follows: the action of  $G = \langle g \rangle$  on  $E'$  is given by

 $g \cdot h =$  $\int g(h), \text{ if } g(h) \in E';$  $g^N(h)$ , if  $g(h) \notin E'$ , where N is the minimal positive integer such that  $g^N(h) \in E'$ ;

the degree function  $d'$  is given by

$$
d'(h) = d(h) - |\{i \mid 1 \le i \le d(h) - 1 \text{ and } g^{i}(h) \notin E'\}|.
$$

Then E' is a connected Brauer G-set, and the groupoids  $\Pi_m(E, E')$  and  $\Pi_m(E', E')$  are isomorphic.

Especially, if we choose  $C = E - U$ , then the fundamental groups of E' and E are isomorphic.

Let  $E = (E, U, \tau, d)$  be a Brauer G-set. A sub-Brauer G-set  $E' = (E', U', \tau', d')$  of E is a Brauer G-set such that E' is a sub-G-set of E and the inclusion  $E' \to E$  is a morphism of Brauer G-sets. That is, E' is a sub-G-set of E, U' is a subset of  $E' \cap U$  such that  $\sigma(U') = U'$  and  $\tau(U') = U'$  ( $\sigma$ is the Nakayama automorphism of E),  $\tau'$  is the restriction of  $\tau$  on U', and d' is the restriction of  $d$  on  $E'$ .

For a set of sub-Brauer G-sets  $\{E_\alpha = (E_\alpha, U_\alpha, \tau_\alpha, d_\alpha)\}\$  of E, define the union (resp. the intersection) of them as  $\cup_{\alpha} E_{\alpha} = (\cup_{\alpha} E_{\alpha}, \cup_{\alpha} U_{\alpha}, \tau', d')$  (resp.  $\cap_{\alpha} E_{\alpha} = (\cap_{\alpha} E_{\alpha}, \cap_{\alpha} U_{\alpha}, \tau'', d''))$ , where  $\tau'$ , (resp.  $\tau''$ ) is the restriction of  $\tau$  to  $\cup_{\alpha} U_{\alpha}$  (resp.  $\cap_{\alpha} U_{\alpha}$ ), and d' (resp. d'') is the restriction of d to  $\cup_{\alpha} E_{\alpha}$  (resp.  $\cap_{\alpha} E_{\alpha}$ ).

Similar to [\[8,](#page-38-5) Proposition 5.2], we have the following analogy of Van Kampen theorem, and we omit the proof of it. Note that in this proposition we do not require the family of sub-Brauer G-sets  ${E_{\alpha}}_{\alpha\in I}$  of E being admissible.

<span id="page-12-1"></span>**Proposition 3.4.** Let E be a Brauer G-set, which is the union of a family of sub-Brauer G-sets  ${E_{\alpha}}_{\alpha\in I}$  which is closed under finite intersections. Let A be a subset of  $\bigcap_{\alpha\in I} E_{\alpha}$  such that for each  $\alpha \in I$ , A meets each connected component of  $E_{\alpha}$ . Then the groupoid  $\Pi_{m}(E, A)$  is the direct limit of groupoids  $\Pi_m(E_\alpha, A)$ .

Now we can calculate the fundamental group of a Brauer G-set with integral f-degree (e.g. modified BGs).

<span id="page-12-2"></span>**Proposition 3.5.** Let  $E = (E, U, \tau, d)$  be a finite connected Brauer G-set of integral f-degree with n vertices  $v_1, \dots, v_n$ , k edges, and l double half-edges. Let  $d_i$  be the f-degree of  $v_i$  for each  $1 \leq i \leq n$ , and let  $r = k - n + 1$ . Then the fundamental group of E is isomorphic to

$$
F\langle a_1, \cdots, a_n, b_1, \cdots, b_r, c_1, \cdots, c_l \rangle / \langle a_1^{d_1} = \cdots = a_n^{d_n}, a_1^{d_1} b_i = b_i a_1^{d_1} \ (1 \le i \le r), a_1^{d_1} c_j = c_j a_1^{d_1}, c_j^2 = 1 \ (1 \le j \le l) \rangle.
$$

*Proof.* Note that if we take  $C = E - U$ , then the modified BG  $E' = (E', E', \tau, d')$  constructed in Lemma [3.3](#page-12-0) also has n vertices, k edges, l double half-edges, and the f-degree of each vertex of  $E'$ is equal to the f-degree of the corresponding vertex of  $E$ . Moreover, according to Lemma [3.3,](#page-12-0) the fundamental groups of E' and E are isomorphic. Therefore we may assume that  $E = U$ .

Let  $\{e_1, \tau(e_1)\}, \dots, \{e_k, \tau(e_k)\}\$ be all edges of E. For each  $1 \le i \le k$ , define a sub-Brauer G-set  $E_i = (E_i, U_i, \tau_i, d_i)$  of E as follows:  $E_i = E$  as  $\langle g \rangle$ -sets; the subset  $U_i$  of  $E_i$  is given by

$$
U_i = E - \bigcup_{1 \le j \le k, j \ne i} \{e_j, \tau(e_j)\};
$$

the involution  $\tau_i$  is the restriction of  $\tau$  on  $U_i$ , and the degree function  $d_i$  is equal to  $d$ . Let  $E' = (E', U', \tau', d')$  be the intersection of all the  $E_i$ 's. For each  $1 \leq i \leq n$ , choose  $h_i \in v_i$ , and let  $A = \{h_1, \dots, h_n\}$  be a subset of E. The family  $\{E', E_1, \dots, E_k\}$  of sub-Brauer G-sets is closed under finite intersections, and the union of them is  $E$ . Moreover,  $A$  meets each connected component of  $E'$  and each connected component of every  $E_i$ .

For each vertex  $v_j$  of E, let  $l_j$  be the number of double half-edges of E which belongs to  $v_j$ . For each  $1 \leq i \leq k$ , if the two half-edges  $e_i, \tau(e_i)$  belong to the same vertex  $v_j$ , then we denote



 $\alpha_{it}$ 

 $\vec{x_{it}}$ 

 $\alpha_{it}$ 

 $\vec{x_{it}}$ 

, and for each  $1 \leq t \leq n$  with  $l \neq j$ ,

 $\mathscr{F}_i$  the fundamental groupoid of quiver

denote  $\mathscr{F}_{it}$  the fundamental groupoid of quiver  $\gamma_{itl} \sim \gamma_{it1}$ . By Lemma [3.1](#page-10-0) and Lemma [3.3,](#page-12-0)  $\Pi_m(E_i, A)$  is isomorphic to the groupoid  $\Sigma_i = \bigsqcup_{1 \le t \le n, t \ne j} (\mathscr{F}_{it}/\langle \alpha_{it}^{d_t} \gamma_{itp} = \gamma_{itp} \alpha_{it}^{d_t}, \gamma_{itp}^2 = 1$  $(1 \leq p \leq l_t) \rangle \cup (\mathscr{F}_i/\langle \alpha_{ij}^{d_j} \beta_j = \beta_j \alpha_{ij}^{d_j}, \alpha_{ij}^{d_j} \gamma_{ijp} = \gamma_{ijp} \alpha_{ij}^{d_j}, \gamma_{itp}^2 = 1 \ (1 \leq p \leq l_i) \rangle).$  If the two half $d_{j, \Omega} = \beta_{j, \Omega} d_{j, \Omega} d_{j, \Omega} = \alpha_{j, \Omega} d_{j}$ edges  $e_i, \tau(e_i)$  belong to two different vertices  $v_{j_1}, v_{j_2}$  with  $j_1 < j_2$ , we denote  $\mathscr{F}_i$  the fundamental



groupoid of quiver

, and for each  $1 \leq t \leq n$  with  $t \neq j_1, j_2$ ,

denote  $\mathscr{F}_{it}$  the fundamental groupoid of quiver  $\gamma_{itl,t}$  $^{\gamma_{it1}}$  . By Lemma [3.2](#page-11-0) and Lemma [3.3,](#page-12-0)  $\Pi_m(E_i, A)$  is isomorphic to the groupoid  $\Sigma_i = \bigsqcup_{1 \le t \le n, t \ne j_1, j_2} (\mathscr{F}_{it}/\langle \alpha_{it}^{d_t} \gamma_{itp} = \gamma_{itp} \alpha_{it}^{d_t}, \gamma_{itp}^2 = 1$  $(1 \leq p \leq l_t)\rangle) \sqcup (\mathscr{F}_i/\langle \beta_i \alpha_{ij_1}^{d_{j_1}}$  $\frac{d_{j_1}}{i j_1} = \alpha_{ij_2}^{d_{j_2}}$  $\frac{d_{j_2}}{i j_2} \beta_i, \, \alpha_{ij_q}^{d_{j_q}}$  $\frac{d_{j_q}}{i j_q} \gamma_{i j_q p} = \gamma_{i j_q p} \alpha^{d_{j_q}}_{i j_q}$  $\sum_{i,j_q}^{a_{j_q}}$ ,  $\gamma_{i,j_q p}^2 = 1$   $(q = 1, 2, 1 \le p \le l_{j_q})\rangle$ .



For each  $1 \leq j \leq n$ , let  $\mathscr{G}_j$  be the fundamental groupoid of quiver . It is straightforward to show  $\Pi_m(E', A)$  is isomorphic to  $\Sigma' = \bigsqcup_{1 \leq j \leq n} \mathscr{G}_j / \langle \alpha_j^{d_j} \rangle$  $_{j}^{d_{j}}\gamma _{jp}=\gamma _{jp}\alpha _{j}^{d_{j}}$  $j^a_j, \; \gamma_{jp}^2 = 1$  $(1 \leq p \leq l_j)$ . The direct system  $\{\Pi_m(E', A) \to \Pi_m(E_i, A)\}_{1 \leq i \leq k}$  is isomorphic to the direct system  $\{\mu_i : \Sigma' \to \Sigma_i\}_{1 \leq i \leq k}$ , where  $\mu_i$  is defined by  $\mu_i(x_j) = x_{ij}$ ,  $\overline{\mu_i}(\alpha_j) = \alpha_{ij}$ ,  $\mu_i(\gamma_{jp}) = \gamma_{ijp}$  for  $1 \leq j \leq n, 1 \leq p \leq l_j.$ 

Define a quiver Q as follows:  $Q_0 = \{v_1, \dots, v_n\}$ ,  $Q_1 = \{\alpha'_j, \beta'_i, \gamma'_{jp} \mid 1 \le j \le n, 1 \le i \le k\}$  $1 \leq p \leq l_j$ ; define  $s(\alpha'_j) = t(\alpha'_j) = s(\gamma'_{jp}) = t(\gamma'_{jp}) = v_j$  for  $1 \leq j \leq n$  and  $1 \leq p \leq l_j$ ; for  $1 \leq i \leq k$ , if the two half-edges  $e_i, \tau(e_i)$  belong to the same vertex  $v_j$ , define  $s(\beta'_i) = t(\beta'_i) = v_j$ ; if the two half-edges  $e_i, \tau(e_i)$  belong to two different vertices  $v_{j_1}, v_{j_2}$  with  $j_1 < j_2$ , define  $s(\beta'_i) = v_{j_1}$ and  $t(\beta_i') = v_{i_2}$ . Let  $\Pi(Q)$  be the fundamental groupoid of the quiver Q, and let  $\Sigma$  be the groupoid  $\Pi(Q)/\langle (\alpha'_{q(i)})^{d_{q(i)}}\beta'_i = \beta'_i(\alpha'_{p(i)})^{d_{p(i)}}, (\alpha'_j)^{d_j}\gamma'_{jp} = \gamma'_{jp}(\alpha'_j)^{d_j}, (\gamma'_{jp})^2 = 1 \mid 1 \le i \le k$ ,  $1 \leq j \leq n, 1 \leq p \leq l_j$ , where  $s(\beta_i') = v_{p(i)}$  and  $t(\beta_i') = v_{q(i)}$ . It can be shown that  $\Sigma$  is the direct limit of the direct system  $\{\mu_i : \Sigma' \to \Sigma_i\}_{1 \leq i \leq k}$ . By Proposition [3.4,](#page-12-1) the groupoid  $\Pi_m(E, A)$  is isomorphic to  $\Sigma$ . The rest of the proof is similar to that of [\[8,](#page-38-5) Proposition 5.9].

## 3.2. The fundamental groups of representation-finite and domestic  $f_{ms}$ -BGAs.

In this subsection, we assume that the field  $k$  is algebraically closed. For a Brauer  $G$ -set  $E$ , we always denote  $\sigma$  the Nakayama automorphism of E.

<span id="page-14-0"></span>**Lemma 3.6.** Let  $E = (E, U, \tau, d)$  be a connected Brauer G-set and let  $\Pi = \langle \sigma \rangle \leq \text{Aut}(E)$ . Then the action of  $\Pi$  on E is free, that is,  $\phi(e) \neq e$  for each  $e \in E$  and each  $\phi \neq 1$  in  $\Pi$ .

*Proof.* Suppose that  $\sigma^n(e) = e$  for some  $e \in E$  and for some integer n. For any  $h \in E$ , since E is connected, we can choose a walk w of E from e to h. Let  $w = (h | \delta_r \cdots \delta_2 \delta_1 | e)$ , where  $\delta_i \in \{g, g^{-1}, \tau\}$ for  $1 \leq i \leq r$ . Then  $\sigma^{n}(h) = \sigma^{n}(\delta_r \cdots \delta_2 \delta_1(e)) = \delta_r \cdots \delta_2 \delta_1(\sigma^{n}(e)) = \delta_r \cdots \delta_2 \delta_1(e) = h$ . Therefore  $\sigma^n = 1$ .  $n = 1.$ 

<span id="page-14-1"></span>**Lemma 3.7.** Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG such that  $\langle \sigma \rangle \leq \text{Aut}(E)$  is not admissible, and let e be a half-edge of E such that  $\tau(e) \in e^{\langle \sigma \rangle}$ . Let r be the smallest positive integer with the property that  $\sigma^r(e) = \tau(e)$ . We have

- (1) The  $\langle \sigma \rangle$ -orbit  $e^{\langle \sigma \rangle}$  of e contains 2r half-edges  $e, \sigma(e), \cdots, \sigma^{2r-1}(e)$ .
- (2) The order of the Nakayama automorphism  $\sigma$  of E is  $2r$ .
- (3) If N is the smallest positive integer with the property that  $g^N \cdot e \in e^{\langle \sigma \rangle}$  and suppose that  $g^N \cdot e = \sigma^p(e)$  for some  $0 \le p \le 2r - 1$ , then  $(p, 2r) = 1$ . Especially, p is odd.
- (4) The f-degree of each vertex of  $E/\langle \sigma \rangle$  is odd.

Proof. Since  $\sigma^{i}(\tau(e)) = \tau(\sigma^{i}(e))$  for any integer i, r is also the smallest positive integer with the property that  $\sigma^r(\tau(e)) = e$ . If there exists  $0 < i < 2r$  such that  $\sigma^i(e) = e$ , then  $0 < i < r$  or  $r < i < 2r$ . If  $0 < i < r$ , then  $\sigma^{r-i}(e) = \sigma^{r-i}(\sigma^i(e)) = \sigma^r(e) = \tau(e)$ , which contradict with the minimality of r. If  $r < i < 2r$ , then  $\sigma^{i-r}(\tau(e)) = \sigma^{i-r}(\sigma^r(e)) = \sigma^i(e) = e$ , where  $0 < i - r < r$ , which also contradict with the minimality of  $r$ . Therefore  $2r$  is the minimal positive integer with the property that  $\sigma^{2r}(e) = e$ , so (1) holds, and (2) follows from Lemma [3.6.](#page-14-0)

Note that N divides  $d(e)$ . Otherwise, let  $d(e) = aN + b$  with  $a, b \in \mathbb{Z}$  and  $0 < b < N$ . We have  $\sigma(e) = g^{d(e)} \cdot e = g^{aN+b} \cdot e = g^b \cdot \sigma^{ap}(e)$  and  $g^b \cdot e = \sigma^{1-ap}(e) \in e^{\langle \sigma \rangle}$ , contradict with the minimality of N. Let  $d(e) = aN$  for some integer a. Then  $\sigma(e) = g^{d(e)} \cdot e = g^{aN} \cdot e = \sigma^{ap}(e)$ . Since 2r is the minimal positive integer with the property that  $\sigma^{2r}(e) = e$ , 2r divides  $1 - ap$  and  $(p, 2r) = 1$ , which implies (3). Since  $2r$  divides  $1 - ap$ ,  $a$  is also odd. Since N equals to the cardinal of the vertex of  $E/\langle \sigma \rangle$  containing  $e^{\langle \sigma \rangle}$ , a equals to the f-degree of the vertex of  $E/\langle \sigma \rangle$  containing  $e^{\langle \sigma \rangle}$ . For every half-edge h of E, since  $\langle \sigma \rangle$  acts freely on E and since the order of  $\sigma$  is  $2r$ ,  $h^{\langle \sigma \rangle}$  contains 2r elements. Using the same method we can show that the f-degree of the vertex of  $E/\langle \sigma \rangle$  which contains  $h^{\langle \sigma \rangle}$  is odd, which implies (4). contains  $h^{\langle \sigma \rangle}$  is odd, which implies (4).

<span id="page-15-0"></span>3.2.1. The fundamental groups of representation-finite  $f_{ms}$ -BGAs.

**Theorem 3.8.** Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG. Then  $A_E$  is representationfinite if and only if  $\Pi(E) \cong \mathbb{Z}$ .

*Proof.* By Theorem [2.29,](#page-8-0)  $A_E$  is representation-finite if and only if  $A_{R_E}$  is representation-finite. Since  $R_E$  is a Brauer graph, we have  $A_{R_E}$  is representation-finite if and only if  $R_E$  is a Brauer tree.

" $\Rightarrow$ " Since  $R_E$  is a Brauer tree, by [\[8,](#page-38-5) Proposition 5.9] we have  $\Pi(R_E) \cong \mathbb{Z}$ .

If the group  $\langle \sigma \rangle$  of automorphisms of E is admissible, then  $R_E = E/\langle \sigma \rangle$  and there exists a covering  $E \to R_E$  of  $f_{ms}$ -BGs. By [\[8,](#page-38-5) Theorem 2.19],  $\Pi(E)$  is isomorphic to a subgroup of  $\Pi(R_E)$ . Since E is finite, the order of the automorphism  $\sigma$  of E is finite, and suppose that  $o(\sigma) = r$ . For any  $e \in E$ , by [\[8,](#page-38-5) Proposition 2.49],  $(e|g^{d(e)r}|e) \neq (e||e)$  in  $\Pi(E)$ , therefore  $\Pi(E) \neq \{1\}$ . Since  $\Pi(E)$  is a subgroup of  $\Pi(R_E)$ , it follows that  $\Pi(E) \cong \mathbb{Z}$ .

If the group  $\langle \sigma \rangle$  of automorphisms of E is not admissible, then  $E/\langle \sigma \rangle$  contains a double halfedge and  $R_E = E/\langle \sigma \rangle$ . Suppose that  $E/\langle \sigma \rangle$  has n vertices, k-edges, and l double half-edges. Then  $R_E$  has 2n vertices and 2k+l edges. Since  $R_E$  is a Brauer tree,  $(2k+l)-2n+1=0$ , and  $E/\langle \sigma \rangle$  is fdegree trivial. Since  $E/\langle \sigma \rangle$  is connected,  $k \geq n-1$ . Therefore  $k = n-1$  and  $l = 1$ . By Proposition [3.5,](#page-12-2)  $\Pi_m(E/\langle \sigma \rangle) \cong F\langle a_1, \cdots, a_n, c_1 \rangle / \langle a_1 = \cdots = a_n, a_1 c_1 = c_1 a_1, c_1^2 = 1 \rangle \cong F\langle a, c \rangle / \langle ac = ca,$  $c^2 = 1$  $\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$ 

Let  $e^{\langle \sigma \rangle}$  be the unique double half-edge of  $E/\langle \sigma \rangle$ . Since  $\tau(e) \in e^{\langle \sigma \rangle}$ , there exists a minimal positive integer r such that  $\tau(e) = \sigma^r(e)$ . By Lemma [3.7](#page-14-1) (1),  $e^{\langle \sigma \rangle} = {\sigma^i(e) \mid 0 \le i \le 2r - 1}.$ By the proof of Proposition [3.5,](#page-12-2) the isomorphism  $f : \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \Pi_m(E/\langle \sigma \rangle, e^{\langle \sigma \rangle})$  is given by  $f(1,0) = [(e^{\langle \sigma \rangle} | g^{d(e)} | e^{\langle \sigma \rangle})]$  and  $f(0,\overline{1}) = [(e^{\langle \sigma \rangle} | \tau | e^{\langle \sigma \rangle})]$ . Similar to f-BC case, the covering  $\phi: E \to E/\langle \sigma \rangle$  of Brauer G-set induces a  $\Pi_m(E/\langle \sigma \rangle, e^{\langle \sigma \rangle})$ -set structure on  $\phi^{-1}(e^{\langle \sigma \rangle}) = e^{\langle \sigma \rangle}$ , and the stabilizer subgroup of e in  $\Pi_m(E/\langle \sigma \rangle, e^{\langle \sigma \rangle})$  is isomorphic to  $\Pi_m(E, e)$  by Remark [2.16,](#page-4-2) which is also isomorphic to  $\Pi(E, e)$  by Lemma [2.12.](#page-4-1) The action of  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  on  $e^{\langle \sigma \rangle}$  via the group isomorphism f is given by  $(a, \overline{0}) \cdot e = \sigma^a(e)$  and  $(a, \overline{1}) \cdot e = \sigma^{a+r}(e)$  for any  $a \in \mathbb{Z}$ . We have  $(a, \overline{0}) \cdot e = e$  if and only if  $2r|a$ , and  $(a, \overline{1}) \cdot e = e$  if and only if  $a = br$  with b odd. Therefore  $\Pi(E, e)$  is isomorphic to the subgroup of  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  generated by  $(r, \overline{1})$ , which is isomorphic to  $\mathbb{Z}$ .

