

# A GENERALIZATION OF DUGAS' CONSTRUCTION ON STABLE AUTO-EQUIVALENCES FOR SYMMETRIC ALGEBRAS

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## Abstract

We give a unified generalization of Dugas' construction on stable auto-equivalences of Morita type from local symmetric algebras to arbitrary symmetric algebras. For group algebras  $kP$  of  $p$ -groups in characteristic  $p$ , we recover all the stable auto-equivalences corresponding to endo-trivial modules over  $kP$  except that  $P$  is generalized quaternion of order  $2^m$ . Moreover, we give many examples of stable auto-equivalences of Morita type for non-local symmetric algebras.

## 1. INTRODUCTION

In [8], Dugas gave two methods to construct stable auto-equivalences (of Morita type) for (finite dimensional) local symmetric algebras. One of particular interests is that such stable auto-equivalences are often not induced by auto-equivalences of the derived category.

The first construction is given as follows.

Let  $A$  be an elementary local symmetric  $k$ -algebra, let  $x \in A$  be a nilpotent element. Set  $R = k[x] \cong k[X]/(X^m)$  for some integer  $m \geq 2$  and  $T_A = k \otimes_R A \cong A/xA$ . Suppose that  ${}_R A$  and  $A_R$  are free modules and that  $\underline{\text{End}}_A(T) \cong k[\psi]/(\psi^2)$ , where  $\psi$  is an endomorphism of  $T$  induced by multiplying some  $y \in A$ . (As Dugas pointed out that the algebra  $\underline{\text{End}}_A(T)$  has a periodic bimodule free resolution of period 2.) Let  $C_\mu$  be the kernel of the multiplication map  $\mu : A \otimes_R A \rightarrow A$ . Then  $-\otimes_A C_\mu : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  is a stable auto-equivalence of  $A$ .

Note that  $\Omega_{A^e}^{-1}(C_\mu) \cong \text{Cone}(\mu)$  in  $\underline{\text{mod}}\text{-}A^e$  and Dugas called the stable auto-equivalence  $-\otimes_A \Omega_{A^e}^{-1}(C_\mu)$  as a spherical stable twist which is analogous to spherical twist constructed on the derived category by Seidel and Thomas. Under the more general condition  $\underline{\text{End}}_A(T) \cong k[\psi]/(\psi^{n+1})$  for some  $n \geq 1$ , Dugas gave a second construction using a double cone construction, and the induced stable auto-equivalence is called  $\mathbb{P}_n$ -stable twist since it is analogous to  $\mathbb{P}_n$ -twist on the derived category of coherent sheaves on a variety by Huybrechts and Thomas.

For group algebras of  $p$ -groups in characteristic  $p$ , Dugas recovered many of the stable auto-equivalences corresponding to endo-trivial modules. He also obtained stable auto-equivalences for local algebras of dihedral and semi-dihedral type, which are not group algebras.

In this note, we give a unified generalization of Dugas' construction by greatly relaxing the conditions on both  $A$  and  $R$  and by adding a new subalgebra  $B$  of  $A$ . The main idea is as follows. For a symmetric  $k$ -algebra  $A$ , consider a triple  $(A, R, B)$ , where  $R, B$  are subalgebras of  $A$  such that  $R$  is also symmetric and  $B$  (as a  $B$ - $B$ -bimodule) has a periodic free resolution of period  $q$ . Then, under some commutativity assumptions between  $R, B$  and  $A$ , we may construct a complex of left-right projective  $A$ - $A$ -bimodules. Using this complex, we can construct a left-right projective  $A$ - $A$ -bimodule  $M_q$  using a multiple cone construction such that the functor  $-\otimes_A M_q$  induces a stable auto-equivalence of  $A$ . The main results are Theorem 3.5 and Theorem 4.1.

Our construction generalizes Dugas' construction in three ways. Firstly, we dropped the condition that the algebra  $A$  is local. Secondly, we don't request the subalgebra  $R$  to be local or Nakayama. Thirdly, we use a subalgebra  $B$  of  $A$  to replace  $\underline{\text{End}}_A(T)$  in Dugas' construction, which is more flexible. For a connection between  $B$  and  $\underline{\text{End}}_A(T)$ , we refer to Remark 3.2 below.

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For group algebras  $kP$  of  $p$ -groups in characteristic  $p$ , we recover all the stable auto-equivalences of  $kP$  corresponding to endo-trivial modules except that  $P$  is generalized quaternion of order  $2^m$ , see Proposition 5.1. Moreover, we can construct many examples of stable auto-equivalences of Morita type (which are not induced by derived equivalences in general) for non-local symmetric algebras, see Section 6.

Our discussion is also related to construct stable equivalences between different algebras. In particular, we will use a method in [11], which gives a way to construct new stable equivalence between non-Morita equivalent algebras from a given stable auto-equivalence.

This paper is organized as follows. In Section 2, we state some general results on triangulated functors, in particular we recall some results that are useful in establishing that a given triangulated functor is an equivalence. We give the constructions of stable auto-equivalences for (not necessarily local) symmetric algebras in Section 3 and Section 4. We show in Section 5 that our construction recovers all the stable auto-equivalences corresponding to endo-trivial modules over a finite  $p$ -group algebra  $kP$  when  $P$  is not generalized quaternion of order  $2^m$ . In Section 6, we construct various examples of stable auto-equivalences for non-local symmetric algebras.

#### DATA AVAILABILITY

The datasets generated during the current study are available from the corresponding author on reasonable request.

## 2. PRELIMINARY

Throughout this section, let  $k$  be a field and let  $\mathcal{T}$  be a Hom-finite triangulated  $k$ -category with suspension [1]. A typical example of this kind of triangulated  $k$ -category is the stable category  $\underline{\text{mod}}\text{-}A$  of finite-dimensional right  $A$ -modules, where  $A$  is a finite-dimensional self-injective  $k$ -algebra. Note that the suspension in  $\underline{\text{mod}}\text{-}A$  is given by the cosyzygy functor  $\Omega_A^{-1}$  and  $\underline{\text{mod}}\text{-}A$  has a Serre functor  $\nu_A \Omega_A$ , where  $\nu_A$  is the Nakayama functor.

We have the following interesting result on triangulated functor.

**Lemma 2.1.** *Let  $\mathcal{T}'$  and  $\mathcal{T}_1, \dots, \mathcal{T}_n$  be indecomposable (Hom-finite) Krull-Schmidt triangulated  $k$ -categories and let  $\mathcal{T} = \mathcal{T}_1 \times \dots \times \mathcal{T}_n$ . Let  $F : \mathcal{T}' \rightarrow \mathcal{T}$  be a fully faithful triangulated functor, which maps some nonzero object  $X$  of  $\mathcal{T}'$  to an object of  $\mathcal{T}_1$ . Then the image of  $F$  is in  $\mathcal{T}_1$ .*

*Proof.* Since  $\mathcal{T}'$  and  $\mathcal{T}$  are Krull-Schmidt and  $F$  is fully faithful,  $F$  sends each indecomposable object  $Y$  of  $\mathcal{T}'$  to an indecomposable object  $FY$  of  $\mathcal{T}$ , therefore  $FY \in \mathcal{T}_i$  for some  $i$ . Let  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ) be the full subcategory of  $\mathcal{T}'$  which is formed by the objects  $Z$  such that  $FZ \in \mathcal{T}_1$  (resp.  $FZ \in \mathcal{T}_2 \times \dots \times \mathcal{T}_n$ ). For each object  $Z$  of  $\mathcal{T}'$ , let  $Z_i$  be the direct sum of indecomposable summands of  $Z$  which belong to  $\mathcal{C}_i$ ,  $i = 1, 2$ . Then  $Z = Z_1 \oplus Z_2$  with  $Z_i \in \mathcal{C}_i$ . For every pair of objects  $A_i \in \mathcal{C}_i$  and for each  $n \in \mathbb{Z}$ , since  $FA_1 \in \mathcal{T}_1$  and  $(FA_2)[n] \in \mathcal{T}_2 \times \dots \times \mathcal{T}_n$ ,  $\mathcal{T}'(A_1, A_2[n]) \cong \mathcal{T}(FA_1, (FA_2)[n]) = 0$ . Since  $\mathcal{T}'$  is indecomposable, either  $\mathcal{C}_1$  or  $\mathcal{C}_2$  is zero. Since  $0 \neq X \in \mathcal{C}_1$ ,  $\mathcal{C}_2$  must be zero. Therefore  $\mathcal{C}_1 = \mathcal{T}'$ .  $\square$

**Remark 2.2.** *We will use Lemma 2.1 in the following situation. Let  $A$  be a self-injective  $k$ -algebra with a decomposition  $A = A_1 \times \dots \times A_n$  into indecomposable algebras. Suppose that  $M$  is a left-right projective  $A$ - $A$ -bimodule and induces a fully faithful functor  $-\otimes_A M : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  on stable category. Suppose that  $X$  is a non-projective  $A_1$ -module such that  $X \otimes_A M$  is a  $A_i$ -module for some  $i$ . Then  $-\otimes_A M$  restricts to a fully faithful functor  $\underline{\text{mod}}\text{-}A_1 \rightarrow \underline{\text{mod}}\text{-}A_i$ .*

Next we recall from [1, 8] some general results that are useful in establishing that a given triangulated functor is an equivalence.

