

FRACTIONAL BRAUER CONFIGURATION ALGEBRAS I: DEFINITIONS AND EXAMPLES

NENGQUN LI AND YUMING LIU*

Abstract

In 2017, Green and Schroll introduced a generalization of Brauer graph algebras which they call Brauer configuration algebras. In the present paper, we further generalize Brauer configuration algebras to fractional Brauer configuration algebras by generalizing Brauer configurations to fractional Brauer configurations. The fractional Brauer configuration algebras are locally bounded but neither finite-dimensional nor symmetric in general. We show that if the fractional Brauer configuration is of type S (resp. of type MS), then the corresponding fractional Brauer configuration algebra is a locally bounded Frobenius algebra (resp. a locally bounded special multiserial Frobenius algebra). Moreover, we show that over an algebraically closed field, the class of finite-dimensional indecomposable representation-finite fractional Brauer configuration algebras in type S coincides with the class of basic indecomposable finite-dimensional standard representation-finite self-injective algebras.

1. INTRODUCTION

In [7], Green and Schroll introduced a generalization of Brauer graph algebras which they call Brauer configuration algebras. As each Brauer graph algebra is defined by a Brauer graph, each Brauer configuration algebra is defined by a Brauer configuration. Both a Brauer graph and a Brauer configuration are given by some combinatorial data, each of which encodes the representation theory of the corresponding algebra. It is known that over an algebraically closed field, the class of Brauer graph algebras coincides with the class of symmetric special biserial algebras (see for example [14]) and is closed under derived equivalence (see [2]) (here and throughout when we say “closed under derived equivalence”, we always mean “closed under derived equivalence up to Morita equivalence”.); and the class of Brauer configuration algebras coincides with the class of symmetric special multiserial algebras (see [8]).

In the present paper, we will give a further generalization of Brauer configurations which we call fractional Brauer configurations and define the corresponding fractional Brauer configuration categories and fractional Brauer configuration algebras.

Our motivation comes from two reasons. The first one is that unlike the Brauer graph algebras case, the class of Brauer configuration algebras is not closed under derived equivalence (see Example 6.7). The second one is related to the class of basic indecomposable finite-dimensional representation-finite self-injective algebras (abbr. RFS algebras). It is known that Brauer graph algebras of finite representation type are equal to Brauer tree algebras and form a subclass of RFS algebras of type A_n (see [15, 3]). We find that by allowing to take suitable fractional multiplicities in defining Brauer trees we can include naturally other subclass of RFS algebras of type A_n , however the algebras in such subclass are not symmetric any more.

Our first step is to define a fractional Brauer configuration. For simplicity, let us show how to define a fractional Brauer graph. A Brauer graph $\Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma)$ is defined by a set Γ_0 of vertices (with a multiplicity function $\mu : \Gamma_0 \rightarrow \mathbb{Z}_+$ on it) and a set Γ_1 of edges with an orientation σ on edges associated to every vertex. By viewing each edge as a set consisting of two half edges, we can redefine a vertex of Γ as an (cyclic-)ordered set of half edges associated to a vertex where

* Corresponding author.

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the (cyclic) order is induced from the original orientation σ . In other words, we can define a Brauer graph in a new way as follows. Starting from a finite G -set E (where $G = \langle g \rangle$ is an infinite cyclic group) whose elements are called half edges, we define the G -orbits as vertices of E and the orientation on a vertex is given by the G -action, and define the edges of E by a partition P on E such that each class $P(e)$ (here $P(e)$ denotes the equivalence class of P containing $e \in E$) consisting of exactly two half edges. We can define the multiplicity function $\mu : E \rightarrow \mathbb{Z}_+$ directly on the set E of half edges such that μ is constant on each G -orbit $G \cdot e$.

Using the above terminology, we recall that if A_Γ is the associated Brauer graph algebra, then the indecomposable projective module associated to an edge $P(e) = \{e, e'\}$ is biserial and one uniserial sequence reads along the G -orbit $G \cdot e = \{e, g \cdot e, g^2 \cdot e, \dots\}$ by $\mu(e)$ times, and another uniserial sequence reads along the G -orbit $G \cdot e'$ by $\mu(e')$ times. We now use a new function $d : E \rightarrow \mathbb{Z}_+$ such that d is constant on each G -orbit, which is called the degree function. And we can define a “modified Brauer graph algebra” by defining the indecomposable projective module associated to an edge $P(e) = \{e, e'\}$ as follows: it is still biserial and one uniserial sequence reads along the G -orbit $G \cdot e = \{e, g \cdot e, g^2 \cdot e, \dots\}$ in $d(e)$ steps, and another uniserial sequence reads along the G -orbit $G \cdot e'$ in $d(e')$ steps. It is easy to see that a “modified Brauer graph algebra” can be well-defined if we request a further condition: $P(e) = P(e')$ if and only if $P(g^{d(e)} \cdot e) = P(g^{d(e')} \cdot e')$ for all $e, e' \in E$. Note that a “modified Brauer graph algebra” is a Frobenius algebra but not symmetric in general (see Example 3.5). In Brauer graph algebra case, the multiplicity function $\mu(e)$ is equal to $\frac{d(e)}{|G \cdot e|}$ and $d(e)$ is an integral multiple of $|G \cdot e|$. However, in “modified Brauer graph algebra” case, $\frac{d(e)}{|G \cdot e|}$ can be taken a fractional value, which will be called the fractional-degree at e and is denoted by $d_f(e)$. The data $E = (E, P, d)$ defined as above will be called a fractional Brauer graph of type MS (see Section 3).

Similarly, a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma)$ is defined by a set Γ_0 of vertices (with a multiplicity function $\mu : \Gamma_0 \rightarrow \mathbb{Z}_+$ on it) and a set Γ_1 of polygons (each polygon is a multiset of vertices) with an orientation σ on polygons associated to every vertex. In order to define a fractional Brauer configuration, we start from a (finite or infinite) G -set E (where $G = \langle g \rangle$ is an infinite cyclic group) whose elements are called angles. By similar ideas as above, we can define vertices of E as G -orbits of the angles and polygons of E by some partition P of the angles (where $P(e)$ is a finite set for each $e \in E$) and associate a suitable degree function $d : E \rightarrow \mathbb{Z}_+$. We also use another partition L of E with the property that $L(e) \subseteq P(e)$ for each $e \in E$. The data $E = (E, P, L, d)$ defined as above will be called a fractional Brauer configuration. The precise definitions are given in Definition 3.3 and Remark 3.4. Here we use angles rather than vertices and polygons as basic elements, since this avoids to use multisets and the angles can be directly used to define the arrows in the associated quiver of E (see Section 4), and by this way it is also easier to define the morphisms between fractional Brauer configurations and the covering theory of fractional Brauer configurations (which are discussed in a forthcoming paper [10]).

We will define various types of fractional Brauer configurations. The types that we are most interested in are fractional Brauer configurations of type S (see Definition 3.13), whose definition use some identification relation for sequences of angles in E (see Definition 3.11). The identification relation is important in our definition since it makes the associated category (or algebra) of E not necessarily multiserial. The fractional Brauer configurations with trivial partition L are called fractional Brauer configurations of type MS (see Definition 3.15) and they form an important subclass of fractional Brauer configurations of type S and include the fractional Brauer graphs of type MS introduced as above.

Our second step is to associate each fractional Brauer configuration $E = (E, P, L, d)$ a locally bounded k -category $\Lambda_E = kQ_E/I_E$ (where Q_E is a locally finite quiver and I_E is an ideal of the path category kQ_E), which will be called the fractional Brauer configuration category of E . Roughly speaking, the vertices set of Q_E is given by polygons $\{P(e) \mid e \in E\}$, and the arrows set is given by $\{L(e) \mid e \in E\}$, where the arrow $L(e)$ has the source $P(e)$ and the terminal

$P(g \cdot e)$. The ideal I_E is defined by three kinds of relations which are similar to (but not equal to) defining relations for Brauer configuration algebras. The details are given in Definition 4.1 and in Definition 4.4. Note that in general Q_E is not the Gabriel quiver of Λ_E and I_E is not an admissible ideal of kQ_E . In type S, it is not hard to represent Λ_E as kQ'_E/I'_E where Q'_E is the Gabriel quiver of Λ_E and I'_E is an admissible ideal of kQ'_E (see Lemma 6.2). Similarly, we can define the fractional Brauer configuration algebra A_E of E , which is a locally bounded k -algebra (see Definition 5.1).

Our third step is to prove various properties of fractional Brauer configuration categories and fractional Brauer configuration algebras. Among them, we state the following:

- (1) If E is a fractional Brauer configuration, then Λ_E is a locally bounded k -category in the sense of Bongartz and Gabriel (see Theorem 4.6).
- (2) If E is a fractional Brauer configuration of type S, then Λ_E is a locally bounded Frobenius category (see Definition 4.7 and Theorem 4.20) so that the corresponding module category $\text{mod}\Lambda_E$ is a Frobenius category in the sense of Happel.
- (3) If E is a fractional Brauer configuration of type MS, then Λ_E is a locally bounded special multiserial Frobenius category (see Proposition 6.5).
- (4) If E is a finite fractional Brauer configuration of type S with integral f-degree, then A_E is a finite-dimensional symmetric algebra (see Proposition 5.5). In particular, if E is a finite fractional Brauer configuration of type MS with integral f-degree, then A_E is a Brauer configuration algebra (see Corollary 6.6).

Both (3) and (4) can be seen as proper generalizations of the following result in [8]: every Brauer configuration algebra is symmetric and special multiserial.

- (5) Over an algebraically closed field, the class of finite-dimensional representation-finite indecomposable fractional Brauer configuration algebras in type S coincides with the class of basic indecomposable finite-dimensional standard representation-finite self-injective algebras (see Theorem 7.14) and therefore this class of algebras is closed under derived equivalence.

This paper is organized as follows. In section 2, we give a quick review on Brauer configuration and Brauer configuration algebra. In Section 3, 4 and 5, we define the fractional Brauer configurations, the associated fractional Brauer configuration categories, and the associated fractional Brauer configuration algebras respectively. In particular, in Section 4 we define the notion of locally bounded Frobenius category and show that the fractional Brauer configuration category in type S is a locally bounded Frobenius category. In Section 6, we determine the Gabriel quivers and admissible relations of fractional Brauer configuration categories in type S over an algebraically closed field. Based on this, we show that the fractional Brauer configuration algebra in type MS is a locally bounded special multiserial Frobenius algebra. In Section 7, based on the description of the standard form of a locally representation-finite category by Bretscher and Gabriel and the criterion to determining when a locally representation-finite category is standard by Martínez-Villa and De La Peña, we describe the class of finite-dimensional indecomposable representation-finite fractional Brauer configuration algebras in type S over an algebraically closed field.

DATA AVAILABILITY

The datasets generated during the current study are available from the corresponding author on reasonable request.

2. A QUICK REVIEW ON BRAUER CONFIGURATION AND BRAUER CONFIGURATION ALGEBRA

Throughout this paper, let k be a field. We recall (in a slightly new way) the definitions of a Brauer configuration and a Brauer configuration algebra from [7, Section 1.1 and Section 2.2].

Definition 2.1. A Brauer configuration (abbr. BC) is a quadruple $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ which is defined as follows.

- (1) Γ_0 is a finite set, called the set of vertices;
- (2) Γ_1 is a finite collection of finite labelled multisets whose elements are in Γ_0 , which is called the set of polygons, such that each vertex $\alpha \in \Gamma_0$ is contained in some polygon $V \in \Gamma_1$ and each polygon contains at least two vertices;
- (3) μ is a function $\Gamma_0 \rightarrow \mathbb{Z}_+$, which is called the multiplicity function;
- (4) \mathfrak{o} is an orientation of Γ , that is, for each vertex $\alpha \in \Gamma_0$, a cyclic order over the set $C_\alpha = \{(V, \alpha, i) \mid V \in \Gamma_1 \text{ such that } \alpha \text{ occurs as a vertex in } V \text{ and } 1 \leq i \leq N_{V,\alpha}\}$, where $N_{V,\alpha}$ denotes the number of times that α occurs as a vertex in V .
- (5) For each polygon V , there exists a vertex $\alpha \in V$ such that $\text{val}(\alpha) \cdot \mu(\alpha) > 1$, where $\text{val}(\alpha)$ denotes the cardinal of the set C_α .

Remark 2.2. (1) We say that a vertex $\alpha \in \Gamma_0$ is truncated if $\text{val}(\alpha)\mu(\alpha) = 1$; that is, α occurs exactly once in exactly one $V \in \Gamma_1$ and $\mu(\alpha) = 1$.

(2) A Brauer configuration is called a Brauer graph (abbr. BG) if each polygon of it contains exactly two vertices.

(3) Note that in the above definition we have rewritten the orientation in Green and Schroll's original definition using the set C_α . We will call the elements of C_α as angles at the vertex α (see and compare Definition 3.1 below) and this viewpoint is critical in our definition of fractional Brauer configuration.

(4) The basic elements in Definition 2.1 are vertices and polygons and the new notion of angles are derived from them.

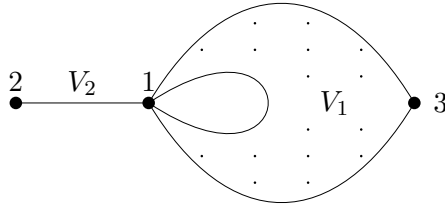
Example 2.3. Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ be a BC defined as follows. Let

$$\Gamma_0 = \{1, 2, 3\},$$

$$\Gamma_1 = \{V_1 = \{1, 1, 3\}, V_2 = \{1, 2\}\}.$$

We choose $\mu(3) = 2$, and $\mu(i) = 1$ for all other vertices. The orientation is given by: $(V_1, 1, 1) < (V_1, 1, 2) < (V_2, 1, 1) < (V_2, 2, 1)$ on C_1 , $(V_2, 2, 1) < (V_2, 2, 1)$ on C_2 , $(V_1, 3, 1) < (V_1, 3, 1)$ on C_3 .

The above BC can be realized by the following diagram:



Green and Schroll associate each BC Γ a quiver algebra kQ'_Γ/I'_Γ in [7]. Since we do not emphasize the truncated vertices in our formulation, the quiver Q_Γ and the ideal I_Γ defined below is slightly different from that in [7], in particular, here I_Γ is not an admissible ideal in general. However, the quotient algebras kQ_Γ/I_Γ and kQ'_Γ/I'_Γ are isomorphic.

The vertices of Q_Γ are in one to one correspondence with the polygons of Γ . For each vertex $\alpha \in \Gamma_0$ and for $(V_1, \alpha, i_1), (V_2, \alpha, i_2) \in C_\alpha$ such that (V_2, α, i_2) is the successor of (V_1, α, i_1) , there is an arrow from v_1 to v_2 in Q_Γ , where v_i is the vertex of Q_Γ which correspond to the polygon V_i of Γ for $i = 1, 2$.

Let a be an arrow of Q_Γ , which corresponds to the triple (V', α, i') being the successor of the triple (V, α, i) in C_α . Let the triple (V'', α, i'') be the successor of the triple (V', α, i') and let a' be an arrow of Q_Γ which corresponds to (V'', α, i'') being the successor of (V', α, i') in C_α . Define a bijection $\iota : (Q_\Gamma)_1 \rightarrow (Q_\Gamma)_1$ by mapping a to a' .

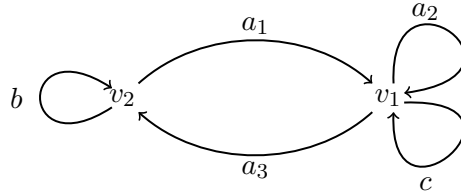
For each arrow a of Q_Γ which corresponds to the triple (V', α, i') being the successor of the triple (V, α, i) in C_α , define $p(a) = \iota^{n-1}(a) \cdots \iota(a)a$ to be a cycle in the quiver Q_Γ , where n is the smallest positive integer such that $\iota^n(a) = a$ (n equals to the cardinal of C_α). Call $p(a)$ a special α -cycle. Let I_Γ be the ideal of the path algebra kQ_Γ generated by the following two types of relations:

- (R1) $p(a)^{\mu(\alpha)} - p(b)^{\mu(\beta)}$, where a, b are arrows of Q_Γ which start at the same vertex, and $p(a)$ (resp. $p(b)$) is a special α -cycle (resp. a special β -cycle);
- (R2) ba , where a, b are arrows of Q_Γ such that $b \neq \iota(a)$.

Definition 2.4. (cf. [7, Definition 2.5]) The Brauer configuration algebra (abbr. BCA) Λ_Γ associated to Γ is defined to be the quotient algebra kQ_Γ/I_Γ .

Note that each Brauer configuration algebra is a special multiserial algebra and in particular a multiserial algebra (see [8]).

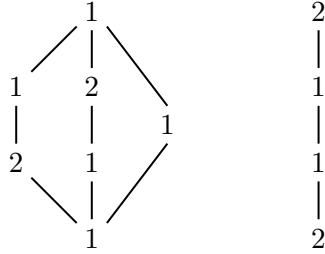
Example 2.5. The BCA corresponding to the BC (which we denote by Γ) in Example 2.3 is given by the following quiver Q_Γ



with relations

$$a_3a_2a_1 = b, a_1a_3a_2 = a_2a_1a_3 = c^2, 0 = ca_1 = a_3a_1 = ca_2 = a_2^2 = ba_3 = a_1b = a_2c = a_3c.$$

Therefore, the indecomposable projective modules have the following structures:



3. FRACTIONAL BRAUER CONFIGURATIONS

In this section we define fractional Brauer configurations, which are generalization of Brauer configurations. According to [15] (see also [13]), a Brauer graph can be seen as a ribbon graph with a multiplicity function. Inspired by this, let us first reformulate the definition of a Brauer configuration as follows.

Definition 3.1. A Brauer configuration (abbr. BC) is a tuple $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$ which is defined as follows.