" $\Leftarrow$ " Suppose that  $A_E$  is not representation-finite, then  $A_E$  has a band. Since  $A_E$  is not a Nakayama algebra, there exists some  $e \in E$  with  $d(e) > 1$ . According to Lemma [2.26,](#page-7-0) E has a band l, which corresponds to a closed walk  $w$  of  $E$  of the form

$$
(e_{2k}|\tau g^{-i_{2k}}|e_{2k-1})(e_{2k-1}|\tau g^{i_{2k-1}}|e_{2k-2})\cdots(e_2|\tau g^{-i_2}|e_1)(e_1|\tau g^{i_1}|e_0),
$$

where  $e_0 = e_{2k}$  and  $0 < i_j < d(e_{j-1})$  for  $1 \le j \le 2k$ . Using w we obtain a special walk

$$
w' = (\sigma^k(e_{2k})|\tau g^{d(e_{2k-1})-i_{2k}}|\sigma^{k-1}(e_{2k-1}))(\sigma^{k-1}(e_{2k-1})|\tau g^{i_{2k-1}}|\sigma^{k-1}(e_{2k-2}))\cdots (\sigma(e_2)|\tau g^{d(e_1)-i_2}|e_1)(e_1|\tau g^{i_1}|e_0)
$$

of E. Let  $e = e_0 = e_{2k}$ . Since E is finite, the order of the automorphism  $\sigma$  of E is finite. Suppose that  $\sigma^r = id_E$ , then the walk  $v = \sigma^{k(r-1)}(w') \cdots \sigma^k(w')w'$  is a closed special walk of E at e. Moreover,  $v^l$  is also a closed special walk of E at e for any positive integer l. Let  $u = (e|g^{rd}(e)|e)$  be a closed walk of E at e. Then  $\overline{uv} = \overline{vu}$ . By [\[8,](#page-38-5) Proposition 2.49], the subgroup of  $\Pi(E, e)$  generated by  $\overline{u}$  and  $\overline{v}$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ . Therefore  $\Pi(E, e)$  is not isomorphic to  $\mathbb{Z}$ , a contradiction. Z, a contradiction.

3.2.2. The fundamental groups of domestic  $f_{ms}$ -BGAs.

<span id="page-16-2"></span>**Lemma 3.9.** Let G be a group and let N be the normal subgroup of G generated by X, where X is a subset of G. Let H be a subgroup of G such that  $N \subseteq H$ , and I be a subset of G which contains exactly one element from each right coset of H. Then N is the normal subgroup of H generated by  $Y = \{bab^{-1} \mid b \in I, a \in X\}.$ 

*Proof.* Since N is the normal subgroup of G generated by X, it is the subgroup of G generated by the set  $X' = \{gag^{-1} | g \in G, a \in X\}$ . Since each element of  $X'$  is conjugate to some element of Y in H, N is the normal subgroup of H generated by Y.

**Definition 3.10.** ([\[9,](#page-38-7) Chapter 6, Section 8]) Let F be a free group on the set S. For any element  $g \neq 1$  of F, express g as a reduced word in the generators:  $g = x_1x_2 \cdots x_k$ , where  $x_i \in S$  or  $x_i^{-1} \in S$  for  $1 \leq i \leq k$ . Define  $g' = x_1x_2 \cdots x_{k-1}$ . A nonempty subset G of F is said to be a Schreier system in F if  $g' \in G$  for any  $g \in G$  with  $g \neq 1$ .

<span id="page-16-1"></span>**Proposition 3.11.** ([\[9,](#page-38-7) Chapter 6, Theorem 8.1]) Let F be a free group on the set S, F' be a  $subgroup of F, and G be a Schreier system in F which contains exactly one element from each$ right coset of F'. Then F' is a free group on the set  $\{gs\Phi(gs)^{-1} \mid g \in G, s \in S, gs\Phi(gs)^{-1} \neq 1\},\$ where the map  $\Phi : F \to G$  assigns each element of F the unique element of G in the same coset of  $F'$ .

<span id="page-16-0"></span>**Lemma 3.12.** Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG and let  $B = E/\langle \sigma \rangle$ . Suppose that the modified BG B has k-edges, l double half-edges, and n vertices  $v_1, \dots, v_n$  of f-degree  $d_1$ ,  $\cdots$ ,  $d_n$  respectively. Then  $A_E$  is domestic if and only if one of the following holds

(1)  $l = 2, k - n + 1 = 0, d_i = 1$  for  $1 \leq i \leq n$ ;

(2)  $l = 0, k - n + 1 = 0, d_i = 2$  for exactly two numbers  $i = i_0, i_1,$  and  $d_i = 1$  for  $i \neq i_0, i_1;$ 

(3)  $l = 0, k - n + 1 = 1, d_i = 1$  for  $1 \leq i \leq n$ .

*Proof.* " $\Rightarrow$ " If the subgroup  $\langle \sigma \rangle$  of Aut(E) is not admissible, by Lemma [3.7,](#page-14-1) the f-degree of each vertex of B is odd. Moreover,  $R_E = \hat{B}$  is a Brauer graph whose Euler character  $\chi(R_E)$  =  $2(k - n + 1) + l - 1$ . By Theorem [2.29,](#page-8-0)  $A_{R_E}$  is domestic. Since the f-degree of each vertex of B is odd, the f-degree of each vertex of  $R_E$  is also odd. Then by Theorem [1.1](#page-0-0) we imply that  $R_E$  is a f-degree trivial BG with  $\chi(R_E) = 1$ . Therefore we have  $k - n + 1 = 0$ ,  $l = 2$ , and  $d_i = 1$  for  $1 \leq i \leq n$ . Then (1) holds.

If the subgroup  $\langle \sigma \rangle$  of Aut(E) is admissible, then  $B = R_E$  is a Brauer graph. By Theorem [2.29,](#page-8-0)  $A_{R_E}$  is domestic. According to Theorem [1.1,](#page-0-0) either (2) or (3) holds.

" $\Leftarrow$ " If (1) holds then  $R_E = B$  is a Brauer graph with trivial f-degree and the underlying graph of  $R_E$  contains a unique cycle. According to Theorem [1.1](#page-0-0) and Theorem [2.29,](#page-8-0)  $A_E$  is domestic. If (2) or (3) holds, then  $R_E = B$ , and by Theorem [1.1,](#page-0-0)  $A_{R_E}$  is domestic. So by Theorem [2.29](#page-8-0)  $A_E$  is also domestic.

<span id="page-16-3"></span>**Proposition 3.13.** Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG such that  $A_E$  is domestic. Then  $\Pi(E) \cong F\langle a, b \rangle / \langle a^2 = b^2 \rangle$  or  $\Pi(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

*Proof.* Suppose that the modified BG  $B = E/\langle \sigma \rangle$  has k-edges, l double half-edges, and n vertices  $v_1, \dots, v_n$  of f-degree  $d_1, \dots, d_n$  respectively. Since  $A_E$  is domestic, one of the conditions (1), (2), (3) in Lemma [3.12](#page-16-0) holds.

Suppose that condition (1) in Lemma [3.12](#page-16-0) holds, that is,  $l = 2$ ,  $k - n + 1 = 0$  and  $d_i = 1$  for  $1 \leq i \leq n$ . Let  $B = (B, B, \tau', d')$ , and let  $e^{\langle \sigma \rangle}$  and  $h^{\langle \sigma \rangle}$  be the two double half-edges of B. Since  $E$  is connected, so is  $B$ . By the proof of Proposition [3.5,](#page-12-2) there is a group isomorphism

$$
f: F\langle a,c_1,c_2\rangle/\langle ac_1=c_1a,ac_2=c_2a,c_1^2=c_2^2=1\rangle \xrightarrow{\sim} \Pi_m(B,e^{\langle \sigma \rangle})
$$

such that  $f(\overline{a}) = [(e^{\langle \sigma \rangle} | g^{d(e)} | e^{\langle \sigma \rangle})], f(\overline{c_1}) = [(e^{\langle \sigma \rangle} | \tau' | e^{\langle \sigma \rangle})], f(\overline{c_2}) = [(w')^{-1} (h^{\langle \sigma \rangle} | \tau' | h^{\langle \sigma \rangle}) w'],$  where w' is a walk of B from  $e^{\langle \sigma \rangle}$  to  $h^{\langle \sigma \rangle}$ . We may assume that w' lifts to a walk w of E from e to h.

Let r be the minimal positive integer with the property that  $\sigma^r(e) = \tau(e)$ . Then by Lemma [3.7](#page-14-1) (1) and (2),  $|e^{\langle \sigma \rangle}| = 2r$  and the order of the Nakayama automorphism  $\sigma$  of E is 2r. By Lemma [3.6,](#page-14-0) we also have  $|h^{\langle\sigma\rangle}| = 2r$ . Then by Lemma [3.7](#page-14-1) (1), r is also the minimal positive integer with the property that  $\sigma^{r}(h) = \tau(h)$ . The covering of Brauer G-sets  $\phi : E \to B$  induces a  $\Pi_{m}(B, e^{\langle \sigma \rangle})$ -set structure on  $\phi^{-1}(e^{\langle \sigma \rangle}) = e^{\langle \sigma \rangle}$ , and  $\Pi(E, e) \cong \Pi_m(E, e)$  is isomorphic to the stabilizer subgroup of e in  $\Pi_m(B, e^{\langle \sigma \rangle})$  by Remark [2.16.](#page-4-2) Using the group isomorphism f we may view  $e^{\langle \sigma \rangle}$  as a  $F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1 a, ac_2 = c_2 a, c_1^2 = c_2^2 = 1 \rangle$ -set. Since

$$
[(w')^{-1}(h^{\langle\sigma\rangle}|\tau'|h^{\langle\sigma\rangle})w']\cdot e = [(w')^{-1}(h^{\langle\sigma\rangle}|\tau'|h^{\langle\sigma\rangle})]\cdot h
$$
  

$$
= [(w')^{-1}]\cdot \tau(h) = [(w')^{-1}]\cdot \sigma^{r}(h) = \sigma^{r}([(w')^{-1}]\cdot h) = \sigma^{r}(e),
$$

we have  $\overline{c_2} \cdot e = \sigma^r(e)$ . Moreover, we have  $\overline{a} \cdot e = \sigma(e)$  and  $\overline{c_1} \cdot e = \sigma^r(e)$ . For each  $1 \leq i \leq 2r$ , identify the element  $\sigma^{i}(e)$  of  $e^{\langle \sigma \rangle}$  with the integer i, and let

$$
\rho: F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1 a, ac_2 = c_2 a, c_1^2 = c_2^2 = 1 \rangle \rightarrow S_{2r}
$$

be the group homomorphism given by the action of  $F\langle a, c_1, c_2\rangle/\langle ac_1 = c_1a, ac_2 = c_2a, c_1^2 = c_2^2 = 1\rangle$ on  $e^{\langle \sigma \rangle}$ . Then  $\rho(\overline{a}) = (1 \ 2 \cdots 2r)$  and  $\rho(\overline{c_1}) = \rho(\overline{c_2}) = (1 \ r + 1)(2 \ r + 2) \cdots (r \ 2r)$ .

Let  $\tilde{\rho}$ :  $F(a, c_1, c_2) \rightarrow S_{2r}$  be the group homomorphism induced by  $\rho$ , and let  $H = \{x \in$  $F\langle a, c_1, c_2 \rangle \mid \tilde{\rho}(x)(1) = 1$  be a subgroup of  $F\langle a, c_1, c_2 \rangle$ . Then  $\Pi(E, e)$  is isomorphic to  $H/N$ , where N is the normal subgroup of  $F\langle a, c_1, c_2 \rangle$  generated by relations  $ac_1 = c_1a, ac_2 = c_2a$ ,  $c_1^2 = c_2^2 = 1$ . Let  $G = \{1, a, a^2, \cdots, a^{2r-1}\}$  be a subset of  $F\langle a, c_1, c_2 \rangle$ . Then G is a Schreier system in  $F\langle a, c_1, c_2 \rangle$  which contains exactly one element from each right coset of H. By Proposition [3.11,](#page-16-1) *H* is a free group on the set  $\{x_i, y_i, z \mid 0 \le i \le 2r - 1\}$ , where

$$
x_i = \begin{cases} a^i c_1 a^{-r-i}, & \text{if } 0 \le i \le r-1; \\ a^i c_1 a^{r-i}, & \text{if } r \le i \le 2r-1; \end{cases}
$$

$$
y_i = \begin{cases} a^i c_2 a^{-r-i}, & \text{if } 0 \le i \le r-1; \\ a^i c_2 a^{r-i}, & \text{if } r \le i \le 2r-1; \end{cases}
$$

and  $z = a^{2r}$ . Since N is the normal subgroup of  $F\langle a, c_1, c_2 \rangle$  generated by  $c_1^2$ ,  $c_2^2$ ,  $c_1ac_1^{-1}a^{-1}$ ,  $c_2ac_2^{-1}a^{-1}$ , by Lemma [3.9,](#page-16-2) N is the normal subgroup of H generated by the set  $\{a^ic_j^2a^{-i},$  $a^{i}c_{j}ac_{j}^{-1}a^{-i-1} | 0 \leq i \leq 2r-1, j = 1, 2$ . A calculation shows that

$$
a^{i}c_{1}^{2}a^{-i} = \begin{cases} x_{i}x_{r+i}, & \text{if } 0 \leq i \leq r-1; \\ x_{i}x_{i-r}, & \text{if } r \leq i \leq 2r-1; \end{cases}
$$

$$
a^{i}c_{2}^{2}a^{-i} = \begin{cases} y_{i}y_{r+i}, & \text{if } 0 \leq i \leq r-1; \\ y_{i}y_{i-r}, & \text{if } r \leq i \leq 2r-1; \end{cases}
$$

$$
a^{i}c_{1}ac_{1}^{-1}a^{-i-1} = \begin{cases} x_{i}x_{i+1}^{-1}, & \text{if } 0 \leq i \leq r-2 \text{ or } r \leq i \leq 2r-2; \\ x_{r-1}x_{r}^{-1}, & \text{if } i = r-1; \\ x_{2r-1}x_{0}^{-1}z^{-1}, & \text{if } i = 2r-1; \end{cases}
$$

$$
a^{i}c_{2}ac_{2}^{-1}a^{-i-1} = \begin{cases} y_{i}y_{i+1}^{-1}, & \text{if } 0 \leq i \leq r-2 \text{ or } r \leq i \leq 2r-2; \\ y_{r-1}zy_{r}^{-1}, & \text{if } i = r-1; \\ y_{2r-1}y_{0}^{-1}z^{-1}, & \text{if } i = 2r-1. \end{cases}
$$

Therefore

$$
\Pi(E, e) \cong H/N = F\langle x_0, \cdots, x_{2r-1}, y_0, \cdots, y_{2r-1}, z \rangle / \langle x_0 = \cdots = x_{r-1} = x_r^{-1} = \cdots
$$
  
=  $x_{2r-1}^{-1}, y_0 = \cdots = y_{r-1} = y_r^{-1} = \cdots = y_{2r-1}^{-1}, z = x_0^{-2} = y_0^{-2} \rangle \cong F\langle x_0, y_0 \rangle / \langle x_0^2 = y_0^2 \rangle$ 

Suppose that condition (2) in Lemma [3.12](#page-16-0) holds, that is,  $l = 0$ ,  $k - n + 1 = 0$ ,  $d_i = 2$  for exactly two numbers  $i = i_0, i_1$ , and  $d_i = 1$  for  $i \neq i_0, i_1$ . Suppose that e, h are two half-edges of E with  $e^{\langle \sigma \rangle} \in v_{i_0}$  and  $h^{\langle \sigma \rangle} \in v_{i_1}$ . By the proof of [\[8,](#page-38-5) Proposition 5.9], there is a group isomorphism

$$
f: F\langle a,b\rangle/\langle a^2 = b^2\rangle \xrightarrow{\sim} \Pi(B, e^{\langle \sigma \rangle})
$$

such that

$$
f(\overline{a}) = (e^{\langle \sigma \rangle} | g^{\frac{d(e)}{2}} | e^{\langle \sigma \rangle}) \text{ and } f(\overline{b}) = (w')^{-1} (h^{\langle \sigma \rangle} | g^{\frac{d(h)}{2}} | h^{\langle \sigma \rangle}) w',
$$

where w' is a walk of B from  $e^{(\sigma)}$  to  $h^{(\sigma)}$ . We may assume that w' lifts to a walk w of E from e to h. Suppose that the  $\langle g \rangle$ -orbit of e contains M half-edges, then  $v_{i_0}$  contains  $N = (M, d(e))$ half-edges. Since  $d_{i_0} = 2$ ,  $d(e) = 2N$ . Since  $1 = (\frac{M}{N}, \frac{d(e)}{N})$  $\frac{N(e)}{N}$ ) =  $(\frac{M}{N}, 2)$ ,  $\frac{M}{N}$  is odd. So 2N divides  $M + N$ . We have

$$
\sigma^{\frac{M+N}{2N}}(e) = g^{\frac{d(e)(M+N)}{2N}} \cdot e = g^{M+N} \cdot e = g^N \cdot e.
$$

Similarly, suppose that the  $\langle g \rangle$ -orbit of h contains  $M'$  half-edges, then  $v_{i_1}$  contains  $N' = (M', d(h))$ half-edges. Moreover,  $2N'$  divides  $M' + N'$  and  $\sigma \frac{M' + N'}{2N'}(h) = g^{N'} \cdot h$ . Since  $\frac{M}{N}$  (resp.  $\frac{M'}{N'}$ ) is the minimal positive integer i (resp. j) such that  $\sigma^{i}(e) = e$  (resp.  $\sigma^{j}(h) = h$ ), by Lemma [3.6,](#page-14-0)  $\frac{M}{N} = o(\sigma) = \frac{M'}{N'}$ .

The covering of  $f_{ms}$ -BGs  $\phi: E \to B$  induces a  $\Pi(B, e^{\langle \sigma \rangle})$ -set structure on  $\phi^{-1}(e^{\langle \sigma \rangle}) = e^{\langle \sigma \rangle}$ , and  $\Pi(E, e)$  is isomorphic to the stabilizer subgroup of e in  $\Pi(B, e^{\langle \sigma \rangle})$ . Denote  $o(\sigma) = r$ , and identify the element  $g^{iN} \cdot e$  of  $e^{\langle \sigma \rangle}$  with the integer i for any  $1 \leq i \leq r$ . Then  $e^{\langle \sigma \rangle} = \{1, 2, \cdots, r\}$  becomes a  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$ -set via the isomorphism  $f : F\langle a,b\rangle/\langle a^2 = b^2\rangle \xrightarrow{\sim} \Pi(B, e^{\langle \sigma \rangle}).$  Since

$$
a \cdot (g^{iN} \cdot e) = \overline{(e^{\langle \sigma \rangle}|g^{\frac{d(e)}{2}}|e^{\langle \sigma \rangle})} \cdot (g^{iN} \cdot e) = g^{\frac{d(e)}{2}}g^{iN} \cdot e = g^Ng^{iN} \cdot e = g^{(i+1)N} \cdot e,
$$

the action of a on  $e^{\langle \sigma \rangle} = \{1, 2, \cdots, r\}$  corresponds to the permutation  $(12 \cdots r)$ . Suppose that the walk w of E from e to h is of the form  $w = (h|\delta_s \cdots \delta_1|e)$  with  $\delta_i \in \{g, g^{-1}, \tau\}$ . Let

$$
\delta_i^{-1} := \begin{cases} g^{-1}, & \text{if } \delta_i = g; \\ g, & \text{if } \delta_i = g^{-1}; \\ \tau, & \text{if } \delta_i = \tau. \end{cases}
$$

Since

$$
b \cdot e = \overline{(w')^{-1}(h^{\langle \sigma \rangle}|g^{\frac{d(h)}{2}}|h^{\langle \sigma \rangle})w'} \cdot e = \delta_1^{-1} \cdots \delta_s^{-1} g^{\frac{d(h)}{2}} \delta_s \cdots \delta_1(e) = \delta_1^{-1} \cdots \delta_s^{-1} g^{N'} \delta_s \cdots \delta_1(e)
$$
  

$$
= \delta_1^{-1} \cdots \delta_s^{-1} g^{N'} \cdot h = \delta_1^{-1} \cdots \delta_s^{-1} \sigma^{\frac{M'+N'}{2N'}}(h) = \delta_1^{-1} \cdots \delta_s^{-1} \sigma^{\frac{M+N}{2N}}(h)
$$
  

$$
= \sigma^{\frac{M+N}{2N}} \delta_1^{-1} \cdots \delta_s^{-1}(h) = \sigma^{\frac{M+N}{2N}}(e) = g^N \cdot e,
$$

and since the action of  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$  on  $e^{\langle\sigma\rangle}$  commutes with the action of  $\langle\sigma\rangle$  on  $e^{\langle\sigma\rangle}$ , the action of b on  $e^{\langle \sigma \rangle} = \{1, 2, \dots, r\}$  also corresponds to the permutation  $(12 \cdots r)$ .