Let  $\mathcal{T}$  be a triangulated category and let  $\mathcal{C}$  be a collection of objects in  $\mathcal{T}$ . For any  $n \in \mathbb{Z}$ , define  $\mathcal{C}[n] := \{X[n] \mid X \in \mathcal{C}\}$ . Moreover, define  $\mathcal{C}^\perp := \{Y \in \mathcal{T} \mid \mathcal{T}(X, Y) = 0 \text{ for any } X \in \mathcal{C}\}$  and  ${}^\perp\mathcal{C} := \{Y \in \mathcal{T} \mid \mathcal{T}(Y, X) = 0 \text{ for any } X \in \mathcal{C}\}$ .

**Definition 2.3.** ([8, Definition 2.1]) Let  $\mathcal{T}$  be a triangulated category. A collection  $\mathcal{C}$  of objects in  $\mathcal{T}$  is called a *spanning class* (resp. *strong spanning class*) if  $(\bigcup_{n \in \mathbb{Z}} \mathcal{C}[n])^\perp = 0$  and  ${}^\perp(\bigcup_{n \in \mathbb{Z}} \mathcal{C}[n]) = 0$  (resp.  $\mathcal{C}^\perp = 0$  and  ${}^\perp \mathcal{C} = 0$ ).

**Remark 2.4.** If  $\mathcal{T}$  is a triangulated category which has a Serre functor, then for any object  $X$  of  $\mathcal{T}$ ,  $\mathcal{C} = \{X\} \cup X^\perp$  is a strong spanning class of  $\mathcal{T}$ .

**Proposition 2.5.** ([1, Theorem 2.3] and [8, Proposition 2.2]) Let  $\mathcal{T}$  and  $\mathcal{T}'$  be triangulated categories, and let  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be a triangulated functor with a left and a right adjoint. Then  $F$  is fully faithful if and only if there exists a strong spanning class  $\mathcal{C}$  of  $\mathcal{T}$  such that  $F$  induces isomorphisms  $\mathcal{T}(X, Y[n]) \rightarrow \mathcal{T}'(FX, F(Y[n]))$  for any  $X, Y \in \mathcal{C}$  and for any  $n = 0, 1$ .

**Proposition 2.6.** ([1, Theorem 3.3]) Let  $\mathcal{T}$  and  $\mathcal{T}'$  be triangulated categories with  $\mathcal{T}$  nonzero,  $\mathcal{T}'$  indecomposable, and let  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be a fully faithful triangulated functor. Then  $F$  is an equivalence of categories if and only if  $F$  has a left adjoint  $G$  and a right adjoint  $H$  such that  $H(Y) \cong 0$  implies  $G(Y) \cong 0$  for any  $Y \in \mathcal{T}'$ .

Combining Propositions 2.5 and 2.6 we have the following consequence for symmetric algebras (see the definition of a symmetric algebra in Section 3):

**Corollary 2.7.** Let  $\Lambda, \Gamma$  be symmetric  $k$ -algebras such that  $\Lambda$  is not semisimple and  $\Gamma$  is indecomposable, and let  $M$  be a left-right projective  $\Lambda$ - $\Gamma$ -bimodule. Denote  $F$  the stable functor induced by the functor  $- \otimes_\Lambda M : \text{mod-}\Lambda \rightarrow \text{mod-}\Gamma$ . If there exists a strong spanning class  $\mathcal{C}$  of  $\underline{\text{mod-}}\Lambda$  such that for any  $X, Y \in \mathcal{C}$  and for any  $n = 0, 1$ , the homomorphism  $F : \underline{\text{Hom}}_\Lambda(X, Y[n]) \rightarrow \underline{\text{Hom}}_\Gamma(FX, F(Y[n]))$  is an isomorphism, then  $F$  is an equivalence.

*Proof.* Since  $\Lambda, \Gamma$  are symmetric, by [8, Lemma 3.2], the functor  $- \otimes_\Gamma DM : \text{mod-}\Gamma \rightarrow \text{mod-}\Lambda$  is both the left and the right adjoint of  $- \otimes_\Lambda M : \text{mod-}\Lambda \rightarrow \text{mod-}\Gamma$ . Therefore the stable functor  $G : \underline{\text{mod-}}\Gamma \rightarrow \underline{\text{mod-}}\Lambda$  induced by  $- \otimes_\Gamma DM$  is both the left and the right adjoint of  $F$ . By Proposition 2.5,  $F$  is fully faithful. Since  $\Lambda$  is not semisimple and  $\Gamma$  is indecomposable,  $\underline{\text{mod-}}\Lambda$  is nonzero and  $\underline{\text{mod-}}\Gamma$  is indecomposable as a triangulated category. Then it follows from Proposition 2.6 that  $F$  is an equivalence.  $\square$

### 3. A CONSTRUCTION OF STABLE AUTO-EQUIVALENCES FOR SYMMETRIC ALGEBRAS

In the following, unless otherwise stated, all algebras considered will be finite dimensional unitary  $k$ -algebras over a field  $k$ , and all their modules will be finite dimensional right modules. By a subalgebra  $B$  of an algebra  $A$ , we mean that  $B$  is a subalgebra of  $A$  with the same identity element.

We denote by  $A^e$  the enveloping algebra of  $A$ , which by definition is  $A^{op} \otimes_k A$ . We let  $D = \text{Hom}_k(-, k)$  be the duality with respect to the ground field  $k$ . Recall that an algebra  $A$  is symmetric if  $A \cong D(A)$  as right  $A^e$ -modules (or equivalently, as  $A$ - $A$ -bimodules). It is well-known that symmetric algebras are self-injective algebras with identity Nakayama functors.

In this section, we make the following

**Assumption 1:** Let  $k$  be a field,  $A$  be a symmetric  $k$ -algebra,  $R$  be a non-semisimple symmetric  $k$ -subalgebra of  $A$  such that  $A_R$  is projective. Let  $B$  be another  $k$ -subalgebra of  $A$ , such that the following conditions hold:

- (a)  $br = rb$  for each  $b \in B$  and  $r \in R$ ;
- (b)  $B \otimes_k (R/\text{rad}R) \xrightarrow{\phi} (R/\text{rad}R) \otimes_R A$ ,  $b \otimes \bar{1} \mapsto \bar{1} \otimes b$  is an isomorphism in  $\underline{\text{mod-}}R$ ;
- (c)  $B$  has a periodic free  $B^e$ -resolution, that is, there exists an exact sequence

$$(1) \quad 0 \rightarrow B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \rightarrow 0$$

of  $B^e$ -modules.

From now on, we fix  $(A, R, B)$  as a triple of algebras satisfying Assumption 1.

**Remark 3.1.** (i) Let  $T_A := (R/\text{rad}R) \otimes_R A_A \cong A/(\text{rad}R)A$ . Since  $R$  is not semisimple,  $R/\text{rad}R$  is non-projective. Since  $B \otimes_k (R/\text{rad}R) \cong T_R$  in  $\underline{\text{mod}}\text{-}R$ ,  $T_R$  is non-projective. Since  $A_R$  is projective,  $T_A$  is also non-projective. Moreover, it shows that  $A$  is not semisimple.

(ii) In most examples of this paper,  $R$  is a subalgebra of  $A$  with the property that  ${}_R A_R \cong {}_R R_R^n \oplus (R \otimes R)^l$  for some positive integers  $n$  and  $l$ .

(iii) The condition (c) implies that  $B$  is a self-injective algebra by [9, Theorem 1.4].

**Remark 3.2.** Since  $B \otimes_k (R/\text{rad}R) \xrightarrow{\phi} (R/\text{rad}R) \otimes_R A \cong A/(\text{rad}R)A$ ,  $b \otimes \bar{1} \mapsto \bar{b}$  is an isomorphism in  $\underline{\text{mod}}\text{-}R$ , we have isomorphisms

$$(2) \quad B \otimes_k \underline{\text{End}}_R(R/\text{rad}R) \cong \underline{\text{Hom}}_R(R/\text{rad}R, B \otimes_k (R/\text{rad}R)) \cong \underline{\text{Hom}}_R(R/\text{rad}R, A/(\text{rad}R)A) \cong \underline{\text{End}}_A(A/(\text{rad}R)A),$$

where the last isomorphism is induced from the adjoint isomorphism given by the adjoint pair  $(F, G)$ , where  $F$  (resp.  $G$ ) is the stable functor  $\underline{\text{mod}}\text{-}R \rightarrow \underline{\text{mod}}\text{-}A$  (resp.  $\underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}R$ ) induced from the induction functor  $- \otimes_R A$  (resp. restriction functor  $- \otimes_A A_R$ ). Moreover, it can be shown that the composition of these isomorphisms is a  $k$ -algebra isomorphism from  $B \otimes_k \underline{\text{End}}_R(R/\text{rad}R)$  to  $\underline{\text{End}}_A(A/(\text{rad}R)A)$ . Especially, if  $R$  is an elementary local symmetric  $k$ -algebra, then our subalgebra  $B$  is isomorphic to  $\underline{\text{End}}_A(T) = \underline{\text{End}}_A(A/(\text{rad}R)A)$ , which give the connection between our construction and Dugas' construction.