- (1) Δ_0 is a finite set, called the set of vertices;
- (2) Δ_1 is a finite set, called the set of angles;
- (3) $\zeta : \Delta_1 \rightarrow \Delta_0$ is a surjective map, which is called the connection map;
- (4) P is a partition of Δ_1 , such that $|P(e)| \geq 2$ for every $e \in \Delta_1$ ($P(e)$ is the equivalence class determined by P which contains the angle e , which is called a polygon);

- (5) \mathfrak{p} is a permutation $\Delta_1 \rightarrow \Delta_1$ whose cycles have underlying sets $\zeta^{-1}(\alpha)$ with $\alpha \in \Delta_0$, which is called an orientation of Δ ;
- (6) $\nu : \Delta_0 \rightarrow \mathbb{Z}_+$ is a function, which is called the multiplicity function;
- (7) For every polygon $P(e)$ of E , there exists some $h \in P(e)$ with $|\zeta^{-1}(\zeta(h))| \cdot \nu(\zeta(h)) > 1$.

A Brauer configuration is called a Brauer graph if each polygon of it contains exactly two angles.

Compare with Definition 2.1, in the above definition we use vertices and angles as the basic elements, and the polygons are derived from them.

Proposition 3.2. *Definition 3.1 and Definition 2.1 are equivalent.*

Proof. Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ be a Brauer configuration in Definition 2.1, construct a tuple $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$ as follows: $\Delta_0 = \Gamma_0$; $\Delta_1 = \bigcup_{\alpha \in \Gamma_0} C_\alpha$, where $C_\alpha = \{(V, \alpha, i) \mid V \in \Gamma_1 \text{ such that } \alpha \text{ occurs as a vertex in } V \text{ and } 1 \leq i \leq N_{V,\alpha}\}$, $N_{V,\alpha}$ denotes the number of times that α occurs as a vertex in V ; $\zeta : \Delta_1 \rightarrow \Delta_0$ is given by $(V, \alpha, i) \mapsto \alpha$; P is a partition of Δ_1 such that for each angle (V, α, i) in Δ_1 , the class $P(V, \alpha, i)$ is equal to $\{(V, \beta, j) \mid \beta \in \Gamma_0 \text{ occurs as a vertex in } V \text{ and } 1 \leq j \leq N_{V,\beta}\}$, which is determined by the polygon V ; \mathfrak{p} is a permutation $\Delta_1 \rightarrow \Delta_1$ such that $\mathfrak{p}(V, \alpha, i) = (V', \alpha, i')$, where (V', α, i') is the successor of the triple (V, α, i) under the cyclic order \mathfrak{o} on C_α ; $\nu = \mu : \Delta_0 \rightarrow \mathbb{Z}_+$. It is straightforward to show that $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$ is a Brauer configuration under Definition 3.1. Note that the condition (5) in Definition 2.1 corresponds to the condition (7) in Definition 3.1.

Conversely, let $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$ be Brauer configuration in Definition 3.1, we define a quadruple $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathfrak{o})$ as follows: $\Gamma_0 = \Delta_0$; $\Gamma_1 = \{[\zeta(P(e))] \mid e \in \Delta_1\}$, where $[\zeta(P(e))]$ represents the finite labelled multiset which is the image of the class $P(e)$ under ζ ; $\mu = \nu : \Gamma_0 \rightarrow \mathbb{Z}_+$; In order to define the orientation \mathfrak{o} , we first define the set C_α of angles at the vertex α (in the sense of Definition 2.1) as follows.

For each vertex $\alpha \in \Gamma_0$, define the set C_α to be $\{(V, \alpha, i) \mid V \in \Gamma_1 \text{ such that } \alpha \text{ appears in } V = [\zeta(P(e))] \text{ for some } e \in \Delta_1 \text{ and } 1 \leq i \leq N_{V,\alpha}\}$, where $N_{V,\alpha}$ denotes the number of times that α appears in V . Note that for a polygon $V = [\zeta(P(e))]$, $\alpha \in V$ if and only if $\zeta^{-1}(\alpha) \cap P(e) \neq \emptyset$, and $N_{V,\alpha}$ is equal to the cardinal of $\zeta^{-1}(\alpha) \cap P(e)$. For each polygon $P(e)$ of Δ with $\zeta^{-1}(\alpha) \cap P(e) \neq \emptyset$, label the elements of $\zeta^{-1}(\alpha) \cap P(e)$ by e_1, \dots, e_s , and define a map $\phi_{P(e)} : \zeta^{-1}(\alpha) \cap P(e) \rightarrow C_\alpha$ by $\phi_{P(e)}(e_i) = ([\zeta(P(e))], \alpha, i)$. Since $\zeta^{-1}(\alpha)$ is a disjoint union of subsets of the form $\zeta^{-1}(\alpha) \cap P(e)$, the maps $\phi_{P(e)} : \zeta^{-1}(\alpha) \cap P(e) \rightarrow C_\alpha$ induce a map $\phi : \zeta^{-1}(\alpha) \rightarrow C_\alpha$, which is bijective. Then the cyclic order \mathfrak{o} on C_α is induced by the permutation \mathfrak{p} on $\zeta^{-1}(\alpha)$. Also the condition (7) in Definition 3.1 corresponds to the condition (5) in Definition 2.1. \square

In view of Definition 3.1, we can also reformulate the definition of a Brauer configuration algebra as follows. For a Brauer configuration $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$, define a quiver Q_Δ as follows:

- The vertices of Q_Δ correspond to the polygons of Δ ;
- The arrows of Q_Δ are given by the angles of Δ by the following way: given an angle $e \in \Delta_1$ with connection vertex $\alpha = \zeta(e) \in \Delta_0$, there is an arrow from the polygon $P(e)$ to the polygon $P(h)$, where $h = \mathfrak{p}(e)$, that is, h is the successor angle of e under the permutation around the vertex α .

In the following, we will identify the angle e with the corresponding arrow in Q_Δ and write $s(e) = P(e), t(e) = P(h)$ for the arrow e . Let e be an arrow of Q_Δ , define $p(e) = \mathfrak{p}^{n-1}(e) \cdots \mathfrak{p}(e)$ to be the cycle in the quiver Q_Δ , where n is the smallest positive integer such that $\mathfrak{p}^n(e) = e$. Let I_Δ be the ideal of the path algebra kQ_Δ generated by the following two types of relations:

- (R1) $p(e)^{\nu(\zeta(e))} - p(h)^{\nu(\zeta(h))}$, where e, h are arrows of Q_Δ with $P(e) = P(h)$;
- (R2) $e_2 e_1$, where e_1, e_2 are arrows of Q_Δ such that $e_2 \neq \mathfrak{p}(e_1)$.

The Brauer configuration algebra Λ_Δ associated to Δ is defined to be kQ_Δ/I_Δ .

We now introduce the notion of fractional Brauer configuration.

Definition 3.3. Let $G = \langle g \rangle$ be an infinite cyclic group. A fractional Brauer configuration (abbr. *f-BC*) is a quadruple $E = (E, P, L, d)$, where E is a G -set, P and L are two partitions of E , and $d : E \rightarrow \mathbb{Z}_+$ is a function, such that the following conditions hold.

- (f1) $L(e) \subseteq P(e)$ and $P(e)$ is a finite set for each $e \in E$.
- (f2) If $L(e_1) = L(e_2)$, then $P(g \cdot e_1) = P(g \cdot e_2)$.
- (f3) If e_1, e_2 belong to same $\langle g \rangle$ -orbit, then $d(e_1) = d(e_2)$.
- (f4) $P(e_1) = P(e_2)$ if and only if $P(g^{d(e_1)} \cdot e_1) = P(g^{d(e_2)} \cdot e_2)$.
- (f5) $L(e_1) = L(e_2)$ if and only if $L(g^{d(e_1)} \cdot e_1) = L(g^{d(e_2)} \cdot e_2)$.
- (f6) The formal sequence $L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e)L(e)$ is not a proper subsequence of the formal sequence $L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h)L(h)$ for all $e, h \in E$.

Remark 3.4. We need some further remarks on the above definition as follows.

- (1) The elements in E are called angles of the *f-BC*. The $\langle g \rangle$ -orbits of E are called vertices of the *f-BC*. In other words, every vertex of the *f-BC* is of the form $G \cdot e$ where $e \in E$ is an angle.
- (2) For each vertex v , the $\langle g \rangle$ -set structure of v gives an order on it, which is a total order when v is infinite and is a cyclic order when v is finite. The order given by $\langle g \rangle$ -action on each vertex of *f-BC* can be regarded as a generalization of the orientation \mathfrak{o} of *BC*.
- (3) The classes $P(e)$ of the partition P are called polygons. Note that we allow the cardinality of $P(e)$ to be 1, that is, $P(e)$ can be a 1-gon, which is different from the *BC* case.
- (4) Condition (f1) means that any two angles e_1, e_2 of the class $L(e)$ lie in the same polygon $P(e)$, and Condition (f2) means that their successors $g \cdot e_1, g \cdot e_2$ also lie in the same polygon $P(g \cdot e)$. The partition L is said to be trivial if $L(e) = \{e\}$ for each $e \in E$.
- (5) The function $d : E \rightarrow \mathbb{Z}_+$ is called degree function. Condition (f3) means that the degree function can be defined on vertices.
- (6) Let E be a *f-BC* and v be a vertex such that v is a finite set, define the fractional-degree (abbr. *f-degree*) $d_f(v)$ of the vertex v to be the rational number $\frac{d(v)}{|v|}$. If the *f-degree* of each vertex of E is an integer, then E is said to have integral *f-degree*. Moreover, if $d_f(v) = 1$ for each vertex v , then E is called *f-degree trivial*.
- (7) Denote σ the map $E \rightarrow E$, $e \mapsto g^{d(e)} \cdot e$, which will be called the Nakayama automorphism of E . This automorphism is guaranteed by the conditions (f3), (f4) and (f5).

Each *BC* $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$ can be seen as a *f-BC* $E = (E, P, L, d)$ with integral *f-degree* and trivial partition L by the following procedure: Let $E = \Delta_1$ such that the $\langle g \rangle$ -action on E is induced by the permutation \mathfrak{p} on Δ_1 ; the partition P of E is just the partition P of Δ_1 ; the partition L of E is defined to be trivial; the degree function d is given by $d(e) = \nu(\zeta(e)) \cdot |\zeta^{-1}(\zeta(e))|$. Then the multiplicity function ν is equal to the *f-degree* d_f .

On the other hand, if we allow the multiplicity function ν in a *BC* $\Delta = (\Delta_0, \Delta_1, \zeta, P, \mathfrak{p}, \nu)$ to take fractional values such that $\nu(v)$ is an integral multiple of $\frac{1}{|\zeta^{-1}(v)|}$ for each vertex $v \in \Delta_0$, then under some suitable conditions we can still form a *f-BC* along the above procedure (see Example 3.5). Note that in *BC* case, a vertex v is truncated (cf. Remark 2.2 (1)) if and only if $d(v) = 1$.

We illustrate our definition of *f-BC* with some examples.

Example 3.5. Let $E = \{1, 1', 2, 2', 3, 3'\}$. Define the group action on E by $g \cdot 1 = 2$, $g \cdot 2 = 3$, $g \cdot 3 = 1$, $g \cdot 1' = 2'$, $g \cdot 2' = 3'$, $g \cdot 3' = 1'$. Define $P(1) = \{1, 1'\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3, 3'\}$ and $L(e) = \{e\}$ for every $e \in E$. The degree d of E is defined by $d(e) = 2$ for every $e \in E$. So the *f-degree* of E has constant value $\frac{2}{3}$.

Example 3.6. Let $E = \{1, 1', 2, 2', 3, 3', 4, 4'\}$. Define the group action on E by $g \cdot 1 = 2$, $g \cdot 2 = 3$, $g \cdot 3 = 1$, $g \cdot 1' = 2'$, $g \cdot 2' = 3'$, $g \cdot 3' = 1'$, $g \cdot 4 = 4$. Define $P(1) = \{1, 1'\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3, 3'\}$, $P(4) = \{4, 4'\}$, $L(1) = \{1, 1'\}$ and $L(e) = \{e\}$ for $e \neq 1, 1'$. The *f-degree* of E is defined to be trivial.

Example 3.7. Let $E = \{a_1, a_2, a_3, a_4\}$. Define the group action on E by $g \cdot a_1 = a_1$, $g \cdot a_2 = a_3$, $g \cdot a_3 = a_2$, $g \cdot a_4 = a_4$. Define $P(a_1) = E$, $L(a_1) = \{a_1, a_2\}$, $L(a_3) = \{a_3, a_4\}$. Define $d(a_1) = d(a_4) = 2$, $d(a_2) = d(a_3) = 4$. So the f -degree of E has constant value 2.

Example 3.8. Let $E = \{1, 1', 1'', 2, 2', 3, 4, 4', 4'', 5, 5', 6\}$. Define the group action on E by $g \cdot 1 = 2$, $g \cdot 2 = 4$, $g \cdot 4 = 5$, $g \cdot 5 = 1$, $g \cdot 1' = 2'$, $g \cdot 2' = 4'$, $g \cdot 4' = 6$, $g \cdot 6 = 1'$, $g \cdot 1'' = 3$, $g \cdot 3 = 4''$, $g \cdot 4'' = 5'$, $g \cdot 5' = 1''$. Define $P(1) = \{1, 1', 1''\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3\}$, $P(4) = \{4, 4', 4''\}$, $P(5) = \{5, 5'\}$, $P(6) = \{6\}$, $L(1) = \{1, 1'\}$, $L(2) = \{2, 2'\}$, $L(4) = \{4, 4''\}$, $L(5) = \{5, 5'\}$, $L(e) = \{e\}$ for other $e \in E$. The f -degree of E is defined to be trivial.

Example 3.9. Let $E = \{i \mid i \in \mathbb{Z}\} \cup \{i' \mid i \in 3\mathbb{Z}\}$. Define the group action on E by

$$g \cdot e = \begin{cases} i+1, & \text{if } e = i \text{ with } i \equiv 0 \pmod{3}; \\ i+2, & \text{if } e = i \text{ with } i \equiv 1 \pmod{3} \text{ or if } e = i' \text{ with } i \equiv 0 \pmod{3}; \\ (i+1)', & \text{if } e = i \text{ with } i \equiv 2 \pmod{3}. \end{cases}$$

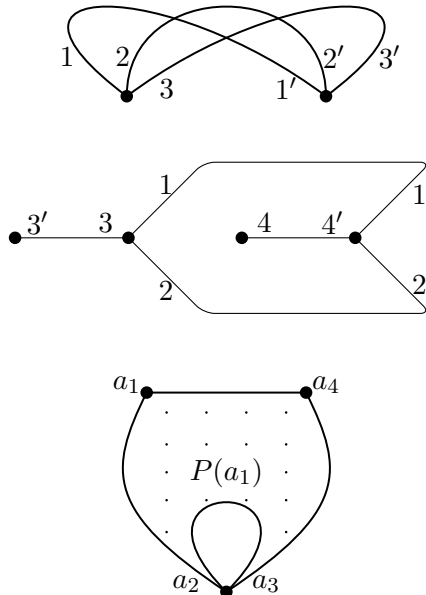
Define

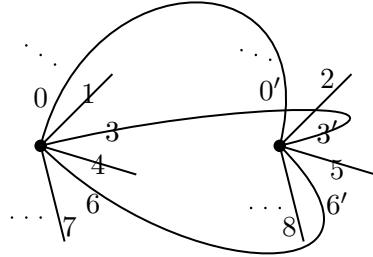
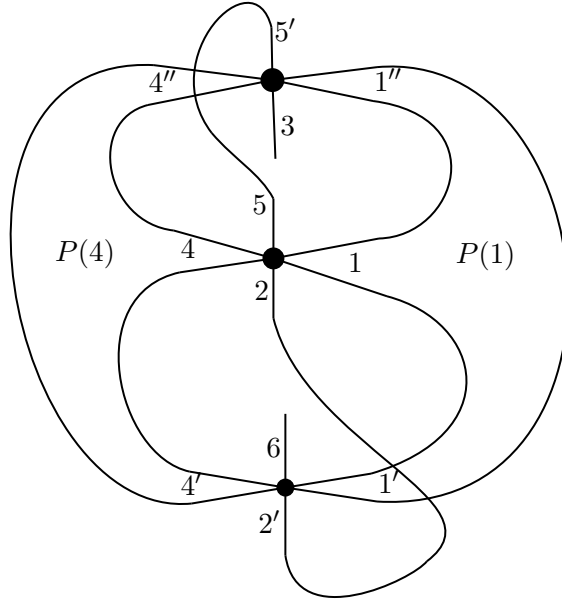
$$P(e) = \begin{cases} \{i, i'\}, & \text{if } e = i \text{ with } i \equiv 0 \pmod{3}; \\ \{e\}, & \text{otherwise}; \end{cases}$$

and $L(e) = \{e\}$ for every $e \in E$. The degree d of E is given by $d(e) = 2$ for every $e \in E$.

Every fractional Brauer configuration $E = (E, P, L, d)$ can be visualized as some diagram $\Gamma(E)$, which consists of polygons such that every two polygons can only intersect on their vertices. Each polygon in $\Gamma(E)$ corresponds to a polygon (of the form $P(e)$) of E , and each $e \in E$ corresponds to an angle of the polygon $P(e)$ in $\Gamma(E)$. Two angles of polygons in $\Gamma(E)$ are connected to the same vertex of $\Gamma(E)$ if and only if the two corresponding elements of E belong to the same $\langle g \rangle$ -orbit of E . For each vertex $v = G \cdot e$ of $\Gamma(E)$, the $\langle g \rangle$ -set structure of E gives an order (which is always taken to be clockwise) to angles of polygons in $\Gamma(E)$ which are connected to v .

Note that if a polygon $P(e)$ has cardinality 1, then $P(e)$ has only one angle and it corresponds to a half edge in $\Gamma(E)$; and if $P(e)$ has cardinality 2, then $P(e)$ has two angles and they correspond to one edge (or two half edges) in $\Gamma(E)$. The f -BCs in Examples 3.5, 3.6, 3.7, 3.8, 3.9 can be visualized by the following diagrams





Definition 3.10. Let E be a f -BC.