The action of  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$  on  $e^{\langle \sigma \rangle} = \{1,2,\cdots,r\}$  defines a group homomorphism  $\rho$ :  $F\langle a,b\rangle/\langle a^2=b^2\rangle \to S_r$ , and let  $\tilde{\rho}: F\langle a,b\rangle \to S_r$  be the homomorphism induced by  $\rho$ . Let

$$
H = \{ x \in F\langle a, b \rangle \mid \tilde{\rho}(x)(r) = r \}
$$

be a subgroup of  $F\langle a, b \rangle$ . Then  $\Pi(E, e)$  is isomorphic to  $H/K$ , where K is the normal subgroup of  $F\langle a, b \rangle$  generated by  $a^2b^{-2}$ . Since  $G = \{1, a, \dots, a^{r-1}\}\$ is a Schreier system in  $F\langle a, b \rangle$  which consists exactly one element from each right coset of  $H$ , by Proposition [3.11,](#page-16-1)  $H$  is a free group on the set  $\{x_i \mid 0 \leq i \leq r\}$ , where

$$
x_i = \begin{cases} a^i b a^{-i-1}, & \text{if } 0 \le i \le r-2; \\ a^{r-1}b, & \text{if } i = r-1; \\ a^r, & \text{if } i = r. \end{cases}
$$

By Lemma [3.9,](#page-16-2) K is the normal subgroup of H generated by  $\{a^{i+2}b^{-2}a^{-i} \mid 0 \le i \le r-1\}$ . A calculation shows that

$$
a^{i+2}b^{-2}a^{-i}=\begin{cases} x_{i+1}^{-1}x_i^{-1},&\text{if }0\leq i\leq r-3;\\ x_rx_{r-1}^{-1}x_{r-2}^{-1},&\text{if }i=r-2;\\ x_rx_0^{-1}x_{r-1}^{-1},&\text{if }i=r-1.\end{cases}
$$

Since  $r = \frac{M}{N}$  is odd, we have  $x_0 = x_1^{-1} = x_2 = \cdots = x_{r-3} = x_{r-2}^{-1}$  and  $x_r = x_{r-2}x_{r-1} = x_{r-1}x_0$  in N  $H/K$ . Therefore  $H/K \cong F\langle x_0, x_{r-1} \rangle / \langle x_0^{-1} x_{r-1} = x_{r-1}x_0 \rangle$ . Let  $y = x_{r-1}$  and  $z = x_{r-1}x_0$ . Then  $x_0^{-1}x_{r-1} = x_{r-1}x_0$  is equivalent to  $y^2 = z^2$ , so  $H/K \cong F\langle y, z \rangle / \langle y^2 = z^2 \rangle$ .

Suppose that condition (3) in Lemma [3.12](#page-16-0) holds, that is,  $l = 0$ ,  $k - n + 1 = 1$  and  $d_i = 1$  for  $1 \leq i \leq n$ . By [\[8,](#page-38-5) Proposition 5.9], we have  $\Pi(B) \cong \mathbb{Z} \oplus \mathbb{Z}$ . Since  $\Pi(E)$  is isomorphic to a nonzero subgroup of  $\Pi(B)$ , either  $\Pi(E) \cong \mathbb{Z}$  or  $\Pi(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ . Since  $A_E$  is not representation-finite, by Theorem [3.8,](#page-15-0) we have  $\Pi(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

Let  $E = (E, U, \tau, d)$  be a finite connected Brauer G-set, where the order of the Nakayama automorphism  $\sigma$  of E is r. Define a equivalence relation  $\approx'$  on the set of walks of E as follows:  $w \approx' v$  if and only if  $w \approx (e|g^{krd(e)}|e)v$  for some integer k, where e is the terminal of v. Denote [[w]] the equivalence class of  $\approx'$  that contains w, and define a group  $\Pi'_m(E, e) = \{[[w]] \mid w \text{ is a }$ closed walk at  $e$ , which is called the reduced fundamental group of  $E$  at  $e$ . Since  $E$  is connected,  $\Pi'_{m}(E, e)$ 's are isomorphic for different  $e \in E$ . Therefore we may simply write  $\Pi'_{m}(E, e)$  as  $\Pi'_{m}(E)$ .

Using Proposition [2.22,](#page-6-1) it is straightforward to show that there is an exact sequence

$$
0 \to \mathbb{Z} \xrightarrow{i} \Pi_m(E, e) \xrightarrow{\pi} \Pi'_m(E, e) \to 0,
$$

<span id="page-19-0"></span>where  $i(1) = [(e|g^{rd(e)}|e)]$  and  $\pi([w]) = [[w]].$ 

**Lemma 3.14.** A covering  $f : E \to E'$  of finite connected Brauer G-sets induces an injective group homomorphism  $f_*: \Pi'_m(E, e) \to \Pi'_m(E', f(e)).$ 

Proof. Define  $f_* : \Pi'_m(E, e) \to \Pi'_m(E', f(e))$  as  $f_*([[w]]) = [[f(w)]]$ . Assume that  $E = (E, U, \tau, d)$ and  $E' = (E', U', \tau', d')$ . Denote  $\sigma$  (resp.  $\sigma'$ ) the Nakayama automorphism of E (resp. E'), and suppose that the order of  $\sigma$  (resp.  $\sigma'$ ) is r (resp. r'). If w, v are closed walks of E at e with  $w \approx' v$ , then  $w \approx (e|g^{krd(e)}|e)v$  for some integer k. Therefore  $f(w) \approx (f(e)|g^{krd(e)}|f(e))f(v)$ , where  $d(e) = d'(f(e))$ . For any  $y \in E'$ , since E' is connected, there exists  $x \in E$  such that  $f(x) = y$ . Then  $(\sigma')^{r}(y) = (\sigma')^{r}(f(x)) = f(\sigma^{r}(x)) = f(x) = y$ . Since  $\sigma'$  acts admissibly on E', we have  $(\sigma')^r = 1$ . So the order of  $\sigma'$  divides r. Therefore  $f(w) \approx' f(v)$  and  $f_*$  is well defined. To show that  $f_*$  is injective, suppose that  $f_*([[w]]) = 1$ , then  $f(w) \approx (f(e)|g^{k'r'd'(f(e))}|f(e))$  for some integer k'. By Proposition [2.15,](#page-4-0)  $w \approx (e|g^{k'r'd'}(fe)|e)$ , where  $d(e) = d'(f(e))$ . Then  $\sigma^{k'r'}(e) = e$ . Since  $\sigma$  acts admissibly on E, r divides  $k'r'$ . Therefore  $w \approx' (e||e)$  and  $[[w]] = 1$ .

<span id="page-19-1"></span>**Lemma 3.15.** If  $E = (E, E, \tau, d)$  is a finite connected  $f_{ms}$ -BG, then  $\Pi'_m(E, e)$  is isomorphic to  $\Pi'_m(E/\langle \sigma \rangle, e^{\langle \sigma \rangle}).$ 

Proof. Let  $f: E \to E/\langle \sigma \rangle$  be the natural projection. According to Lemma [3.14,](#page-19-0)  $f_*: \Pi'_m(E, e) \to$  $\Pi'_{m}(E/\langle \sigma \rangle, e^{\langle \sigma \rangle})$  is injective. For any  $[[w']] \in \Pi'_{m}(E/\langle \sigma \rangle, e^{\langle \sigma \rangle}),$  lift w' to a walk w of E with source e. Since the terminal of w belongs to  $e^{\langle \sigma \rangle}$ , we may assume that  $t(w) = \sigma^n(e)$  for some integer n. Let  $v = (e|g^{-nd(e)}|t(w))w$  be a closed walk of E at e. Then  $f(v) = (e^{\langle \sigma \rangle}|g^{-nd(e)}|e^{\langle \sigma \rangle})w'$ , where  $d(e)$ is equal to the degree of  $e^{\langle \sigma \rangle}$  in  $E/\langle \sigma \rangle$ . Since the order of the Nakayama automorphism of  $E/\langle \sigma \rangle$ is 1, we have  $f(v) \approx' w'$ . Then  $f_*([[v]]) = [[f(v)]] = [[w']]$  and  $f_* : \Pi'_m(E, e) \to \Pi'_m(E/\langle \sigma \rangle, e^{\langle \sigma \rangle})$  is also surjective.  $\Box$ 

<span id="page-19-2"></span>For a group  $\Pi$ , denote  $\Pi$  the abelianization of  $\Pi$ .

**Proposition 3.16.** For a finite connected  $f_{ms}$ -BG  $E = (E, E, \tau, d)$ , the follow are equivalent: (a)  $A_E$  is domestic;

- (b)  $\Pi'_{m}(E) \cong \mathbb{Z}$  or  $\Pi'_{m}(E) \cong F\langle a,b\rangle/\langle a^{2} = b^{2} = 1 \rangle;$
- (c)  $\widetilde{\Pi'_{m}(E)} \cong \mathbb{Z}$  or  $\widetilde{\Pi'_{m}(E)} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$

*Proof.* Suppose that the modified BG  $B = E/(\sigma)$  has k-edges, l double half-edges, and n vertices  $v_1, \dots, v_n$  of f-degree  $d_1, \dots, d_n$  respectively. According to Lemma [3.12,](#page-16-0)  $A_E$  is domestic if and only if B satisfies one of the conditions in Lemma [3.12.](#page-16-0) We choose some  $e^{\langle \sigma \rangle} \in B$ .

 $\Gamma(a) \Rightarrow (b)$ " Suppose that B satisfies condition (1) in Lemma [3.12,](#page-16-0) that is,  $l = 2$ ,  $k - n + 1 = 0$ ,  $d_i = 1$  for  $1 \leq i \leq n$ . By the proof of Proposition [3.5,](#page-12-2) there exists an isomorphism

$$
F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1 a, ac_1 = c_1 a, c_1^2 = c_2^2 = 1 \rangle \rightarrow \Pi_m(B, e^{\langle \sigma \rangle})
$$

which maps  $\overline{a}$  to  $[(e^{\langle \sigma \rangle} | g^{d(e)} | e^{\langle \sigma \rangle})]$ . By the exact sequence

$$
0 \to \mathbb{Z} \xrightarrow{i} \Pi_m(B, e^{\langle \sigma \rangle}) \xrightarrow{\pi} \Pi'_m(B, e^{\langle \sigma \rangle}) \to 0
$$

we see that  $\Pi'_m(B, e^{\langle \sigma \rangle})$  is isomorphic to  $F\langle c_1, c_2 \rangle / \langle c_1^2 = c_2^2 = 1 \rangle$ . By Lemma [3.15](#page-19-1) we have  $\Pi'_{m}(E, e) \cong F\langle c_1, c_2\rangle/\langle c_1^2 = c_2^2 = 1\rangle$ . Using the same method, it can be shown that  $\Pi'_{m}(E, e) \cong$  $F\langle a,b\rangle/\langle a^2 = b^2 = 1\rangle$  if B satisfies condition (2) in Lemma [3.12,](#page-16-0) and  $\Pi'_m(E) \cong \mathbb{Z}$  if B satisfies condition (3) in Lemma [3.12.](#page-16-0)

 $f'(b) \Rightarrow (c)$ " By a straightforward calculation.

 $C(n) \Rightarrow (a)$ " Let  $r = k - n + 1$ . By the proof of Proposition [3.5,](#page-12-2) there exists an isomorphism

$$
f: F\langle a_1, \cdots, a_n, b_1, \cdots, b_r, c_1, \cdots, c_l \rangle / \langle a_1^{d_1} = \cdots = a_n^{d_n}, a_1^{d_1} b_i = b_i a_1^{d_1} (1 \le i \le r),
$$
  

$$
a_1^{d_1} c_j = c_j a_1^{d_1}, c_j^2 = 1(1 \le j \le l) \rangle \rightarrow \Pi_m(B, e^{\langle \sigma \rangle})
$$

such that  $f(a_1^{d_1}) = [(e^{\langle \sigma \rangle} | g^{d(e)} | e^{\langle \sigma \rangle})]$ . By the exact sequence

$$
0 \to \mathbb{Z} \xrightarrow{i} \Pi_m(B, e^{\langle \sigma \rangle}) \xrightarrow{\pi} \Pi'_m(B, e^{\langle \sigma \rangle}) \to 0
$$

we see that  $\Pi'_m(B, e^{\langle \sigma \rangle})$  is isomorphic to

$$
F\langle a_1,\cdots,a_n,b_1,\cdots,b_r,c_1,\cdots,c_l\rangle/\langle a_1^{d_1}=\cdots=a_n^{d_n}=1,c_j^2=1(1\leq j\leq l)\rangle.
$$

Therefore the abelianization of  $\Pi'_{m}(B, e^{\langle \sigma \rangle})$  is isomorphic to

$$
\mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^l.
$$

Since  $\Pi'_{m}(B)$  is isomorphic to  $\Pi'_{m}(E)$  by Lemma [3.15,](#page-19-1) we see that the group

$$
\mathbb{Z}^r\oplus\mathbb{Z}/d_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_n\mathbb{Z}\oplus\left(\mathbb{Z}/2\mathbb{Z}\right)
$$

l

is isomorphic to Z or  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . Therefore there are only four possible cases: (*i*)  $r = 1$ ,  $l = 0$  and  $d_1 = \cdots = d_n = 1$ ;

(ii)  $r = 0$ ,  $l = 0$ ,  $d_i = 2$  for exactly two numbers  $i = i_0, i_1$ , and  $d_i = 1$  for  $i \neq i_0, i_1$ ; (iii)  $r = 0, l = 1, d_i = 2$  for exactly one number  $i = i_0$ , and  $d_i = 1$  for  $i \neq i_0$ ; (iv)  $r = 0$ ,  $l = 2$ , and  $d_1 = \cdots = d_n = 1$ .

According to Lemma [3.7,](#page-14-1) case *(iii)* can not happen. Since case *(i)*, *(ii)*, *(iv)* correspond to condition (3), (2), (1) in Lemma [3.12,](#page-16-0) respectively,  $A_E$  is domestic.

<span id="page-20-0"></span>**Lemma 3.17.** The center of the group  $F\langle a,b\rangle/\langle a^2 = b^2 \rangle$  is an infinite cyclic group generated by  $a^2$ .

*Proof.* It is straightforward to show that each element of  $F\langle a, b \rangle/\langle a^2 = b^2 \rangle$  of one of the following type: (1)  $\overline{a^{2m}(ab)^n a}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ ; (2)  $\overline{a^{2m}(ba)^n b}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ ; (3)  $\overline{a^{2m}(ab)^n}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ ; (4)  $\overline{a^{2m}(ba)^n}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ . Let E be the Brauer graph

$$
\boxed{2} \bullet \qquad \qquad \bullet \boxed{2}
$$

and let e be a half-edge of E. By the proof of  $[8, \text{Lemma } 5.8]$ , there exists an isomorphism

$$
f: F\langle a, b \rangle / \langle a^2 = b^2 \rangle \to \Pi(E, e)
$$

such that  $f(\overline{a}) = \overline{(e|g|e)}, f(\overline{b}) = \overline{(e|\tau g\tau|e)}$ . For any  $m \in \mathbb{Z}$  and  $n \in \mathbb{N}$ , if  $\overline{a^{2m}(ab)^n a}$  belong to the center of  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$ , then so does  $\overline{(ab)^n a}$ . Therefore

$$
\overline{(e|(\tau g\tau g)^{n+1}|e)} = f(\overline{b(ab)^n a}) = f(\overline{(ab)^n ab}) = \overline{(e|(\overline{g\tau g\tau})^{n+1}|e)}.
$$

Since  $(e|(\tau g \tau g)^{n+1}|e)$  and  $(e|(\tau g \tau g \tau)^{n+1}|e)$  are homotopic special walks of E, by [\[8,](#page-38-5) Proposition 2.49] they are equal, a contradiction. Therefore any element of  $F\langle a, b \rangle / \langle a^2 = b^2 \rangle$  of type (1) does not belong to the center of  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$ . For elements of type (2) we also have the same conclusion. For elements of type  $(3)$  or type  $(4)$ , it can be shown that they belong to the center of  $F\langle a, b \rangle / \langle a^2 = b^2 \rangle$  if and only if  $n = 0$ . Therefore center of  $\overline{F}\langle a, b \rangle / \langle a^2 = b^2 \rangle$  is generated by  $\overline{a^2}$ . Since  $f(\overline{a^2}) = \overline{(e|g^2|e)}$ , by [\[8,](#page-38-5) Proposition 2.49], the order of  $\overline{a^2}$  is infinite.

<span id="page-21-0"></span>**Lemma 3.18.** For any positive integers m,  $n \geq 2$ , the center of  $F\langle a,b\rangle/\langle a^m = b^n = 1 \rangle$  is trivial.

*Proof.* Let  $E$  be the Brauer graph

$$
m \bullet \hspace{1.5cm} \bullet \hspace{1.5cm} \cdots \hspace{1.5cm} \bullet
$$

Suppose that  $e$  is the half-edge of  $E$  on the left of the diagram, by the proof of Lemma [3.2,](#page-11-0) there exists an isomorphism

$$
f: F\langle a,b\rangle/\langle a^m = b^n \rangle \to \Pi_m(E,e)
$$
  
such that  $f(\overline{a}) = [(e|g|e)], f(\overline{b}) = [(e|\tau g\tau|e)].$  Therefore f induces an isomorphism  

$$
\widetilde{f}: F\langle a,b\rangle/\langle a^m = b^n = 1 \rangle \to \Pi'_m(E,e).
$$

It is obvious that every element of  $F\langle a,b\rangle/\langle a^m = b^n = 1\rangle$  is of the form  $\overline{a^{i_k}b^{j_k}a^{i_{k-1}}b^{j_{k-1}}\cdots a^{i_1}b^{j_1}a^{i_0}},$ where  $0 \le i_0, i_k < m, 0 < i_l < m$  for  $1 \le l \le k-1$ , and  $0 < j_l < n$  for  $1 \le l \le k$ . We need show that such an expression is unique: Suppose that  $a^{i_k}b^{j_k}a^{i_{k-1}}b^{j_{k-1}}\cdots a^{i_1}b^{j_1}a^{i_0}$  and  $a^{p_r}b^{q_r}a^{p_{r-1}}b^{q_{r-1}}\cdots a^{p_1}b^{q_1}a^{p_0}$  are two such expressions with

$$
\overline{a^{i_k}b^{j_k}a^{i_{k-1}}b^{j_{k-1}}\cdots a^{i_1}b^{j_1}a^{i_0}} = \overline{a^{p_r}b^{q_r}a^{p_{r-1}}b^{q_{r-1}}\cdots a^{p_1}b^{q_1}a^{p_0}}.
$$

Then

$$
[[w]] = \widetilde{f}(a^{i_k}b^{j_k}a^{i_{k-1}}b^{j_{k-1}}\cdots a^{i_1}b^{j_1}a^{i_0}) = \widetilde{f}(a^{p_r}b^{q_r}a^{p_{r-1}}b^{q_{r-1}}\cdots a^{p_1}b^{q_1}a^{p_0}) = [[v]],
$$

where

$$
w = (e|g^{i_k}\tau g^{j_k}\tau g^{i_{k-1}}\tau g^{j_{k-1}}\tau \cdots g^{i_1}\tau g^{j_1}\tau g^{i_0}|e),
$$
  

$$
v = (e|g^{p_r}\tau g^{q_r}\tau g^{p_{r-1}}\tau g^{q_{r-1}}\tau \cdots g^{p_1}\tau g^{q_1}\tau g^{p_0}|e)
$$

are two special walks of E. Since  $w \approx' v$ , we have  $w \approx (e|g^{mN}|e)v$  for some integer N. By Proposition [2.22,](#page-6-1)  $N = 0$  and  $w = v$ . Therefore  $k = r$ ,  $i_l = p_l$  for  $0 \le l \le k$ , and  $j_l = q_l$  for  $1 \leq l \leq k$ .

If x belongs to the center of the group  $F\langle a,b\rangle/\langle a^m = b^n = 1\rangle$ , write x as the standard form  $a^{i_k}b^{j_k}a^{i_{k-1}}b^{j_{k-1}}\cdots a^{i_1}b^{j_1}a^{i_0}$  as above. Since x commutes with  $\overline{a}$  and  $\overline{b}$ , it can be shown that  $\underline{x}$ must equal to 1.

<span id="page-21-1"></span>**Proposition 3.19.** Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG. If  $\Pi(E) \cong F\langle a, b \rangle / \langle a^2 = b^2 \rangle$ or  $\Pi(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ , then  $A_E$  is domestic.