**Remark 3.3.** Since  $A$  is symmetric,  ${}_A A$  is isomorphic to  $D(A_A)$  as  $A$ -modules, and  ${}_R A$  is isomorphic to  $D(A_R)$  as  $R$ -modules. Since  $A_R$  is projective and  $R$  is self-injective,  $A_R$  is injective and therefore  ${}_R A \cong D(A_R)$  is projective.

Let  $\text{lrp}(A)$  be the category of left-right projective  $A$ - $A$ -bimodules, and let  $\underline{\text{lrp}}(A)$  be the stable category of  $\text{lrp}(A)$  obtained by factoring out the morphisms that factor through a projective  $A^e$ -module. Since  $A^e$  is self-injective (even symmetric),  $\underline{\text{lrp}}(A)$  becomes a triangulated category. Let  $\text{sum-}B^e$  be the full subcategory of  $\text{mod-}B^e$  consists of finite direct sum of copies of  $B \otimes_k B$ . For each  $B^e$ -module homomorphism  $f : B \otimes_k B \rightarrow B \otimes_k B$ ,  $1 \otimes 1 \mapsto \sum b_i \otimes b'_i$ , applies the functor  $A \otimes_B - \otimes_B A$ , we have an  $A^e$ -homomorphism  $\tilde{f} : A \otimes_k A \rightarrow A \otimes_k A$ ,  $1 \otimes 1 \mapsto \sum b_i \otimes b'_i$ . Since  $\tilde{f}$  is induced from a  $B^e$ -homomorphism and the elements of  $B$  commute with the elements of  $R$  under multiplication,  $\tilde{f}$  induces an  $A^e$ -homomorphism  $H(f) : A \otimes_R A \rightarrow A \otimes_R A$ , which makes the diagram

$$\begin{array}{ccc} B \otimes_k B & \xrightarrow{f} & B \otimes_k B \\ \downarrow & & \downarrow \\ A \otimes_R A & \xrightarrow{H(f)} & A \otimes_R A \end{array}$$

commutes. In general, for each  $B^e$ -homomorphism  $f : (B \otimes_k B)^n \rightarrow (B \otimes_k B)^m$  in  $\text{sum-}B^e$ , let  $H(f)$  be the unique  $A^e$ -homomorphism  $(A \otimes_R A)^n \rightarrow (A \otimes_R A)^m$  such that the diagram

$$\begin{array}{ccc} (B \otimes_k B)^n & \xrightarrow{f} & (B \otimes_k B)^m \\ \downarrow & & \downarrow \\ (A \otimes_R A)^n & \xrightarrow{H(f)} & (A \otimes_R A)^m \end{array}$$

commutes, where the vertical morphisms are the obvious morphisms. Then we have defined a functor  $H : \text{sum-}B^e \rightarrow \text{lrp}(A)$ .

Applying  $H$  to the complex  $(B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0}$  in Equation (1) we get a complex

$$(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \dots \rightarrow (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0}.$$

Let  $\tilde{d}_0$  be the composition  $(A \otimes_k A)^{m_0} \xrightarrow{A \otimes_B \delta_0 \otimes_B A} A \otimes_B A \xrightarrow{\mu} A$ , where  $\mu$  is the morphism given by multiplication. Since the elements of  $B$  commute with the elements of  $R$  under multiplication,  $\tilde{d}_0$  induces an  $A^e$ -homomorphism  $(A \otimes_R A)^{m_0} \xrightarrow{d_0} A$ . It can be shown that  $d_0 d_1 = 0$ , so the sequence

$$(3) \quad (A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \cdots \rightarrow (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A$$

is again a complex.

**Lemma 3.4.** *There exist triangles*

$$\begin{aligned} M_1 &\xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow, \\ M_2 &\xrightarrow{i_2} (A \otimes_R A)^{m_1} \xrightarrow{f_1} M_1 \rightarrow, \\ &\quad \cdots, \\ M_q &\xrightarrow{i_q} (A \otimes_R A)^{m_{q-1}} \xrightarrow{f_{q-1}} M_{q-1} \rightarrow \end{aligned}$$

in the triangulated category  $\underline{\text{lrp}}(A)$  such that  $i_p f_p = d_p$  for  $1 \leq p \leq q-1$ .

*Proof.* Let  $i_1 : M_1 \rightarrow (A \otimes_R A)^{m_0}$  be the kernel of  $d_0 : (A \otimes_R A)^{m_0} \rightarrow A$ . Since  $d_0$  is surjective,  $0 \rightarrow M_1 \xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow 0$  is an exact sequence, which induces a triangle  $M_1 \xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow$  in  $\underline{\text{lrp}}(A)$ . Since  $d_0 d_1 = 0$ , there exists a morphism  $f_1 : (A \otimes_R A)^{m_1} \rightarrow M_1$  such that  $d_1 = i_1 f_1$ . Let  $v_1 : P_1 \rightarrow M_1$  be the projective cover of  $M_1$  as an  $A^e$ -module, and let  $\begin{bmatrix} i_2 \\ u_1 \end{bmatrix} : M_2 \rightarrow (A \otimes_R A)^{m_1} \oplus P_1$  be the kernel of  $[f_1 \ v_1] : (A \otimes_R A)^{m_1} \oplus P_1 \rightarrow M_1$ . Since the morphism  $[f_1 \ v_1]$  is surjective, the short exact sequence  $0 \rightarrow M_2 \xrightarrow{\begin{bmatrix} i_2 \\ u_1 \end{bmatrix}} (A \otimes_R A)^{m_1} \oplus P_1 \xrightarrow{[f_1 \ v_1]} M_1 \rightarrow 0$  induces a triangle  $M_2 \xrightarrow{i_2} (A \otimes_R A)^{m_1} \xrightarrow{f_1} M_1 \rightarrow$  in  $\underline{\text{lrp}}(A)$ . Since  $i_1 f_1 d_2 = d_1 d_2 = 0$  and  $i_1$  is injective,  $f_1 d_2 = 0$ . Since the morphism  $\begin{bmatrix} d_2 \\ 0 \end{bmatrix} : (A \otimes_R A)^{m_2} \rightarrow (A \otimes_R A)^{m_1} \oplus P_1$  satisfies  $[f_1 \ v_1] \begin{bmatrix} d_2 \\ 0 \end{bmatrix} = f_1 d_2 = 0$ , there exists a morphism  $f_2 : (A \otimes_R A)^{m_2} \rightarrow M_2$  such that  $d_2 = i_2 f_2$  and  $u_1 f_2 = 0$ .

Using the same method, we can construct morphisms  $i_p : M_p \rightarrow (A \otimes_R A)^{m_{p-1}}$  for  $1 \leq p \leq q$ , and morphisms  $f_{p'} : (A \otimes_R A)^{m_{p'}} \rightarrow M_{p'}$ ,  $u_{p'} : M_{p'+1} \rightarrow P_{p'}$ ,  $v_{p'} : P_{p'} \rightarrow M_{p'}$  for  $1 \leq p' \leq q-1$  with  $P_{p'}$  projective as  $A^e$ -modules, such that the following conditions hold:

- (i)  $i_p f_p = d_p$  for  $1 \leq p \leq q-1$ ;
- (ii)  $u_p f_{p+1} = 0$  for  $1 \leq p \leq q-2$ ;

(iii)  $0 \rightarrow M_1 \xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow 0$  and  $0 \rightarrow M_{p+1} \xrightarrow{\begin{bmatrix} i_{p+1} \\ u_p \end{bmatrix}} (A \otimes_R A)^{m_p} \oplus P_p \xrightarrow{[f_p \ v_p]} M_p \rightarrow 0$  are short exact sequences for  $1 \leq p \leq q-1$ .