- (1) Let n be a positive integer. We call $p = (e_n, \dots, e_2, e_1)$ a sequence of angles of length n in E , if e_i 's are angles in E and $P(g \cdot e_i) = P(e_{i+1})$ for all $1 \leq i \leq n-1$. Moreover, for every $e \in E$, we call $(\)_e$ a sequence of angles of length 0 in E at e , or a trivial sequence of E at e .
- (2) A sequence of the form $p = (g^{n-1} \cdot e, \dots, g \cdot e, e)$ with $e \in E$ and $0 \leq n \leq d(e)$ is called a standard sequence (associated to the vertex $G \cdot e$) of E (we define $p = (\)_e$ when $n = 0$). For $n > 0$, the source (resp. terminal) of standard sequence $p = (g^{n-1} \cdot e, \dots, g \cdot e, e)$ is defined to be e (resp. $g^n \cdot e$); both the source and the terminal of the trivial sequence $(\)_e$ are defined to be e . The composition of two standard sequences is defined in a natural way.
- (3) A standard sequence of the form $(g^{d(e)-1} \cdot e, \dots, g \cdot e, e)$ with $e \in E$ is called a full sequence of E .

For a standard sequence $p = (g^{n-1} \cdot e, \dots, g \cdot e, e)$, we can define two associated standard sequences (which can be called the left complement and the right complement of p respectively)

$$\hat{\ } p = \begin{cases} (g^{d(e)-1} \cdot e, \dots, g^{n+1} \cdot e, g^n \cdot e), & \text{if } 0 < n < d(e); \\ (\)_{g^{d(e)}(e)}, & \text{if } n = d(e); \\ (g^{d(e)-1} \cdot e, \dots, g \cdot e, e), & \text{if } n = 0 \text{ and } p = (\)_e, \end{cases}$$

and

$$p^\wedge = \begin{cases} (g^{-1} \cdot e, g^{-2} \cdot e, \dots, g^{n-d(e)} \cdot e), & \text{if } 0 < n < d(e); \\ ()_e, & \text{if } n = d(e); \\ (g^{-1} \cdot e, \dots, g^{-d(e)} \cdot e), & \text{if } n = 0 \text{ and } p = ()_e. \end{cases}$$

Note that for a standard sequence p of E , both ${}^\wedge pp$ and pp^\wedge are full sequences of E .

For a sequence $p = (e_n, \dots, e_2, e_1)$ in E , we can associate a formal sequence as follows:

$$L(p) = \begin{cases} L(e_n) \cdots L(e_2)L(e_1), & \text{if } p = (e_n, \dots, e_2, e_1) \text{ is a sequence of length } > 0; \\ 1_{P(e)}, & \text{if } p = ()_e \text{ is a trivial sequence at } e. \end{cases}$$

Moreover, for a set \mathcal{X} of sequences, define $L(\mathcal{X}) = \{L(p) \mid p \in \mathcal{X}\}$.

Definition 3.11. Let E be a f -BC, p, q be two sequences of angles in E . Denote $p \equiv q$ if the associated formal sequences $L(p)$ and $L(q)$ are equal. In this case we say that p, q are identical.

Remark 3.12. (1) Any standard sequence which is identical to a trivial sequence $()_e$ is also a trivial sequence $()_{e'}$ such that $P(e) = P(e')$.

(2) By (f6) we see that any standard sequence which is identical to a full sequence p is also a full sequence q with ${}^\wedge p \equiv {}^\wedge q$.

(3) By (f4) and (f5), for standard sequences p, q , ${}^\wedge p \equiv {}^\wedge q$ if and only if $p^\wedge \equiv q^\wedge$.

For a set \mathcal{X} of standard sequences, denote ${}^\wedge \mathcal{X} = \{{}^\wedge p \mid p \in \mathcal{X}\}$ (resp. $\mathcal{X}^\wedge = \{p^\wedge \mid p \in \mathcal{X}\}$), and denote $[\mathcal{X}] = \{\text{standard sequence } q \mid q \text{ is identical to some } p \in \mathcal{X}\}$.

Definition 3.13. A f -BC E is said to be of type S (or E is a f_s -BC in short) if it satisfies additionally the following condition:

$$(f7) \text{ For standard sequences } p \equiv q, [{}^\wedge p]^\wedge = [{}^\wedge q]^\wedge \text{ or } [{}^\wedge p^\wedge] = [{}^\wedge q^\wedge].$$

In particular, if E is a f_s -BC such that each polygon of E has exactly two elements, then we call E a fractional Brauer graph of type S (abbr. f_s -BG).

Remark 3.14. (1) For a standard sequence p , we have $[{}^\wedge p^\wedge] = [{}^\wedge[{}^\wedge p]^\wedge] = [{}^\wedge[{}^\wedge p]^\wedge] = [{}^\wedge p]^\wedge$. It follows that the conditions $[{}^\wedge p]^\wedge = [{}^\wedge q]^\wedge$ and $[{}^\wedge p^\wedge] = [{}^\wedge q^\wedge]$ in Definition 3.13 are equivalent.

(2) For standard sequences $p \equiv q$, the equation $[{}^\wedge p]^\wedge = [{}^\wedge q]^\wedge$ is always true if p is a trivial or a full sequence.

In Example 3.5, the partition L is trivial. So for non-trivial standard sequences p, q of E , $p \equiv q$ implies $p = q$. Then (f7) holds and E is a f_s -BC and indeed a f_s -BG.

In Example 3.6, the only case for $p \equiv q$ and $p \neq q$ for non-trivial standard sequences p, q is $p = (1)$ and $q = (1')$. Since $[{}^\wedge p]^\wedge = [{}^\wedge q]^\wedge = \{(1), (1')\}$, E is a f_s -BC and indeed a f_s -BG.

In Example 3.7, let $p = (a_1)$ and $q = (a_2)$ be identical standard sequences in E , then $[{}^\wedge p]^\wedge = \{(a_1), (a_2), (a_3, a_2, a_3)\}$ and $[{}^\wedge q]^\wedge = \{(a_1), (a_2)\}$. Therefore E is a f -BC but not a f_s -BC.

In Example 3.8, let $p = (2, 1)$ and $q = (2', 1')$ be identical standard sequences in E , then $[{}^\wedge p]^\wedge = \{(2, 1), (2', 1'), (3, 1'')\}$ and $[{}^\wedge q]^\wedge = \{(2, 1), (2', 1')\}$. Therefore E is a f -BC but not a f_s -BC.

In Example 3.9, the partition L is trivial. Then (f7) holds and E is a f_s -BC.

Definition 3.15. A f -BC $E = (E, P, L, d)$ is said to be of type MS (or E is a f_{ms} -BC in short) if partition L of E is trivial. If E is a f -BC of type MS such that $P(e)$ contains exactly two elements for each $e \in E$, then E is said to be a fractional Brauer graph of type MS (abbr. f_{ms} -BG).

Since identical non-trivial standard sequences in a f -BC of type MS must be equal, it is of type S . The above Example 3.5 and Example 3.9 are f -BCs of type MS .

We summarize our discussion by the following diagram:

$$\begin{array}{ccccccc}
 BC & \Longrightarrow & f_{ms}\text{-}BC & \Longrightarrow & f_s\text{-}BC & \Longrightarrow & f\text{-}BC \\
 \Uparrow & & \Uparrow & & \Uparrow & & \\
 BG & \Longrightarrow & f_{ms}\text{-}BG & \Longrightarrow & f_s\text{-}BG & &
 \end{array}$$

4. FRACTIONAL BRAUER CONFIGURATION CATEGORY

In this section, we will associate every f-BC E a k -category Λ_E which we call a fractional Brauer configuration category. It turns out that Λ_E is a locally bounded k -category in the sense of Bongartz and Gabriel.

4.1. The quiver of a fractional Brauer configuration category.

Definition 4.1. For a f-BC $E = (E, P, L, d)$, we associate a quiver Q_E as follows.

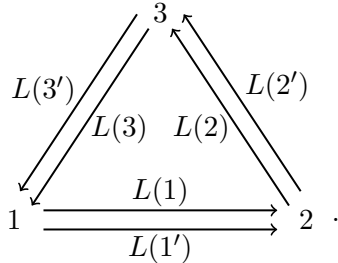
- The set $(Q_E)_0$ of vertices is given by $\{P(e) \mid e \in E\}$;
- The set $(Q_E)_1$ of arrows is given by $\{L(e) \mid e \in E\}$, where the arrow $L(e)$ has the source $P(e)$ and the terminal $P(g \cdot e)$.

Denote $s : (Q_E)_1 \rightarrow (Q_E)_0$ and $t : (Q_E)_1 \rightarrow (Q_E)_0$ by two maps with s sending each arrow to its source and t sending each arrow to its terminal.

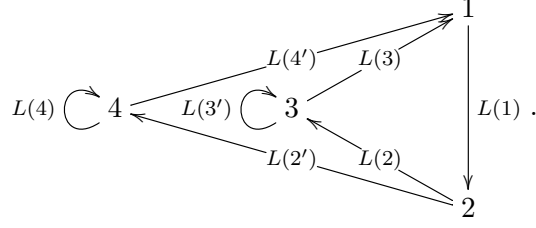
Remark 4.2. We have the following interpretations of Q_E in terms of the conditions of f-BC.

- (1) (f1) and (f2) ensure that the definition of the source and the terminal of an arrow $L(e)$ of Q_E are independent to the choice of the representative e of the class $L(e)$.
- (2) Partition L means that the arrows of Q_E are obtained by identifying $L(e)$ with $L(e')$ in Q_E if e and e' belong to the same partition class of L . Moreover, since $P(e)$ (hence also $L(e)$) is a finite set for each e , the quiver Q_E is locally finite, that is, the number of arrows starting or stopping at any vertex is finite.
- (3) (f4) (resp. (f5)) ensures that the Nakayama automorphism σ of E induces a permutation on $(Q_E)_0$ (resp. $(Q_E)_1$). (f4) and (f5) together with (f3) ensure that σ commutes with s and t so that it induces an automorphism of the quiver Q_E . We also denote this automorphism by σ .
- (4) $p = (e_n, \dots, e_2, e_1)$ is a sequence of E if and only if $L(p) = L(e_n) \cdots L(e_2)L(e_1)$ is a path of Q_E , and we write a path of Q_E from right to left. For a trivial sequence $p = ()_e$, $L(p) = 1_{P(e)}$ is the trivial path of Q_E at vertex $P(e)$.
- (5) Every finite (resp. infinite) $\langle g \rangle$ -orbit $G \cdot e$ corresponds to a cycle $L(g^{|G \cdot e| - 1} \cdot e) \cdots L(g \cdot e)L(e)$ of length $|G \cdot e|$ (resp. an infinite path $\cdots L(g \cdot e)L(e)L(g^{-1} \cdot e) \cdots$) in the quiver Q_E .

We usually denote a polygon in a f-BC E by $P(i) = \{i, i', i'', \dots\}$ and denote the corresponding vertex in Q_E by i . In Example 3.5, Q_E is the following quiver



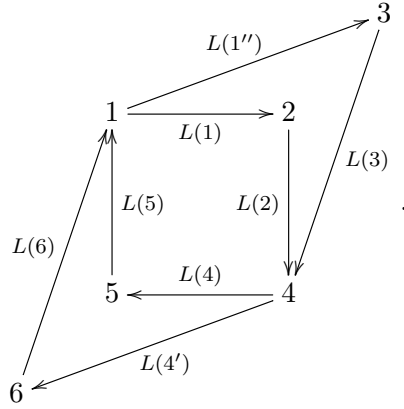
In Example 3.6, Q_E is the following quiver



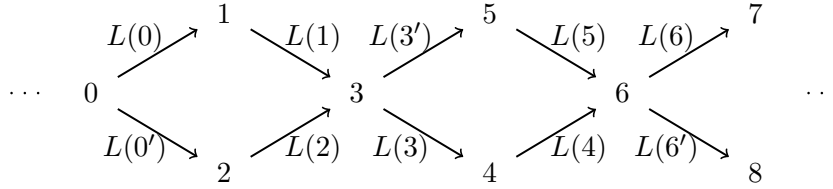
In Example 3.7, Q_E is the following quiver

$$L(a_1) \circlearrowleft a \circlearrowright L(a_3) .$$

In Example 3.8, Q_E is the following quiver



In Example 3.9, Q_E is the following quiver



Lemma 4.3. *Let $E = (E, P, L, d)$ be a f -BC. For a path $p = L(e_n) \cdots L(e_2)L(e_1)$ of Q_E , the following conditions are equivalent:*

- (1) *There exists some $e \in E$ such that $p = L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e)$.*
- (2) $\bigcap_{i=1}^n g^{n-i} \cdot L(e_i) \neq \emptyset$.

Proof. If there exists some $e \in E$ such that $p = L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e)$, then $g^{n-i} \cdot L(e_i) = g^{n-i} \cdot L(g^{i-1} \cdot e)$ for each $1 \leq i \leq n$ and $g^{n-1} \cdot e \in \bigcap_{i=1}^n g^{n-i} \cdot L(e_i)$. Therefore $\bigcap_{i=1}^n g^{n-i} \cdot L(e_i) \neq \emptyset$. Conversely, if $\bigcap_{i=1}^n g^{n-i} \cdot L(e_i) \neq \emptyset$, let $h \in \bigcap_{i=1}^n g^{n-i} \cdot L(e_i)$ and let $e = g^{1-n} \cdot h$. Since $e \in g^{1-i} \cdot L(e_i)$ for each $1 \leq i \leq n$, $L(g^{i-1} \cdot e) = L(e_i)$ for each $1 \leq i \leq n$. Therefore $p = L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e)$. \square

The above quiver Q_E defines a path category kQ_E whose objects are the vertices of Q_E and whose morphisms are generated by the paths of Q_E .

4.2. The relations of a fractional Brauer configuration category.

Definition 4.4. For a f -BC $E = (E, P, L, d)$, we define an ideal I_E of the path category kQ_E which is generated by the following three types of relations:

- (fR1) $L(g^{d(e)-1-k} \cdot e) \cdots L(g \cdot e)L(e) - L(g^{d(h)-1-k} \cdot h) \cdots L(g \cdot h)L(h)$, where $k \geq 0$, $P(e) = P(h)$ and $L(g^{d(e)-i} \cdot e) = L(g^{d(h)-i} \cdot h)$ for $1 \leq i \leq k$.
- (fR2) Paths of the form $L(e_n) \cdots L(e_2)L(e_1)$ with $\bigcap_{i=1}^n g^{n-i} \cdot L(e_i) = \emptyset$ for $n > 1$.
- (fR3) Paths of the form $L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e)$ for $n > d(e)$.

We call the quotient category $\Lambda_E = kQ_E/I_E$ the fractional Brauer configuration category (abbr. f -BCC) of E . Moreover, if E is a f_s -BC (resp. f_s -BG, f_{ms} -BC, f_{ms} -BG), then we call Λ_E a f_s -BCC (resp. f_s -BGC, f_{ms} -BCC, f_{ms} -BGC).

Remark 4.5. We have the following interpretations to the ideal I_E :

- (fR1) means that $L(p) - L(q)$ is contained in I_E if p, q are standard sequences of E with $\wedge p \equiv \wedge q$. Note that (fR1) is equal to (R1) in BCA case, but if the partition L is nontrivial, then (fR1) contains more elements in general.
- (fR2) means that each path of Q_E not of the form $L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e)$ is contained in I_E (see Lemma 4.3). Unfortunately, Conditions (f1) – (f6) do not ensure that all full sequences define nonzero paths in kQ_E/I_E (see Examples 3.7, 3.8), for this to be true, we need the condition (f7) (see Remark 4.16). Note that (fR2) is equal to (R2) in BCA case, but if the partition L is nontrivial, then (fR2) contains more elements in general.
- (fR3) defines the maximal possible paths in kQ_E which are not contained in I_E . Together with (f3) in Definition 3.3 it is not hard to see that, for each vertex $x = P(e) \in (Q_E)_0$, there is a natural number $N_x = (\max\{d(h) \mid h \in P(e)\} + 1)$ such that I_E contains each path of length $\geq N_x$ which starts or stops at x .
- Condition (f6) in Definition 3.3 ensures that I_E is contained in the ideal of kQ_E generated by arrows. Otherwise, there exists a relation $r = L(g^{d(e)-1-k} \cdot e) \cdots L(g \cdot e)L(e) - L(g^{d(h)-1-k} \cdot h) \cdots L(g \cdot h)L(h)$ of type (fR1) such that the length of one of the paths $L(g^{d(e)-1-k} \cdot e) \cdots L(g \cdot e)L(e)$ and $L(g^{d(h)-1-k} \cdot h) \cdots L(g \cdot h)L(h)$ is 0. If the length of $L(g^{d(e)-1-k} \cdot e) \cdots L(g \cdot e)L(e)$ is 0, then $k = d(e)$. Since r is nonzero, $k < d(h)$. So $L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e)L(e)$ is a proper subsequence of $L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h)L(h)$, which contradicts the condition (f6).
- Condition (f7) holds when the partition L is trivial and in this case E is a f_{ms} -BC.

In Example 3.5, I_E is generated by the following relations:

$$\begin{aligned} &L(2)L(1) - L(2')L(1'), L(3)L(2) - L(3')L(2'), L(1)L(3) - L(1')L(3'), \\ &L(2')L(1), L(2)L(1'), L(3')L(2), L(3)L(2'), L(1')L(3), L(1)L(3'). \end{aligned}$$

In Example 3.6, I_E is generated by the following relations:

$$\begin{aligned} &L(3') - L(2)L(1)L(3), L(4) - L(2')L(1)L(4'), L(3)L(2) - L(4')L(2'), L(3')L(2), \\ &L(3)L(3'), L(4)L(2'), L(4')L(4), L(2)L(1)L(4'), L(2')L(1)L(3), L(1)L(3)L(2)L(1). \end{aligned}$$

In Example 3.7, let $p = (a_1)$, $q = (a_3, a_2, a_3)$ be standard sequences, then $\wedge p \equiv \wedge q$. Therefore $L(a_1) = L(a_3)L(a_1)L(a_3)$ in Λ_E . Similarly, $L(a_3) = L(a_1)L(a_3)L(a_1)$ in Λ_E . We have

$$(1) \quad L(a_1) = L(a_3)L(a_1)L(a_3) = L(a_3)L(a_3)L(a_1)L(a_3)L(a_3)$$

in Λ_E . Since

$$L(a_3) \cap g \cdot L(a_3) \cap g^2 \cdot L(a_1) = \{a_3, a_4\} \cap \{a_2, a_4\} \cap \{a_1, a_2\} = \emptyset,$$

$L(a_3)L(a_3)L(a_1) = 0$ in Λ_E . Therefore $L(a_1) = 0$ in Λ_E . Similarly, $L(a_3) = 0$ in Λ_E . So I_E is generated by $L(a_1)$, $L(a_3)$ and $\Lambda_E \cong k$.