*Proof.* If  $\Pi(E) \cong F\langle a,b\rangle/\langle a^2 = b^2\rangle$ , by Lemma [2.12,](#page-4-1)  $\Pi_m(E) \cong F\langle a,b\rangle/\langle a^2 = b^2\rangle$ . Since there exists an exact sequence

 $0 \to \mathbb{Z} \stackrel{i}{\to} \Pi_m(E, e) \stackrel{\pi}{\to} \Pi'_m(E, e) \to 0,$ 

where  $i(1) = [(e|g^{rd(e)}|e)]$  belongs to the center of  $\Pi_m(E, e)$ , by Lemma [3.17](#page-20-0) we see that  $\Pi'_m(E) \cong$  $F\langle a,b\rangle/\langle a^2 = b^2, a^{2N} = 1\rangle$  for some positive integer N. Then the abelianization of  $\Pi'_m(E)$ is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}$ . Suppose that  $E/\langle \sigma \rangle$  has k-edges, l double half-edges, and n

vertices  $v_1, \dots, v_n$  of f-degree  $d_1, \dots, d_n$ , respectively. By Lemma [3.15,](#page-19-1)  $\Pi'_m(E) \cong \Pi'_m(E/\langle \sigma \rangle)$ , which is isomorphic to

$$
F\langle a_1,\cdots,a_n,b_1,\cdots,b_r,c_1,\cdots,c_l\rangle/\langle a_1^{d_1}=\cdots=a_n^{d_n}=c_1^2=\cdots=c_l^2=1\rangle
$$

by Proposition [3.5,](#page-12-2) where  $r = k - n + 1$ . The abelianization of

$$
F\langle a_1, \cdots, a_n, b_1, \cdots, b_r, c_1, \cdots, c_l \rangle / \langle a_1^{d_1} = \cdots = a_n^{d_n} = c_1^2 = \cdots = c_l^2 = 1 \rangle
$$

is  $\mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^l$ , which is isomorphic to  $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}$ . Suppose that  $N > 1$ , then there are two possible cases:

(1)  $r = l = 0, d_{i_0} = 2N, d_{i_1} = 2$  for some  $1 \leq i_0, i_1 \leq n$ , and  $d_i = 1$  for  $i \neq i_0, i_1$ ; (2)  $r = 0, l = 1, d_{i_0} = 2N$  for some  $1 \le i_0 \le n$ , and  $d_i = 1$  for  $i \ne i_0$ . If case (1) occurs, then

$$
\Pi'_{m}(E) \cong \Pi'_{m}(E/\langle \sigma \rangle) \cong F\langle a_1, a_2 \rangle / \langle a_1^{2N} = a_2^2 = 1 \rangle.
$$

So by Lemma [3.18,](#page-21-0) the center of  $\Pi'_m(E)$  is trivial. But we also have  $\Pi'_m(E) \cong F\langle a,b\rangle/\langle a^2 = b^2,$  $a^{2N} = 1$ , therefore the center of  $\Pi'_{m}(E)$  contains an element  $a^{2}$ , which is not equal to the identity element, a contradiction. If case (2) occurs, since  $E/\langle \sigma \rangle$  contains a double half-edge, the automorphism group  $\langle \sigma \rangle$  of E is not admissible. By Lemma [3.7,](#page-14-1) the f-degree of each vertex of  $E/\langle \sigma \rangle$  is odd, a contradiction. Therefore  $N = 1$ , and  $\Pi'_{m}(E) \cong F\langle a, b \rangle / \langle a^2 = b^2 = 1 \rangle$ . By Proposition [3.16,](#page-19-2)  $A_E$  is domestic.

If  $\Pi(E) \cong \mathbb{Z} \oplus \mathbb{Z}$ , by Lemma [2.12,](#page-4-1)  $\Pi_m(E)$  is also isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ . By the exact sequence

$$
0 \to \mathbb{Z} \to \Pi_m(E) \to \Pi'_m(E) \to 0,
$$

we see that  $\Pi'_m(E) \cong \mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$  for some positive integer N. Suppose that  $E/\langle \sigma \rangle$  has k-edges, l double half-edges, and n vertices  $v_1, \dots, v_n$  of f-degree  $d_1, \dots, d_n$ , respectively. By Lemma [3.15,](#page-19-1)  $\Pi'_m(E) \cong \Pi'_m(E/\langle \sigma \rangle)$ , which is isomorphic to

$$
F\langle a_1,\cdots,a_n,b_1,\cdots,b_r,c_1,\cdots,c_l\rangle/\langle a_1^{d_1}=\cdots=a_n^{d_n}=c_1^2=\cdots=c_l^2=1\rangle
$$

by Proposition [3.5,](#page-12-2) where  $r = k - n + 1$ . The abelianization of

$$
F\langle a_1,\cdots,a_n,b_1,\cdots,b_r,c_1,\cdots,c_l\rangle/\langle a_1^{d_1}=\cdots=a_n^{d_n}=c_1^2=\cdots=c_l^2=1\rangle
$$

is  $\mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^l$ , which is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$ . We have four possible cases:

(1)  $N = 1, r = 1, l = 0, d_1 = \cdots = d_n = 1;$ 

(2)  $N = 2, r = 1, l = 0, d_i = 2$  for some  $1 \le i \le n$ , and  $d_i = 1$  for  $j \ne i$ .

(3)  $N = 2, r = 1, l = 1, d_1 = \cdots = d_n = 1;$ 

(4)  $N > 2$ ,  $r = 1$ ,  $l = 0$ ,  $d_i = N$  for some  $1 \le i \le n$ , and  $d_j = 1$  for  $j \ne i$ .

Since  $\Pi'_m(E)$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$ , it is abelian. However,  $\Pi'_m(E)$  is also isomorphic to

$$
F\langle a_1,\cdots,a_n,b_1,\cdots,b_r,c_1,\cdots,c_l\rangle/\langle a_1^{d_1}=\cdots=a_n^{d_n}=c_1^2=\cdots=c_l^2=1\rangle,
$$

which is isomorphic to  $F\langle a, b \rangle/\langle a^2 = 1 \rangle$ ,  $F\langle b, c \rangle/\langle c^2 = 1 \rangle$ , and  $F\langle a, b \rangle/\langle a^N = 1 \rangle$  for case (2), (3), and (4), respectively. We see that  $\Pi'_m(E)$  is non-abelian in these cases, a contradiction. Therefore only case (1) can occurs. In this case,  $\Pi'_{m}(E) \cong \mathbb{Z}$ , and by Proposition [3.16,](#page-19-2)  $A_{E}$  is domestic.  $\Box$ 

**Theorem 3.20.** For a finite connected  $f_{ms}$ -BG E,  $A_E$  is domestic if and only if  $\Pi(E) \cong$  $F\langle a,b\rangle/\langle a^2=b^2\rangle$  or  $\Pi(E)\cong\mathbb{Z}\oplus\mathbb{Z}$ .

*Proof.* By Proposition [3.13](#page-16-3) and Proposition [3.19.](#page-21-1)

## 4. Representation-finite fractional Brauer graph algebras of type MS

In this section we assume that  $k$  is an algebraically closed field. We abbreviate indecomposable, basic, representation-finite self-injective algebra over  $k$  (not isomorphic to the underlying field  $k$ ) by RFS algebra.

## 4.1. Review on configurations of stable translation-quivers of tree class  $A_n$ .

**Definition 4.1.** ([\[10,](#page-38-8) Definition 2.3]) Let Γ be a stable translation quiver and let  $k(\Gamma)$  be the mesh category of Γ. A (combinatorial) configuration C is a set of vertices of Γ which satisfies the following conditions:

- (1) For any  $e, f \in \mathcal{C}$ , Hom $_{k(\Gamma)}(e, f) = \begin{cases} 0 & (e \neq f), \\ k & (e = f). \end{cases}$  $k \quad (e = f).$
- (2) For any  $e \in \Gamma_0$ , there exists some  $\hat{f} \in \mathcal{C}$  such that  $\text{Hom}_{k(\Gamma)}(e, f) \neq 0$ .

Let  $\Pi$  be an admissible group of automorphisms of  $\mathbb{Z}A_n$ , and let  $\mathscr{C}$  be a  $\Pi$ -stable configuration of ZA<sub>n</sub>. By [\[10,](#page-38-8) Theorem 5], there exists an RFS algebra  $A_{\mathscr{C},\Pi}$  of class  $A_n$  such that  $\text{ind}A_{\mathscr{C},\Pi}$  is isomorphic to the mesh category of the translation quiver  $(\mathbb{Z}A_n)_{\mathscr{C}}/\Pi$ , where ind $A_{\mathscr{C},\Pi}$  denotes the category of a chosen set of representatives of non-isomorphic indecomposable finitely generated  $A_{\mathscr{C},\Pi}$ -modules. Together with [\[10,](#page-38-8) Theorem 4.1], we have

**Proposition 4.2.** ([\[10\]](#page-38-8)) The map  $\mathscr{C} \mapsto A_{\mathscr{C},\Pi}$  is a bijection between the isomorphism classes of  $\Pi$ -stable configurations of  $\mathbb{Z}A_n$  and the isomorphism classes of RFS algebras of class  $A_n$  with admissible group Π.

<span id="page-23-2"></span>**Definition 4.3.** ([\[10,](#page-38-8) Definition 2.6]) A Brauer relation of order n is an equivalence relation on the set  $\sqrt[n]{1} = \{e^{\frac{2n\pi}{n}i} \mid m \in \mathbb{Z}\}\subseteq \mathbb{C}$  such that the convex hulls of distinct equivalence classes are disjoint.

If B is a Brauer relation of order n, we denote  $\beta_{\mathcal{B}}$  the permutation of  $\sqrt[n]{1}$  assigning to each point  $s$  its successor in the equivalence class of  $s$  endowed with the anti-clockwise orientation, see [\[10,](#page-38-8) Section 2.6].

<span id="page-23-0"></span>**Proposition 4.4.** ([\[10,](#page-38-8) Proposition 2.6]) Let  $\beta$  be a Brauer relation of order n and denote by  $\mathscr{C}_{\beta}$ the set of vertices  $(i, j)$  of  $\mathbb{Z}A_n$  such that  $e_n(i + j) = \beta_{\mathcal{B}}(e_n(i))$ , where  $e_n(m) = e^{\frac{2m\pi}{n}i}$ . The map  $\mathcal{B} \mapsto \mathscr{C}_{\mathcal{B}}$  is a bijection between the Brauer relations of order n and the configurations of  $\mathbb{Z}A_n$ .

For any integer p, let  $\Delta_p = \{(p, i) \mid 1 \leq i \leq n\} \subseteq (\mathbb{Z}A_n)_0$  be the "going up diagonal" and  $\nabla_p = \{(p-i,i) \mid 1 \leq i \leq n\} \subseteq (\mathbb{Z}A_n)_0$  be the "going down diagonal" of  $\mathbb{Z}A_n$ . If  $\mathscr{C}$  is a configuration of  $\mathbb{Z}A_n$ , define two permutations  $\alpha_{\mathscr{C}}$  and  $\beta_{\mathscr{C}}$  of  $\mathbb{Z}$  (see [\[10,](#page-38-8) Section 3.4]) as follows:  $\alpha_{\mathscr{C}}(p) = n + x + 1$ , where  $(x, y)$  is the unique point of  $\mathscr{C}$  in  $\nabla_p$ , and  $\beta_{\mathscr{C}}(p) = p + i$ , where  $(p, i)$ is the unique point of  $\mathscr C$  in  $\Delta_p$ . It is straightforward to show that  $\alpha_{\mathscr C}\beta_{\mathscr C}(p) = p + n + 1$  for any  $p \in \mathbb{Z}$ .

Given any admissible group of automorphisms  $\Pi$  of  $\mathbb{Z}A_n$  and any  $\Pi$ -stable configuration  $\mathscr C$  of  $\mathbb{Z}A_n$ , the RFS algebra  $A_{\mathscr{C},\Pi}$  of class  $A_n$  can be described as follows (see [\[10,](#page-38-8) Section 6.2]): Let  $\alpha = \alpha_{\mathscr{C}}$  and  $\beta = \beta_{\mathscr{C}}$  be the permutations of Z associated with  $\mathscr{C}$ . Let  $c_r$  be the unique point of  $\mathscr{C}$ on  $\nabla_r$ . Define an action of  $\Pi$  on  $\mathbb Z$  by setting  $c_{gr} = gc_r$  for any  $g \in \Pi$  and  $r \in \mathbb Z$ . Let  $\widetilde{Q} = \widetilde{Q}_{\mathscr{C}}$  be the quiver with  $\tilde{Q}_0 = \mathbb{Z}$  and  $\tilde{Q}_1 = \{\alpha_r, \beta_r \mid r \in \mathbb{Z}\}\$ , where  $\alpha_r : r \to \alpha(r)$  and  $\beta_r : r \to \beta(r)$ . For any  $g \in \Pi$ , it can be shown that either  $g\alpha = \alpha g$  and  $g\beta = \beta g$ , or  $g\alpha = \beta g$  and  $g\beta = \alpha g$ , depending on whether  $g$  is a translation or a translation-reflection. Therefore  $g$  induces an isomorphism of Q. Denote  $Q_{\mathscr{C},\Pi}$  the residue quiver  $Q/\Pi$ , and let  $\overline{\alpha_r}, \overline{\beta_r}$  the residue classes of  $\alpha_r, \beta_r$  modulo  $\Pi$ .

<span id="page-23-1"></span>**Theorem 4.5.** ([\[10,](#page-38-8) Theorem 6.2])  $A_{\mathscr{C},\Pi}$  is isomorphic to the algebra defined by the quiver  $Q_{\mathscr{C},\Pi}$ and the relations  $\beta_{\alpha(r)}\overline{\alpha}_r = \overline{\alpha}_{\beta(r)}\beta_r = 0$  and  $\overline{\alpha}_{\alpha^{a_r-1}(r)}\cdots\overline{\alpha}_{\alpha(r)}\overline{\alpha}_r = \beta_{\beta^{b_r-1}(r)}\cdots\beta_{\beta(r)}\beta_r$ , where  $r \in \mathbb{Z}$  and  $a_r$ ,  $b_r$  are defined by  $\alpha^{a_r}(r) = r + n = \beta^{b_r}(r)$ .

Next we will give a Brauer relation of order  $n$  from a Brauer tree with  $n$  edges and trivial f-degree. Let  $B = (B, P, L, d)$  be such a Brauer tree and fix  $e \in B$ . Note that we can view B as a Brauer G-set with  $\tau$  the involution on B. Then for each half-edge b of B, there exists a unique integer  $0 \leq i \leq 2n-1$  such that  $b = (\tau g)^{i}(e)$ . Denote  $\alpha_i$  the arrow  $L((\tau g)^{i-1}(e))$  in  $Q_B$ , where  $1 \leq i \leq 2n$ . Call the arrows  $\alpha_{2j-1}$   $(1 \leq j \leq n)$  the *β*-arrows and the arrows  $\alpha_{2j}$   $(1 \leq j \leq n)$  the  $\alpha$ -arrows of  $Q_B$ . Note that for each half-edge h of B,  $L(h)$  is a  $\beta$ -arrow if and only if  $L(\tau(h))$ is an  $\alpha$ -arrow, and  $L(h)$  is a  $\beta$ -arrow if and only if  $L(g \cdot h)$  is a  $\beta$ -arrow. Moreover, call a path of  $Q_B$  a β-path (resp.  $\alpha$ -path) if each arrow of this path is a β-arrow (resp.  $\alpha$ -arrow). Define an equivalence relation B on  $\sqrt[n]{1} = \{e^{\frac{2m\pi}{n}i} \mid m \in \mathbb{Z}\}\subseteq \mathbb{C}$  as follows:  $e^{\frac{2k\pi}{n}i}$  and  $e^{\frac{2l\pi}{n}i}$  are equivalent if and only if the vertices  $P((\tau g)^{2k}(e))$  and  $P((\tau g)^{2l}(e))$  of  $Q_B$  can be connected by a  $\beta$ -path (the elements in  $\sqrt[n]{1}$  are in one-to-one correspondence with the vertices of  $Q_B$ , which is given by the map  $e^{\frac{2m\pi}{n}i} \mapsto P((\tau g)^{2m}(e))$ ). Moreover, we denote  $\beta_{\mathcal{B}}$  the permutation of  $\sqrt[n]{1}$  assigning to each point s its successor in the equivalence class of s endowed with the anti-clockwise orientation.

<span id="page-24-0"></span>**Proposition 4.6.** B is a Brauer relation of order n. Moreover, for each  $e^{\frac{2k\pi}{n}} \in \sqrt[n]{1}$ , suppose the terminal of the  $\beta$ -arrow  $\alpha_{2k+1}$  of  $Q_B$  is  $P((\tau g)^{2l}(e))$ , then  $\beta_{\mathcal{B}}(e^{\frac{2k\pi}{n}i})=e^{\frac{2l\pi}{n}i}$ .

*Proof.* For  $e^{\frac{2k\pi}{n}i}$ ,  $e^{\frac{2l\pi}{n}i} \in \sqrt[n]{1}$ , suppose that  $0 \leq k, l < n$ . Let

$$
l' = \begin{cases} l, & \text{if } l > k; \\ l+n, & l \leq k, \end{cases}
$$

and define  $(e^{\frac{2k\pi}{n}i}, e^{\frac{2l\pi}{n}i}) = \{e^{\frac{2r\pi}{n}i} \in \sqrt[n]{1} \mid k < r < l'\}.$ 

For any  $\beta$ -arrow  $\alpha_{2k+1}$  of  $Q_B$ , it can be shown that  $g(\tau g)^{2k}(e) = (\tau g)^{2(k+l)}(e)$ , where l is the number of vertices of B which can be connected to the vertex  $(\tau g)^{2k}(e)^{\langle g \rangle}$  of B via a path of B that contains the edge  $P((\tau g)^{2k+1}(e))$  (here we consider B as a graph). That is, l is the number of edges in the dotted circle in Figure 1 below.



Figure 1

So the terminal of the  $\beta$ -arrow  $\alpha_{2k+1}$  is  $P((\tau g)^{2(k+l)}(e))$ , and we have  $\beta_{\mathcal{B}}(e^{\frac{2k\pi}{n}i})=e^{\frac{2(k+l)\pi}{n}i}$ . To show that B is a Brauer relation of order n, it suffices to show that for any  $e^{\frac{2r\pi}{n}i} \in (e^{\frac{2k\pi}{n}i}, e^{\frac{2(k+l)\pi}{n}i}),$  $e^{\frac{2(r+s)\pi}{n}i}$  also belongs to  $(e^{\frac{2k\pi}{n}i}, e^{\frac{2(k+l)\pi}{n}i})$ , where  $1 \leq s \leq n$  is the integer such that  $g(\tau g)^{2r}(e) =$  $(\tau g)^{2(r+s)}(e).$ 

Since  $e^{\frac{2r\pi}{n}i} \in (e^{\frac{2k\pi}{n}i}, e^{\frac{2(k+l)\pi}{n}i})$ , we may assume that  $k < r < k+l$ . So the half-edge  $(\tau g)^{2r}(e)$ of B belongs to the dotted circle in Figure 1 (Note that the set of half-edges in the dotted circle in Figure 1 is  $\{(\tau g)^{2k+i}(e) \mid 1 \leq i \leq 2l\}$ ). Since  $(\tau g)^{2r}(e) \neq (\tau g)^{2(k+l)}(e)$ , the halfedge  $(\tau g)^{2(r+s)}(e) = g(\tau g)^{2r}(e)$  also belongs to the dotted circle in Figure 1. So  $(\tau g)^{2(r+s)}(e) \in$  $2(r+s)(\rho) = a(\tau a)^{2r}(\rho)$  also belongs to the dotted circle in Figure 1. So  $(\tau a)^{2(r+s)}$  $\{(\tau g)^{2k+i}(e) \mid 1 \leq i \leq 2l\}$ . Suppose  $(\tau g)^{2(r+s)}(e) = (\tau g)^{2k+i}(e)$  for some  $1 \leq i \leq 2l$ , then  $2n|(2(r+s)-(2k+i))$ , so we have  $i=2j$  for some  $1 \leq j \leq l$  and  $e^{\frac{2(r+s)\pi}{n}i} = e^{\frac{2(k+j)\pi}{n}i}$ . If  $j=l$ , then  $g(\tau g)^{2r}(e) = (\tau g)^{2(r+s)}(e) = (\tau g)^{2(k+l)}(e) = g(\tau g)^{2k}(e)$  (the last identity follows from Figure 1) and  $(\tau g)^{2r}(e) = (\tau g)^{2k}(e)$ . Therefore  $(\tau g)^{2(r-k)}(e) = e$ , and  $2n|2(r-k)$ , a contradiction. So  $1 \leq j < l$  and  $e^{\frac{2(r+s)\pi}{n}i} = e^{\frac{2(k+j)\pi}{n}i} \in (e^{\frac{2k\pi}{n}i}, e^{\frac{2(k+l)\pi}{n}i}).$ 

# 4.2. AR-quivers of representation-finite  $f_{ms}$ -BGAs: the main statements.

Let  $E = (E, E, \tau, d)$  be a finite connected  $f_{ms}$ -BG with Nakayama automorphism  $\sigma$  such that  $\Lambda_E$  is representation-finite and let  $R_E$  be the reduced form of E (see Section 4). According to the proof of Theorem [3.8,](#page-15-0)  $E/\langle \sigma \rangle$  is one of the following forms: (a) a Brauer tree; (b) a modified BG of trivial f-degree with a unique double half-edge, which has  $p + 1$  vertices and  $2p + 1$  half-edges  $(p \geq 0)$ . For case (a), choose a half-edge h of  $E/\langle \sigma \rangle$  which belongs to the unique exceptional vertex of  $E/\langle \sigma \rangle$ ; and for case (b), choose h to be the unique double half-edge of  $E/\langle \sigma \rangle$ . Then for case (a) we have  $\Pi_m(E/\langle \sigma \rangle, h) = \langle x \rangle \cong \mathbb{Z}$ , where  $x = [(h|g^l|h)]$  and l is the cardinal of the  $\langle g \rangle$ -orbit of h, and for case (b) we have  $\Pi_m(E/\langle \sigma \rangle, h) = \langle x, y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , where  $x = [(h|g^l|h)]$ with l the cardinal of the  $\langle g \rangle$ -orbit of h and  $y = [(h|\tau|h)]$  (Here  $\tau$  denotes the involution of the modified BG  $E/\langle \sigma \rangle$ ).

For case (a), suppose that the f-degree of the exceptional vertex of  $E/\langle \sigma \rangle$  is m, and suppose that  $B = B_{(E/(\sigma),h)}$  is a Brauer tree with trivial f-degree given in [\[8,](#page-38-5) Example 2.43]. Let n be the number of edges of B. Then  $E/\langle \sigma \rangle$  has  $\frac{n}{m}$  edges. There exists a covering  $p : B \to E/\langle \sigma \rangle$ such that the image of the fundamental group of B in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  is  $\langle x^m \rangle$ . Choose a half-edge e of B with  $p(e) = h$ , then the pair  $(B, e)$  defines a Brauer relation of B order n (cf. the remarks before Proposition [4.6\)](#page-24-0). Let  $\mathscr{C} = \mathscr{C}_{\mathcal{B}}$  be the configuration of  $\mathbb{Z}A_n$ corresponding to  $\beta$  (see Proposition [4.4\)](#page-23-0). Suppose the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  via the homomorphism induced by the covering  $E \to E/\langle \sigma \rangle$  is  $\langle x^r \rangle$ , then we have

<span id="page-25-0"></span>**Theorem 4.7.** For case (a), the configuration  $\mathscr{C}$  of  $\mathbb{Z}A_n$  is  $\tau^{\frac{n}{m}}$ -stable and the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E$  is isomorphic to  $(\mathbb{Z} A_n)_\mathscr{C}/\langle \tau^\frac{nr}{m}\rangle,$  where  $\tau$  denotes the automorphism of  $\mathbb{Z} A_n$  induced from the translation of the translation quiver  $\mathbb{Z}A_n$  and the positive integers n, m, r are defined as above.