Since each  $P_p$  is a projective  $A^e$ -module, these short exact sequences induce triangles

$$\begin{aligned} M_1 &\xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow, \\ M_2 &\xrightarrow{i_2} (A \otimes_R A)^{m_1} \xrightarrow{f_1} M_1 \rightarrow, \\ &\quad \cdots, \\ M_q &\xrightarrow{i_q} (A \otimes_R A)^{m_{q-1}} \xrightarrow{f_{q-1}} M_{q-1} \rightarrow \end{aligned}$$

in  $\underline{\text{lrp}}(A)$ . □

**Theorem 3.5.** *Let  $(A, R, B)$  be the triple that satisfies Assumption 1. If  $M_q$  is the  $A$ - $A$ -bimodule defined in Lemma 3.4, then  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  is a stable auto-equivalence of  $A$ .*

*Proof.* Let  $F = - \otimes_R A_A$  and  $G = - \otimes_A A_R$  be the induction and the restriction functors respectively. Since  $A$  and  $R$  are symmetric and  ${}_R A_A$  is left-right projective, both  $(F, G)$  and  $(G, F)$  are adjoint pairs. Since both  $F$  and  $G$  map projectives to projectives, they induce stable functors (which are also denoted by  $F$  and  $G$ ). Moreover,  $G$  is both the left and the right adjoint of  $F$  as stable functors. Let  $T_A = F(R/\text{rad}R) = (R/\text{rad}R) \otimes_R A_A \cong A/(\text{rad}R)A$ . According to Remark 3.1,  $T_A$  is a nonzero object in  $\underline{\text{mod}}\text{-}A$ . Since the elements of  $B$  commute with the elements of  $R$  under multiplication,  $T \cong A/(\text{rad}R)A$  becomes a  $B$ - $A$ -bimodule.

Under the above notations, we now prove that  $- \otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  is a stable auto-equivalence of  $A$ . We will consider two cases.

*Case 1: Assume that  $A$  (as an algebra) is indecomposable.*

Choose a strong spanning class  $\mathcal{C} = \{T\} \cup T^\perp$  of  $\underline{\text{mod}}\text{-}A$ , where  $T^\perp = \{X \in \underline{\text{mod}}\text{-}A \mid \underline{\text{Hom}}_A(T, X) = 0\}$ . According to Corollary 2.7, it suffices to show that  $- \otimes_A M_q$  induces bijections between  $\underline{\text{Hom}}_A(X, Y[i])$  and  $\underline{\text{Hom}}_A(X \otimes_A M_q, (Y[i]) \otimes_A M_q)$  for all  $X, Y \in \mathcal{C}$  and for all  $i = 0, 1$ . We will divide the proof of Case 1 into four steps.

*Step 1.1: To show that  $- \otimes_A M_q$  induces a bijection between  $\underline{\text{Hom}}_A(T, T)$  and  $\underline{\text{Hom}}_A(T \otimes_A M_q, T \otimes_A M_q)$ .*

Since  $\phi : B \otimes_k (R/\text{rad}R) \rightarrow A/(\text{rad}R)A$ ,  $b \otimes \bar{1} \mapsto \bar{b}$  is an isomorphism in  $\underline{\text{mod}}\text{-}R$ ,  $\phi \otimes 1 : B \otimes_k T \cong B \otimes_k (R/\text{rad}R) \otimes_R A \rightarrow A/(\text{rad}R)A \otimes_R A = T \otimes_R A$  is an isomorphism in  $\underline{\text{mod}}\text{-}A$ . Applying the functors  $- \otimes_B T_A$  and  $T \otimes_A -$  to the complex  $0 \rightarrow B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \rightarrow 0$  and the complex  $(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \dots \rightarrow (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A$  respectively, we get a commutative diagram in  $\text{mod}\text{-}A$ :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & T & \xrightarrow{\delta_q \otimes 1} & (B \otimes_k T)^{m_{q-1}} & \xrightarrow{\delta_{q-1} \otimes 1} & \dots & \longrightarrow & (B \otimes_k T)^{m_1} & \xrightarrow{\delta_1 \otimes 1} & (B \otimes_k T)^{m_0} & \xrightarrow{\delta_0 \otimes 1} & T & \longrightarrow & 0 \\ & & & & (\phi \otimes 1)^{m_{q-1}} \downarrow & & & & (\phi \otimes 1)^{m_1} \downarrow & & (\phi \otimes 1)^{m_0} \downarrow & & & \parallel & & \\ & & & & (T \otimes_R A)^{m_{q-1}} & \xrightarrow{1 \otimes d_{q-1}} & \dots & \longrightarrow & (T \otimes_R A)^{m_1} & \xrightarrow{1 \otimes d_1} & (T \otimes_R A)^{m_0} & \xrightarrow{1 \otimes d_0} & T & & & \end{array}$$

Since  $0 \rightarrow B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \rightarrow 0$  is split exact as a complex of right  $B$ -modules, the first row of this commutative diagram is also split exact. Therefore we have split exact sequences  $0 \rightarrow K_1 \xrightarrow{j_1} (B \otimes_k T)^{m_0} \xrightarrow{\delta_0 \otimes 1} T \rightarrow 0$ ,  $0 \rightarrow K_2 \xrightarrow{j_2} (B \otimes_k T)^{m_1} \xrightarrow{p_1} K_1 \rightarrow 0$ ,  $\dots$ ,  $0 \rightarrow K_{q-1} \xrightarrow{j_{q-1}} (B \otimes_k T)^{m_{q-2}} \xrightarrow{p_{q-2}} K_{q-2} \rightarrow 0$ ,  $0 \rightarrow T \xrightarrow{\delta_q \otimes 1} (B \otimes_k T)^{m_{q-1}} \xrightarrow{p_{q-1}} K_{q-1} \rightarrow 0$  in  $\text{mod}\text{-}A$  such that  $j_l p_l = \delta_l \otimes 1$  for  $1 \leq l \leq q-1$ .

There is a commutative diagram

$$\begin{array}{ccccc} K_1 & \xrightarrow{j_1} & (B \otimes_k T)^{m_0} & \xrightarrow{\delta_0 \otimes 1} & T & \longrightarrow \\ & & (\phi \otimes 1)^{m_0} \downarrow & & \parallel & \\ T \otimes_A M_1 & \xrightarrow{1 \otimes i_1} & (T \otimes_R A)^{m_0} & \xrightarrow{1 \otimes d_0} & T & \longrightarrow \end{array}$$

in  $\underline{\text{mod}}\text{-}A$ , where its two rows are triangles and  $(\phi \otimes 1)^{m_0}$  is an isomorphism in  $\underline{\text{mod}}\text{-}A$ . Therefore we have an isomorphism  $\underline{\alpha}_1 : K_1 \rightarrow T \otimes_A M_1$  in  $\underline{\text{mod}}\text{-}A$  such that  $(\phi \otimes 1)^{m_0} j_1 = (1 \otimes i_1) \underline{\alpha}_1$ . Since  $j_1$  is a split monomorphism in  $\underline{\text{mod}}\text{-}A$ , so does  $1 \otimes i_1$ . Since

$$(4) \quad \underline{(1 \otimes i_1) \alpha_1 p_1} = \underline{(\phi \otimes 1)^{m_0} j_1 p_1} = \underline{(\phi \otimes 1)^{m_0} (\delta_1 \otimes 1)} = \underline{(1 \otimes d_1) (\phi \otimes 1)^{m_1}} = \underline{(1 \otimes i_1) (1 \otimes f_1) (\phi \otimes 1)^{m_1}}$$

in  $\underline{\text{mod}}\text{-}A$  and since  $\underline{1 \otimes i_1}$  is a split monomorphism in  $\underline{\text{mod}}\text{-}A$ , we have  $\underline{\alpha_1 p_1} = \underline{(1 \otimes f_1)(\phi \otimes 1)^{m_1}}$  in  $\underline{\text{mod}}\text{-}A$ . Then we have a commutative diagram

$$\begin{array}{ccccc} K_2 & \xrightarrow{j_2} & (B \otimes_k T)^{m_1} & \xrightarrow{p_1} & K_1 & \longrightarrow \\ & & \downarrow (\phi \otimes 1)^{m_1} & & \downarrow \alpha_1 & \\ T \otimes_A M_2 & \xrightarrow{1 \otimes i_2} & (T \otimes_R A)^{m_1} & \xrightarrow{1 \otimes f_1} & T \otimes_A M_1 & \longrightarrow \end{array}$$

in  $\underline{\text{mod}}\text{-}A$ , whose rows are triangles and vertical morphisms are isomorphisms. So we have an isomorphism  $\underline{\alpha_2} : K_2 \rightarrow T \otimes_A M_2$  in  $\underline{\text{mod}}\text{-}A$  such that  $\underline{(\phi \otimes 1)^{m_1} j_2} = \underline{(1 \otimes i_2) \alpha_2}$ . Inductively, we have isomorphisms  $\underline{\alpha_l} : K_l \rightarrow T \otimes_A M_l$  in  $\underline{\text{mod}}\text{-}A$  for  $1 \leq l \leq q$  (let  $K_q = T$ ), such that

$$\begin{array}{ccccc} K_1 & \xrightarrow{j_1} & (B \otimes_k T)^{m_0} & \xrightarrow{\delta_0 \otimes 1} & T & \longrightarrow \\ & & \downarrow (\phi \otimes 1)^{m_0} & & \parallel & \\ T \otimes_A M_1 & \xrightarrow{1 \otimes i_1} & (T \otimes_R A)^{m_0} & \xrightarrow{1 \otimes d_0} & T & \longrightarrow \end{array}$$

is an isomorphism of triangles and

$$\begin{array}{ccccc} K_{l+1} & \xrightarrow{j_{l+1}} & (B \otimes_k T)^{m_l} & \xrightarrow{p_l} & K_l & \longrightarrow \\ & & \downarrow (\phi \otimes 1)^{m_l} & & \downarrow \alpha_l & \\ T \otimes_A M_{l+1} & \xrightarrow{1 \otimes i_{l+1}} & (T \otimes_R A)^{m_l} & \xrightarrow{1 \otimes f_l} & T \otimes_A M_l & \longrightarrow \end{array}$$

are isomorphisms of triangles for  $1 \leq l \leq q-1$  (let  $j_q = \delta_q \otimes 1 : T \rightarrow (B \otimes_k T)^{m_{q-1}}$ ).