In Example 3.8, I_E is generated by the following relations:

$$L(5)L(4) - L(6)L(4'), L(2)L(1) - L(3)L(1''), L(4')L(3), L(1'')L(6).$$

In Example 3.9, I_E is generated by the following relations:

$$L(3i+1)L(3i) - L(3i+2)L(3i'), L(3i')L(3i-2), L(3i)L(3i-1),$$

where $i \in \mathbb{Z}$.

Recall from [4, Definition 2.1] that a locally bounded category is a k -category Λ satisfying the following three conditions:

- For each $x \in \Lambda$, the endomorphism algebra $\Lambda(x, x)$ is local.
- Distinct objects of Λ are not isomorphic.
- For each $x \in \Lambda$, $\sum_{y \in \Lambda} \dim_k \Lambda(x, y) < \infty$ and $\sum_{y \in \Lambda} \dim_k \Lambda(y, x) < \infty$.

For the definition of the radical of a locally bounded category, we refer to [4, Section 2].

Theorem 4.6. *Let $E = (E, P, L, d)$ be a f -BC, and let $\Lambda_E = kQ_E/I_E$ be the associated f -BCC. Then we have the following.*

- (1) Λ_E is locally bounded k -category.
- (2) Let J be the ideal of Λ_E generated by the arrows of Q_E . Then J is the radical of Λ_E .
- (3) The Nakayama automorphism σ of E induces an automorphism of the category Λ_E which is also denoted by σ .

Proof. (1) follows from the facts that Q_E is a locally finite quiver and that for each vertex $x \in (Q_E)_0$, there is a natural number N_x such that I_E contains each path of length $\geq N_x$ which starts or stops at x (cf. [4, Page 337]). Alternatively, we give a direct proof as follows.

For $e \in E$, each nonzero path in Λ_E starting at $P(e)$ is of the form $L(g^{n-1} \cdot h) \cdots L(g \cdot h)L(h)$, where $h \in P(e)$ and $0 \leq n \leq d(h)$. Since $\bigoplus_{a \in (Q_E)_0} \Lambda_E(P(e), a)$ is generated by such paths as a k -space, and since $P(e)$ is finite, $\dim_k(\bigoplus_{a \in (Q_E)_0} \Lambda_E(P(e), a)) < \infty$. Similarly, $\dim_k(\bigoplus_{a \in (Q_E)_0} \Lambda_E(a, P(e))) < \infty$.

For each morphism $f : x \rightarrow y$ in J , to show that f is in the radical of Λ_E , we need to show that $id_x - f'f$ is invertible for all $f' \in \Lambda_E(y, x)$. Let $x = P(e)$, then each path starting at x of length larger than $\max\{d(h) \mid h \in P(e)\}$ is zero in Λ_E . Therefore $f'f$ is nilpotent and $id_x - f'f$ is invertible.

For each morphism $f : x \rightarrow y$ with $f \notin J$, there exists $\lambda \in k^*$ such that $f = \lambda id_x + f'$ with $f' \in J$. Since f' is nilpotent, f is invertible. Therefore f is not in the radical of Λ_E . We have proved that J is the radical of Λ_E , that is (2).

For each object x in Λ_E , $\Lambda(x, x)$ is a finite dimensional algebra, and $\Lambda(x, x)/(\text{rad}\Lambda(x, x)) = \Lambda(x, x)/J(x, x) \cong k$. Therefore $\Lambda(x, x)$ is local. For different objects x, y in Λ_E , $\Lambda(x, y) = J(x, y)$ is contained in the radical of Λ_E , thus each morphism $f : x \rightarrow y$ is not an isomorphism, so $x \not\cong y$. Therefore Λ_E is locally bounded.

By Remark 4.2 (3), σ induces an automorphism of the quiver Q_E . Moreover, it is straightforward to show that the automorphism σ of Q_E also induces an automorphism of the category Λ_E . This proves (3). \square

4.3. The category Λ_E in type S is a locally bounded Frobenius category.

Definition 4.7. *We call a locally bounded category Λ to be Frobenius if for every object x of Λ , there exists some objects y, z of Λ such that $\Lambda(-, x) \cong D\Lambda(y, -)$ and $\Lambda(x, -) \cong D\Lambda(-, z)$, where D denotes the usual k -duality on vector spaces.*

Note that such objects y, z in the definition above are also unique by Yoneda's lemma. Denote by $\text{mod}\Lambda$ the category of finitely generated Λ -modules (here, a finitely generated Λ -module means it is isomorphic to a quotient of a finite direct sum of representable contravariant functors from Λ to $\text{mod}k$). According to [4, Section 2], the indecomposable projective modules in $\text{mod}\Lambda$ are

isomorphic to $\Lambda(-, x)$ ($x \in \Lambda$) and the indecomposable injective modules in $\text{mod}\Lambda$ are isomorphic to $D\Lambda(y, -)$ ($y \in \Lambda$). Therefore if Λ is a locally bounded Frobenius category, then the category $\text{mod}\Lambda$ is a Frobenius category in the sense of Happel [9]. This demonstrates our terminology of locally bounded Frobenius category.

Definition 4.8. Let $E = (E, P, L, d)$ be a f -BC and let

$$\mathcal{E} = \{L(p) \mid p \text{ is a standard sequence of } E\},$$

which is a set of paths of Q_E . Define a relation R on \mathcal{E} as follows: for $u, v \in \mathcal{E}$, uRv if and only if $u = L(p)$ and $v = L(q)$ for some standard sequences p, q of E with $\wedge p \equiv \wedge q$.

- Remark 4.9.**
- (1) If $u, v \in \mathcal{E}$ and uRv , then u, v have the same source and the same terminal.
 - (2) For $u, v \in \mathcal{E}$ with uRv , if $u = L(p)$ such that p is a trivial sequence $(\)_e$, then $v = L(q)$ for some trivial sequence $q = (\)_{e'}$ such that $P(e) = P(e')$, and therefore $u = v$.
 - (3) For paths u, v of \mathcal{E} of length ≥ 1 , uRv if and only if $u - v$ is a relation of I_E of type (fR1).

Lemma 4.10. Let $E = (E, P, L, d)$ be a f -BC and let

$$L(g^{n-1} \cdot e) \cdots L(g \cdot e)L(e) = L(g^{n-1} \cdot h) \cdots L(g \cdot h)L(h)$$

be a path of Q_E . Then $n > d(e)$ (resp. $n < d(e)$) if and only if $n > d(h)$ (resp. $n < d(h)$).

Proof. Suppose $n > d(e)$ and $n \leq d(h)$. Then $L(g^{d(e)-1} \cdot e) \cdots L(g \cdot e)L(e)$ is a proper subsequence of $L(g^{d(h)-1} \cdot h) \cdots L(g \cdot h)L(h)$, which contradicts to (f6). Therefore $n > d(e)$ if and only if $n > d(h)$. Similarly $n < d(e)$ if and only if $n < d(h)$. \square

Remark 4.11. Let \mathcal{B}_1 (resp. \mathcal{B}_2) be the set of paths of Q_E which are relations of type (fR2) (resp. of type (fR3)) in I_E . By Lemma 4.10, we know that the set of paths in Q_E is a disjoint union of $\mathcal{B}_1, \mathcal{B}_2$ and \mathcal{E} .

Lemma 4.12. For a f -BC $E = (E, P, L, d)$, (f7) is equivalent to the following condition:

(f7') For any standard sequence p of E , the set of paths $v \in \mathcal{E}$ with $vRL(p)$ is $\{L(p') \mid \wedge p \equiv \wedge p'\}$.

Proof. Suppose that (f7) holds. For a standard sequence p of E , if v is a path in \mathcal{E} with $vRL(p)$, then there exists standard sequences q, r of E with $L(p) = L(q)$, $v = L(r)$ and $\wedge q \equiv \wedge r$. Then $p \equiv q$ and $r \in [\wedge q]^\wedge \subseteq [[\wedge q]^\wedge] = [[\wedge p]^\wedge]$. So there exists some standard sequence l of E with $\wedge p \equiv \wedge l$ and $l \equiv r$, and therefore $v = L(l)$ belongs to the set $\{L(p') \mid \wedge p \equiv \wedge p'\}$. Then the condition (f7') holds.

Conversely, suppose that (f7') holds. For standard sequences p, q of E with $p \equiv q$ and for $r \in [[\wedge p]^\wedge]$, there exists some standard sequence t of E with $L(r) = L(t)$ and $\wedge t \equiv \wedge p$. Since $L(q) = L(p)$, $L(r)RL(q)$. Then by (f7') we have $L(r) = L(s)$ for some standard sequence s of E with $\wedge q \equiv \wedge s$. Therefore $r \equiv s$ and $r \in [[\wedge q]^\wedge]$, which implies that $[[\wedge p]^\wedge] \subseteq [[\wedge q]^\wedge]$. Similarly, we have $[[\wedge q]^\wedge] \subseteq [[\wedge p]^\wedge]$, and the condition (f7) holds. \square

Lemma 4.13. If $E = (E, P, L, d)$ is a f_s -BC, that is, E is a f -BC satisfying moreover (f7), then R is an equivalence relation on \mathcal{E} .

Proof. By definition, R is reflexive and symmetric. Suppose uRv and vRw for $u, v, w \in \mathcal{E}$. If $u = L(p)$ for some standard sequence p of E , by Lemma 4.12, $v = L(q)$ for some standard sequence q of E with $\wedge p \equiv \wedge q$, and $w = L(r)$ for some standard sequence r of E with $\wedge q \equiv \wedge r$. Therefore $\wedge p \equiv \wedge r$ and uRw . \square

Lemma 4.14. Let $E = (E, P, L, d)$ be a f_s -BC, u, v be paths of Q_E such that $u - v$ is a relation of I_E of type (fR1). For each path w of Q_E whose source (resp. terminal) is equal to the terminal (resp. source) of u , one of the following holds: (1) wu, vw (resp. uw, vw) are relations of I_E of type (fR2) or type (fR3); (2) $wu - vw$ (resp. $uw - vw$) is a relation of I_E of type (fR1).

Proof. Since $u - v$ is a relation of I_E of type $(fR1)$, u, v are paths of length ≥ 1 in \mathcal{E} with uRv . Let w be a path of Q_E whose source is equals to the terminal of u . We may assume that $l(w) > 0$. If wu is neither a relation of I_E of type $(fR2)$ nor a relation of I_E of type $(fR3)$, then $wu \in \mathcal{E}$. Let $wu = L(p)$, where p is a non-trivial standard sequence of E . Write $p = p_2p_1$, where $u = L(p_1)$ and $w = L(p_2)$. Since $l(w), l(u) > 0$, both p_1 and p_2 are non-trivial standard sequences of E . Since uRv , by Lemma 4.12, $v = L(q_1)$ for some standard sequence q_1 of E with ${}^\wedge p_1 \equiv {}^\wedge q_1$. Write ${}^\wedge p_1 = p_3p_2$, where p_3 is a standard sequence of E . Moreover, write ${}^\wedge q_1 = q_3q_2$ such that $q_2 \equiv p_2$ and $q_3 \equiv p_3$. Then $p_3 = {}^\wedge(p_2p_1)$ and $q_3 = {}^\wedge(q_2q_1)$. Since $wu = L(p_2p_1)$ and $wv = L(q_2q_1)$ with ${}^\wedge(p_2p_1) = p_3 \equiv q_3 = {}^\wedge(q_2q_1)$, $(wu)R(wv)$ and $wu - wv$ is a relation of I_E of type $(fR1)$. \square

For a set \mathcal{A} of paths of Q_E and for every two vertices x, y of Q_E , denote ${}_y\mathcal{A}_x$ be the subset of \mathcal{A} consists of paths with source x and terminal y . For a subset S of any k -vector space V , we denote by kS the k -subspace of V generated by S .

Lemma 4.15. *Let $E = (E, P, L, d)$ be a f_s -BC. For every two vertices x, y of Q_E , $I_E(x, y) = k_{{}_y(\mathcal{B}_1)_x} \oplus k_{{}_y(\mathcal{B}_2)_x} \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}_x \text{ and } uRv\}$. In particular, the relations of types $(fR1), (fR2), (fR3)$ generate I_E as a k -vector space in this case.*

Proof. Each element η of $I_E(x, y)$ is of the form $\sum \lambda_i u_i r_i v_i$, where $\lambda_i \in k^*$, u_i, v_i are paths of Q_E , and r_i 's are relations of I_E of type $(fR1)$, type $(fR2)$, or of type $(fR3)$. If r_i is a relation of I_E of type $(fR2)$ or of type $(fR3)$, it is straightforward to show that $u_i r_i v_i$ is also a relation of I_E of type $(fR2)$ or of type $(fR3)$. Therefore $u_i r_i v_i \in {}_y(\mathcal{B}_1)_x \sqcup {}_y(\mathcal{B}_2)_x$. If r_i is a relation of I_E of type $(fR1)$, by Lemma 4.14, either $u_i r_i$ is a relation of I_E of type $(fR1)$ or $u_i r_i \in k_{{}_y(\mathcal{B}_1)_z} \oplus k_{{}_y(\mathcal{B}_2)_z}$, where z is the source of r_i . Therefore either $u_i r_i v_i$ is a relation of I_E of type $(fR1)$ or $u_i r_i v_i \in k_{{}_y(\mathcal{B}_1)_x} \oplus k_{{}_y(\mathcal{B}_2)_x}$, which also follows by Lemma 4.14. Then we implies $u_i r_i v_i \in k_{{}_y(\mathcal{B}_1)_x} \oplus k_{{}_y(\mathcal{B}_2)_x} \oplus k\{u - v \mid u, v \in \mathcal{E} \text{ and } uRv\}$. So $I_E(x, y) \subseteq k_{{}_y(\mathcal{B}_1)_x} \oplus k_{{}_y(\mathcal{B}_2)_x} \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}_x \text{ and } uRv\}$. The fact that $k_{{}_y(\mathcal{B}_1)_x} \oplus k_{{}_y(\mathcal{B}_2)_x} \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}_x \text{ and } uRv\}$ is contained in $I_E(x, y)$ follows from the definition of I_E . \square

Remark 4.16. *Lemma 4.15 shows that if E is a f_s -BC, then every path in \mathcal{E} is not contained in I_E , and therefore it is nonzero in Λ_E . This shows that Condition (f7) ensures that the nonzero paths precisely correspond to standard sequences of E .*

Proposition 4.17. *Let $E = (E, P, L, d)$ be a f_s -BC, \mathcal{C} be a set of representatives of paths in \mathcal{E} under equivalence relation R . Then for every two vertices x, y of Q_E , the image of ${}_y\mathcal{C}_x$ in $\Lambda(x, y)$ forms a k -basis of $\Lambda(x, y)$.*

Proof. It follows from Lemma 4.15 and the fact that ${}_y(\mathcal{B}_1)_x \sqcup {}_y(\mathcal{B}_2)_x \sqcup {}_y\mathcal{E}_x$ forms a basis of $kQ_E(x, y)$. \square

Note that for $u, v \in \mathcal{E}$ with uRv , we have $u = v$ in Λ_E , therefore the image of ${}_y\mathcal{C}_x$ in $\Lambda_E(x, y)$ is independent to the choice of representatives of paths in \mathcal{E} under equivalence relation R .

Let $E = (E, P, L, d)$ be a f_s -BC. For every two objects x, y of Λ_E and for $u \in {}_y\mathcal{C}_x$, let $u = L(p)$ for some standard sequence p . Define ${}^\wedge u$ to be the unique path in ${}_{\sigma(x)}\mathcal{C}_y$ such that ${}^\wedge u RL({}^\wedge p)$, and u^\wedge to be the unique path in ${}_x\mathcal{C}_{\sigma^{-1}(y)}$ such that $u^\wedge RL(p^\wedge)$.

Note that ${}^\wedge(-)$ is a well defined map from ${}_y\mathcal{C}_x$ to ${}_{\sigma(x)}\mathcal{C}_y$: if $u = L(p) = L(q)$ for standard sequences p, q of E , then $p \equiv q$, therefore $L({}^\wedge p)RL({}^\wedge q)$. Similarly, $(-)^^\wedge$ is a well defined map from ${}_y\mathcal{C}_x$ to ${}_x\mathcal{C}_{\sigma^{-1}(y)}$.

Lemma 4.18. *Let $E = (E, P, L, d)$ be a f_s -BC. For every two objects x, y of Λ_E , $(-)^^\wedge : {}_{\sigma(x)}\mathcal{C}_y \rightarrow {}_y\mathcal{C}_x$ is the inverse of ${}^\wedge(-) : {}_y\mathcal{C}_x \rightarrow {}_{\sigma(x)}\mathcal{C}_y$.*

Proof. For $u \in {}_y\mathcal{C}_x$ with $u = L(p)$ for some standard sequence p , ${}^\wedge u$ is a path in \mathcal{C} with ${}^\wedge u RL({}^\wedge p)$. Then there exists standard sequences q_1, q_2 of E with ${}^\wedge u = L(q_1)$, $L({}^\wedge p) = L(q_2)$ and ${}^\wedge q_1 \equiv {}^\wedge q_2$. By Remark 3.12 (3), $q_1^\wedge \equiv q_2^\wedge$, and therefore $L(q_1^\wedge) = L(q_2^\wedge)$. Since $L({}^\wedge p) = L(q_2)$, $L(p)RL(q_2^\wedge)$.