For case (b), let  $B = R_E = E/\langle \sigma \rangle$  and  $e = h_1 \in (E/\langle \sigma \rangle)_1$ . Then B is a Brauer tree with trivial f-degree, which has  $n = 2p + 1$  edges. Let B be the Brauer relation given by the pair  $(B, e)$ , and  $\mathscr{C} = \mathscr{C}_{\mathcal{B}}$  be the configuration of  $\mathbb{Z}A_n$  corresponding to B. According to Theorem [3.8,](#page-15-0) the fundamental group of E is isomorphic to  $\mathbb{Z}$ . Then the the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h) = \langle x, y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$  via the homomorphism induced by the covering  $E \to E/(\sigma)$  is either generated by  $x^r$  for some  $r \in \mathbb{Z}_+$ , or generated by  $x^r y$  for some  $r \in \mathbb{Z}_+$ . We have

<span id="page-25-1"></span>**Theorem 4.8.** For case (b), configuration  $\mathscr C$  defined above is symmetric (cf. [\[10,](#page-38-8) Section 3.2]). If the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  is generated by  $x^r$  for some  $r \in \mathbb{Z}_+$ , then the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E$  is isomorphic to  $(\mathbb{Z}A_n)_{\mathscr{C}}/\langle \tau^{nr} \rangle$ . If the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  is generated by  $x^r y$  for some  $r \in \mathbb{Z}_+$ , then the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E$  is isomorphic to  $(\mathbb{Z}A_n)_{\mathscr{C}}/\langle \tau^{nr}\phi \rangle$ , where  $\phi$  is the automorphism of  $(\mathbb{Z}A_n)_{\mathscr{C}}$  which induces an involution of  $\mathbb{Z}A_n$  (cf. [\[10,](#page-38-8) Proposition 3.2]).

**Remark 4.9.** In fact, it is not hard to show that the class of representation-finite  $f_{ms}$ -BGAs coincides with the class of RFS algebras of class  $A_n$ . On the one hand, let  $A_E$  be a representationfinite  $f_{ms}$ -BGA. Since  $A_E$  is self-injective special biserial, for every vertex x of the stable AR-quiver of  $A_E$ , there are at most two arrows starting at x. Therefore  $A_E$  cannot be an RFS algebras of class  $D_n$  or class  $E_n$ . On the other hand, let A be a RFS algebra of class  $A_n$ . According to

Riedtmann's description of quivers with relations of RFS algebras of class  $A_n$  (see Theorem [4.5\)](#page-23-1), A is special biserial. Therefore, by the construction of its corresponding  $f$ -BC  $E = (E, P, L, d)$  (cf. remarks before [\[7,](#page-38-0) Proposition 7.13]), each polygon  $P(e)$  of E contains at most two half-edges and the partition L is trivial. Then E is a  $f_{ms}$ -BC such that each polygon of E contains at most two half-edges. By adding a half-edge to every single half-edge of E, we obtain a  $f_{ms}$ -BG E' such that the corresponding algebras of  $E$  and  $E'$  are isomorphic.

## 4.3. Proofs of the main statements.

Let  $B = (B, P, L, d)$  be a Brauer tree with trivial f-degree which has n edges and fix  $e \in B$ ,  $\mathcal{B}$ be the Brauer relation given by the pair  $(B, e)$ , and  $\mathscr{C} = \mathscr{C}_{\mathcal{B}}$  be the configuration of  $\mathbb{Z}A_n$  which corresponds to B. Let  $\widetilde{Q} = \widetilde{Q}_{\mathscr{C}}$  be the quiver given by the configuration  $\mathscr{C}$  (see the remarks before Theorem [4.5\)](#page-23-1). Denote  $\tau$  the involution of B as a Brauer G-set. For each edge  $P(h)$  of B, there exists a unique number  $i \in \{0, 1, \dots, n-1\}$  such that  $P(h) = P((\tau g)^{2i}(e))$ , and we define  $f(P(h)) = i.$ 

Let  $\mathscr X$  be a subset of B such that  $\mathscr X$  meets each vertex of B in exactly one half-edge, and such that for any  $h \in \mathscr{X}$ ,  $f(P(h)) \leq f(P(g^k \cdot h))$  for every integer k. Define a  $f_{ms}$ -BG  $\mathbb{Z}B$  =  $(\mathbb{Z}B, \widetilde{P}, \widetilde{L}, \widetilde{d})$  as follows:  $\mathbb{Z}B = \{(h, k) \mid h \in B, k \in \mathbb{Z}\}\$ , where the G-set structure of  $\mathbb{Z}B$  is given by

$$
g \cdot (h,k) = \begin{cases} (g \cdot h,k), \text{ if } g \cdot h \notin \mathcal{X}; \\ (g \cdot h,k+1), \text{ if } g \cdot h \in \mathcal{X}; \end{cases}
$$

 $\widetilde{P}(h,k) = \{(h',k) \mid h' \in P(h)\}, \ \widetilde{L}(h,k) = \{(h,k)\}, \ \widetilde{d}(h,k) = d(h)$  for every  $(h,k) \in \mathbb{Z}B$ . It is easy to show that  $\mathbb{Z}B$  is the universal cover of B.

Define  $f: (Q_{\mathbb{Z}B})_0 \to \mathbb{Z}$  as the function given by  $f(\widetilde{P}(h,k)) = f(P(h)) + nk$  for each edge  $\widetilde{P}(h,k)$ of ZB. Note that for each half-edge  $(h, k)$  of ZB, we have  $1 \leq f(\widetilde{P}(g \cdot (h, k))) - f(\widetilde{P}(h, k)) \leq n$ . For each half-edge  $(h, k)$  of  $\mathbb{Z}B$ , call the arrow  $\tilde{L}(h, k)$  of  $Q_{\mathbb{Z}B}$  a  $\beta$ -arrow (resp. an  $\alpha$ -arrow) of  $Q_{\mathbb{Z}B}$  if  $L(h)$  is a  $\beta$ -arrow (resp. an  $\alpha$ -arrow) of  $Q_B$  (cf. the remarks before Proposition [4.6\)](#page-24-0).

<span id="page-26-1"></span>**Lemma 4.10.** Let  $(h,k) \in \mathbb{Z}B$  with  $\widetilde{L}(h,k)$  a  $\beta$ -arrow of  $Q_{\mathbb{Z}B}$ . Then  $f(\widetilde{P}((\tau g)^2(h,k))) =$  $f(\widetilde{P}(h,k)) + n + 1$ , where  $\tau$  denotes the involution of  $\mathbb{Z}B$  as a Brauer G-set.

Proof. Note that  $1 \leq f(\widetilde{P}(g \cdot (h,k))) - f(\widetilde{P}(h,k)) \leq n$  and  $1 \leq f(\widetilde{P}(g \tau g(h,k))) - f(\widetilde{P}(\tau g(h,k))) \leq$ *n*. Since  $P(g \cdot (h,k)) = P(\tau g(h,k))$  and  $P(g \tau g(h,k)) = P((\tau g)^2(h,k)), 2 \le f(P((\tau g)^2(h,k)))$  $f(\widetilde{P}(h,k)) \leq 2n$ . Since  $\widetilde{L}(h,k)$  a  $\beta$ -arrow of  $Q_{\mathbb{Z}B}$ ,  $L(h)$  a  $\beta$ -arrow of  $Q_B$ . Then  $f(P((\tau g)^2(h)))$  –  $f(P(h)) \equiv 1 \pmod{n}$ . Since  $f(\tilde{P}((\tau g)^2(h,k))) - f(\tilde{P}(h,k)) \equiv f(P((\tau g)^2(h)))- f(P(h)) \pmod{n}$ , we have  $f(\widetilde{P}((\tau g)^2(h,k))) - f(\widetilde{P}(h,k)) = n+1.$ 

<span id="page-26-2"></span>**Lemma 4.11.** There exists an isomorphism of quivers  $Q_{\mathbb{Z}B} \to \tilde{Q}$  which maps each vertex v of  $Q_{\mathbb{Z}B}$  to the vertex  $f(v)$  of Q.

*Proof.* For each integer r, there exists a unique half-edge  $(h, k)$  of  $\mathbb{Z}B$  such that  $L(h, k)$  is a  $\beta$ arrow of  $Q_{\mathbb{Z}B}$  and  $f(\widetilde{P}(h,k)) = r$ . Define  $f(\widetilde{L}(h,k)) = \beta_r$  and  $f(\widetilde{L}(\tau(h),k)) = \alpha_r$ . We need to show that f defines a quiver isomorphism  $Q_{\mathbb{Z}B} \to \tilde{Q}$ .

For each arrow  $\gamma$  of  $Q_{\mathbb{Z}B}$ , by the definition of f we have  $f(s(\gamma)) = s(f(\gamma))$ . To show  $f: Q_{\mathbb{Z}B} \to$ Q is a morphism of quivers, it suffices to show that  $f(t(\gamma)) = t(f(\gamma))$ .

Since  $1 \leq f(\widetilde{P}(g \cdot (h,k))) - f(\widetilde{P}(h,k)) \leq n$  for each half-edge  $(h,k)$  of  $\mathbb{Z}B$ , we have

<span id="page-26-0"></span>
$$
(1) \qquad f(\widetilde{P}(g \cdot (h,k))) - f(\widetilde{P}(h,k)) = \begin{cases} f(P(g \cdot h)) - f(P(h)), & \text{if } f(P(g \cdot h)) > f(P(h)); \\ f(P(g \cdot h)) - f(P(h)) + n, & \text{if } f(P(g \cdot h)) \le f(P(h)). \end{cases}
$$

For each arrow  $\gamma = \widetilde{L}(h,k)$  of  $Q_{\mathbb{Z}B}$ , if  $f(\widetilde{L}(h,k)) = \beta_r$ , then  $L(h)$  is a  $\beta$ -arrow of  $Q_B$  and  $f(P(h)) + nk = f(\widetilde{P}(h,k)) = r$ . f maps the terminal  $\widetilde{P}(g \cdot (h,k))$  of  $\widetilde{L}(h,k)$  to  $f(\widetilde{P}(g \cdot (h,k)))$ . By Equation [\(1\)](#page-26-0) we have

$$
f(\widetilde{P}(g \cdot (h,k))) - r = \begin{cases} f(P(g \cdot h)) - f(P(h)), & \text{if } f(P(g \cdot h)) > f(P(h)); \\ f(P(g \cdot h)) - f(P(h)) + n, & \text{if } f(P(g \cdot h)) \le f(P(h)). \end{cases}
$$

Since  $L(h)$  is a  $\beta$ -arrow of  $Q_B$ ,  $e^{\frac{2\pi f(P(g\cdot h))}{n}i}$  is the successor of  $e^{\frac{2\pi f(P(h))}{n}i}$  in the Brauer relation  $\mathscr{B}$ , so  $f(P(g \cdot h)) - f(P(h)) \equiv \beta(f(P(h))) - f(P(h)) = \beta(r) - r \pmod{n}$ . Since  $1 \le \beta(r) - r \le n$ ,

$$
\beta(r) - r = \begin{cases} f(P(g \cdot h)) - f(P(h)), & \text{if } f(P(g \cdot h)) > f(P(h)); \\ f(P(g \cdot h)) - f(P(h)) + n, & \text{if } f(P(g \cdot h)) \le f(P(h)). \end{cases}
$$

So we have  $f(\widetilde{P}(g \cdot (h,k))) = \beta(r)$ , which is the terminal of  $\beta_r$  in  $\widetilde{Q}$ .

If  $f(\tilde{L}(h,k)) = \alpha_r$ , then  $\tilde{L}(h,k)$  is a  $\alpha$ -arrow of  $Q_{\mathbb{Z}B}$ . Therefore  $\tilde{L}(\tau(h,k))$  and  $\tilde{L}(g^{-1}\tau(h,k))$  are β-arrows of  $Q_{\mathbb{Z}B}$ . According to Lemma [4.10,](#page-26-1)  $f(\widetilde{P}(g \cdot (h,k))) - f(\widetilde{P}(g^{-1} \cdot (\tau(h),k))) = n+1$ . Suppose that  $f(\widetilde{P}(g^{-1}\cdot(\tau(h),k)))=t$ , then  $f(\widetilde{L}(g^{-1}(\tau(h),k)))=\beta_t$ , and therefore  $r=f(\widetilde{P}(h,k))=t$  $f(\widetilde{P}(\tau(h), k)) = \beta(t)$ , where the last identity follows from last paragraph. So  $f(\widetilde{P}(g \cdot (h, k)))$  $n + 1 + t = \alpha \beta(t) = \alpha(r)$ , which is the terminal of  $\alpha_r$  in Q.

By the arguments above,  $f: Q_{\mathbb{Z}B} \to \tilde{Q}$  is a morphism of quivers. Clearly f is an isomorphism.  $\Box$ 

Let  $\widetilde{\Lambda}$  be the k-category  $k\widetilde{Q}/\widetilde{I}$ , where  $\widetilde{I}$  is the ideal of  $k\widetilde{Q}$  generated by the following relations: (a)  $\beta_{\alpha(r)} \alpha_r = \alpha_{\beta(r)} \beta_r = 0;$ 

(b)  $\alpha_{\alpha^{a_r-1}(r)} \cdots \alpha_{\alpha(r)} \alpha_r = \beta_{\beta^{b_r-1}(r)} \cdots \beta_{\beta(r)} \beta_r$ , where  $r \in \mathbb{Z}$  and  $a_r$ ,  $b_r$  are defined by  $\alpha^{a_r}(r) =$  $r + n = \beta^{b_r}(r)$ .

If  $\Pi$  is an admissible automorphism group of  $\mathbb{Z}A_n$  which stabilize  $\mathscr{C}$ , then each  $g \in \Pi$  induces an automorphism of  $Q$  (see the remarks before Theorem [4.5\)](#page-23-1), which also induces a k-linear automorphism  $\tilde{q}$  of  $\tilde{\Lambda}$ . Denote  $\Pi$  the group of automorphisms of  $\tilde{\Lambda}$  formed by  $\tilde{q}$  with  $q \in \Pi$ . It can be shown that  $\Pi$  acts freely on  $\Lambda$ . Let  $\Lambda = \Lambda/\Pi$  be the quotient category (see [\[5,](#page-38-9) Section 3]) and  $A = \bigoplus_{x,y \in \Lambda} \Lambda(x,y)$ . By Theorem [4.5,](#page-23-1) A is isomorphic to  $A_{\mathscr{C},\Pi}$ .

**Lemma 4.12.** The quiver isomorphism  $Q_{\mathbb{Z}B} \to \tilde{Q}$  in Lemma [4.11](#page-26-2) induces an isomorphism  $\Lambda_{\mathbb{Z}B} \to$  $\Lambda$  of k-categories.

*Proof.* By definition  $\Lambda_{\mathbb{Z}B} = kQ_{\mathbb{Z}B}/I_{\mathbb{Z}B}$ , where  $I_{\mathbb{Z}B}$  is generated by the following relations:  $(a')\ \widetilde{\mathcal{L}}(\tau g(h,k))\widetilde{\mathcal{L}}(h,k)=0;$ 

 $(b') \ \widetilde{L}(g^{d(h)-1} \cdot (h,k)) \cdots \widetilde{L}(g \cdot (h,k)) \widetilde{L}(h,k) = \widetilde{L}(g^{d(\tau(h))-1} \cdot (\tau(h),k)) \cdots \widetilde{L}(g \cdot (\tau(h),k)) \widetilde{L}(\tau(h),k).$ Since for each  $(h, k) \in \mathbb{Z}B$ ,  $\widetilde{L}(h, k)$  is an  $\alpha$ -arrow (resp. a  $\beta$ -arrow) of  $Q_{\mathbb{Z}B}$  if and only if  $\widetilde{L}(\tau g(h, k))$ is a β-arrow (resp. an  $\alpha$ -arrow) of  $Q_{\mathbb{Z}B}$ , the quiver isomorphism  $Q_{\mathbb{Z}B} \to \tilde{Q}$  maps the relations of type  $(a')$  in  $I_{\mathbb{Z}B}$  to the relations of type  $(a)$  in  $\tilde{I}$ . Since  $f(\tilde{P}(g^{d(h)} \cdot (h,k))) = f(\tilde{P}(h,k+1)) =$  $f(P(h,k)) + n$ , we see that the quiver isomorphism  $Q_{\mathbb{Z}B} \to \tilde{Q}$  maps the relations of type  $(b')$  in  $I_{\mathbb{Z}B}$  to the relations of type (b) in I.

**Proof of Theorem [4.7.](#page-25-0)** Step 1: To show that the configuration  $\mathscr{C}$  in Theorem [4.7](#page-25-0) is  $\tau^{\frac{n}{m}}$ -stable. Since the Brauer tree  $E/\langle \sigma \rangle$  has  $\frac{n}{m}$  edges, we have  $(\tau g)^{\frac{2n}{m}}(h) = h$ . Then  $p((\tau g)^{\frac{2n}{m}}(e)) =$  $(\tau g)^{\frac{2n}{m}}(p(e)) = (\tau g)^{\frac{2n}{m}}(h) = h = p(e)$ . Since the fundamental group  $\Pi(E/\langle \sigma \rangle, h) \cong \mathbb{Z}$  of  $E/\langle \sigma \rangle$  is abelian, the covering  $p : B \to E/\langle \sigma \rangle$  is regular. Therefore there exists an automorphism  $\nu$  of B such that  $\nu(e) = (\tau g)^{\frac{2n}{m}}(e)$ . Since each half-edge b of B is of the form  $(\tau g)^i(e)$  for some integer *i*, we have  $\nu(b) = (\tau g)^{\frac{2n}{m}}(b)$  for every  $b \in B$ . For any  $i \in \mathbb{Z}$ , let j be the unique integer in [1, *n*] such that  $g(\tau g)^{2i}(e) = (\tau g)^{2(i+j)}(e)$ . Then  $(i, j)$  is the unique vertex of  $\mathbb{Z}A_n$  in  $\Delta_i \cap \mathscr{C}$ . Since  $g(\tau g)^{2(i+\frac{n}{m})}(e)=g(\tau g)^{2i}(\nu(e))=\nu(g(\tau g)^{2i}(e))=\nu((\tau g)^{2(i+j)}(e))=(\tau g)^{2(i+j+\frac{n}{m})}(e),\ (i+\frac{n}{m},j)\in{\mathbb N}$  $\mathscr{C}$ , and  $\mathscr{C}$  is  $\tau^{\frac{n}{m}}$ -stable.

Step 2: To show that  $\Lambda_E$  is isomorphic to the k-category  $\tilde{\Lambda}/\tilde{\Pi}$ , where  $\tilde{\Pi}$  is induced by the group of automorphisms  $\Pi$  of  $\mathbb{Z}A_n$  generated by  $\tau^{\frac{nr}{m}}$ .

Since the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  via the homomorphism induced by the covering  $E \to E/\langle \sigma \rangle$  is  $\langle x^r \rangle$ , where  $x = [(h|g^l|h)]$  and l is the cardinal of the  $\langle g \rangle$ -orbit of h, E is isomorphic to  $\mathbb{Z}B/\langle \mu \rangle$ , where  $\mu$  is the automorphisms of  $\mathbb{Z}B$  such that  $\mu(e, 0) = g^{lr}(e, 0)$ . Let u be the automorphism of  $Q_{\mathbb{Z}B}$  induced by  $\mu$ . Then  $\Lambda_E$  is isomorphic to  $\Lambda_{\mathbb{Z}B}/\langle \tilde{u} \rangle$ , where  $\tilde{u}$  denotes the automorphism of  $\Lambda_{\mathbb{Z}B}$  induced by u. Let  $f : Q_{\mathbb{Z}B} \to \tilde{Q}$  be the isomorphism of quivers in Lemma [4.11.](#page-26-2) To show that  $\Lambda_E$  is isomorphic to  $\tilde{\Lambda}/\tilde{\Pi}$ , it suffices to show the diagram

$$
Q_{\mathbb{Z}B} \xrightarrow{f} \widetilde{Q}
$$
  
\n
$$
u \downarrow \qquad \qquad \downarrow v
$$
  
\n
$$
Q_{\mathbb{Z}B} \xrightarrow{f} \widetilde{Q}
$$

commutes, where v is the automorphism of  $\tilde{Q}$  given by  $v(i) = i + \frac{nr}{m}$  for any  $i \in \tilde{Q}_0$  and  $v(\alpha_i) =$  $\alpha_{i+\frac{nr}{m}}, v(\beta_i) = \beta_{i+\frac{nr}{m}}$  for any  $i \in \mathbb{Z}$ .