Since  $\underline{\alpha_q} : T \rightarrow T \otimes_A M_q$  is an isomorphism in  $\underline{\text{mod}}\text{-}A$ , to show  $- \otimes_A M_q$  induces a bijection between  $\underline{\text{Hom}}_A(T, T)$  and  $\underline{\text{Hom}}_A(T \otimes_A M_q, T \otimes_A M_q)$ , it suffices to show that for each  $\underline{f} \in \underline{\text{End}}_A(T)$ , the diagram

$$\begin{array}{ccc} T & \xrightarrow{\underline{f}} & T \\ \downarrow \underline{\alpha_q} & & \downarrow \underline{\alpha_q} \\ T \otimes_A M_q & \xrightarrow{\underline{f \otimes 1}} & T \otimes_A M_q \end{array}$$

is commutative. We have an isomorphism  $\underline{\text{End}}_A(T) \cong \underline{\text{Hom}}_R(R/\text{rad}R, T_R) \cong \underline{\text{Hom}}_R(R/\text{rad}R, B \otimes_k (R/\text{rad}R))$ , where the second isomorphism is induced from the isomorphism  $\phi : B \otimes_k (R/\text{rad}R) \rightarrow A/(\text{rad}R)A$ ,  $b \otimes \bar{1} \mapsto \bar{b}$  in  $\underline{\text{mod}}\text{-}R$ . For  $\underline{f} \in \underline{\text{End}}_A(T)$ , suppose the isomorphism  $\underline{\text{End}}_A(T) \rightarrow \underline{\text{Hom}}_R(R/\text{rad}R, B \otimes_k (R/\text{rad}R))$  maps  $\underline{f}$  to  $\underline{g}$ , where  $g(\bar{1}) = \sum_j \beta_j \otimes \bar{r}_j$  with  $\beta_j \in B$ ,  $r_j \in R$ . Then  $\underline{f} = \underline{h}$ , where  $h : T_A \rightarrow T_A$ ,  $\bar{1} \mapsto \sum_j \beta_j r_j$ . Consider the diagram

$$\begin{array}{ccccc} & & T & & \\ & \swarrow \underline{h} & \downarrow \underline{\alpha_q} & \searrow (\phi \otimes 1)^{m_{q-1}} (\delta_q \otimes 1) & \\ T & & T \otimes_A M_q & \xrightarrow{1 \otimes i_q} & (T \otimes_R A)^{m_{q-1}} \\ \downarrow \underline{\alpha_q} & \swarrow \underline{h \otimes 1} & \downarrow \underline{(\phi \otimes 1)^{m_{q-1}} (\delta_q \otimes 1)} & \searrow \underline{(h \otimes 1)^{m_{q-1}}} & \\ T \otimes_A M_q & \xrightarrow{1 \otimes i_q} & (T \otimes_R A)^{m_{q-1}} & & \end{array}$$

Figure 1

in  $\underline{\text{mod}}\text{-}A$ , where  $(\phi \otimes 1)^{m_{q-1}}(\delta_q \otimes 1)$  denotes the composition  $T \xrightarrow{\delta_q \otimes 1} (B \otimes_k T)^{m_{q-1}} \xrightarrow{(\phi \otimes 1)^{m_{q-1}}} (T \otimes_R A)^{m_{q-1}}$ . Since

$$\begin{array}{ccccc} T & \xrightarrow{\delta_q \otimes 1} & (B \otimes_k T)^{m_{q-1}} & \xrightarrow{p_{q-1}} & K_{q-1} & \longrightarrow \\ \downarrow \alpha_q & & \downarrow (\phi \otimes 1)^{m_{q-1}} & & \downarrow \alpha_{q-1} & \\ T \otimes_A M_k & \xrightarrow{1 \otimes i_q} & (T \otimes_R A)^{m_{q-1}} & \xrightarrow{1 \otimes f_{q-1}} & T \otimes_A M_{q-1} & \longrightarrow \end{array}$$

is an isomorphism of triangles in  $\underline{\text{mod}}\text{-}A$ , and since  $\delta_q \otimes 1$  is a split monomorphism in  $\underline{\text{mod}}\text{-}A$ ,  $1 \otimes i_q$  is also a split monomorphism in  $\underline{\text{mod}}\text{-}A$ . Since the bottom face, the front face, the back face of Figure 1 are commutative, and since  $1 \otimes i_q$  is a split monomorphism, to show the left face of Figure 1 commutes, it suffices to show the diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & T \\ \downarrow \delta_q \otimes 1 & & \downarrow \delta_q \otimes 1 \\ (B \otimes_k T)^{m_{q-1}} & & (B \otimes_k T)^{m_{q-1}} \\ \downarrow (\phi \otimes 1)^{m_{q-1}} & & \downarrow (\phi \otimes 1)^{m_{q-1}} \\ (T \otimes_R A)^{m_{q-1}} & \xrightarrow{(h \otimes 1)^{m_{q-1}}} & (T \otimes_R A)^{m_{q-1}} \end{array}$$

Figure 2

is commutative in  $\underline{\text{mod}}\text{-}A$ .

Since  $\delta_q : B \rightarrow (B \otimes_k B)^{m_{q-1}}$  is a  $B^e$ -homomorphism, we may write  $\delta_q$  as  $(\delta_q^1, \dots, \delta_q^{m_{q-1}})'$ , where  $\delta_q^i : B \rightarrow B \otimes_k B$ ,  $1 \mapsto \sum_l b_{il} \otimes b'_{il}$  for  $1 \leq i \leq m_{q-1}$ . To show that the diagram in Figure 2 commutes, it suffices to show for each  $1 \leq i \leq m_{q-1}$ , the diagram

$$\begin{array}{ccc} T & \xrightarrow{h} & T \\ \downarrow \delta_q^i \otimes 1 & & \downarrow \delta_q^i \otimes 1 \\ B \otimes_k T & & B \otimes_k T \\ \downarrow \phi \otimes 1 & & \downarrow \phi \otimes 1 \\ T \otimes_R A & \xrightarrow{h \otimes 1} & T \otimes_R A \end{array}$$

Figure 3

is commutative in  $\text{mod}\text{-}A$ .

For  $\bar{1} \in T = A/(\text{rad}R)A$ ,  $(h \otimes 1)(\phi \otimes 1)(\delta_q^i \otimes 1)(\bar{1}) = (h \otimes 1)(\phi \otimes 1)(\sum_l b_{il} \otimes \bar{b}'_{il}) = (h \otimes 1)(\sum_l \bar{b}_{il} \otimes b'_{il}) = \sum_l (\sum_j \beta_j r_j) \bar{b}_{il} \otimes b'_{il} = \sum_j (\sum_l \bar{\beta}_j \bar{b}_{il} \otimes b'_{il}) r_j$ , where the last identity follows from the fact that the elements of  $B$  commute with the elements of  $R$  under multiplication. Moreover,  $(\phi \otimes 1)(\delta_q^i \otimes 1)h(\bar{1}) = (\phi \otimes 1)(\delta_q^i \otimes 1)(\sum_j \beta_j r_j) = (\phi \otimes 1)(\sum_l b_{il} \otimes b'_{il}(\sum_j \beta_j r_j)) = \sum_l \bar{b}_{il} \otimes b'_{il}(\sum_j \beta_j r_j) = \sum_j (\sum_l \bar{b}_{il} \otimes b'_{il} \beta_j) r_j$ . Since  $\delta_q^i : B \rightarrow B \otimes_k B$  is a  $B^e$ -homomorphism,  $\sum_l \beta_j b_{il} \otimes b'_{il} = \beta_j (\sum_l b_{il} \otimes b'_{il}) = \beta_j \delta_q^i(1) = \delta_q^i(\beta_j) = \delta_q^i(1) \beta_j = (\sum_l b_{il} \otimes b'_{il}) \beta_j = \sum_l b_{il} \otimes b'_{il} \beta_j$  in  $B \otimes_k B$ . Since  $\sum_l \bar{\beta}_j \bar{b}_{il} \otimes b'_{il} \in T \otimes_R A$  (resp.  $\sum_l \bar{b}_{il} \otimes b'_{il} \beta_j \in T \otimes_R A$ ) is the image of  $\sum_l \beta_j b_{il} \otimes b'_{il}$  (resp.  $\sum_l b_{il} \otimes b'_{il} \beta_j$ ) under the composition of morphisms  $B \otimes_k B \rightarrow A \otimes_k A \rightarrow A \otimes_R A \rightarrow T \otimes_R A$ ,



$\sum_l \overline{\beta_j b_{il}} \otimes b'_{il} = \sum_l \overline{b_{il}} \otimes b'_{il} \beta_j$  in  $T \otimes_R A$ . Therefore  $(h \otimes 1)(\phi \otimes 1)(\delta_q^i \otimes 1)(\overline{1}) = \sum_j (\sum_l \overline{\beta_j b_{il}} \otimes b'_{il}) r_j = \sum_j (\sum_l \overline{b_{il}} \otimes b'_{il} \beta_j) r_j = (\phi \otimes 1)(\delta_q^i \otimes 1)h(\overline{1})$  and the diagram in Figure 3 commutes.