By the definition of $(-)^{\wedge}$, we have $(\wedge u)^{\wedge} RL(q_1^{\wedge})$. So $(\wedge u)^{\wedge} RL(p)$, where $L(p) = u$. Since both u and $(\wedge u)^{\wedge}$ belong to ${}_y\mathcal{C}_x$, $(\wedge u)^{\wedge} = u$. It can be shown similarly that $\wedge(u^{\wedge}) = u$. \square

For each object x of Λ_E , let w be the unique path in ${}_{\sigma(x)}\mathcal{C}_x$ which corresponds to a full sequence. Define a linear form $\epsilon : \Lambda_E(x, \sigma(x)) \rightarrow k$ by sending w to 1 and sending other $u \in {}_{\sigma(x)}\mathcal{C}_x$ to zero. For every objects x, y of Λ_E , let $\langle -, - \rangle : \Lambda_E(y, \sigma(x)) \times \Lambda_E(x, y) \rightarrow k$ be the bilinear form defined by the composition $\Lambda_E(y, \sigma(x)) \times \Lambda_E(x, y) \xrightarrow{\text{multi}} \Lambda_E(x, \sigma(x)) \xrightarrow{\epsilon} k$.

Lemma 4.19. *For every two objects x, y of Λ_E , the bilinear form $\langle -, - \rangle : \Lambda_E(y, \sigma(x)) \times \Lambda_E(x, y) \rightarrow k$ defined by $\langle a, b \rangle = \epsilon(ab)$ is non-degenerate.*

Proof. It suffices to show for every $u \in {}_y\mathcal{C}_x$ and $v \in {}_{\sigma(x)}\mathcal{C}_y$,

$$(2) \quad \langle v, u \rangle = \begin{cases} 1, & v = \wedge u; \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $u \in {}_y\mathcal{C}_x$ and $v \in {}_{\sigma(x)}\mathcal{C}_y$, either $vu = 0$ in Λ_E or $(vu)Rw$ for some $w \in {}_{\sigma(x)}\mathcal{C}_x$, so the value of $\langle v, u \rangle$ can only take 0 or 1. When $v = \wedge u$, suppose that $u = L(p)$ for some standard sequence p . Then $\wedge u = L(\wedge p)$ in Λ_E and $\langle v, u \rangle = \epsilon(\wedge uu) = \epsilon(L(\wedge p)L(p)) = \epsilon(L(\wedge pp)) = 1$. Conversely, suppose that $\langle v, u \rangle = 1$. Since $\epsilon(vu) = \langle v, u \rangle = 1$, $vu \neq 0$ in Λ_E . By Lemma 4.15, $vu \in \mathcal{E}$. By the definition of ϵ , $vu = L(q)$ for some full sequence q of E . Write $q = q_2q_1$ such that $L(q_1) = u$ and $L(q_2) = v$. Then $\wedge u RL(\wedge q_1)$ and $L(\wedge q_1) = L(q_2) = v$ imply $\wedge u Rv$. Since both $\wedge u$ and v belong to \mathcal{C} , we have $\wedge u = v$. \square

We now show that the associated locally bounded category of a f_s -BC is a Frobenius category.

Theorem 4.20. *Let E be a f_s -BC. Then $\Lambda_E(-, \sigma(x)) \cong D\Lambda_E(x, -)$ for all object x of Λ_E . Therefore the associated f -BCC Λ_E is a locally bounded Frobenius category.*

Proof. For each two objects x, y of Λ_E , by Lemma 4.19, the bilinear form $\langle -, - \rangle : \Lambda_E(y, \sigma(x)) \times \Lambda_E(x, y) \rightarrow k$ is non-degenerate. Therefore it induces an isomorphism $\Lambda_E(y, \sigma(x)) \cong D\Lambda_E(x, y)$. Moreover, by the definition of the bilinear form, this isomorphism is natural at y , so we have $\Lambda_E(-, \sigma(x)) \cong D\Lambda_E(x, -)$. Replace x by $\sigma^{-1}(x)$, we have $\Lambda_E(-, x) \cong D\Lambda_E(\sigma^{-1}(x), -)$. Therefore Λ_E is a locally bounded Frobenius category. \square

5. FRACTIONAL BRAUER CONFIGURATION ALGEBRA

Definition 5.1. *Let E be a f -BC, and let $A_E = \bigoplus_{x, y \in (Q_E)_0} \Lambda_E(x, y)$ be the algebra corresponding to the category Λ_E . We call A_E the fractional Brauer configuration algebra (abbr. f -BCA) of E . If moreover E is a f_s -BC (resp. f_s -BG, f_{ms} -BC, f_{ms} -BG), then we call A_E a f_s -BCA (resp. f_s -BGA, f_{ms} -BCA, f_{ms} -BGA).*

By definition, A_E is a locally bounded algebra (that is, there is a complete set of pairwise orthogonal primitive idempotents $\{1_x \mid x \in (Q_E)_0\}$ such that $(A_E)1_x$ and $1_x(A_E)$ are finite-dimensional over k for all $x \in (Q_E)_0$) since the category Λ_E is locally bounded and A_E is isomorphic to kQ_E/I_E . If moreover E is a finite set, then A_E is a finite-dimensional algebra.

When E is a finite f_s -BC, we may extend the linear forms $\epsilon : \Lambda_E(x, \sigma(x)) \rightarrow k$ (the definition of ϵ is given before Lemma 4.19) to a linear form of A_E , which is also denoted by ϵ . Let $\langle -, - \rangle : A_E \times A_E \rightarrow k$ be the bilinear form defined by the composition $A_E \times A_E \xrightarrow{\text{multi}} A_E \xrightarrow{\epsilon} k$. For each $x, y \in (Q_E)_0$, the restriction of this bilinear form to $\Lambda_E(y, \sigma(x)) \times \Lambda_E(x, y)$ is non-degenerate by Lemma 4.19 and the restriction of $\langle -, - \rangle : A_E \times A_E \rightarrow k$ to $\Lambda_E(y', x') \times \Lambda_E(x, y)$ is zero, whenever $y \neq y'$ or $x' \neq \sigma(x)$. Therefore the bilinear form $\langle -, - \rangle : A_E \times A_E \rightarrow k$ is non-degenerate with the property that $\langle ab, c \rangle = \langle a, bc \rangle$ for all $a, b, c \in A_E$. Then we have:

Proposition 5.2. *If E is a finite f_s -BC, then A_E is a finite-dimensional Frobenius algebra with the linear form $\epsilon : A_E \rightarrow k$ defined as above.*

In Example 3.5, E is a f_{ms} -BG and A_E is a finite-dimensional special biserial Frobenius algebra. The structures of indecomposable projective modules of A_E are

$$P_1 = \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 2 \quad 2 \\ \diagup \quad \diagdown \\ 3 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 3 \quad 3 \\ \diagup \quad \diagdown \\ 1 \end{array}, \quad P_3 = \begin{array}{c} 3 \\ \diagdown \quad \diagup \\ 1 \quad 1 \\ \diagup \quad \diagdown \\ 2 \end{array}.$$

In Example 3.6, E is a f_s -BG and A_E is a finite-dimensional symmetric algebra but is not multiserial. The structures of indecomposable projective modules of A_E are

$$P_1 = \begin{array}{c} 1 \\ | \\ 2 \\ \diagdown \quad \diagup \\ 3 \quad 4 \\ \diagup \quad \diagdown \\ 1 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 3 \quad 4 \\ \diagup \quad \diagdown \\ 1 \\ | \\ 2 \end{array}, \quad P_3 = \begin{array}{c} 3 \\ | \\ 1 \\ | \\ 2 \\ | \\ 3 \end{array}, \quad P_4 = \begin{array}{c} 4 \\ | \\ 1 \\ | \\ 2 \\ | \\ 4 \end{array}.$$

In Example 3.7, E is a f-BC but not a f_s -BC (see remarks after Remark 3.14), and $A_E \cong k$ (see remarks after Remark 4.5).

In Example 3.8, E is a f-BC but not a f_s -BC (see remarks after Remark 3.14). The structures of indecomposable projective modules of A_E are

$$P_1 = \begin{array}{c} 1 \\ \diagdown \quad \diagup \\ 2 \quad 3 \\ \diagup \quad \diagdown \\ 4 \\ | \\ 5 \end{array}, \quad P_2 = \begin{array}{c} 2 \\ | \\ 4 \\ \diagdown \quad \diagup \\ 5 \quad 6 \\ \diagup \quad \diagdown \\ 1 \\ | \\ 2 \end{array}, \quad P_3 = \begin{array}{c} 3 \\ | \\ 4 \\ | \\ 5 \end{array}, \quad P_4 = \begin{array}{c} 4 \\ \diagdown \quad \diagup \\ 5 \quad 6 \\ \diagup \quad \diagdown \\ 1 \\ | \\ 2 \end{array}, \quad P_5 = \begin{array}{c} 5 \\ | \\ 1 \\ \diagdown \quad \diagup \\ 2 \quad 3 \\ \diagup \quad \diagdown \\ 4 \\ | \\ 5 \end{array}, \quad P_6 = \begin{array}{c} 6 \\ | \\ 1 \\ | \\ 2 \end{array}.$$

Therefore A_E is finite-dimensional but not self-injective.

In Example 3.9, E is a f_{ms} -BC and A_E is a locally bounded special biserial Frobenius algebra. The structures of indecomposable projective modules of A_E are

$$P_{3i} = \begin{array}{c} 3i \\ \diagdown \quad \diagup \\ 3i+1 \quad 3i+2 \\ \diagup \quad \diagdown \\ 3i+3 \end{array}, \quad P_{3i+1} = \begin{array}{c} 3i+1 \\ | \\ 3i+3 \\ | \\ 3i+4 \end{array}, \quad P_{3i+2} = \begin{array}{c} 3i+2 \\ | \\ 3i+3 \\ | \\ 3i+5 \end{array},$$

where $i \in \mathbb{Z}$.

Let E be a finite f_s -BC. The automorphism σ of Λ_E induces an automorphism of A_E , which is also denoted by σ . Note that the automorphism σ preserves the bilinear form $\langle -, - \rangle$ on A_E , that is, $\langle \sigma(a), \sigma(b) \rangle = \langle a, b \rangle$ for all $a, b \in A_E$.

Lemma 5.3. *If E is a finite f_s -BC, then $\epsilon(\sigma(a)b) = \epsilon(ba)$ for all $a, b \in A_E$.*

Proof. Let $a = \sum_{x,y \in (Q_E)_0} y a_x$ and $b = \sum_{x,y \in (Q_E)_0} y b_x$, where ${}_y a_x, {}_y b_x \in \Lambda_E(x, y)$. Then $\epsilon(ba) = \sum_{x,y \in (Q_E)_0} \epsilon({}_{\sigma(x)} b_y {}_y a_x)$ and $\epsilon(\sigma(a)b) = \sum_{x,y \in (Q_E)_0} \epsilon({}_{\sigma(y)} \sigma(a)_{\sigma(x)} {}_{\sigma(x)} b_y)$. To show $\epsilon(\sigma(a)b) = \epsilon(ba)$, it suffices to show that $\epsilon({}_{\sigma(x)} b_y {}_y a_x) = \epsilon({}_{\sigma(y)} \sigma(a)_{\sigma(x)} {}_{\sigma(x)} b_y)$ for each $x, y \in (Q_E)_0$. We may assume that ${}_y a_x \in {}_y \mathcal{C}_x$ and ${}_{\sigma(x)} b_y \in {}_{\sigma(x)} \mathcal{C}_y$. For $u \in {}_y \mathcal{C}_x$ and $v \in {}_{\sigma(x)} \mathcal{C}_y$, by the proof of Lemma

4.19,

$$(3) \quad \epsilon(vu) = \begin{cases} 1, & v = \wedge u; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, it can be shown straightforward that $\sigma(u) = \wedge(\wedge u)$ in Λ_E . If $v = \wedge u$, then $\epsilon(\sigma(u)v) = \epsilon(\wedge(\wedge u)\wedge u) = 1 = \epsilon(vu)$. If $v \neq \wedge u$, by Lemma 4.18, $\wedge v \neq \wedge(\wedge u)$. Then $\epsilon(\sigma(u)v) = \epsilon(\wedge(\wedge u)v) = 0 = \epsilon(vu)$. Therefore $\epsilon_{(\sigma(x)b_y y a_x)} = \epsilon_{(\sigma(y)\sigma(a)_{\sigma(x)}\sigma(x)b_y)}$. \square

Proposition 5.4. *If E is a finite f_s -BC, then the automorphism σ of A_E is equal to the inverse of the usual Nakayama automorphism of the self-injective algebra A_E .*

Proof. Let ν_A be the Nakayama automorphism of A_E . Then $\epsilon(ab) = \epsilon(b\nu_A(a))$ for all $a, b \in A_E$. By Lemma 5.3, $\epsilon(ab) = \epsilon(b\sigma^{-1}(a))$ for all $a, b \in A_E$. Since $\langle a, b \rangle = \epsilon(ab)$ and $\langle -, - \rangle : A_E \times A_E \rightarrow k$ is non-degenerate, $\sigma^{-1} = \nu_A$. Therefore $\sigma = \nu_A^{-1}$. \square

Proposition 5.5. *If E is a finite f_s -BC of integral f -degree (for example, if E is a finite f_{ms} -BC of integral f -degree or if E is a finite f_s -BG of integral f -degree), then A_E is a finite-dimensional symmetric algebra.*

Proof. Since E has integral f -degree, the automorphism σ of A_E is identity. By Lemma 5.3, the linear form ϵ of A_E is symmetric. \square

Remark 5.6. (1) *There exist self-injective algebras which are not Morita equivalent to f_s -BCA. For example, if k is an algebraically closed field, then every basic indecomposable nonstandard representation-finite self-injective algebra is not isomorphic to f_s -BCA (see Theorem 7.14). Another example which is not Morita equivalent to f_s -BCA is the algebra $k\langle x, y \rangle / \langle x^2, y^2, xy - \lambda yx \rangle$, where $\lambda \neq 1$.*

(2) *By Corollary 6.6, a f_{ms} -BCA with integral f -degree is equal to a BCA at least over an algebraically closed field. However, the f_s -BGAs with integral f -degree beyond the scope of BCAs, they are also symmetric but not special biserial in general and will be studied in a forthcoming paper [11].*

6. THE GABRIEL QUIVER AND ADMISSIBLE RELATIONS OF A FRACTIONAL BRAUER CONFIGURATION CATEGORY IN TYPE S

Throughout this section k will be an algebraically closed field. Let Λ be a locally bounded k -category. According to [4, Section 2.1], Λ is isomorphic to the form kQ/I , where Q is a locally finite quiver and I is an admissible ideal of path category kQ , that is, for each $x \in Q_0$, there exists a positive integer N_x with $kQ^{\geq N_x}(x, -) \subseteq I(x, -) \subseteq kQ^{\geq 2}(x, -)$ and $kQ^{\geq N_x}(-, x) \subseteq I(-, x) \subseteq kQ^{\geq 2}(-, x)$, where $kQ^{\geq n}$ denotes the ideal of kQ generated by paths of length $\geq n$. Moreover, such quiver Q is uniquely determined by Λ , which is called the Gabriel quiver of Λ .

By [4, Section 2.1], the Gabriel quiver Q and the admissible ideal I of Λ can be constructed as follows. The vertices of Q are the objects of Λ . Denote rad_Λ the radical of the k -category Λ . For every objects x, y of Λ , choose morphisms f_1, \dots, f_m in $\text{rad}_\Lambda(x, y)$ such that the images of f_1, \dots, f_m in $\text{rad}_\Lambda(x, y)/\text{rad}_\Lambda^2(x, y)$ form a k -basis of it. Then there are m arrows $\alpha_1, \dots, \alpha_m$ from x to y in Q . Moreover, the functor $\rho : kQ \rightarrow \Lambda$ sending each vertex of Q to the associated object in Λ and sending each arrow α_i to the morphism f_i is full, which induces an isomorphism $kQ/I \rightarrow \Lambda$, where $I = \ker \rho$ is the admissible ideal.

In this section, we will determine the Gabriel quiver and admissible relations of $\Lambda_E = kQ_E/I_E$, where E is a f_s -BC. As the following lemma shows that, the Gabriel quiver and the admissible ideal are obtained from a reduction procedure from the quiver Q_E and the ideal I_E .

Let $E = (E, P, L, d)$ be a f_s -BC. Let \mathcal{E} be the set of paths of the quiver Q_E with an equivalence relation R defined in Definition 4.8. Note that each arrow α of Q_E from x to y belongs to ${}_y\mathcal{E}_x$.

Definition 6.1. (1) We call an arrow α of Q_E from x to y reduced, if there exists a path $p \in {}_y\mathcal{E}_x$ of length ≥ 2 such that αRp .

(2) Denote ${}_y\mathcal{N}_x$ the set of representatives of non-reduced arrows of Q_E from x to y under the equivalence relation R , and define a sub-quiver Q'_E of Q_E :

$$(Q'_E)_0 = (Q_E)_0, \quad (Q'_E)_1 = \bigsqcup_{x,y \in (Q_E)_0} {}_y\mathcal{N}_x.$$

(3) Denote $\rho : kQ'_E \rightarrow \Lambda_E = kQ_E/I_E$ the natural k -linear functor, and let $I'_E := \ker \rho$.

Note that by the property (5) in Definition 2.1, if E is a BC (viewed as a f-BC) and e is an angle such that the corresponding vertex $G \cdot e$ is truncated, then the arrow $L(e)$ of Q_E is reduced.

Lemma 6.2. Q'_E is the Gabriel quiver of Λ_E , the functor $\rho : kQ'_E \rightarrow \Lambda_E$ is dense and full, and the kernel I'_E of ρ is an admissible ideal of kQ'_E . In particular, the f_s -BC category Λ_E is isomorphic to the category kQ'_E/I'_E .