Let  $r = am + b$  with  $a, b \in \mathbb{N}$  and  $0 \leq b < m$ . Then  $g^{lr} \cdot (e, 0) = g^{lb} g^{lma} \cdot (e, 0) = g^{lb} \cdot (e, a)$ . It can be shown that  $f(P(g^{lb} \cdot e)) = \frac{bn}{m}$ , so  $f(\widetilde{P}(g^{lr} \cdot (e, 0))) = f(\widetilde{P}(g^{lb} \cdot (e, a))) = an + \frac{bn}{m} = n(a + \frac{b}{m}) = \frac{nr}{m}$ . For each vertex  $\widetilde{P}(x, k)$  of  $Q_{\mathbb{Z}B}$  with  $\widetilde{L}(x, k)$  a  $\beta$ -arrow of  $Q_{\mathbb{Z}B}$ , assume that  $f(\widetilde{P}(x, k)) = i$ . Denote σ the Nakayama automorphism of ZB. Since  $\tilde{L}((\sigma^{-1}(\tau g)^2)^i(e,0))$  is a β-arrow of  $Q_{\mathbb{Z}B}$  with  $f(\tilde{P}((\sigma^{-1}(\tau g)^2)^i(e, 0))) = i$  by Lemma [4.10,](#page-26-1) we have  $(\sigma^{-1}(\tau g)^2)^i(e, 0) = (x, k)$ . So  $u(\tilde{P}(x, k)) =$  $\widetilde{P}(\mu(x,k)) = \widetilde{P}(\mu((\sigma^{-1}(\tau g)^{2})^{i}(e,0))) = \widetilde{P}((\sigma^{-1}(\tau g)^{2})^{i}\mu(e,0)) = \widetilde{P}((\sigma^{-1}(\tau g)^{2})^{i}g_{\sim}^{lr}(e,0)).$  Since  $\widetilde{L}(g^{lr} \cdot (e, 0))$  is a  $\beta$ -arrow of  $Q_{\mathbb{Z}B}$ , by Lemma [4.10,](#page-26-1)  $f(\widetilde{P}((\sigma^{-1}(\tau g)^2)^i g^{lr}(e, 0))) = f(\widetilde{P}(g^{lr}_{\sim}(e, 0))) +$  $i = \frac{nr}{m} + i = v(i)$ . Therefore  $fu(\widetilde{P}(x,k)) = vf(\widetilde{P}(x,k))$ . Similarly we have  $fu(\widetilde{L}(x,k)) =$  $v f(\widetilde{L}(x,k))$  and  $fu(\widetilde{L}(\tau(x,k))) = vf(\widetilde{L}(\tau(x,k))),$  so the diagram above is commutative.

Let  $A = \bigoplus_{x,y \in \widetilde{\Lambda}/\widetilde{\Pi}} (\widetilde{\Lambda}/\widetilde{\Pi})(x,y)$ . According to Theorem [4.5,](#page-23-1)  $A \cong A_{\mathscr{C},\Pi}$ , where  $\Pi$  is generated by the automorphism  $\tau^{\frac{nr}{m}}$  of  $\mathbb{Z}A_n$ . By Step 2,  $\Lambda_E$  is isomorphic to  $\widetilde{\Lambda}/\widetilde{\Pi}$ , so  $\Gamma_{\Lambda_E} \cong \Gamma_A \cong$  $(\mathbb{Z}A_n)_{\mathscr{C}}/\langle \tau^{\frac{nr}{m}}\rangle.$  $\Box$ 

## **Proof of Theorem [4.8.](#page-25-1)** Step 1: To show that the configuration  $\mathscr{C}$  in Theorem [4.8](#page-25-1) is symmetric.

We need to show that for any  $(i, j) \in \mathscr{C}$ , the vertex  $(i+j-p-1, n+1-j)$  of  $\mathbb{Z}A_n$  also belongs to  $\mathscr C$ .

Let  $\iota$  be the automorphism of  $B = \widehat{E}/\langle \sigma \rangle$  which maps  $x_i \in (E/\langle \sigma \rangle)_i$  to  $x_{3-i} \in (E/\langle \sigma \rangle)_{3-i}$ for each  $x \in E/\langle \sigma \rangle$  and  $i = 1, 2$ . Denote  $\tau$  the involution of B as a Brauer G-set. Since  $u(e) = \tau(e) = (\tau g)^n(e)$  and since each half-edge b of B is of the form  $(\tau g)^i(e)$  for some integer *i*, we have  $\iota(b) = (\tau g)^n(b)$  for any  $b \in B$ . Since  $(i, j) \in \mathscr{C}$ , we have  $g(\tau g)^{2i}(e) = (\tau g)^{2(i+j)}(e)$ . Therefore

$$
(\tau g)^{2(i+j)+n}(e) = \iota((\tau g)^{2(i+j)}(e)) = \iota(g(\tau g)^{2i}(e)) = g \cdot \iota((\tau g)^{2i}(e)) = g(\tau g)^{2i+n}(e)
$$

and

$$
\tau(\tau g)^{2(i+j)+n}(e) = (\tau g)^{2i+n+1}(e) = (\tau g)^{2(i+p+1)}(e).
$$

Since

$$
\tau(\tau g)^{2(i+j)+n}(e) = g(\tau g)^{2(i+j)+n-1}(e) = g(\tau g)^{2(i+j+p)}(e),
$$

 $\beta_{\mathcal{B}}(e^{\frac{2(i+j+p)\pi}{n}i})=e^{\frac{2(i+p+1)\pi}{n}i}$ , where  $\beta_{\mathcal{B}}$  is the permutation of  $\sqrt[n]{1}$  assigning to each point s its successor in the equivalence class of s endowed with the anti-clockwise orientation (see the remarks after Definition [4.3\)](#page-23-2). Since  $(i+p+1)-(i+j+p) = 1-j \equiv n+1-j \pmod{n}$  and  $1 \leq n+1-j \leq n$ , we have  $(i+j+p, n+1-j) \in \mathscr{C}$ . Since  $\mathscr{C}$  is  $\tau^n$ -stable,  $(i+j-p-1, n+1-j)$  also belongs to  $\mathscr{C}$ .

Step 2: To show that the automorphism  $\psi$  of  $\mathbb{Z}B$  which maps  $(e, 0)$  to  $(\tau(e), 0)$  induces an automorphism u of  $Q_{\mathbb{Z}B}$  such that the diagram



commutes, where f is the isomorphism of quivers in Lemma  $4.11$  and v is the automorphism of Q induced by the automorphism  $\phi$  of  $(\mathbb{Z}A_n)\mathscr{C}$  in Theorem [4.8](#page-25-1) (see the paragraph before Theorem  $(4.5).$  $(4.5).$  $(4.5).$ 

For each  $r \in \widetilde{Q}_0$ , let  $(i, j)$  be the unique vertex of  $\mathbb{Z}A_n$  which belongs to  $\nabla_r \cap \mathscr{C}$ , and let  $(x, k) \in \mathbb{Z}$ B such that  $f(P(x, k)) = i$  and  $\tilde{L}(x, k)$  is a  $\beta$ -arrow of  $Q_{\mathbb{Z}B}$ . By the definition of f,  $f(\widetilde{L}(x,k)) = \beta_i$ , so  $f(\widetilde{P}(g \cdot (x,k))) = \beta(i) = r$ . Denote  $\sigma$  the Nakayama automorphism of ZB. By Lemma [4.10,](#page-26-1)  $f(\tilde{P}((\sigma^{-1}(\tau g)^2)^i(e, 0))) = i$ . Since both  $\tilde{L}(x, k)$  and  $\tilde{L}((\sigma^{-1}(\tau g)^2)^i(e, 0))$  are  $\beta$ -arrows of  $Q_{\mathbb{Z}B}$  and since  $f(P(x,k)) = f(P((\sigma^{-1}(\tau g)^2)^i(e,0))) = i$ , we have  $(x, k) = (\sigma^{-1}(\tau g)^2)^i(e, 0)$ . Then  $g \cdot (x, k) = g(\sigma^{-1}(\tau g)^2)^i(e, 0)$  and  $\psi(g(x, k)) = g(\sigma^{-1}(\tau g)^2)^i \psi(e, 0) = g(\sigma^{-1}(\tau g)^2)^i \tau(e, 0) =$  $\sigma^{-i}(g\tau)^{2i+1}(e,0)$ . Therefore

$$
u(\widetilde{P}(g \cdot (x,k))) = \widetilde{P}(\psi(g \cdot (x,k))) = \widetilde{P}(\sigma^{-i}(g\tau)^{2i+1}(e,0))
$$
  
= 
$$
\widetilde{P}(\tau \sigma^{-i}(g\tau)^{2i+1}(e,0)) = \widetilde{P}(\sigma^{-i}(\tau g)^{2i}\tau g\tau(e,0)).
$$

Since  $\tau g\tau(e,0)$  is a  $\beta$ -arrow of  $Q_{\mathbb{Z}B}$ , by Lemma [4.10,](#page-26-1)  $f(P(\sigma^{-i}(\tau g)^{2i}\tau g\tau(e,0))) = f(P(\tau g\tau(e,0))) +$ i. Since  $\tau(e) = (\tau g)^n(e)$ ,  $\tau g \tau(e) = (\tau g)^{n+1}(e) = (\tau g)^{2(p+1)}(e)$ . So

$$
f(P(\tau g\tau(e,0))) \equiv f(P(\tau g\tau(e))) = p + 1 \pmod{n}.
$$

Since

$$
f(\widetilde{P}(\tau g\tau(e,0))) = f(\widetilde{P}(\tau g\tau(e,0))) - f(\widetilde{P}((e,0))) = f(\widetilde{P}(g\tau(e,0))) - f(\widetilde{P}(\tau(e,0))),
$$
  

$$
1 \leq f(\widetilde{P}(\tau g\tau(e,0))) \leq n. \text{ Therefore } f(\widetilde{P}(\tau g\tau(e,0))) = p+1, \text{ and}
$$

$$
fu(\widetilde{P}(g \cdot (x,k))) = f(\widetilde{P}(\sigma^{-i}(\tau g)^{2i}\tau g\tau(e,0))) = f(\widetilde{P}(\tau g\tau(e,0))) + i = p + 1 + i.
$$

Since  $\phi(i, j) = (i + j - p - 1, n + 1 - j), v(r) = (i + j - p - 1) + (n + 1 - j) = i + p + 1$ . Then  $f u f^{-1}(r) = i + p + 1 = v(r)$ , and the above diagram is commutative on vertices. Since u maps each  $\alpha$ -arrow (resp.  $\beta$ -arrow) of  $Q_{\mathbb{Z}B}$  to a  $\beta$ -arrow (resp. an  $\alpha$ -arrow) of  $Q_{\mathbb{Z}B}$  and v maps each α-arrow (resp. β-arrow) of Q to a β-arrow (resp. an α-arrow) of Q, we see that the above diagram is also commutative on arrows.

Step 3: To show that the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E$  is isomorphic to  $(\mathbb{Z} A_n)_{\mathscr{C}}/\langle \tau^{nr} \rangle$  if the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  is generated by rx for some  $r \in \mathbb{Z}_+$ .

Since the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  is generated by rx for some  $r \in \mathbb{Z}_+$ , according to Proposition [2.17](#page-5-1) E is isomorphic to  $\mathbb{Z}B/\langle \sigma^r \rangle$  (here  $\sigma$ denotes the Nakayama automorphism of  $\mathbb{Z}B$ ). Therefore  $\Lambda_E$  is isomorphic to  $\Lambda_{\mathbb{Z}B}/\Pi$ , where  $\Pi$ is a group of automorphisms of  $\Lambda_{\mathbb{Z}B}$  generated by the automorphism of  $\Lambda_{\mathbb{Z}B}$  which is induced by the automorphism w of  $Q_{\mathbb{Z}B}$ , where w is induced by the automorphism  $\sigma^r$  of  $\mathbb{Z}B$ . Let t be the automorphism of  $\tilde{Q}$  induced by the automorphism  $\tau^{nr}$  of  $\mathbb{Z}A_n$  which stabilize  $\mathscr{C}$ . Then  $t(i) = i + nr$  and  $t(\alpha_i) = \alpha_{i+n_r}$ ,  $t(\beta_i) = \beta_{i+n_r}$  for each  $i \in \mathbb{Z}$ . We have a commutative diagram



# where f is the isomorphism of quivers in Lemma [4.11.](#page-26-2) Therefore the quiver isomorphism  $f$ :  $Q_{\mathbb{Z}B} \to \tilde{Q}$  induces an isomorphism  $\Lambda_{\mathbb{Z}B}/\Pi \to \tilde{\Lambda}/\tilde{\Pi}$  of k-categories, where  $\tilde{\Pi}$  is generated by

the automorphism of  $\widetilde{\Lambda}$  induced by the automorphism t of  $\widetilde{Q}$ . Let  $A = \bigoplus_{x,y \in \widetilde{\Lambda}/\widetilde{\Pi}} (\widetilde{\Lambda}/\widetilde{\Pi})(x, y)$ . According to Theorem [4.5,](#page-23-1)  $A \cong A_{\mathscr{C},\langle \tau^{nr} \rangle}$ . Therefore the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E \cong \Lambda_{\mathbb{Z}B}/\Pi$  is isomorphic to  $(\mathbb{Z}A_n)_{\mathscr{C}}/\langle \tau^{nr} \rangle$ .

Step 4: To show that if the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$ is generated by  $x^r y$  for some  $r \in \mathbb{Z}_+$ , then the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E$  is isomorphic to  $(\mathbb{Z}A_n)_{\mathscr{C}}/\langle \tau^{nr}\phi\rangle$ , where  $\phi$  is the automorphism of  $(\mathbb{Z}A_n)_{\mathscr{C}}$  which induces an involution of  $\mathbb{Z}A_n$ .

Let  $\psi$  be the automorphism of ZB which maps  $(e, 0)$  to  $(\tau(e), 0)$ . Since the image of the fundamental group of E in  $\Pi(E/\langle \sigma \rangle, h) \cong \Pi_m(E/\langle \sigma \rangle, h)$  is generated by  $x^r y$ , according to Proposition [2.17](#page-5-1) E is isomorphic to  $ZB/\langle \sigma^r \psi \rangle$  (here  $\sigma$  denotes the Nakayama automorphism of  $ZB$ ). So  $\Lambda_E$ is isomorphic to  $\Lambda_{\mathbb{Z}_B}/\Pi$ , where  $\Pi$  is a group of automorphisms of  $\Lambda_{\mathbb{Z}_B}$  generated by the automorphism of  $\Lambda_{\mathbb{Z}B}$  which is induced by the automorphism wu of  $Q_{\mathbb{Z}B}$ , where w (resp. u) is the automorphism of  $Q_{\mathbb{Z}B}$  induced by the automorphism  $\sigma^r$  (resp.  $\psi$ ) of  $\mathbb{Z}B$ . Let t (resp. v) be the automorphism of  $\tilde{Q}$  induced by the automorphism  $\tau^{nr}$  (resp.  $\phi$ ) of  $(\mathbb{Z}A_n)_{\mathscr{C}}$ . By Figure 2 and Figure 3, we have a commutative diagram



So the quiver isomorphism  $f: Q_{\mathbb{Z}B} \to \tilde{Q}$  induces an isomorphism  $\Lambda_{\mathbb{Z}B}/\Pi \to \tilde{\Lambda}/\tilde{\Pi}$  of k-categories, where  $\Pi$  is generated by the automorphism of  $\Lambda$  induced by the automorphism tv of  $Q$ . Let  $A = \bigoplus_{x,y \in \widetilde{\Lambda}/\widetilde{\Pi}} (\widetilde{\Lambda}/\widetilde{\Pi})(x,y)$ . According to Theorem [4.5,](#page-23-1)  $A \cong A_{\mathscr{C},\langle \tau^{nr}\phi \rangle}$ . Therefore the AR-quiver  $\Gamma_{\Lambda_E}$  of  $\Lambda_E \cong \Lambda_{\mathbb{Z}B}/\Pi$  is isomorphic to  $(\mathbb{Z}A_n)_{\mathscr{C}}/\langle \tau \rangle$  $\Box$  $\Box$ 

# 5. Domestic fractional Brauer graph algebras of type MS

In this section we assume that  $k$  is an algebraically closed field. For a finite dimensional selfinjective k-algebra A, denote  ${}_{s}\Gamma_{A}$  the stable AR-quiver of A.

## 5.1. Exceptional tubes of representation-infinite  $f_{ms}$ -BGAs.

Let  $E = (E, P, L, d)$  be a finite connected  $f_{ms}$ -BG with  $A_E$  representation-infinite. According to [\[7,](#page-38-0) Section 6],  $A_E \cong kQ'_E/I'_E$  with  $I'_E$  admissible, where  $Q_E$  is the sub-quiver of  $Q_E$  given by  $(Q'_E)_0 = (Q_E)_0$  and  $(Q'_E)_1 = \{L(e) \mid e \in E \text{ with } d(e) > 1\}$ , and  $I'_E$  is generated by the following three types of relations

 $(fR1') L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e) L(e) - L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h) L(h)$ , where  $e, h \in E$  and  $d(e), d(h) > 1$ ;  $(fR2') L(e_1)L(e_2)$ , where  $e, h \in E$ ,  $d(e), d(h) > 1$ , and  $e_1 \neq g \cdot e_2$ ;<br> $(fR3') L(g^{d(e)}, e) \dots L(g, e) L(e)$ , where  $e \in E$  and  $d(e) > 1$ 

$$
(fR3') L(g^{d(e)} \cdot e) \cdots L(g \cdot e)L(e)
$$
, where  $e \in E$  and  $d(e) > 1$ .

We will consider a module of  $A_E$  as a representation of the quiver with relations  $(Q'_E, I'_E)$ . We call an  $A_E$ -module M a string module if it can be seen as a string module over the quotient algebra  $A_E/\text{soc}(A_E)$  (for the definition of string and string modules, see [\[3,](#page-38-6) II.2 and II.3]).

For every  $e \in E$ , denotes  $M_e$  the uniserial  $A_E$ -module given by the direct string  $L(g^{d(e)-2})$ .  $e) \cdots L(g \cdot e) L(e)$  (define  $M_e$  as the simple  $A_E$ -module at  $P(e)$  if  $d(e) = 1$ ). Since  $A_E$  representationinfinite, it is not a Nakayama algebra. Then it is straightforward to show that for each edge  $P(e) = \{e, e'\}$  of E, either  $d(e) > 1$  or  $d(e') > 1$ . Therefore  $M_{e_1}$  is not isomorphic to  $M_{e_2}$  for every  $e_1 \neq e_2$ .

Recall that a connected component of  $\binom{\Gamma_{A_E}}{\Gamma_{A_E}}$  consisting of string modules of the form  $\mathbb{Z}A_{\infty}/\langle\tau^n\rangle$  $(n \geq 1)$  is called an exceptional tube. The following result should be compared with [\[2,](#page-38-3) Lemma 4.4] for BGAs.

<span id="page-31-0"></span>**Lemma 5.1.** Let  $E = (E, P, L, d)$  be a finite connected  $f_{ms}$ -BG with  $A_E$  representation-infinite. Then a string module M of  $A_E$  is at the mouth of an exceptional tube in the stable AR-quiver of  $A_E$  if and only if  $M \cong M_e$  for some  $e \in E$ .

*Proof.* Note that for each edge  $P(e) = \{e, e'\}$  of E, either  $d(e) > 1$  or  $d(e') > 1$ . By [\[4,](#page-38-4) Theorem 2.1 and Theorem 2.2, an indecomposable string module  $M$  of  $A_E$  is at the mouth of an exceptional tube in the stable AR-quiver of  $A_E$  if and only if there exists precisely one irreducible morphism  $N \to M$  in  $_s \Gamma_{A_E}$  for some indecomposable  $A_E$ -module N.

" $\Rightarrow$ " Let M be at the mouth of an exceptional tube in  $\binom{s}{k}$ , which is given by a string w. Since there is only one irreducible morphism  $N \to M$  with N indecomposable non-projective, it is straightforward to show that w is a direct (or an inverse) string. Suppose that  $w = L(g^{r-1} \cdot$  $e) \cdots L(g \cdot e)L(e)$  is a direct string, where  $0 \le r \le d(e) - 1$ . If  $r = d(e) - 1$ , then  $M = M_e$ . If  $r <$  $d(e)-1$ , let N be the string module given by the string  $w'=L(h)^{-1}\cdots L(g^{d(h)-2}\cdot h)^{-1}L(g^r\cdot e)w$ , where  $g^{r+1} \cdot e$  and  $g^{d(h)-1} \cdot h$  are two half-edges of E which form an edge of E. Then w' is obtained from w by adding a co-hook, and there exists an irreducible morphism  $N \to M$ . Since there do not exists another irreducible morphism in  $_s\Gamma_{A_E}$  with terminal M, the projective cover of M is uniserial, and the string w is trivial. Let  $e'$  be the half-edge of E such that e and  $e'$  form an edge of E. Since the projective cover of M is uniserial,  $d(e') = 1$ , so we have  $M = M_{e'}$ .

" $\Leftarrow$ " Let  $M = M_e$  for some  $e \in E$ , and let  $P(e) = \{e, e'\}$ . If the projective cover of M is uniserial, then either  $d(e) = 1$  or  $d(e') = 1$ . In both case it is straightforward to show that there is only one irreducible morphism  $N \to M$  with N indecomposable non-projective. If the projective cover of M is not uniserial, then there exists an AR-sequence  $0 \to M_h \to N \to M \to 0$ , where N is the string module given by the string

$$
L(g^{d(e)-2} \cdot e) \cdots L(g \cdot e) L(e) L(e')^{-1} L(g^{d(h)-2} \cdot h) \cdots L(g \cdot h) L(h)
$$

with  $g \cdot e'$  and  $g^{d(h)-1} \cdot h$  two half-edges of E which form an edge of E. Then  $N \to M$  is the only irreducible morphism in  $_s\Gamma_{A_E}$  with terminal M.

<span id="page-31-1"></span>**Lemma 5.2.** Let  $E = (E, P, L, d)$  be a finite connected  $f_{ms}$ -BG with  $A_E$  representation-infinite. Denote  $\tau$  the involution of E as a Brauer G-set. Then for every  $e \in E$ ,  $\mathrm{DTr}(M_e) \cong M_{\sigma^{-1}(g\tau)^2(e)}$ , where DTr denotes the AR-translation of  $A_E$  and  $\sigma$  denotes the Nakayama automorphism of E.