*Step 1.2:* To show that  $-\otimes_A M_q$  induces a bijection between  $\underline{\text{Hom}}_A(T, T[1])$  and  $\underline{\text{Hom}}_A(T \otimes_A M_q, T[1] \otimes_A M_q)$ .

Since the functor  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  commutes with the functor  $[1] = \Omega_A^{-1} : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  up to natural isomorphism, it suffices to show  $-\otimes_A M_q$  induces a bijection between  $\underline{\text{Hom}}_A(\Omega_A T, T)$  and  $\underline{\text{Hom}}_A(\Omega_A T \otimes_A M_q, T \otimes_A M_q)$ .

There is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \otimes_k \text{rad}R & \longrightarrow & B \otimes_k R & \longrightarrow & B \otimes_k (R/\text{rad}R) \longrightarrow 0 \\ & & \downarrow \nu & & \downarrow \mu & & \downarrow \phi \\ 0 & \longrightarrow & (\text{rad}R)A & \longrightarrow & A & \longrightarrow & A/(\text{rad}R)A \longrightarrow 0 \end{array}$$

in  $\underline{\text{mod}}\text{-}R$  with exact rows, where  $\mu$  and  $\nu$  are induced by the multiplication of  $A$ . Since  $R$  is symmetric and  $A_R$  is projective,  $\underline{\nu} = \Omega_R(\phi)$  is an isomorphism in  $\underline{\text{mod}}\text{-}R$ . Therefore  $B \otimes_k \Omega_A T = B \otimes_k (\text{rad}R)A \cong B \otimes_k \text{rad}R \otimes_R A \xrightarrow{\nu \otimes 1} (\text{rad}R)A \otimes_R A = \Omega_A T \otimes_R A$  is an isomorphism in  $\underline{\text{mod}}\text{-}A$ .

Since the elements of  $B$  commute with the elements of  $R$  under multiplication,  $\Omega_A T = (\text{rad}R)A$  becomes a  $B$ - $A$ -bimodule. Applies the functors  $-\otimes_B (\Omega_A T)_A$  and  $\Omega_A T \otimes_A -$  to the complex  $0 \rightarrow B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \rightarrow 0$  and the complex  $(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \dots \rightarrow (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A$  respectively, we get a commutative diagram in  $\underline{\text{mod}}\text{-}A$ :

$$\begin{array}{ccccccc} 0 \longrightarrow \Omega_A T \xrightarrow{\delta_q \otimes 1} (B \otimes_k \Omega_A T)^{m_{q-1}} \xrightarrow{\delta_{q-1} \otimes 1} \dots \longrightarrow (B \otimes_k \Omega_A T)^{m_1} \xrightarrow{\delta_1 \otimes 1} (B \otimes_k \Omega_A T)^{m_0} \xrightarrow{\delta_0 \otimes 1} \Omega_A T \longrightarrow 0 \\ \quad \quad \quad \downarrow (\nu \otimes 1)^{m_{q-1}} \quad \quad \quad \downarrow (\nu \otimes 1)^{m_1} \quad \quad \quad \downarrow (\nu \otimes 1)^{m_0} \quad \quad \quad \downarrow \parallel \\ (\Omega_A T \otimes_R A)^{m_{q-1}} \xrightarrow{1 \otimes d_{q-1}} \dots \longrightarrow (\Omega_A T \otimes_R A)^{m_1} \xrightarrow{1 \otimes d_1} (\Omega_A T \otimes_R A)^{m_0} \xrightarrow{1 \otimes d_0} \Omega_A T \end{array}$$

By the same argument as in Step 1.1, we have isomorphisms of split triangles

$$\begin{array}{ccccc} L_{l+1} & \xrightarrow{\iota_{l+1}} & (B \otimes_k \Omega_A T)^{m_l} & \xrightarrow{q_l} & L_l \longrightarrow \\ \downarrow \beta_{l+1} & & \downarrow (\nu \otimes 1)^{m_l} & & \downarrow \beta_l \\ \Omega_A T \otimes_A M_{l+1} & \xrightarrow{1 \otimes \iota_{l+1}} & (\Omega_A T \otimes_R A)^{m_l} & \xrightarrow{1 \otimes f_l} & \Omega_A T \otimes_A M_l \longrightarrow \end{array}$$

in  $\underline{\text{mod}}\text{-}A$  for  $0 \leq l \leq q-1$ , where  $L_0 = L_q = \Omega_A T$ ,  $q_0 = \delta_0 \otimes 1 : (B \otimes_k \Omega_A T)^{m_0} \rightarrow \Omega_A T$ ,  $f_0 = d_0 : (A \otimes_R A)^{m_0} \rightarrow A$ ,  $\iota_q = \delta_q \otimes 1 : \Omega_A T \rightarrow (B \otimes_k \Omega_A T)^{m_{q-1}}$ .

To show  $-\otimes_A M_q$  induces a bijection between  $\underline{\text{Hom}}_A(\Omega_A T, T)$  and  $\underline{\text{Hom}}_A(\Omega_A T \otimes_A M_q, T \otimes_A M_q)$ , it suffices to show that for each  $f \in \underline{\text{Hom}}_A(\Omega_A T, T)$ , the diagram

$$\begin{array}{ccc} \Omega_A T & \xrightarrow{f} & T \\ \downarrow \beta_q & & \downarrow \alpha_q \\ \Omega_A T \otimes_A M_q & \xrightarrow{f \otimes 1} & T \otimes_A M_q \end{array}$$

is commutative. We have isomorphisms

$$(5) \quad \underline{\text{Hom}}_A(\Omega_A T, T) = \underline{\text{Hom}}_A(F(\text{rad}R), T) \cong \underline{\text{Hom}}_R(\text{rad}R, T_R) \cong \underline{\text{Hom}}_R(\text{rad}R, B \otimes_k (R/\text{rad}R)),$$

where the second isomorphism is induced from the isomorphism  $\phi : B \otimes_k (R/\text{rad}R) \rightarrow A/(\text{rad}R)A$ ,  $b \otimes \overline{1} \mapsto \overline{b}$  in  $\underline{\text{mod}}\text{-}R$ . Choose a  $k$ -basis  $x_1, \dots, x_n$  of  $B$ , then each  $g \in \underline{\text{Hom}}_R(\text{rad}R, B \otimes_k (R/\text{rad}R))$  can be written as a column vector  $(g_1, \dots, g_n)'$ , where  $g_i \in \underline{\text{Hom}}_R(\text{rad}R, R/\text{rad}R)$  for  $1 \leq i \leq n$ .