Proof. The fact that ρ is dense is clear since Q'_E and Q_E have the same vertices. To show that ρ is full, it suffices to show that for each $x, y \in (Q_E)_0$, the image of ${}_y\mathcal{N}_x$ in $J(x, y)/J^2(x, y)$ forms a k -basis of $J(x, y)/J^2(x, y)$, where J is the ideal of Λ_E generated by the arrows of Q_E (J is also the radical of Λ_E by Theorem 4.6).

By Proposition 4.17, the set of paths

$${}_y\mathcal{D}_x = \begin{cases} {}_y\mathcal{C}_x - \{1_x\}, & \text{if } x = y; \\ {}_y\mathcal{C}_x, & \text{otherwise} \end{cases}$$

forms a k -basis of $J(x, y)$, where \mathcal{C} is a set of representatives of paths in \mathcal{E} under equivalence relation R . We may assume that ${}_y\mathcal{N}_x$ is contained in ${}_y\mathcal{D}_x$. For each $p \in {}_y\mathcal{D}_x - {}_y\mathcal{N}_x$, either p is a reduced arrow of Q_E or p is a path of length ≥ 2 , therefore $p \in J^2(x, y)$. So the image of ${}_y\mathcal{N}_x$ in $J(x, y)/J^2(x, y)$ generates the whole space. Suppose that $\sum_{i=1}^n \lambda_i \alpha_i \in J^2(x, y)$, where $\lambda_i \in k$ and $\alpha_i \in {}_y\mathcal{N}_x$ for each $1 \leq i \leq n$, then we may assume that $\sum_{i=1}^n \lambda_i \alpha_i = \sum_{j=1}^m \mu_j p_j$ in Λ_E , where $\mu_j \in k$ and p_j is a path of Q_E of length ≥ 2 for each $1 \leq j \leq m$. According to Lemma 4.15, we may assume that each p_j belongs to ${}_y\mathcal{E}_x$. For each $1 \leq j \leq m$, let q_j be the path of ${}_y\mathcal{D}_x$ such that $p_j R q_j$. Since $l(p_j) \geq 2$, $q_j \notin {}_y\mathcal{N}_x$. Since ${}_y\mathcal{D}_x$ forms a k -basis of $J(x, y)$, we have $\sum_{i=1}^n \lambda_i \alpha_i = \sum_{j=1}^m \mu_j q_j = 0$ in Λ_E . Then $\lambda_i = 0$ for each $1 \leq i \leq n$ and the image of ${}_y\mathcal{N}_x$ in $J(x, y)/J^2(x, y)$ is linearly independent. \square

For each set \mathcal{A} of paths of Q_E , denote \mathcal{A}' the subset of \mathcal{A} formed by the paths in \mathcal{A} which are also paths of Q'_E . Then for every vertices x, y of Q'_E , ${}_y(\mathcal{B}'_1)_x \sqcup {}_y(\mathcal{B}'_2)_x \sqcup {}_y\mathcal{E}'_x$ forms a basis of kQ'_E (cf. Remark 4.11). Similar to Lemma 4.15, we have

Lemma 6.3. Let $E = (E, P, L, d)$ be a f_s -BC. For every vertices x, y of Q'_E , $I'_E(x, y) = k_{{}_y(\mathcal{B}'_1)_x} \oplus k_{{}_y(\mathcal{B}'_2)_x} \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}'_x \text{ and } uRv\}$.

Proof. By Lemma 4.15 we have $I_E(x, y) = k_{{}_y(\mathcal{B}_1)_x} \oplus k_{{}_y(\mathcal{B}_2)_x} \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}_x \text{ and } uRv\}$. Since I'_E is the kernel of the k -linear functor $\rho : kQ'_E \rightarrow \Lambda_E = kQ_E/I_E$, the subspace $k_{{}_y(\mathcal{B}'_1)_x} \oplus k_{{}_y(\mathcal{B}'_2)_x} \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}'_x \text{ and } uRv\}$ of $kQ'_E(x, y)$ is contained in $I'_E(x, y)$.

Conversely, for each $r \in I'_E(x, y)$, we may write

$$r = \sum_{p \in {}_y(\mathcal{B}'_1)_x \sqcup {}_y(\mathcal{B}'_2)_x} \lambda_p p + \sum_{q \in {}_y\mathcal{E}'_x} \mu_q q,$$

where $\lambda_p, \mu_q \in k$. Then

$$\sum_{p \in {}_y(\mathcal{B}'_1)_x \sqcup {}_y(\mathcal{B}'_2)_x} \lambda_p p + \sum_{q \in {}_y\mathcal{E}'_x} \mu_q q \in I_E(x, y).$$

Since each $p \in {}_y(\mathcal{B}'_1)_x \sqcup {}_y(\mathcal{B}'_2)_x$ belongs to $I_E(x, y)$, $\sum_{q \in {}_y\mathcal{E}'_x} \mu_q q \in I_E(x, y)$. By Lemma 4.15 we imply that $\sum_{q \in {}_y\mathcal{E}'_x} \mu_q q$ belongs to the subspace $k\{u - v \mid u, v \in {}_y\mathcal{E}_x \text{ and } uRv\}$ of $I_E(x, y)$. Therefore for each $q \in {}_y\mathcal{E}'_x$,

$$\sum_{l \in {}_y\mathcal{E}'_x, lRq} \mu_l = 0.$$

So $\sum_{q \in {}_y\mathcal{E}'_x} \mu_q q \in k\{u - v \mid u, v \in {}_y\mathcal{E}'_x \text{ and } uRv\}$ and r belongs to the subspace $k_y(\mathcal{B}'_1)_x \oplus k_y(\mathcal{B}'_2)_x \oplus k\{u - v \mid u, v \in {}_y\mathcal{E}'_x \text{ and } uRv\}$ of $kQ'_E(x, y)$. \square

Note that Examples 3.5, 3.6 and 3.9 are of type S. In Examples 3.5 and 3.9, Q'_E is the same as Q_E respectively, and in Example 3.6, Q'_E is obtained from Q_E by removing the loops.

Definition 6.4. (cf. [8]) We call a locally bounded category Λ special multiserial if $\Lambda \cong kQ/I$ for some locally finite quiver Q and some admissible ideal I , such that for each arrow α of Q , there exists at most one arrow β (resp. γ) of Q such that $\beta\alpha \notin I$ (resp. $\alpha\gamma \notin I$).

Proposition 6.5. *If E is a f_{ms} -BC, then Λ_E is a locally bounded special multiserial Frobenius category. In particular, if E is a f_{ms} -BG, then Λ_E is a locally bounded special biserial Frobenius category.*

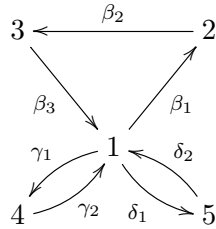
Proof. By Lemma 6.2, $\Lambda_E \cong kQ'_E/I'_E$, where Q'_E is a locally finite quiver and I'_E is an admissible ideal of kQ'_E . Let $\alpha = L(e)$ be an arrow of Q'_E . If $\beta\alpha \notin I'_E$ for some arrow β of Q'_E , by Lemma 6.3 we have $\beta\alpha \notin \mathcal{B}'_1$, so $\beta\alpha = L(g \cdot h)L(h)$ for some $h \in E$. Since the partition L of E is trivial, $e = h$ and $\beta = L(g \cdot e)$. Similarly, if $\alpha\gamma \notin I'_E$ for some arrow γ of Q'_E , then $\gamma = L(g^{-1} \cdot e)$. So there exists at most one arrow β (resp. γ) of Q'_E such that $\beta\alpha \notin I'_E$ (resp. $\alpha\gamma \notin I'_E$), and Λ_E is special multiserial. The fact that Λ_E is Frobenius follows from Theorem 4.20. \square

Corollary 6.6. *If E is a finite f_{ms} -BC with integral f -degree, then A_E is a BCA.*

Proof. By Propositions 6.5 and 5.5, A_E is a finite-dimensional special multiserial symmetric algebra. Now by the result of [8], over an algebraically closed field, the class of symmetric special multiserial algebras coincides with the class of BCAs. \square

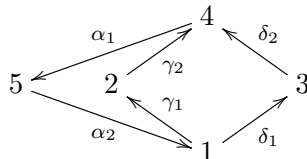
One reason for us to define f-BCAs is that BCAs are not closed under derived equivalence.

Example 6.7. *Let $E = \{1, 1', 1'', 2, 2', 3, 3', 4, 4', 5, 5'\}$. Define the group action on E by $g \cdot 1 = 2$, $g \cdot 2 = 3$, $g \cdot 3 = 1$, $g \cdot 1' = 4'$, $g \cdot 4' = 1'$, $g \cdot 1'' = 5'$, $g \cdot 5' = 1''$, $g \cdot 2' = 2'$, $g \cdot 3' = 3'$, $g \cdot 4 = 4$, $g \cdot 5 = 5$. Define $P(1) = \{1, 1', 1''\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3, 3'\}$, $P(4) = \{4, 4'\}$, $P(5) = \{5, 5'\}$, and $L(e) = \{e\}$ for every $e \in E$. The f -degree of E is defined to be trivial. Then E is a Brauer configuration, and A_E is a BCA, which is given by the quiver*



with relations $0 = \beta_1\gamma_2 = \gamma_1\beta_3 = \beta_1\delta_2 = \delta_1\beta_3 = \gamma_1\delta_2 = \delta_1\gamma_2 = \beta_2\beta_1\beta_3\beta_2$, $\beta_3\beta_2\beta_1 = \gamma_2\gamma_1 = \delta_2\delta_1$.

A_E is a representation-finite self-injective algebra of type $(D_5, 1, 1)$, which is derived equivalent to the algebra B , where B is given by the quiver



with relations $\gamma_2\gamma_1 = \delta_2\delta_1$, $0 = \delta_1\alpha_2\alpha_1\gamma_2 = \gamma_1\alpha_2\alpha_1\delta_2 = \alpha_2\alpha_1\gamma_2\gamma_1\alpha_2 = \alpha_1\gamma_2\gamma_1\alpha_2\alpha_1$.

Since B is not multiserial, it is not a BCA. However, B is a f_s -BCA, whose fractional Brauer configuration $E' = (E', P', L', d')$ is given as follows: $E' = \{1, 1', 2, 3', 4, 4', 5, 5'\}$. The action of $\langle g \rangle$ on E' is given by $g \cdot 1 = 2$, $g \cdot 2 = 4$, $g \cdot 4 = 5$, $g \cdot 5 = 1$, $g \cdot 1' = 3'$, $g \cdot 3' = 4'$, $g \cdot 4' = 5'$, $g \cdot 5' = 1'$. The partition P' on E' is given by $P'(1) = \{1, 1'\}$, $P'(2) = \{2\}$, $P'(3') = \{3'\}$, $P'(4) = \{4, 4'\}$, $P'(5) = \{5, 5'\}$. The partition L' on E' is given by $L'(4) = \{4, 4'\}$, $L'(5) = \{5, 5'\}$, and $L'(e) = \{e\}$ for other $e \in E'$. The f -degree of E is defined to be trivial.

7. FRACTIONAL BRAUER CONFIGURATION ALGEBRAS AND REPRESENTATION-FINITE SELF-INJECTIVE ALGEBRAS

Throughout this section k will be an algebraically closed field.

Let Q be a locally finite connected quiver without double arrows, I be an admissible ideal of path category kQ . The pair (Q, I) is called a quiver with relations.

Recall from [4, Definition 2.2] that a locally bounded category Λ is called locally representation-finite if for every object x of Λ , the number of isomorphism classes of finitely generated indecomposable Λ -module l such that $l(x) \neq 0$ is finite. Denote the Gabriel quiver of Λ by Q_Λ .

For a translation quiver Γ , we let $k\Gamma$ be its path category, and let $k(\Gamma)$ be the mesh category of Γ , which is a factor category of $k\Gamma$ by the mesh ideal. For a locally bounded category Λ , we denote by $\text{ind}\Lambda$ the category formed by chosen representatives of the finitely generated indecomposable modules.

Definition 7.1. ([4, Definition 5.1]) A locally representation-finite category Λ is said to be standard if $k(\Gamma_\Lambda) \cong \text{ind}\Lambda$, where Γ_Λ is the AR-quiver of Λ .

Definition 7.2. ([12, Definition 1.3]) Let (Q, I) be a quiver with relations. A relation $\rho = \sum_{i=1}^n \lambda_i u_i \in I(x, y)$ with $\lambda_i \in k^*$ and u_i a path from x to y , is a minimal relation if $n \geq 2$ and for every non-empty proper subset K of $\{1, \dots, n\}$, $\sum_{i \in K} \lambda_i u_i \notin I(x, y)$.

For a quiver with relations (Q, I) , [12, Lemma 2.3] introduces two conditions:

(D) If $\rho = \sum_{i=1}^n \lambda_i u_i \in I$ is a minimal relation, then for every two different $i, j \in \{1, \dots, n\}$ there exists $c \in k^*$ such that $u_i + cu_j \in I$.

(C) Let $x, y, z \in Q_0$, u be a path of Q with source x and terminal y , and v, w be two paths of Q with source y and terminal z , such that $vu, wu \notin I$. Then for $\lambda \in k^*$, $vu + \lambda wu \in I$ if and only if $v + \lambda w \in I$.

According to [12, Corollary 3.9], a locally representation-finite k -category Λ is standard if and only if $\Lambda \cong k(Q, I)$ for some quiver with relations (Q, I) which satisfies conditions (D) and (C).

Proposition 7.3. Let $E = (E, P, L, d)$ be a f_s -BC such that Λ_E is locally representation-finite. Then (Q'_E, I'_E) satisfies conditions (D) and (C), hence $\Lambda_E \cong kQ'_E/I'_E$ is standard.

Proof. By Lemma 6.3, any relation $r \in I'_E(x, y)$ can be written as $\sum_{i=1}^n \lambda_i u_i + \sum_{j=1}^m \mu_j v_j + \sum_{k=1}^d \nu_k (p_k - q_k)$, where $\lambda_i, \mu_j, \nu_k \in k^*$, $u_i \in {}_y(\mathcal{B}'_1)_x$, $v_j \in {}_y(\mathcal{B}'_2)_x$, and $p_k, q_k \in {}_y\mathcal{E}'_x$ with $p_k R q_k$. If r is minimal, then $n, m = 0$. Therefore r is of the form $\sum_{i=1}^N \sum_{j=1}^{d_i} c_{ij} w_{ij}$, where $c_{ij} \in k^*$, $w_{ij} \in {}_y\mathcal{E}'_x$, and $w_{ij} R w_{i', j'}$ if and only if $i = i'$. Since $r \in I'_E(x, y)$, by Lemma 6.3 we have $\sum_{j=1}^{d_i} c_{ij} = 0$ for each i , and therefore $\sum_{j=1}^{d_i} c_{ij} w_{ij} \in I'_E(x, y)$ for each i . Since r is minimal, $N = 1$. For every two different $i, j \in \{1, \dots, d_1\}$, since $w_{1i} R w_{1j}$, $w_{1i} - w_{1j} \in I'_E$. Then (Q'_E, I'_E) satisfies condition (D).

Let x, y, z be vertices of Q'_E and u, v, w be paths of Q'_E with $u \in kQ'(x, y)$, $v, w \in kQ'(y, z)$, such that $vu, wu \notin I'_E$. If $vu + \lambda wu \in I'_E$ for $\lambda \in k^*$, by Lemma 6.3 we have $\lambda = -1$ and $vu, wu \in {}_z\mathcal{E}'_x$ with $(vu)R(wu)$. Let $vu = L(p)$ and $wu = L(q)$ for some standard sequences p, q of E with $\wedge p \equiv \wedge q$, we may write $p = p_2 p_1$ and $q = q_2 q_1$, where $L(p_1) = L(q_1) = u$, $L(p_2) = v$, $L(q_2) = w$. Then $p_2^\wedge \equiv p_1 p^\wedge \equiv q_1 q^\wedge = q_2^\wedge$. So $\wedge p_2 \equiv \wedge q_2$ and $v R w$. Therefore $v + \lambda w = v - w \in I'_E$. Then (Q'_E, I'_E) satisfies condition (C). \square

Let Λ be a locally representation-finite category. For any objects a, b of Λ , $\Lambda(a, b)$ is a uniserial bimodule over $\Lambda(b, b)$ and $\Lambda(a, a)$ (see [6, Section 2.4]). Denote

$$\mathcal{R}^0\Lambda(a, b) = \Lambda(a, b) \supseteq \mathcal{R}^1\Lambda(a, b) \supseteq \mathcal{R}^2\Lambda(a, b) \supseteq \cdots \supseteq \mathcal{R}^i\Lambda(a, b) \supseteq \cdots$$

the radical series of $\Lambda(a, b)$ as a $\Lambda(b, b)$ - $\Lambda(a, a)$ -bimodule. A morphism $\mu \in \Lambda(a, b)$ is said to have level n if $\mu \in \mathcal{R}^n\Lambda(a, b) - \mathcal{R}^{n+1}\Lambda(a, b)$. We denote by $I = \text{ind}\Lambda$ the category formed by chosen representatives of the isomorphism classes of indecomposable finite-dimensional Λ -modules, such that $a^* = \Lambda(-, a)$ is chosen as a representative for each object a of Λ . Note that $(-)^*$ induces a fully faithful functor $\Lambda \rightarrow I = \text{ind}\Lambda$. Denote

$$\text{rad}_I^0 = I \supseteq \text{rad}_I^1 \supseteq \text{rad}_I^2 \supseteq \cdots \supseteq \text{rad}_I^p \supseteq \cdots$$

the radical series of the category I . Since $\text{rad}_I^p(a^*, b^*)$ is a bimodule over $I(b^*, b^*)$ and $I(a^*, a^*)$, it is equal to some $\mathcal{R}^n\Lambda(a, b)^*$. Let $g(a, b, n) = \sup\{p \mid \text{rad}_I^p(a^*, b^*) = \mathcal{R}^n\Lambda(a, b)^*\}$. Define the grade $g(\mu)$ of a morphism $\mu \in \Lambda(a, b)$ of level n by $g(\mu) = g(a, b, n)$. See [5, Section 1.1].