*Proof.* If  $d(e)$ ,  $d(\tau(e)) > 1$ , then there are two arrows  $L(e)$ ,  $L(\tau(e))$  of  $Q'_E$  starting at  $P(e)$ . There exists an AR-sequence  $0 \to M_{\sigma^{-1}(g\tau)^2(e)} \to N \to M_e \to 0$ , where N is the string module given by the string

$$
L(g^{d(e)-2} \cdot e) \cdots L(g \cdot e) L(e) L(\tau(e))^{-1} L(g^{d(h)-2} \cdot h) \cdots L(g \cdot h) L(h)
$$

with  $h = \sigma^{-1}(g\tau)^2(e)$ .

If  $d(e) = 1$  and  $d(\tau(e)) > 1$ , then  $M_e$  is the simple  $A_E$ -module at the vertex  $P(e)$  of  $Q'_E$ . There exists an AR-sequence  $0 \to M_{\sigma^{-1}(g\tau)^2(e)} \to N \to M_e \to 0$ , where N is the string module given by the string

$$
L(\tau(e))^{-1}L(g^{d(h)-2}\cdot h)\cdots L(g\cdot h)L(h)
$$

with  $h = \sigma^{-1}(g\tau)^2(e)$ .

If  $d(\tau(e)) = 1$ , then the projective cover P of  $M_e$  is uniserial and  $M_e \cong P/\text{soc}(P)$ . So there exists an AR-sequence  $0 \to \text{rad}(P) \to P \oplus \text{rad}(P)/\text{soc}(P) \to M_e \to 0$ , where  $\text{rad}(P) \cong M_{q \cdot e}$ . Since  $d(\tau(e)) = 1$ ,  $g\tau(e) = \sigma\tau(e) = \tau\sigma(e)$ , and  $\sigma^{-1}(g\tau)^2(e) = \sigma^{-1}(g\tau)(\tau\sigma)(e) = \sigma^{-1}g\sigma(e) = g \cdot e$ . Then  $DTr(M_e) \cong rad(P) \cong M_{\sigma^{-1}(g\tau)^2(e)}$ .

<span id="page-32-1"></span>The following result should be compared with [\[2,](#page-38-3) Theorem 4.5] for BGAs.

**Proposition 5.3.** Let  $E = (E, P, L, d)$  be a finite connected  $f_{ms}$ -BG with  $A_E$  representationinfinite. Denote  $\tau$  the involution of E as a Brauer G-set,  $\sigma$  the Nakayama automorphism of E, and  $\sigma^{-1}(g\tau)^2 : E \to E$ ,  $e \mapsto \sigma^{-1}(g\tau)^2(e)$  the permutation on E. Then

- (1) There is a bijection between the set of exceptional tubes in the stable  $AR\text{-}quiver$ <sub>s</sub> $\Gamma_{A_E}$  of  $A_E$  and the set of  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of E.
- (2) The rank of an exceptional tube of  ${}_s\Gamma_{A_E}$  is equal to the length of the associated  $\langle \sigma^{-1}(g\tau)^2 \rangle$ orbit of E.

*Proof.* According to Lemma [5.1,](#page-31-0) there is a bijection between  $E$  and the set of string modules of  $A_E$  at the mouth of an exceptional tube of  ${}_s\Gamma_{A_E}$ . Moreover, by Lemma [5.2,](#page-31-1) the action of the AR-translation DTr on this set of modules corresponds to the permutation  $\sigma^{-1}(g\tau)^2$  on E.  $\Box$ 

## 5.2. Construction of domestic  $f_{ms}$ -BGAs.

Let E be a finite connected  $f_{ms}$ -BG with  $A_E$  domestic. By Theorem [2.29](#page-8-0) so does  $A_{R_E}$ , where

$$
R_E = \begin{cases} E/\langle \sigma \rangle, & \text{if } \langle \sigma \rangle \text{ is admissible;} \\ \widehat{E/\langle \sigma \rangle}, & \text{otherwise.} \end{cases}
$$

Suppose that the modified BG  $E/\langle \sigma \rangle$  has k-edges, l double half-edges, and n vertices  $v_1, \dots, v_n$ of f-degree  $d_1, \dots, d_n$ , respectively. By Lemma [3.12,](#page-16-0) there are only three possible cases:

(1)  $l = 2, k - n + 1 = 0, d_i = 1$  for  $1 \leq i \leq n$ ;

(2)  $l = 0, k - n + 1 = 0, d_i = 2$  for exactly two numbers  $i = i_0, i_1,$  and  $d_i = 1$  for  $i \neq i_0, i_1;$ 

(3)  $l = 0, k - n + 1 = 1, d_i = 1$  for  $1 \leq i \leq n$ .

<span id="page-32-0"></span>**Lemma 5.4.** In case (1), E is determined by  $E/\langle \sigma \rangle$  and the order of the Nakayama automorphism  $\sigma$  of E up to isomorphism.

*Proof.* Suppose that E and E' are two  $f_{ms}$ -BGs such that  $E/\langle \sigma \rangle \cong E'/\langle \sigma \rangle$  is a modified BG as in case (1) and the Nakayama automorphisms of  $E$  and  $E'$  have the same order. According to Lemma [3.7](#page-14-1) (2), the order of the Nakayama automorphisms of E is even, say  $2r$ .

By Proposition [3.5,](#page-12-2) the fundamental group of  $E/\langle \sigma \rangle$  is isomorphic to

$$
F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1 a, ac_2 = c_2 a, c_1^2 = c_2^2 = 1 \rangle,
$$

and by the proof of Proposition [3.13,](#page-16-3) the image of the fundamental group of E in  $F\langle a, c_1, c_2\rangle/\langle ac_1 =$  $c_1a, ac_2 = c_2a, c_1^2 = c_2^2 = 1$  is the subgroup of  $F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1a, ac_2 = c_2a, c_1^2 = c_2^2 = 1 \rangle$ formed by elements x which satisfies  $\rho(x)(1) = 1$ , where  $\rho : F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1 a, ac_2 = c_2 a,$  $c_1^2 = c_2^2 = 1$ )  $\rightarrow$   $S_{2r}$  is the group homomorphism given by  $\rho(\overline{a}) = (1 \ 2 \ \cdots \ 2r)$  and  $\rho(\overline{c_1}) = \rho(\overline{c_2}) = (1 \ 2 \ \cdots \ 2r)$  $(r+1)(2r+2)\cdots(r)$ . Since the same things also hold for E', the image of the fundamental groups of E and E' in  $F\langle a, c_1, c_2 \rangle / \langle ac_1 = c_1 a, ac_2 = c_2 a, c_1^2 = c_2^2 = 1 \rangle$  are equal. By Proposition [2.17,](#page-5-1) E and E' are isomorphic.  $\square$ 

Now we construct the  $f_{ms}$ -BG E with  $E/\langle \sigma \rangle$  in case (1). Suppose that the order of the Nakayama automorphism of E is 2r, where r is a positive integer. Let  $B = E/\langle \sigma \rangle = (B, B, \tau, d)$ and fix some  $b \in B$ . Since the diagram obtained by deleting the two double half-edges of the diagram of B is a tree, each element c of B can be expressed uniquely as the form  $(g\tau)^{j_c}(b)$ , where  $0 \leq j_c \leq 2n-1$ . For every vertex v of B, let  $b_v$  be the half-edge in v with  $j_{b_v}$  smallest. Define a  $f_{ms}$ -BG  $E_r = (E_r, E_r, \tau, d)$  as follows:  $E_r = \{(c, j) | c \in B, j \in \{1, 2, \cdots, 2r\} = \mathbb{Z}/2r\mathbb{Z}\}$ ; for every  $(c, j) \in B$ , define

$$
g \cdot (c, j) = \begin{cases} (g \cdot c, j), & \text{if } g \cdot c \neq b_v \text{ for any vertex } v \text{ of } B; \\ (g \cdot c, j + 1), & \text{if } g \cdot c = b_v \text{ for some vertex } v \text{ of } B, \end{cases}
$$

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$$
\tau(c,j) = \begin{cases} (\tau(c),j), \text{ if } c \text{ is not a double half-edge of } B; \\ (c,j+r), \text{ if } c \text{ is a double half-edge of } B, \end{cases}
$$

<span id="page-33-1"></span>and  $d(c, j) = d(c)$ .

**Proposition 5.5.** In case (1), E is isomorphic to  $E_r$ .

*Proof.* Since  $B = E/\langle \sigma \rangle$  is f-degree trivial,  $d(c) = |G \cdot c|$  for each  $c \in B$ . For every  $(c, j) \in E_r$ , since  $d(c, j) = d(c)$ , it can be shown that the Nakayama automorphism  $\sigma$  of  $E_r$  is given by  $\sigma(c, j) = (c, j + 1)$ . So the covering  $E_r \to B$ ,  $(c, j) \mapsto c$  induces an isomorphism  $E_r/\langle \sigma \rangle \cong B$ . Since the order of the Nakayama automorphism of  $E_r$  is  $2r$ , by Lemma [5.4,](#page-32-0)  $E_r$  is isomorphic to  $E.$ 

<span id="page-33-0"></span>**Lemma 5.6.** In case (2), E is determined by  $E/\langle \sigma \rangle$  and the order of the Nakayama automorphism  $\sigma$  of E up to isomorphism.

*Proof.* The proof is similar to that of Lemma [5.4.](#page-32-0) Suppose that E and E' are two  $f_{ms}$ -BGs such that  $E/\langle \sigma \rangle \cong E'/\langle \sigma \rangle$  is a  $f_{ms}$ -BG as in case (2) and the Nakayama automorphisms of E and  $E'$  have the same order. According to the proof of Proposition [3.13,](#page-16-3) the order of the Nakayama automorphisms of E is odd, say  $2r - 1$ .

By [\[8,](#page-38-5) Proposition 5.9], the fundamental group of  $E/\langle \sigma \rangle$  is isomorphic to  $F\langle a, b \rangle/\langle a^2 = b^2 \rangle$ , and by the proof of Proposition [3.13,](#page-16-3) the image of the fundamental group of E in  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$ is the subgroup of  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$  formed by elements x which satisfies  $\rho(x)(1) = 1$ , where  $\rho: F\langle a,b\rangle/\langle a^2 = b^2 \rangle \rightarrow S_{2r-1}$  is the group homomorphism given by  $\rho(\overline{a}) = \rho(\overline{b}) = (1 \ 2 \cdots 2r-1).$ Since the same things also hold for  $E'$ , the images of the fundamental groups of  $E$  and  $E'$  in  $F\langle a,b\rangle/\langle a^2 = b^2\rangle$  are equal. By [\[8,](#page-38-5) Proposition 2.32], E and E' are isomorphic.

Now we construct the  $f_{ms}$ -BG E with  $E/\langle \sigma \rangle$  in case (2). This construction is similar to that in case (1). Suppose that the order of the Nakayama automorphism of E is  $2r-1$ , where r is a positive integer. Let  $B = E/\langle \sigma \rangle = (B, B, \tau, d)$  and fix some  $b \in B$ . Since the diagram of B is a tree, each element c of B can be expressed uniquely as the form  $(g\tau)^{j_c}(b)$ , where  $0 \leq j_c \leq 2n-3$ . For every vertex v of B, let  $b_v$  be the half-edge in v such that  $j_{b_v}$  is smallest. Define a  $f_{ms}$ -BG  $E'_{r} = (E'_{r}, E'_{r}, \tau, d)$  as follows:  $E'_{r} = \{(c, j) | c \in B, j \in \{1, 2, \cdots, 2r - 1\} = \mathbb{Z}/(2r - 1)\mathbb{Z}\}$ ; for every  $(c, j) \in E'_r$ , define

$$
g \cdot (c, j) = \begin{cases} (g \cdot c, j), \text{ if } g \cdot c \neq b_v \text{ for any vertex } v \text{ of } B; \\ (g \cdot c, j + 1), \text{ if } g \cdot c = b_{v_i} \text{ for some vertex } v_i \text{ of } B \text{ with } i \neq i_0, i_1; \\ (g \cdot c, j + r), \text{ if } g \cdot c = b_{v_i} \text{ for some vertex } v_i \text{ of } B \text{ with } i = i_0 \text{ or } i = i_1, \end{cases}
$$

<span id="page-33-2"></span> $\tau(c, j) = (\tau(c), j)$ , and  $d(c, j) = d(c)$ .

**Proposition 5.7.** In case (2), E is isomorphic to  $E'_r$ .

*Proof.* For every  $(c, j) \in E'_r$ , since

$$
d(c, j) = \begin{cases} |G \cdot c|, & \text{if } c \notin v_{i_0}, v_{i_1}; \\ 2|G \cdot c|, & \text{if } c \in v_{i_0} \text{ or } c \in v_{i_1} \end{cases}
$$

we have  $\sigma(c, j) = (c, j+1)$ . Therefore  $E'_r/\langle \sigma \rangle \cong B$ . Since the order of the Nakayama automorphism of  $E'_r$  is  $2r-1$ , by Lemma [5.6,](#page-33-0)  $E'_r$  is isomorphic to  $E$ . □

,

Let  $B = E/\langle \sigma \rangle = (B, B, \tau, d)$  be the  $f_{ms}$ -BG in case (3). Then the diagram of B is a graph with a unique cycle. Suppose the length of this cycle is  $m$ , and there are  $p$  edges outside this cycle and q edges inside this cycle. Then we have  $m + p + q = n$ . Denote  $q\tau$  the permutation on B mapping each  $c \in B$  to  $(g\tau)(c)$ . It is straightforward to show that B has exactly two  $\langle g\tau \rangle$ -orbits, one of length  $m + 2p$  containing every half-edge outside the unique cycle of B, and the other of length  $m + 2q$  containing every half-edge inside the unique cycle of B. We call a half-edge of B outer (resp. inner) if it belongs to the first (resp. second)  $\langle q\tau \rangle$ -orbit.

Now we construct the  $f_{ms}$ -BG E with  $E/\langle \sigma \rangle$  in case (3). Fix an outer half-edge  $b \in B = E/\langle \sigma \rangle$ belonging to the unique cycle of B. Then  $\tau(b)$  is an inner half-edge of B. For every outer (resp. inner) half-edge c of B, there exists a unique integer  $0 \leq j_c \leq m+2p-1$  (resp.  $0 \leq j_c' \leq m+2q-1$ ) such that  $c = (g\tau)^{j_c}(b)$  (resp.  $c = (g\tau)^{j_c'}(\tau(b))$ ). For every vertex v of B, define a half-edge  $b_v \in v$ as follows: if v contains an outer half-edge, define  $b_v$  as the outer half-edge in v with  $j_{b_v}$  smallest; if v does not contain any outer half-edge, define  $b_v$  as the inner half-edge in v with  $j'_{b_v}$  smallest.

Suppose that the order of the Nakayama automorphism  $\sigma$  of E is r. For every integer  $1 \leq l \leq r$ , define a  $f_{ms}$ -BG  $E_{rl} = (E_{rl}, E_{rl}, \tau, d)$  as follows:  $E_{rl} = \{(c, j) | c \in B, j \in \{1, \cdots, r\} = \mathbb{Z}/r\mathbb{Z}\}$ ; for every  $(c, j) \in E_{rl}$ , define

$$
g \cdot (c, j) = \begin{cases} (g \cdot c, j), & \text{if } g \cdot c \neq b_v \text{ for any vertex } v \text{ of } B; \\ (g \cdot c, j + 1), & \text{if } g \cdot c = b_v \text{ for some vertex } v \text{ of } B, \end{cases}
$$

$$
\tau(c, j) = \begin{cases} (\tau(c), j), & \text{if } c \neq b \text{ and } c \neq \tau(b); \\ (\tau(c), j + l), & \text{if } c = b; \\ (\tau(c), j - l), & \text{if } c = \tau(b), \end{cases}
$$

<span id="page-34-0"></span>and  $d(c, j) = d(c)$ .

**Proposition 5.8.** In case (3), E is isomorphic to some  $E_{rl}$  with  $1 \le l \le r$ .

We need some preparations before we prove Proposition [5.8.](#page-34-0)

According to the proof of [\[8,](#page-38-5) Proposition 5.9], we have an isomorphism  $u : \Pi(B, b) \to \mathbb{Z} \oplus \mathbb{Z}$  with  $u((b|g^{d(b)}|b)) = (1,0)$  and  $u((b|(g\tau)^{m+2p}|b)) = (0,1)$ . Let H be the image of the composition map  $\Pi(E, e) \to \Pi(B, b) \stackrel{u}{\to} \mathbb{Z} \oplus \mathbb{Z}$ , where  $e \in E$  is a preimage of b in E (the subgroup H of  $\mathbb{Z} \oplus \mathbb{Z}$  does not depend on the choice of e). Let  $A = \{(a, 0) | a \in \mathbb{Z}\}, B = \{(0, a) | a \in \mathbb{Z}\}$  be two subgroups of  $\mathbb{Z} \oplus \mathbb{Z}$ . Since E is connected and since the order of the Nakayama automorphism  $\sigma$  of E is r, the closed walk  $(b|g^{kd(b)}|b)$  of B at b lifts to a closed walk of E at e if and only if  $r \mid k$ . Therefore  $H \cap A = rA$ . Moreover, since the covering  $E \to B = E/\langle \sigma \rangle$  is r-sheeted,  $[\mathbb{Z} \oplus \mathbb{Z} : H] = r$ .

<span id="page-34-1"></span>**Lemma 5.9.** H is a free abelian subgroup of  $\mathbb{Z} \oplus \mathbb{Z}$  generated by  $(r, 0)$  and  $(i, 1)$ , where i is some integer with  $0 \leq i \leq r-1$ .

*Proof.* We have  $(A + H)/H \cong A/(A \cap H) \cong \mathbb{Z}/r\mathbb{Z}$ , and  $[A + H : H] = r$ . Since  $[\mathbb{Z} \oplus \mathbb{Z} : H] = r$ ,  $A + H = \mathbb{Z} \oplus \mathbb{Z}$ . Since  $(0,1) \in A + H$ , there exists some  $h \in H$  such that  $(0,1) \in A + h$ . So  $h = (i, 1)$  for some  $i \in \mathbb{Z}$ . Let H' be the subgroup of H generated by  $(r, 0)$  and  $(i, 1)$ . Since  $[\mathbb{Z} \oplus \mathbb{Z} : H'] = r$ ,  $H' = H$ . Then H is generated by  $(r, 0)$  and  $(i, 1)$ , and we may choose i to be an integer such that  $0 \leq i \leq r-1$ .

**Proof of Proposition [5.8.](#page-34-0)** Since the f-degree of B is trivial, the Nakayama automorphism  $\sigma$ of  $E_{rl}$  is given by  $\sigma(c, j) = (c, j + 1)$ . Then the covering  $p : E_{rl} \to B$ ,  $(c, j) \mapsto c$  induces an isomorphism  $\underline{E_{rl}}/\langle \sigma \rangle \cong B$ . Let  $u : \Pi(B, b) \to \mathbb{Z} \oplus \mathbb{Z}$  be the isomorphism with  $u(\overline{(b|g^{d(b)}|b)}) =$  $(1,0)$  and  $u(\overline{(b|(g\tau)^{m+2p}|b)}) = (0,1)$ . In order to calculate the image of the composition map  $\Pi(E_{rl},(b,1)) \longrightarrow^{\mathcal{P}*} \Pi(B,b) \longrightarrow^{\mathcal{U}} \mathbb{Z} \oplus \mathbb{Z}$ , we first consider the action of  $\mathbb{Z} \oplus \mathbb{Z}$  on  $p^{-1}(b) = \{(b,j) \mid j \in$  $\{1, \dots, r\} = \mathbb{Z}/r\mathbb{Z}\}\$ via the isomorphism  $u : \Pi(B, b) \to \mathbb{Z} \oplus \mathbb{Z}$ .

We have  $(1,0)\cdot(b,j) = g^{d(b)}\cdot(b,j) = \sigma(b,j) = (b,j+1)$ . Since the number of outer half-edges of B of the form  $b_v$  with v a vertex of B is  $m+p$ , we have  $(0,1)\cdot (b,j) = (g\tau)^{m+2p}(b,j) =$  $(b, j+l+m+p)$ . Let  $K_l$  be the image of the composition map  $\Pi(E_{rl},(b,1)) \stackrel{p_*}{\longrightarrow} \Pi(B,e) \stackrel{u}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z}$ . Then  $K_l = \{x \in \mathbb{Z} \oplus \mathbb{Z} \mid x \cdot (b, 1) = (b, 1)\}\$ , which is the subgroup of  $\mathbb{Z} \oplus \mathbb{Z}$  generated by  $(r, 0)$  and  $(-(l+m+p), 1)$ . According to Lemma [5.9,](#page-34-1) there exists some  $1 \leq l \leq r$  such that  $K_l = H$ . By [\[8,](#page-38-5) Proposition 2.32].  $F_{nl}$  is isomorphic to  $F$ . Proposition 2.32,  $E_{rl}$  is isomorphic to E.

We remark that the  $f_{ms}$ -BGs constructed in this subsection may not be pairwise non-isomorphic, although every finite connected  $f_{ms}$ -BG E with  $A_E$  domestic is isomorphic to one of the  $f_{ms}$ -BG constructed in this subsection.

## 5.3. Stable AR-components of domestic  $f_{ms}$ -BGAs.

In this subsection, we always denote  $\tau$  the involution and  $\sigma$  the Nakayama automorphism for a Brauer G-set if there is no confusion.