$n$ . For  $\underline{f} \in \underline{\text{Hom}}_A(\Omega_{AT}, T)$ , suppose the isomorphism  $\underline{\text{Hom}}_A(\Omega_{AT}, T) \rightarrow \underline{\text{Hom}}_R(\text{rad}R, B \otimes_k (R/\text{rad}R))$  maps  $\underline{f}$  to  $\underline{g}$ , where  $\underline{g} = (g_1, \dots, g_n)'$  with  $g_i \in \text{Hom}_R(\text{rad}R, R/\text{rad}R)$ . Suppose for each  $r \in \text{rad}R$ ,  $g_i(r) = \overline{\gamma_i}$  with  $\gamma_i \in R$ . Then  $\underline{f} = \underline{h}$ , where  $h \in \text{Hom}_A(\Omega_{AT}, T)$  with  $h(r) = \sum_{i=1}^n x_i \gamma_i$  for each  $r \in \text{rad}R$ . Consider the diagram

$$\begin{array}{ccccc}
& & \Omega_{AT} & & \\
& \swarrow \underline{h} & \downarrow \underline{\beta}_q & \searrow & \\
T & & \Omega_{AT} \otimes_A M_q & & (\Omega_{AT} \otimes_R A)^{m_{q-1}} \\
\downarrow \underline{\alpha}_q & & \downarrow \underline{h \otimes 1} & \xrightarrow{\underline{1 \otimes i}_q} & \downarrow \underline{(h \otimes 1)^{m_{q-1}}} \\
T \otimes_A M_q & \xrightarrow{\underline{1 \otimes i}_q} & (T \otimes_R A)^{m_{q-1}} & & 
\end{array}$$

Figure 4

in  $\underline{\text{mod}}\text{-}A$ , where  $(\phi \otimes 1)^{m_{q-1}}(\delta_q \otimes 1)$  denotes the composition  $T \xrightarrow{\delta_q \otimes 1} (B \otimes_k T)^{m_{q-1}} \xrightarrow{(\phi \otimes 1)^{m_{q-1}}} (T \otimes_R A)^{m_{q-1}}$  and  $(\nu \otimes 1)^{m_{q-1}}(\delta_q \otimes 1)$  denotes the composition  $\Omega_{AT} \xrightarrow{\delta_q \otimes 1} (B \otimes_k \Omega_{AT})^{m_{q-1}} \xrightarrow{(\nu \otimes 1)^{m_{q-1}}} (\Omega_{AT} \otimes_R A)^{m_{q-1}}$ . Since the bottom face, the front face, the back face of Figure 4 are commutative, and since  $\underline{1 \otimes i}_q$  is a split monomorphism in  $\underline{\text{mod}}\text{-}A$ , to show the left face of Figure 4 commutes, it suffices to show the diagram

$$\begin{array}{ccc}
\Omega_{AT} & \xrightarrow{\underline{h}} & T \\
\downarrow \underline{\delta}_q \otimes 1 & & \downarrow \underline{\delta}_q \otimes 1 \\
(B \otimes_k \Omega_{AT})^{m_{q-1}} & & (B \otimes_k T)^{m_{q-1}} \\
\downarrow \underline{(\nu \otimes 1)^{m_{q-1}}} & & \downarrow \underline{(\phi \otimes 1)^{m_{q-1}}} \\
(\Omega_{AT} \otimes_R A)^{m_{q-1}} & \xrightarrow{\underline{(h \otimes 1)^{m_{q-1}}}} & (T \otimes_R A)^{m_{q-1}}
\end{array}$$

Figure 5

is commutative in  $\underline{\text{mod}}\text{-}A$ .

Since  $\delta_q : B \rightarrow (B \otimes_k B)^{m_{q-1}}$  is a  $B^e$ -homomorphism, we may write  $\delta_q$  as  $(\delta_q^1, \dots, \delta_q^{m_{q-1}})'$ , where  $\delta_q^i : B \rightarrow B \otimes_k B$ ,  $1 \mapsto \sum_l b_{il} \otimes b'_{il}$  for  $1 \leq i \leq m_{q-1}$ . To show the diagram in Figure 5 commutes, it suffices to show for each  $1 \leq i \leq m_{q-1}$ , the diagram

$$\begin{array}{ccc}
\Omega_{AT} & \xrightarrow{h} & T \\
\downarrow \delta_q^i \otimes 1 & & \downarrow \delta_q^i \otimes 1 \\
B \otimes_k \Omega_{AT} & & B \otimes_k T \\
\downarrow \nu \otimes 1 & & \downarrow \phi \otimes 1 \\
\Omega_{AT} \otimes_R A & \xrightarrow{h \otimes 1} & T \otimes_R A
\end{array}$$

Figure 6

is commutative in  $\text{mod-}A$ .

For each  $r \in \text{rad}R \subseteq (\text{rad}R)A = T$ ,  $(h \otimes 1)(\nu \otimes 1)(\delta_q^i \otimes 1)(r) = (h \otimes 1)(\nu \otimes 1)(\sum_l \overline{b_{il}} \otimes b'_{il}r) = (h \otimes 1)(\nu \otimes 1)(\sum_l \overline{b_{il}r} \otimes b'_{il}) = (h \otimes 1)(\sum_l \overline{b_{il}r} \otimes b'_{il}) = (h \otimes 1)(\sum_l r \overline{b_{il}} \otimes b'_{il}) = \sum_l (\sum_{j=1}^n x_j \gamma_j) \overline{b_{il}} \otimes b'_{il} = \sum_{j=1}^n (\sum_l \overline{x_j b_{il}} \otimes b'_{il}) \gamma_j$  and  $(\phi \otimes 1)(\delta_q^i \otimes 1)h(r) = (\phi \otimes 1)(\delta_q^i \otimes 1)(\sum_{j=1}^n x_j \gamma_j) = (\phi \otimes 1)(\sum_l \overline{b_{il}} \otimes b'_{il}(\sum_{j=1}^n x_j \gamma_j)) = \sum_l \sum_{j=1}^n \overline{b_{il}} \otimes b'_{il} x_j \gamma_j = \sum_{j=1}^n (\sum_l \overline{b_{il}} \otimes b'_{il} x_j) \gamma_j$ . Here we use the fact that the elements of  $B$  commute with the elements of  $R$  under multiplication. Since  $\delta_q^i : B \rightarrow B \otimes_k B$  is a  $B^e$ -homomorphism,  $\sum_l x_j \overline{b_{il}} \otimes b'_{il} = x_j (\sum_l \overline{b_{il}} \otimes b'_{il}) = x_j \delta_q^i(1) = \delta_q^i(x_j) = \delta_q^i(1)x_j = (\sum_l \overline{b_{il}} \otimes b'_{il})x_j = \sum_l \overline{b_{il}} \otimes b'_{il} x_j$  in  $B \otimes_k B$ . Since  $\sum_l \overline{x_j b_{il}} \otimes b'_{il} \in T \otimes_R A$  (resp.  $\sum_l \overline{b_{il}} \otimes b'_{il} x_j \in T \otimes_R A$ ) is the image of  $\sum_l x_j \overline{b_{il}} \otimes b'_{il}$  (resp.  $\sum_l \overline{b_{il}} \otimes b'_{il} x_j$ ) under the composition of morphisms  $B \otimes_k B \rightarrow A \otimes_k A \rightarrow A \otimes_R A \rightarrow T \otimes_R A$ ,  $\sum_l \overline{x_j b_{il}} \otimes b'_{il} = \sum_l \overline{b_{il}} \otimes b'_{il} x_j$  in  $T \otimes_R A$ . Therefore  $(h \otimes 1)(\nu \otimes 1)(\delta_q^i \otimes 1)(r) = \sum_{j=1}^n (\sum_l \overline{x_j b_{il}} \otimes b'_{il}) \gamma_j = \sum_{j=1}^n (\sum_l \overline{b_{il}} \otimes b'_{il} x_j) \gamma_j = (\phi \otimes 1)(\delta_q^i \otimes 1)h(r)$  and the diagram in Figure 6 commutes.

*Step 1.3: To show that  $-\otimes_A M_q$  induces bijections between  $\underline{\text{Hom}}_A(X, Y[i])$  and  $\underline{\text{Hom}}_A(X \otimes_A M_q, Y[i] \otimes_A M_q)$  for  $X, Y \in T^\perp$  and  $i = 0, 1$ .*

For each  $X \in \underline{\text{mod-}}A$ ,  $\underline{\text{Hom}}_A(T, X) = \underline{\text{Hom}}_A(F(R/\text{rad}R), X) \cong \underline{\text{Hom}}_R(R/\text{rad}R, X_R)$ . Since  $R$  is symmetric,  $T^\perp = \{X \in \underline{\text{mod-}}A \mid X_R \text{ projective}\}$ . Since  $A_R$  is projective,  $T^\perp$  is closed under  $[n] = \Omega_A^{-n} : \underline{\text{mod-}}A \rightarrow \underline{\text{mod-}}A$  for all  $n \in \mathbb{Z}$ . Therefore it is suffice to show that  $-\otimes_A M_q$  is fully faithful when is restricted to  $T^\perp$ . Since there exists a triangle  $\Omega_{A^e}(A) \xrightarrow{w_1} M_1 \xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A$  in  $\underline{\text{lrp}}(A)$ , and since  $X \otimes_A (A \otimes_R A)^{m_0} = 0$  in  $\underline{\text{mod-}}A$  for  $X \in T^\perp$ ,  $w_1$  induces a natural isomorphism between functors  $-\otimes_A \Omega_{A^e}(A) : T^\perp \rightarrow \underline{\text{mod-}}A$  and  $-\otimes_A M_1 : T^\perp \rightarrow \underline{\text{mod-}}A$ . Similarly, the functors  $-\otimes_A (M_i[-1]) : T^\perp \rightarrow \underline{\text{mod-}}A$  and  $-\otimes_A M_{i+1} : T^\perp \rightarrow \underline{\text{mod-}}A$  are natural isomorphic for  $1 \leq i \leq q-1$ . Therefore  $-\otimes_A M_q : T^\perp \rightarrow \underline{\text{mod-}}A$  is natural isomorphic to  $\Omega_A^q(-) \cong -\otimes_A \Omega_{A^e}^q(A) : T^\perp \rightarrow \underline{\text{mod-}}A$ , which implies that  $-\otimes_A M_q$  is fully faithful when is restricted to  $T^\perp$ .