Remark 7.4. According to [6, Section 2.4], $\mathcal{R}^{n+1}\Lambda(a, b)$ has codimension ≤ 1 in $\mathcal{R}^n\Lambda(a, b)$. Let $g = g(a, b, n)$, then

$$(4) \quad \begin{aligned} \mathcal{R}^{n+1}\Lambda(a, b)^* &= (\mathcal{R}^n\Lambda(a, b)\text{rad}\Lambda(a, a) + \text{rad}\Lambda(b, b)\mathcal{R}^n\Lambda(a, b))^* \\ &= \text{rad}_I^g(a^*, b^*)\text{rad}I(a^*, a^*) + \text{rad}I(b^*, b^*)\text{rad}_I^g(a^*, b^*) \subseteq \text{rad}_I^{g+1}(a^*, b^*). \end{aligned}$$

Since $\text{rad}_I^g(a^*, b^*) = \mathcal{R}^n\Lambda(a, b)^*$, we imply that $\text{rad}_I^{g+1}(a^*, b^*)$ has codimension ≤ 1 in $\text{rad}_I^g(a^*, b^*)$. In case $g < \infty$, $\text{rad}_I^{g+1}(a^*, b^*) \neq \text{rad}_I^g(a^*, b^*)$, so $\text{rad}_I^{g+1}(a^*, b^*)$ has codimension 1 in $\text{rad}_I^g(a^*, b^*)$.

Let Λ be a locally representation-finite category. According to [6, Section 2.1], the universal cover $\tilde{\Lambda}$ of Λ is defined to be the full subcategory of $k(\tilde{\Gamma}_\Lambda)$ formed by projective vertices of $\tilde{\Gamma}_\Lambda$, where $\tilde{\Gamma}_\Lambda$ denotes the universal cover of the AR-quiver Γ_Λ of Λ . Moreover, there exists a covering functor $F : \tilde{\Lambda} \rightarrow \Lambda$, which is given by the commutative diagram

$$\begin{array}{ccc} k(\tilde{\Gamma}_\Lambda) & \xrightarrow{E} & I \\ \text{incl.} \downarrow & & \downarrow (-)^* \\ \tilde{\Lambda} & \xrightarrow{F} & \Lambda, \end{array}$$

where $E : k(\tilde{\Gamma}_\Lambda) \rightarrow I$ is a well-behaved functor ([4, Section 3.1]).

Since $\tilde{\Gamma}_\Lambda$ is simply connected, it does not contains oriented cycles. Therefore $k(\tilde{\Gamma}_\Lambda)(x, x) \cong k$ for each object x of $k(\tilde{\Gamma}_\Lambda)$. Since $\tilde{\Lambda}$ is a full subcategory of $k(\tilde{\Gamma}_\Lambda)$, $\tilde{\Lambda}(x, x) \cong k$ for each object x of $\tilde{\Lambda}$. Since $\tilde{\Gamma}_\Lambda$ is a Riedtmann-quiver ([4, Theorem 2.9]), by [4, Proposition 2.4], $\tilde{\Lambda}$ is locally representation-finite. By [5, Section 1.3], $\tilde{\Lambda}$ is schurian, that is, $\dim_k \tilde{\Lambda}(x, y) \leq 1$ for all objects x, y of $\tilde{\Lambda}$.

Let M be a locally bounded Schurian category. A path

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_p = y$$

of Q_M is said to be nonzero if the composition

$$M(x_{p-1}, x_p) \times \cdots \times M(x_0, x_1) \rightarrow M(x_0, x_p)$$

is nonzero (see [5, Section 1.3]).

In next two lemmas, we use the following notations from [5]: denote $\bar{\alpha}$ an arrow in Q_Λ and denote α a representative of $\bar{\alpha}$, that is, a morphism of Λ which belongs to $\text{rad}_\Lambda(x, y) - \text{rad}_\Lambda^2(x, y)$ with x (resp. y) the source (resp. terminal) of $\bar{\alpha}$.

Lemma 7.5. ([5, Lemma 1.4]) *Let Λ be a locally representation-finite category, $a_0 \xrightarrow{\bar{\alpha}_1} a_1 \xrightarrow{\bar{\alpha}_2} \cdots \rightarrow a_{n-1} \xrightarrow{\bar{\alpha}_n} a_n$ be a path of Q_Λ and $\alpha_i \in \text{rad}_\Lambda(a_{i-1}, a_i) - \text{rad}_\Lambda^2(a_{i-1}, a_i)$ a representative of $\bar{\alpha}_i$. The following statements are equivalent:*

- (1) $g(\alpha_n \cdots \alpha_2 \alpha_1) = \sum_{i=1}^n g(\alpha_i)$;
- (2) $\alpha'_n \cdots \alpha'_2 \alpha'_1 \neq 0$ for all representatives α'_i of $\overline{\alpha}_i$;
- (3) $\overline{\alpha_n \cdots \alpha_2 \alpha_1}$ is the projection of a nonzero path of $\tilde{\Lambda}$, where $\tilde{\Lambda}$ is the universal cover of Λ .

If a path u of Q_Λ satisfies one of the statements of Lemma 7.5, then it is called a stable path. For a stable path $a_0 \xrightarrow{\overline{\alpha}_1} a_1 \xrightarrow{\overline{\alpha}_2} \cdots \rightarrow a_{n-1} \xrightarrow{\overline{\alpha}_n} a_n$ of Q_Λ , the level of $\alpha_n \cdots \alpha_2 \alpha_1$ is independent to the choices of the representatives α_i of $\overline{\alpha}_i$. Define the level of the stable path $a_0 \xrightarrow{\overline{\alpha}_1} a_1 \xrightarrow{\overline{\alpha}_2} \cdots \rightarrow a_{n-1} \xrightarrow{\overline{\alpha}_n} a_n$ to be the level of the morphism $\alpha_n \cdots \alpha_2 \alpha_1$.

Definition 7.6. (see [5, Section 1.5]) *Let Λ be a locally representation-finite category. A stable contour of Λ is a pair (v, w) of stable paths of Q_Λ which have the same source, the same terminal and the same level.*

Lemma 7.7. *Let Λ be a locally representation-finite category, (v, w) be a pair of paths of Q_Λ with the same source a and the same terminal b , \tilde{a} (resp. \tilde{b}) be a point of the universal cover $\tilde{\Lambda}$ which lies over a (resp. b), \tilde{v} be the path of $Q_{\tilde{\Lambda}}$ with source \tilde{a} (resp. with terminal \tilde{b}) which lies over v , and \tilde{w} be the path of $Q_{\tilde{\Lambda}}$ with source \tilde{a} (resp. with terminal \tilde{b}) which lies over w . Then (v, w) is a stable contour of Λ if and only if \tilde{v} and \tilde{w} are nonzero paths of $\tilde{\Lambda}$ with the same terminal (resp. with the same source).*

Proof. We only prove the case that \tilde{v} and \tilde{w} have the same source \tilde{a} .

" \Rightarrow " Let $v = \overline{\alpha_n \cdots \alpha_2 \alpha_1}$ and $w = \overline{\beta_m \cdots \beta_2 \beta_1}$, where each $\overline{\alpha}_i$ and each $\overline{\beta}_j$ is an arrow of Q_Λ . Let \tilde{v} be the path

$$\tilde{v} = x_0 \xrightarrow{\overline{\gamma}_1} x_1 \xrightarrow{\overline{\gamma}_2} \cdots x_{n-1} \xrightarrow{\overline{\gamma}_n} x_n$$

and \tilde{w} be the path

$$\tilde{w} = y_0 \xrightarrow{\overline{\delta}_1} y_1 \xrightarrow{\overline{\delta}_2} \cdots y_{m-1} \xrightarrow{\overline{\delta}_m} y_m,$$

where each $\overline{\gamma}_i$ and each $\overline{\delta}_j$ is an arrow of $Q_{\tilde{\Lambda}}$. For each $1 \leq i \leq n$ (resp. $1 \leq j \leq m$), choose a representative $\gamma_i \in \text{rad}_{\tilde{\Lambda}}^2(x_{i-1}, x_i) - \text{rad}_{\tilde{\Lambda}}^2(x_{i-1}, x_i)$ of $\overline{\gamma}_i$ (resp. a representative $\delta_j \in \text{rad}_{\tilde{\Lambda}}^2(y_{j-1}, y_j) - \text{rad}_{\tilde{\Lambda}}^2(y_{j-1}, y_j)$ of $\overline{\delta}_j$), then $\alpha_i = F(\gamma_i)$ (resp. $\beta_j = F(\delta_j)$) is a representative of $\overline{\alpha}_i$ (resp. $\overline{\beta}_j$), where $F : \tilde{\Lambda} \rightarrow \Lambda$ is the covering functor. Since (v, w) is a stable contour, $g(\alpha_n \cdots \alpha_2 \alpha_1) = g(\beta_m \cdots \beta_2 \beta_1)$. Let $g = g(\alpha_n \cdots \alpha_2 \alpha_1)$.

Since $E : k(\tilde{\Gamma}_\Lambda) \rightarrow I$ is a covering functor, it induces an isomorphism

$$\bigoplus_{Ez=b^*} \text{rad}_{k(\tilde{\Gamma}_\Lambda)}^g(\tilde{a}, z) / \text{rad}_{k(\tilde{\Gamma}_\Lambda)}^{g+1}(\tilde{a}, z) \rightarrow \text{rad}_I^g(a^*, b^*) / \text{rad}_I^{g+1}(a^*, b^*).$$

By Remark 7.4, $\dim_k(\text{rad}_I^g(a^*, b^*) / \text{rad}_I^{g+1}(a^*, b^*)) = 1$, so there exists a unique object z of $k(\tilde{\Gamma}_\Lambda)$ such that $Ez = b^*$ and $\text{rad}_{k(\tilde{\Gamma}_\Lambda)}^g(\tilde{a}, z) / \text{rad}_{k(\tilde{\Gamma}_\Lambda)}^{g+1}(\tilde{a}, z) \neq 0$. Since $E(\gamma_n \cdots \gamma_1) = F(\gamma_n \cdots \gamma_1)^* = (\alpha_n \cdots \alpha_1)^* \in \text{rad}_I^g(a^*, b^*) - \text{rad}_I^{g+1}(a^*, b^*)$, $\gamma_n \cdots \gamma_1 \in \text{rad}_{k(\tilde{\Gamma}_\Lambda)}^g(\tilde{a}, x_n) - \text{rad}_{k(\tilde{\Gamma}_\Lambda)}^{g+1}(\tilde{a}, x_n)$. Therefore $x_n = z$, and $y_m = z$ for the same reason. Moreover, since $\gamma_n \cdots \gamma_1 \neq 0$ (resp. $\delta_m \cdots \delta_1 \neq 0$), \tilde{v} (resp. \tilde{w}) is a nonzero path of $Q_{\tilde{\Lambda}}$.

" \Leftarrow " Since \tilde{v} and \tilde{w} are nonzero paths of $\tilde{\Lambda}$, v and w are stable. Let \tilde{v} be the path

$$\tilde{v} = x_0 \xrightarrow{\overline{\gamma}_1} x_1 \xrightarrow{\overline{\gamma}_2} \cdots x_{n-1} \xrightarrow{\overline{\gamma}_n} x_n = \tilde{b}$$

and \tilde{w} be the path

$$\tilde{w} = y_0 \xrightarrow{\overline{\delta}_1} y_1 \xrightarrow{\overline{\delta}_2} \cdots y_{m-1} \xrightarrow{\overline{\delta}_m} y_m = \tilde{b},$$

where each $\overline{\gamma}_i$ and each $\overline{\delta}_j$ is an arrow of $Q_{\tilde{\Lambda}}$. For each $1 \leq i \leq n$ (resp. $1 \leq j \leq m$), choose a representative $\gamma_i \in \text{rad}_{\tilde{\Lambda}}^2(x_{i-1}, x_i) - \text{rad}_{\tilde{\Lambda}}^2(x_{i-1}, x_i)$ of $\overline{\gamma}_i$ (resp. a representative $\delta_j \in \text{rad}_{\tilde{\Lambda}}^2(y_{j-1}, y_j) - \text{rad}_{\tilde{\Lambda}}^2(y_{j-1}, y_j)$ of $\overline{\delta}_j$), and let $\alpha_i = F(\gamma_i)$ (resp. $\beta_j = F(\delta_j)$), where $F : \tilde{\Lambda} \rightarrow \Lambda$ is the covering

functor. Since $\tilde{\Lambda}$ is schurian and \tilde{v}, \tilde{w} are nonzero paths of $\tilde{\Lambda}$, there exists some $\lambda \in k^*$ such that $\delta_m \cdots \delta_1 = \lambda \gamma_n \cdots \gamma_1$. Therefore $\beta_m \cdots \beta_1 = \lambda \alpha_n \cdots \alpha_1$ and the stable paths v and w have the same level. \square

Remark 7.8. *If (v, w) is a stable contour with $v \neq w$, then both v and w are paths of Q_Λ of length ≥ 2 . The reason is as follows: by Lemma 7.7, the pair (v, w) lifts to a pair (\tilde{v}, \tilde{w}) of nonzero paths of $Q_{\tilde{\Lambda}}$ with same source \tilde{a} and same terminal \tilde{b} . Suppose $\tilde{\Lambda} \cong kQ_{\tilde{\Lambda}}/I_{\tilde{\Lambda}}$ for some admissible ideal $I_{\tilde{\Lambda}}$ of $kQ_{\tilde{\Lambda}}$. Since $\dim_k \tilde{\Lambda}(\tilde{a}, \tilde{b}) = 1$ and \tilde{v}, \tilde{w} are nonzero paths of $\tilde{\Lambda}$, there exists some $\lambda \in k^*$ with $\tilde{v} - \lambda \tilde{w} \in I_{\tilde{\Lambda}}$. Therefore both \tilde{v} and \tilde{w} are paths of length ≥ 2 , and the same things are true for v and w .*

Definition 7.9. ([5, Section 3.1]) The standard form $\bar{\Lambda}$ of a locally representation-finite category Λ is the full subcategory of the mesh category $k(\Gamma_\Lambda)$ of the AR-quiver Γ_Λ of Λ formed by projective vertices. Similarly, the standard form \bar{A} of a basic representation-finite algebra A is defined to be the algebra $\bigoplus_{p,q} k(\Gamma_A)(p, q)$, where p, q range over all projective vertices of Γ_A .

It can be shown that for a locally representation-finite category Λ (resp. a basic representation-finite algebra A), $\bar{\Lambda}$ (resp. \bar{A}) is standard and the AR-quivers of Λ and $\bar{\Lambda}$ (resp. A and \bar{A}) are isomorphic. For a locally representation-finite category Λ , the standard form $\bar{\Lambda}$ of Λ is described as follows.

Proposition 7.10. ([5, Theorem 3.1]) Let Λ be a locally representation-finite category. For any two objects x, y of Λ , let $I_\Lambda(x, y)$ be the subspace of $kQ_\Lambda(x, y)$ which is generated by the non-stable paths and by the differences $v - w$, where (v, w) ranges over the stable contours with source x and terminal y . Then I_Λ is an ideal of the path category kQ_Λ and the standard form $\bar{\Lambda}$ of Λ is isomorphic to kQ_Λ/I_Λ .

We abbreviate indecomposable, basic, representation-finite self-injective algebra over k (not isomorphic to the underlying field k) by RFS algebra.

Lemma 7.11. *Let A be an RFS algebra, $\bar{A} = kQ_A/I_A$ be the standard form of A , v, w be two paths of Q_A with the same source (resp. terminal) which belong to $\text{soc}\bar{A} - \{0\}$. Then (v, w) is a stable contour of A .*

Proof. Since \bar{A} is self-injective and $v, w \in \text{soc}\bar{A} - \{0\}$, they also have the same terminal. Since v and w are nonzero in \bar{A} , they are stable paths. Since v and w belong to $\text{soc}\bar{A} - \{0\}$, which have the same source, there exists some $\lambda \in k^*$ such that $w = \lambda v$ in \bar{A} . By the definition of I_A , we imply that $\lambda = 1$ and (v, w) is a stable contour of A . \square

Lemma 7.12. *Let A be an RFS algebra of Loewy length > 2 , $\bar{A} = kQ_A/I_A$ be the standard form of A . Let $\alpha_n \cdots \alpha_2 \alpha_1$ be a path of Q_A which belongs to $\text{soc}\bar{A} - \{0\}$, p (resp. q) be a path of Q_A such that $p\alpha_n \cdots \alpha_2$ (resp. $\alpha_{n-1} \cdots \alpha_1 q$) belongs to $\text{soc}\bar{A} - \{0\}$, then $l(p) = 1$ (resp. $l(q) = 1$) and p (resp. q) is unique.*

Proof. Since $\alpha_n \cdots \alpha_2 \notin \text{soc}\bar{A}$, $l(p) \geq 1$. Suppose $l(p) \geq 2$, write $p = \alpha_{n+l} \cdots \alpha_{n+1}$ where $l \geq 2$. Since $\alpha_{n+1} \alpha_n \cdots \alpha_2 \notin \text{soc}\bar{A}$, there exists arrows $\alpha'_1, \alpha'_0, \dots, \alpha'_{2-k}$ ($k \geq 1$) of Q_A such that $\alpha_{n+1} \alpha_n \cdots \alpha_2 \alpha'_1 \cdots \alpha'_{2-k} \in \text{soc}\bar{A} - \{0\}$. Similarly, since $\alpha_n \cdots \alpha_2 \alpha'_1 \cdots \alpha'_{2-k} \notin \text{soc}\bar{A}$, there exists arrows $\alpha'_{1-k}, \dots, \alpha'_{2-k-l}$ ($l \geq 1$) of Q_A such that $\alpha_n \cdots \alpha_2 \alpha'_1 \cdots \alpha'_{2-k-l} \in \text{soc}\bar{A} - \{0\}$. According to Lemma 7.11, the pair of paths $(\alpha_n \cdots \alpha_2 \alpha_1, \alpha_n \cdots \alpha_2 \alpha'_1 \cdots \alpha'_{2-k-l})$ is a stable contour of A . By Lemma 7.7, $(\alpha_1, \alpha'_1 \cdots \alpha'_{2-k-l})$ is also a stable contour of A . Since $l(\alpha'_1 \cdots \alpha'_{2-k-l}) \geq 2$, $\alpha_1 \neq \alpha'_1 \cdots \alpha'_{2-k-l}$. But it contradicts to Remark 7.8.