Let E be a finite connected  $f_{ms}$ -BG such that  $E/(\sigma)$  is a Brauer G-set in case (1), that is,  $B = E/\langle \sigma \rangle$  is a f-degree trivial modified BG with *n* vertices, *n* – 1 edges and 2 double half-edges. Denote  $\sigma^{-1}(g\tau)^2$  the permutation on E mapping each  $e \in E$  to  $\sigma^{-1}(g\tau)^2(e)$ . Suppose that the order of the Nakayama automorphism  $\sigma$  of E is 2r.

<span id="page-35-0"></span>**Lemma 5.10.** For every  $e \in E$ , the length of the  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of e is n.

*Proof.* According to Proposition [5.5,](#page-33-1) E is isomorphic to  $E_r$  (the definition of  $E_r$  is given be-fore Proposition [5.5\)](#page-33-1). For every  $(c, j) \in E_r$ , let N be the minimal positive integer such that  $(\sigma^{-1}(g\tau)^2)^N(c,j) = (c,j)$ . Note that the Nakayama automorphism  $\sigma$  of B is identity. So n is the minimal positive integer such that  $({\sigma}^{-1}(g\tau)^2)^n(c) = c$ . Since there exists a covering of Brauer G-sets  $p: E_r \to B$  which maps each  $(c', j') \in E_r$  to  $c' \in B$ , we have

$$
(\sigma^{-1}(g\tau)^2)^N(c) = (\sigma^{-1}(g\tau)^2)^N(p(c,j)) = p((\sigma^{-1}(g\tau)^2)^N(c,j)) = p(c,j) = c.
$$

Therefore  $n \mid N$ .

Since  $\{(g\tau)(c), (g\tau)^2(c), \cdots, (g\tau)^{2n}(c)\}=B$ , there are exactly *n* numbers  $i \in \{1, 2, \cdots, 2n\}$ such that  $(g\tau)^i(c)$  is of the form  $b_v$  for some vertex v of B. Moreover, since B contains two double half-edges and since  $\{c, (g\tau)(c), \cdots, (g\tau)^{2n-1}(c)\} = B$ , there are exactly two numbers  $i \in \{0, 1, \dots, 2n-1\}$  such that  $(g\tau)^i(c)$  is a double half-edge of B. So we have  $(g\tau)^{2n}(c, j) =$  $((g\tau)^{2n}(c), j+n+2r) = (c, j+n)$ . Therefore  $(\sigma^{-1}(g\tau)^2)^n(c, j) = \sigma^{-n}(g\tau)^{2n}(c, j) = \sigma^{-n}(c, j+n) =$  $(c, j)$  and  $N | n$ . Then we have  $N = n$ .

**Proposition 5.11.** Let E be a finite connected  $f_{ms}$ -BG such that  $E/\langle \sigma \rangle$  is a Brauer G-set in case (1). Suppose that the order of the Nakayama automorphism of E is  $2r$ . Then  $\binom{1}{s}$   $\Gamma_{A_E}$  is a disjoint union of  $4r$  components of the form  $\mathbb{Z}A_{\infty}/\langle \tau^n \rangle$ ,  $2r$  components of the form  $\mathbb{Z}\widetilde{A}_{n,n}$ , and infinitely many components of the form  $\mathbb{Z}A_{\infty}/\langle \tau \rangle$ .

*Proof.* Since E has  $4nr$  half-edges, by Lemma [5.10,](#page-35-0) E contains  $4r \langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits, each of length n. By Proposition [5.3,](#page-32-1)  $_{s}\Gamma_{A_{E}}$  contains 4r exceptional tubes, where the rank of each tube is n. Now the result follows from [\[4,](#page-38-4) Theorem 2.1].

Let E be a finite connected  $f_{ms}$ -BG such that  $E/(\sigma)$  is a Brauer G-set in case (2), that is,  $B = E/\langle \sigma \rangle$  is a Brauer graph whose underlying diagram is a tree with n vertices  $v_1, \dots, v_n$  of f-degree  $d_1, \dots, d_n$ , respectively, such that  $d_i = 2$  for exactly two numbers  $i = i_0, i_1$  and  $d_i = 1$  for  $i \neq i_0$ ,  $i_1$ . Denote  $\sigma^{-1}(g\tau)^2$  the permutation on E mapping each  $e \in E$  to  $\sigma^{-1}(g\tau)^2(e)$ . Suppose that the order of the Nakayama automorphism  $\sigma$  of E is  $2r - 1$ .

<span id="page-35-1"></span>**Lemma 5.12.** For every  $e \in E$ , the length of the  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of e is  $n-1$ .

*Proof.* According to Proposition [5.7,](#page-33-2) E is isomorphic to  $E'_r$  (the definition of  $E'_r$  is given be-fore Proposition [5.7\)](#page-33-2). For every  $(c, j) \in E'_r$ , let N be the minimal positive integer such that  $(\sigma^{-1}(g\tau)^2)^N(c,j) = (c,j)$ . Note that the Nakayama automorphism  $\sigma$  of B is identity. So  $n-1$ is the minimal positive integer such that  $({\sigma}^{-1}(g\tau)^2)^{n-1}(c) = c$ . Since there exists a covering of Brauer G-sets  $p: E'_r \to B$  which maps each  $(c', j') \in E'_r$  to  $c' \in B$ , we have

$$
(\sigma^{-1}(g\tau)^2)^N(c) = (\sigma^{-1}(g\tau)^2)^N(p(c,j)) = p((\sigma^{-1}(g\tau)^2)^N(c,j)) = p(c,j) = c.
$$

Therefore  $(n-1)$  | N.

Since  $\{(g\tau)(c), (g\tau)^2(c), \cdots, (g\tau)^{2n-2}(c)\}=B$ , for each vertex v of B, there is exactly one number  $i \in \{1, 2, \dots, 2n-2\}$  such that  $(g\tau)^i(c) = b_v$ . So we have  $(g\tau)^{2n-2}(c, j) = ((g\tau)^{2n-2}(c), j+n-1)$  $(2+2r) = (c, j+n-2+2r)$ . Since  $\sigma(c', j') = (c, j'+1)$  for every  $(c', j') \in E'_r$ ,  $(\sigma^{-1}(g\tau)^2)^{n-1}(c, j) =$  $\sigma^{-n+1}(g\tau)^{2n-2}(c,j) = \sigma^{-n+1}(c,j+n-2+2r) = (c,j+2r-1) = (c,j)$  and  $N \mid (n-1)$ . Then we have  $N = n - 1$ . **Proposition 5.13.** Let E be a finite connected  $f_{ms}$ -BG such that  $E/\langle \sigma \rangle$  is a Brauer G-set in case (2) and suppose that the order of the Nakayama automorphism of E is  $2r - 1$ . Then  $_s \Gamma_{A_E}$  is a disjoint union of  $4r-2$  components of the form  $\mathbb{Z}A_{\infty}/\langle\tau^{n-1}\rangle$ ,  $2r-1$  components of the form  $\mathbb{Z}A_{n-1,n-1}$ , and infinitely many components of the form  $\mathbb{Z}A_{\infty}/\langle \tau \rangle$ .

*Proof.* Since E has  $(4r-2)(n-1)$  half-edges, by Lemma [5.12,](#page-35-1) E contains  $4r-2 \langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits, each of length  $n-1$ . By Proposition [5.3,](#page-32-1)  ${}_{s}\Gamma_{A_{E}}$  contains  $4r-2$  exceptional tubes, where the rank of each tube is  $n-1$ . Now the result follows from [\[4,](#page-38-4) Theorem 2.1]. □

Let E be a finite connected  $f_{ms}$ -BG such that  $E/\langle \sigma \rangle$  is a Brauer G-set in case (3), that is, B is a Brauer graph with trivial f-degree whose diagram contains a unique cycle. Suppose that the length of this cycle is  $m$ , and suppose that there are  $p$  edges outside this cycle and  $q$  edges inside this cycle, where  $n = m + p + q$ . Fix an outer half-edge b of B which belongs to the unique cycle of B, and suppose that the order of the Nakayama automorphism of E is  $r$ . According to Proposition [5.8,](#page-34-0) E is isomorphic to some  $E_{rl}$  with  $1 \leq l \leq r$ , where the definition of  $E_{rl}$ 's are given before Proposition [5.8.](#page-34-0)

<span id="page-36-0"></span>Denote  $\sigma^{-1}(g\tau)^2$  the permutation on E mapping each  $e \in E$  to  $\sigma^{-1}(g\tau)^2(e)$ .

**Lemma 5.14.** Under the assumptions above, when m is odd,  $E \cong E_{rl}$  contains  $(r, m + 2l)$  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of length  $\frac{r(m+2p)}{(r,m+2l)}$  and  $(r,m+2l)$   $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of length  $\frac{r(m+2q)}{(r,m+2l)}$ , and when m is even,  $E \cong E_{rl}$  contains  $(2r, m + 2l)$   $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of length  $\frac{r(m+2p)}{(2r,m+2l)}$  and  $(2r, m + 2l)$  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of length  $\frac{r(m+2q)}{(2r,m+2l)}$ , where  $(a, b)$  denotes the greatest common divisor of a and b.

*Proof.* Denote  $f: E_{rl} \to B$  the covering of Brauer G-sets which maps each  $(c, j) \in E_{rl}$  to  $c \in B$ . Note that B is a disjoint union of two  $\langle g\tau \rangle$ -orbits  $b^{\langle g\tau \rangle}$  and  $\tau(b)^{\langle g\tau \rangle}$ , where the length of  $b^{\langle g\tau \rangle}$  is  $m+2p$  and the length of  $\tau(b)^{\langle g\tau\rangle}$  is  $m+2q$  (here  $g\tau$  denotes the permutation on B mapping each  $c \in B$  to  $g\tau(c)$ ).

When m is odd, since both  $m+2p$  and  $m+2q$  are odd, the  $\langle g\tau \rangle$ -orbits  $b^{\langle g\tau \rangle}$  and  $\tau(b)^{\langle g\tau \rangle}$  of B are also  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of B (note that the Nakayama automorphism  $\sigma$  of B is identity). Since the projection of each  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of  $E_{rl}$  under f is a  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of B, each  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of  $E_{rl}$  is of the form  $(b, j)^{(\sigma^{-1}(g\tau)^2)}$  or of the form  $(\tau(b), j)^{(\sigma^{-1}(g\tau)^2)}$ , where  $j \in \{1, \cdots, r\} = \mathbb{Z}/r\mathbb{Z}$ .

Let N be the minimal positive integer such that  $(\sigma^{-1}(g\tau)^2)^N(b, j) = (b, j)$ . Since  $m + 2p$  is the minimal positive integer such that  $({\sigma}^{-1}(g\tau)^2)^{m+2p}(b) = b$ , we have

$$
(\sigma^{-1}(g\tau)^2)^N(b) = (\sigma^{-1}(g\tau)^2)^N(f(b,j)) = f((\sigma^{-1}(g\tau)^2)^N(b,j)) = f(b,j) = b.
$$

Therefore  $(m+2p) \mid N$ . Note that  $b^{(g\tau)} = \{b, g\tau(b), \cdots, (g\tau)^{m+2p-1}(b)\}\)$  contains  $m+p$  half-edges of the form  $b_v$ , and  $b \in b^{\langle g\tau \rangle}$ ,  $\tau(b) \notin b^{\langle g\tau \rangle}$ . Therefore

$$
(g\tau)^{m+2p}(b,j) = ((g\tau)^{m+2p}(b), j+m+p+l) = (b, j+m+p+l),
$$

and

 $(\sigma^{-1}(g\tau)^2)^{m+2p}(b, j) = \sigma^{-(m+2p)}(g\tau)^{2(m+2p)}(b, j) = (b, j+2(m+p+l)-(m+2p)) = (b, j+m+2l).$ Since  $N' = \frac{r}{(r m)}$  $\frac{r}{(r,m+2l)}$  is the minimal positive integer such that  $((\sigma^{-1}(g\tau)^2)^{m+2p})^{N'}(b,j) = (b,j),$  $N = (m+2p)N' = \frac{r(m+2p)}{(r(m+2p))}$  $\frac{r(m+2p)}{(r,m+2l)}$ . Moreover, two half-edges  $(b, j_1)$ ,  $(b, j_2)$  of  $E_{rl}$  belong to the same  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit if and only if  $(r, m+2l)$  divides  $j_1-j_2$ . Therefore there are  $(r, m+2l)$   $\langle \sigma^{-1}(g\tau)^2 \rangle$ orbits of  $E_{rl}$  of the form  $(b, j)^{\langle \sigma^{-1}(gr)^2 \rangle}$ , each of length  $\frac{r(m+2p)}{(r,m+2l)}$ .

Let M be the minimal positive integer such that  $({\sigma}^{-1}(g\tau)^2)^M(\tau(b),j) = (\tau(b),j)$ . Since  $m+2q$ is the minimal positive integer such that  $({\sigma}^{-1}(g\tau)^2)^{m+2q}(\tau(b)) = \tau(b)$ ,  $m + 2q$  divides M. Since  $\tau(b)^{\langle g\tau\rangle} = \{\tau(b), g\tau(\tau(b)), \cdots, (g\tau)^{m+2q-1}(\tau(b))\}$  contains q half-edges of the form  $b_v$ , and since  $b \notin \tau(b)^{\langle g\tau \rangle}, \tau(b) \in \tau(b)^{\langle g\tau \rangle},$  we have

$$
(g\tau)^{m+2q}(\tau(b),j) = ((g\tau)^{m+2q}(\tau(b)),j+q-l) = (\tau(b),j+q-l).
$$

$$
(\sigma^{-1}(g\tau)^2)^{m+2q}(\tau(b),j) = \sigma^{-(m+2q)}(g\tau)^{2(m+2q)}(\tau(b),j)
$$
  
=  $(\tau(b),j+2(q-l)-(m+2q)) = (\tau(b),j-(m+2l)).$ 

Then  $M' = \frac{r}{(r,m+2l)}$  is the minimal positive integer such that  $((\sigma^{-1}(g\tau)^2)^{m+2q})^{M'}(\tau(b),j)$  $(\tau(b), j)$ , and  $M = (m + 2q)M' = \frac{r(m+2q)}{(r(m+2q))}$  $\frac{r(m+2q)}{(r,m+2l)}$ . Moreover, two half-edges  $(\tau(b), j_1)$ ,  $(\tau(b), j_2)$  of  $E_{rl}$ belong to the same  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit if and only if  $(r, m + 2l)$  divides  $j_1 - j_2$ . Therefore there are  $(r, m+2l) \langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of  $E_{rl}$  of the form  $(\tau(b), j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , each of length  $\frac{r(m+2q)}{(r,m+2l)}$ .

When m is even, since both  $m + 2p$  and  $m + 2q$  are even, the  $\langle g\tau \rangle$ -orbits  $b^{\langle g\tau \rangle}$  of B splits into two  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits  $b^{\langle \sigma^{-1}(g\tau)^2 \rangle}$  and  $g\tau(b)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , each of length  $\frac{m}{2} + p$ , and the  $\langle g\tau \rangle$ -orbits  $\tau(b)^{\langle g\tau\rangle}$  of B splits into two  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits  $\tau(b)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$  and  $(g \cdot b)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , each of length  $\frac{m}{2} + q$  (note that the Nakayama automorphism  $\sigma$  of B is identity). Since the projection of each  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of  $E_{rl}$  under f is a  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of  $B$ , each  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit of  $E_{rl}$  is equal to one of the following  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of  $E_{rl}$ :  $(b, j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ ,  $(g\tau(b), j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ ,  $(\tau(b), j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ ,  $(g \cdot b, j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , where  $j \in \{1, \cdots, r\} = \mathbb{Z}/r\mathbb{Z}$ .

Let N be the minimal positive integer such that  $({\sigma}^{-1}(g\tau)^2)^N(b,j) = (b,j)$ . Since  $\frac{m}{2} + p$  is the minimal positive integer such that  $(\sigma^{-1}(g\tau))^2)^{\frac{m}{2}+p}(b) = b$ , we have

$$
(\sigma^{-1}(g\tau)^2)^N(b) = (\sigma^{-1}(g\tau)^2)^N(f(b,j)) = f((\sigma^{-1}(g\tau)^2)^N(b,j)) = f(b,j) = b.
$$

Therefore  $(\frac{m}{2} + p) \mid N$ . Similar to the case m odd, we have  $(g\tau)^{m+2p}(b, j) = (b, j + m + p + l)$ . Then

$$
(\sigma^{-1}(g\tau)^2)^{\frac{m}{2}+p}(b,j) = \sigma^{-(\frac{m}{2}+p)}(g\tau)^{m+2p}(b,j) = (b,j+m+p+l-(\frac{m}{2}+p)) = (b,j+\frac{m}{2}+l).
$$

Since  $N' = \frac{r}{(r, \frac{m}{2} + l)}$  is the minimal positive integer such that  $((\sigma^{-1}(g\tau)^2)^{\frac{m}{2} + p})^{N'}(b, j) = (b, j)$ ,

$$
N = \left(\frac{m}{2} + p\right)N' = \frac{r(m+2p)}{(2r, m+2l)}.
$$

Moreover, two half-edges  $(b, j_1)$ ,  $(b, j_2)$  of  $E_{rl}$  belong to the same  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit if and only if  $(r, \frac{m}{2} + l)$  divides  $j_1 - j_2$ . Therefore there are  $(r, \frac{m}{2} + l) \langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of  $E_{rl}$  of the form  $(b, j)^{(\sigma^{-1}(g\tau)^2)}$ , each of length  $\frac{r(m+2p)}{(2r,m+2l)}$ . Similarly it can be shown that there are  $(r, \frac{m}{2} + l)$  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of  $E_{rl}$  of the form  $(g\tau(b), j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , each of length  $\frac{r(m+2p)}{(2r,m+2l)}$ .

Let M be the minimal positive integer such that  $(\sigma^{-1}(g\tau)^2)^M(\tau(b),j) = (\tau(b),j)$ . Since  $\frac{m}{2} + q$  is the minimal positive integer such that  $(\sigma^{-1}(g\tau)^2)^{\frac{m}{2}+q}(\tau(b)) = \tau(b)$ , we have  $(\frac{m}{2}+q) | M$ . Similar to the case m odd, we have  $(g\tau)^{m+2q}(\tau(b),j)=(\tau(b),j+q-l)$ . Then

$$
(\sigma^{-1}(g\tau)^2)^{\frac{m}{2}+q}(\tau(b),j) = \sigma^{-(\frac{m}{2}+q)}(g\tau)^{m+2q}(\tau(b),j) = (\tau(b),j+q-l-(\frac{m}{2}+q)) = (\tau(b),j-(\frac{m}{2}+l)).
$$
  
Since  $M' = \frac{r}{(\tau,\frac{m}{2}+l)}$  is the minimal positive integer such that  $((\sigma^{-1}(g\tau)^2)^{\frac{m}{2}+q})^{M'}(\tau(b),j) = (\tau(b),j),$ 

$$
M = (\frac{m}{2} + q)M' = \frac{r(m + 2q)}{(2r, m + 2l)}.
$$

Moreover, two half-edges  $(\tau(b), j_1)$ ,  $(\tau(b), j_2)$  of  $E_{rl}$  belong to the same  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbit if and only if  $(r, \frac{m}{2} + l)$  divides  $j_1 - j_2$ . Therefore there are  $(r, \frac{m}{2} + l) \langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of  $E_{rl}$  of the form  $(\tau(b), j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , each of length  $\frac{r(m+2q)}{(2r,m+2l)}$ . Similarly it can be shown that there are  $(r, \frac{m}{2} + l)$  $\langle \sigma^{-1}(g\tau)^2 \rangle$ -orbits of  $E_{rl}$  of the form  $(g \cdot b, j)^{\langle \sigma^{-1}(g\tau)^2 \rangle}$ , each of length  $\frac{r(m+2q)}{(2r,m+2l)}$ .

By Lemma [5.14,](#page-36-0) Proposition [5.3](#page-32-1) and [\[4,](#page-38-4) Theorem 2.1], we have

So

**Proposition 5.15.** Let  $E \cong E_{rl}$  be a finite connected  $f_{ms}$ -BG such that  $E/\langle \sigma \rangle$  is a Brauer G-set in case (3), where the length of the unique cycle of  $B = E/\langle \sigma \rangle$  is m and the number of edges of B outside (resp. inside) this cycle is p (resp. q). If m is odd, then  $_s\Gamma_{A_E}$  is a disjoint union of  $(r, m + 2l)$  components of the form  $\mathbb{Z}A_{\infty}/\langle \tau^{\frac{r(m+2p)}{(r,m+2l)}} \rangle$ ,  $(r, m + 2l)$  components of the form  $\mathbb{Z}A_{\infty}/\langle \tau^{\frac{r(m+2q)}{(r,m+2l)}} \rangle$ ,  $(r, m + 2l)$  components of the form  $\mathbb{Z}\widetilde{A}_{\frac{r(m+2p)}{(r,m+2l)},\frac{r(m+2q)}{(r,m+2l)}}$ , and infinitely many components of the form  $\mathbb{Z}A_{\infty}/\langle\tau\rangle$ . If m is even, then  ${}_{s}\Gamma_{A_{E}}$  is a disjoint union of  $(2r, m+2l)$ components of the form  $\mathbb{Z}A_{\infty}/\langle \tau^{\frac{r(m+2p)}{(2r,m+2l)}} \rangle$ ,  $(2r, m + 2l)$  components of the form  $\mathbb{Z}A_{\infty}/\langle \tau^{\frac{r(m+2q)}{(2r,m+2l)}} \rangle$ ,  $(2r, m + 2l)$  components of the form  $\mathbb{Z}\widetilde{A}_{\frac{r(m+2p)}{(2r,m+2l)},\frac{r(m+2q)}{(2r,m+2l)}}$ , and infinitely many components of the form  $\mathbb{Z}A_{\infty}/\langle \tau \rangle$ .

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