To show p is unique, let $\alpha_{n+1}, \alpha'_{n+1}$ be two arrows of Q_A such that both $\alpha_{n+1} \alpha_n \cdots \alpha_2$ and $\alpha'_{n+1} \alpha_n \cdots \alpha_2$ belong to $\text{soc}\bar{A} - \{0\}$. Similarly, we can show that $(\alpha_{n+1}, \alpha'_{n+1})$ is a stable contour of A . Then by Remark 7.8 we have $\alpha_{n+1} = \alpha'_{n+1}$. \square

Let A be an RFS algebra of Loewy length > 2 with the standard form $\bar{A} = kQ_A/I_A$. For each path $v = \alpha_n \cdots \alpha_2 \alpha_1$ of Q_A which belongs to $\text{soc}\bar{A} - \{0\}$, let α_{n+1} (resp. α_0) be the unique arrow of Q_A such that $\alpha_{n+1}\alpha_n \cdots \alpha_2 \in \text{soc}\bar{A} - \{0\}$ (resp. $\alpha_{n-1} \cdots \alpha_1 \alpha_0 \in \text{soc}\bar{A} - \{0\}$). Define $v[1] = \alpha_{n+1}\alpha_n \cdots \alpha_2$ and $v[-1] = \alpha_{n-1} \cdots \alpha_1 \alpha_0$. Define $v[n] = (v[n-1])[1]$ and $v[-n] = (v[1-n])[-1]$ for each $n > 1$ inductively. Let $E = \{e_v \mid v \text{ is a path of } Q_A \text{ which belongs to } \text{soc}\bar{A} - \{0\}\}$ be a set, define the G -set structure on E by setting $g^n \cdot e_v = e_{v[n]}$ for each $e_v \in E$ and for each $n \in \mathbb{Z}$, where $G = \langle g \rangle$ is an infinite cyclic group. For each path v of Q_A which belongs to $\text{soc}\bar{A} - \{0\}$, let α_v be the initial arrow of v . Define two partitions P, L on E as follows: for every $e_v \in E$, $P(e_v) = \{e_w \mid s(v) = s(w)\}$ and $L(e_v) = \{e_w \mid \alpha_v = \alpha_w\}$, where $s(p)$ denotes the source of the path p . Define a function $d : E \rightarrow \mathbb{Z}_+$ by setting $d(e_v)$ to be the length of v .

Proposition 7.13. *Let A be an RFS algebra of Loewy length > 2 with the standard form $\bar{A} = kQ_A/I_A$ and let $E = (E, P, L, d)$ as above. Then E is a finite f_s -BC and the corresponding f_s -BCA A_E is isomorphic to \bar{A} .*

Proof. Step 1: To show that $E = (E, P, L, d)$ is a finite f_s -BC.

(f1) Since I_A is an admissible ideal and Q_A is finite, the number of nonzero paths of Q_A is finite. Therefore E and each class $P(e_v)$ are finite sets. By definition we have $L(e_v) \subseteq P(e_v)$ for any $e_v \in E$.

(f2) If $L(e_v) = L(e_w)$, then $\alpha_v = \alpha_w$, therefore $s(v[1]) = t(\alpha_v) = t(\alpha_w) = s(w[1])$ and $P(g \cdot e_v) = P(e_{v[1]}) = P(e_{w[1]}) = P(g \cdot e_w)$.

(f3) Since $l(v) = l(v[n])$ for any path v which belongs to $\text{soc}\bar{A} - \{0\}$ and for any $n \in \mathbb{Z}$, $d(e_v) = d(e_w)$ for any $e_v, e_w \in E$ which belong to the same $\langle g \rangle$ -orbit.

(f4) If $P(e_v) = P(e_w)$, then $s(v) = s(w)$. Since A is self-injective and $v, w \in \text{soc}\bar{A} - \{0\}$, v and w have the same terminal. Therefore $v[l(v)]$ and $w[l(w)]$ have the same source. Since $\sigma(e_v) = e_{v[l(v)]}$ (resp. $\sigma(e_w) = e_{w[l(w)]}$), $P(\sigma(e_v)) = P(\sigma(e_w))$. Similarly we have $P(\sigma^{-1}(e_v)) = P(\sigma^{-1}(e_w))$.

(f5) If $L(e_v) = L(e_w)$, then $\alpha_v = \alpha_w$. Let $v = v'\alpha_v$ and $w = w'\alpha_w$. By Lemma 7.11, (v, w) is a stable contour of A , and by Lemma 7.7, (v', w') is also a stable contour of A . Let β be the arrow of Q_A such that $\beta v' \in \text{soc}\bar{A} - \{0\}$. Since (v', w') is a stable contour of A , $v' - w' \in I_A$. Therefore $v' = w'$ in \bar{A} , and $\beta w' = \beta v' \in \text{soc}\bar{A} - \{0\}$. Thus $v[1] = \beta v'$ and $w[1] = \beta w'$. Since $\alpha_{v[l(v)]} = \beta = \alpha_{w[l(w)]}$, $L(\sigma(e_v)) = L(\sigma(e_w))$. Similarly, we have $L(\sigma^{-1}(e_v)) = L(\sigma^{-1}(e_w))$.

(f6) For $e_v, e_w \in E$, if $d(e_v) \leq d(e_w)$ and $L(e_{v[i]}) = L(e_{w[i]})$ for each $0 \leq i < d(e_v)$, then $w = w'v$ for some path w' of Q_A . Since $v, w \in \text{soc}\bar{A} - \{0\}$, we have $w = v$.

(f7) Note that each standard sequence $(g^{n-1} \cdot e_v, \dots, g \cdot e_v, e_v)$ of E can be regarded as a pair of paths (v'', v') of Q_A such that $l(v') = n$ and $v''v' = v$. Conversely, each pair of paths (v'', v') of Q_A with $v''v' \in \text{soc}\bar{A} - \{0\}$ corresponds to a standard sequence $(g^{n-1} \cdot e_v, \dots, g \cdot e_v, e_v)$ of E , where $v = v''v'$ and $n = l(v')$. Therefore we may consider a standard sequence of E as such a pair. Two standard sequences (v'', v') and (w'', w') of E are identical if and only if $v' = w'$. Therefore for each standard sequence (v'', v') of E , we have $[\wedge(v'', v')] = \{(w, v'') \mid ww'' \in \text{soc}\bar{A} - \{0\}\}$. For identical standard sequences (u, v) , (w, v) of E , we have $[\wedge(u, v)]^\wedge = \{(u, v') \mid uv' \in \text{soc}\bar{A} - \{0\}\}$ and $[[\wedge(u, v)]^\wedge] = \{(u', v') \mid u'v', uv' \in \text{soc}\bar{A} - \{0\}\}$. Also we have $[[\wedge(w, v)]^\wedge] = \{(w', v') \mid w'v', wv' \in \text{soc}\bar{A} - \{0\}\}$. By Lemma 7.11, (uv, wv) is a stable contour of A , then by Lemma 7.7 (u, w) is also a stable contour of A . Therefore $u = w$ in \bar{A} , and $uw' \in \text{soc}\bar{A} - \{0\}$ if and only if $wv' \in \text{soc}\bar{A} - \{0\}$. Thus $[[\wedge(u, v)]^\wedge] = [[\wedge(w, v)]^\wedge]$.

Step 2: To show $\bar{A} \cong A_E$.

Define a quiver morphism $f : Q_E \rightarrow Q_A$ which maps each vertex $P(e_v)$ to $s(v)$ and maps each arrow $L(e_v)$ to α_v . It is straightforward to show that f is a quiver isomorphism. To show f induces an isomorphism between A_E and \bar{A} , it suffices to show that $\tilde{f}(I_E) = I_A$, where \tilde{f} is the isomorphism $kQ_E \rightarrow kQ_A$ of path categories induced by f .

If $L(e_{v[m-1-k]}) \cdots L(e_{v[1]})L(e_v) - L(e_{w[m'-1-k]}) \cdots L(e_{w[1]})L(e_w)$ is a relation of I_E of type (fR1), where $s(v) = s(w)$, $m = l(v)$, $m' = l(w)$, and $L(e_{v[m-i]}) = L(e_{w[m'-i]})$ for $1 \leq i \leq k$,

then we may assume that $v = pq$ and $w = pq'$, where p is a path of length k . By Lemma 7.11 and Lemma 7.7, (q, q') is a stable contour of A . Therefore $\tilde{f}(L(e_{v[m-1-k]}) \cdots L(e_{v[1]})L(e_v) - L(e_{w[m'-1-k]}) \cdots L(e_{w[1]})L(e_w)) = q - q' \in I_A$.

If $L(e_{v_n}) \cdots L(e_{v_2})L(e_{v_1})$ is a relation of I_E of type (fR2), then the path $p = f(L(e_{v_n}) \cdots L(e_{v_2})L(e_{v_1})) = \alpha_{v_n} \cdots \alpha_{v_2} \alpha_{v_1}$ of Q_A is not a subpath of any path v of Q_A which belongs to $\text{soc}\bar{A} - \{0\}$. Therefore p is non-stable and it belongs to I_A .

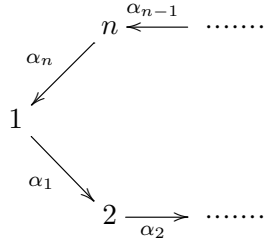
If $L(e_{v[n]}) \cdots L(e_{v[1]})L(e_v)$ is a relation of I_E of type (fR3), where $n \geq l(v)$, then $f(L(e_{v[n]}) \cdots L(e_{v[1]})L(e_v))$ is a path of Q_A which contains v as a proper subpath. Therefore $f(L(e_{v[n]}) \cdots L(e_{v[1]})L(e_v)) = 0$ in \bar{A} . So we have $\tilde{f}(I_E) \subseteq I_A$.

For any non-stable path p of A , we have $p = 0$ in \bar{A} . Then p is not a subpath of any path of Q_A which belongs to $\text{soc}\bar{A} - \{0\}$. Let q be the preimage of p under \tilde{f} . If q is not a relation of I_E of type (fR2), then $q = L(e_{v[n-1]}) \cdots L(e_{v[1]})L(e_v)$ for some $e_v \in E$ and some $n > 0$. Since p is not a subpath of v , $n > l(v)$. Therefore q is a relation of I_E of type (fR3).

For any stable contour (v, w) of A , since $v \neq 0$ in \bar{A} , there exists some path u of Q_A such that $uv \in \text{soc}\bar{A} - \{0\}$. Since $v = w$ in \bar{A} , $wv \in \text{soc}\bar{A} - \{0\}$. Let $p = uv$, $q = ww$, then $L(e_{p[m-k-1]}) \cdots L(e_{p[1]})L(e_p) - L(e_{q[m'-k-1]}) \cdots L(e_{q[1]})L(e_q)$ is a relation of I_E of type (fR1), where $m = l(p)$, $m' = l(q)$, $k = l(u)$, which is the preimage of $v - w$ under \tilde{f} . Therefore the ideal I_A of Q_A is contained in $\tilde{f}(I_E)$. \square

Theorem 7.14. *The class of standard RFS algebras is equal to the class of finite-dimensional indecomposable representation-finite f_s -BCAs.*

Proof. Let A be a standard RFS algebra. Then A and its standard form \bar{A} are isomorphic. If the Loewy length of A is larger than 2, by Proposition 7.13, A is a f_s -BCA. If the Loewy length of A is 2, then A is given by the quiver



with relations $\alpha_{i+1}\alpha_i = 0$, where $i \in \{0, 1, \dots, n-1\} = \mathbb{Z}/n\mathbb{Z}$. Define a f_s -BC $E = (E, P, L, d)$ as follows: $E = \{e_0, e_1, \dots, e_{n-1}\}$ with the G -set structure given by $g \cdot e_i = e_{i+1}$, where $i \in \{0, 1, \dots, n-1\} = \mathbb{Z}/n\mathbb{Z}$; the partitions P and L are given by $P(e) = L(e) = \{e\}$ for every $e \in E$; the degree function d is given by $d(e) = 1$ for every $e \in E$. Then A is isomorphic to A_E , which implies that A is a f_s -BCA.

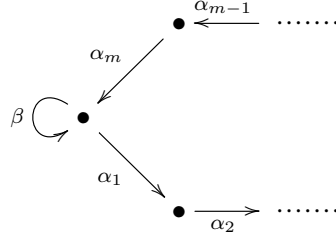
Conversely, let A_E be a finite-dimensional indecomposable representation-finite f_s -BCA. By Proposition 7.3 and Proposition 5.2, A_E is standard and self-injective. Moreover, by Proposition 4.17 and Theorem 4.6, the radical of A is nonzero, so A is not isomorphic to k . Therefore A_E is a standard RFS algebra. \square

Corollary 7.15. *The class of finite-dimensional representation-finite f_s -BCAs is closed under derived equivalence.*

Proof. It follows from Theorem 7.14 and the fact that the class of standard RFS algebras is closed under derived equivalence (by [3] and [1]). \square

We illustrate the discussion in this section with an example.

Example 7.16. *Let $A = kQ/I$, where Q is the quiver*

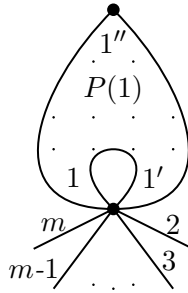


($m \geq 2$), and I is the ideal of kQ generated by $\alpha_m \cdots \alpha_2 \alpha_1 - \beta^2$, $\alpha_1 \alpha_m$, $\alpha_i \cdots \alpha_1 \beta \alpha_m \cdots \alpha_i$ ($1 \leq i \leq m$). A is a standard RFS algebra of type $(D_{3m}, \frac{1}{3}, 1)$. Let $\bar{A} = kQ/I_A$ be the standard form of A , where I_A is the ideal of kQ generated by non-stable paths and by differences $v - w$ with (v, w) being a stable contour of A . It is straightforward to show that $(\alpha_m \cdots \alpha_2 \alpha_1, \beta^2)$ is a stable contour of A and the paths $\alpha_1 \alpha_m$, $\alpha_i \cdots \alpha_1 \beta \alpha_m \cdots \alpha_i$ ($1 \leq i \leq m$) are non-stable paths. So the ideal I is contained in I_A . Since A is standard, \bar{A} is isomorphic to A , and $\dim_k A = \dim_k \bar{A}$. Therefore $I = I_A$.

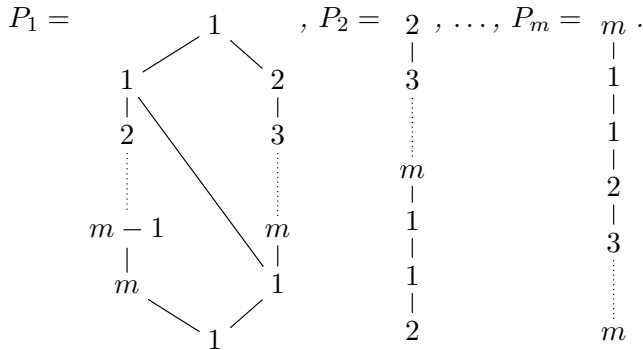
We now construct the f_s -BC $E = (E, P, L, d)$ associated to \bar{A} (see the paragraph before Proposition 7.13). The set of paths of Q which belong to $\text{soc} \bar{A} - \{0\}$ is

$$\{\beta \alpha_m \cdots \alpha_1, \beta^3, \alpha_m \cdots \alpha_1 \beta, \alpha_1 \beta \alpha_m \cdots \alpha_2, \alpha_2 \alpha_1 \beta \alpha_m \cdots \alpha_3, \dots, \alpha_{m-1} \cdots \alpha_1 \beta \alpha_m\}.$$

Denote $1 = e_{\alpha_m \cdots \alpha_1 \beta}$, $1' = e_{\beta \alpha_m \cdots \alpha_1}$, $1'' = e_{\beta^3}$, and $i = e_{\alpha_{i-1} \cdots \alpha_1 \beta \alpha_m \cdots \alpha_i}$ for $2 \leq i \leq m$, then $E = \{1, 1', 1'', 2, \dots, m\}$ with the G -set structure given by $g \cdot 1 = 1'$, $g \cdot 1' = 2$, $g \cdot 2 = 3, \dots, g \cdot (m-1) = m$, $g \cdot m = 1$, $g \cdot 1'' = 1''$. The partition P is given by $P(1) = \{1, 1', 1''\}$ and $P(e) = \{e\}$ for other $e \in E$. The partition L is given by $L(1) = \{1, 1''\}$ and $L(e) = \{e\}$ for other $e \in E$. The degree function d is given by $d(1'') = 3$ and $d(e) = m + 1$ for other $e \in E$. E can be visualized by the following diagram:



Let A_E be the associated f -BCA. Then we have the following structure of the indecomposable projective A_E -modules:



Note that there is a standard sequence $p = (1'', 1')$ associated to the vertex $G \cdot 1''$ and there is a standard sequence $q = (m, \dots, 3, 2, 1')$ associated to the vertex $G \cdot 1$. Since $L(1) = \{1, 1''\}$, the left complements $\hat{\wedge} p$ and $\hat{\wedge} q$ are identical, we have a relation $L(p) - L(q)$ of type $(fR1)$, which corresponds to the relation $\beta^2 - \alpha_m \cdots \alpha_1$ in \bar{A} .

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NENGQUN LI AND YUMING LIU
 SCHOOL OF MATHEMATICAL SCIENCES
 LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS
 BEIJING NORMAL UNIVERSITY
 BEIJING 100875
 P.R.CHINA
Email address: ymliu@bnu.edu.cn
Email address: wd0843@163.com