*Step 1.4: To show that  $-\otimes_A M_q$  induces bijections between  $\underline{\text{Hom}}_A(T, X[i])$  (resp.  $\underline{\text{Hom}}_A(X, T[i])$ ) and  $\underline{\text{Hom}}_A(T \otimes_A M_q, X[i] \otimes_A M_q)$  (resp.  $\underline{\text{Hom}}_A(X \otimes_A M_q, T[i] \otimes_A M_q)$ ) for  $X \in T^\perp$  and for  $i = 0, 1$ .*

For each  $X \in \underline{\text{mod-}}A$ , we have

$$\underline{\text{Hom}}_A(X, T) = \underline{\text{Hom}}_A(X, F(R/\text{rad}R)) \cong \underline{\text{Hom}}_R(X_R, R/\text{rad}R).$$

Therefore  ${}^\perp T = \{X \in \underline{\text{mod-}}A \mid X_R \text{ is projective}\} = T^\perp$ . Since  $T^\perp = {}^\perp T$  is closed under  $[n] = \Omega_A^{-n} : \underline{\text{mod-}}A \rightarrow \underline{\text{mod-}}A$  for all  $n \in \mathbb{Z}$ ,  $\underline{\text{Hom}}_A(T, X[i]) = 0$  and  $\underline{\text{Hom}}_A(X, T[i]) = 0$  for  $X \in T^\perp$  and for  $i = 0, 1$ . Since  $T \otimes_A M_q \cong T$  in  $\underline{\text{mod-}}A$  and  $Y \otimes_A M_q \cong Y[-q]$  in  $\underline{\text{mod-}}A$  for every  $Y \in T^\perp$ ,  $\underline{\text{Hom}}_A(T \otimes_A M_q, X[i] \otimes_A M_q) = 0$  and  $\underline{\text{Hom}}_A(X \otimes_A M_q, T[i] \otimes_A M_q) = 0$  for  $X \in T^\perp$  and for  $i = 0, 1$ .

By Step 1.1 ~ Step 1.4, we have shown that  $-\otimes_A M_q : \underline{\text{mod-}}A \rightarrow \underline{\text{mod-}}A$  is a stable auto-equivalence of  $A$  when  $A$  is indecomposable.

*Case 2: Assume that  $A$  is decomposable.*

Let  $A = A_1 \times \cdots \times A_p \times A_{p+1} \times \cdots \times A_r$  be the decomposition of  $A$  into indecomposable blocks, where  $A_{p+1}, \dots, A_r$  are all semisimple blocks of  $A$ . Let  $T_A = (R/\text{rad}R) \otimes_A A \cong A/(\text{rad}R)A$  and suppose  $A_1, \dots, A_m$  ( $m \leq p$ ) be all indecomposable blocks of  $A$  such that there exists an indecomposable non-projective summand of  $T_A$  which belongs to the block. Then  $\underline{\text{mod-}}A_i$  is contained in  $T^\perp$  for each  $m+1 \leq i \leq p$ . Let  $\mathcal{C} = \{T\} \cup T^\perp$  be a strong spanning class of  $\underline{\text{mod-}}A$ .

Similar to Case 1, the following statements are still true:

- (i)  $T^\perp = {}^\perp T$  is closed under  $[n] = \Omega_A^{-n} : \underline{\text{mod-}}A \rightarrow \underline{\text{mod-}}A$  for all  $n \in \mathbb{Z}$ ;
- (ii)  $T \otimes_A M_q \cong T$  in  $\underline{\text{mod-}}A$  and  $X \otimes_A M_q \cong X[-q]$  in  $\underline{\text{mod-}}A$  for every  $X \in T^\perp$ ;

(iii)  $-\otimes_A M_q$  induces bijections between  $\underline{\text{Hom}}_A(X, Y[i])$  and  $\underline{\text{Hom}}_A(X \otimes_A M_q, (Y[i]) \otimes_A M_q)$  for all  $X, Y \in \mathcal{C}$  and for all  $i = 0, 1$ .

Since the functor  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  has both left and right adjoints, by statement (iii) and Proposition 2.5 it is fully faithful.

Let  $T \cong \bigoplus_{i=1}^m T_i$  in  $\underline{\text{mod}}\text{-}A$ , where  $T_i \in \underline{\text{mod}}\text{-}A_i$ . Then  $T_i \neq 0$  in  $\underline{\text{mod}}\text{-}A_i$  for each  $1 \leq i \leq m$ . Since the functor  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  is fully faithful and since  $\underline{\text{mod}}\text{-}A_i$  is an indecomposable triangulated category for  $1 \leq i \leq p$ , by Lemma 2.1, for each  $1 \leq i \leq m$ ,  $T_i \otimes_A M_q \in \underline{\text{mod}}\text{-}A_{\sigma(i)}$  for some  $1 \leq \sigma(i) \leq p$ . Since  $T \otimes_A M_q \cong T$  in  $\underline{\text{mod}}\text{-}A$ , we implies that  $\sigma$  is a permutation of  $\{1, \dots, m\}$  and  $T_i \otimes_A M_q \cong T_{\sigma(i)}$  for each  $1 \leq i \leq m$ . By Lemma 2.1,  $-\otimes_A M_q$  induces functors  $\underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\sigma(i)}$  for each  $1 \leq i \leq m$ . Since  $X \otimes_A M_q \cong X[-q]$  in  $\underline{\text{mod}}\text{-}A$  for every  $X \in T^\perp$  and since  $\underline{\text{mod}}\text{-}A_i$  is contained in  $T^\perp$  for each  $m+1 \leq i \leq p$ ,  $-\otimes_A M_q$  induces functors  $\underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_i$  for each  $m+1 \leq i \leq p$ .

Let  $\tau$  be a permutation of  $\{1, \dots, p\}$  such that  $\tau(i) = \sigma(i)$  for  $1 \leq i \leq m$  and  $\tau(i) = i$  for  $m+1 \leq i \leq p$ . Since  $-\otimes_A M_q$  induces functors  $\underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\tau(i)}$  for each  $1 \leq i \leq p$ , to show  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  is a triangulated equivalence, it suffices to show each  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\tau(i)}$  is a triangulated equivalence for each  $1 \leq i \leq p$ .

Let  $1 = \sum_{i=1}^r e_i$ , where  $e_i \in A_i$ . For each  $1 \leq i \leq p$ ,  $-\otimes_A M_q$  is natural isomorphic to  $-\otimes_A e_i M_q$  as functors from  $\underline{\text{mod}}\text{-}A_i$  to  $\underline{\text{mod}}\text{-}A_{\tau(i)}$ . For each  $X \in \underline{\text{mod}}\text{-}A_i$ ,  $X \otimes_A e_i M_q \cong \bigoplus_{j=1}^p (X \otimes_A e_i M_q e_j)$  in  $\underline{\text{mod}}\text{-}A$ . Since  $X \otimes_A e_i M_q \in \underline{\text{mod}}\text{-}A_{\tau(i)}$ ,  $X \otimes_A e_i M_q e_j = 0$  in  $\underline{\text{mod}}\text{-}A_j$  for  $j \neq \tau(i)$ . Then  $-\otimes_A M_q$  is natural isomorphic to  $-\otimes_A e_i M_q e_{\tau(i)}$  as functors from  $\underline{\text{mod}}\text{-}A_i$  to  $\underline{\text{mod}}\text{-}A_{\tau(i)}$  for each  $1 \leq i \leq p$ . Since  $e_i M_q e_{\tau(i)}$  is a summand of  $e_i M_i$  as left  $A_i$ -module, and since  $e_i M_i$  is projective as a left  $A_i$ -module, so is  $e_i M_q e_{\tau(i)}$ . Similarly,  $e_i M_q e_{\tau(i)}$  is also projective as a right  $A_{\tau(i)}$ -module. Therefore  $e_i M_q e_{\tau(i)}$  is a left-right projective  $A_i$ - $A_{\tau(i)}$ -bimodule. Since both  $A_i$  and  $A_{\tau(i)}$  are symmetric,  $-\otimes_A D(e_i M_q e_{\tau(i)}) : \underline{\text{mod}}\text{-}A_{\tau(i)} \rightarrow \underline{\text{mod}}\text{-}A_i$  is both the left and the right adjoint of  $-\otimes_A e_i M_q e_{\tau(i)} : \underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\tau(i)}$ . Since  $-\otimes_A e_i M_q e_{\tau(i)} : \underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\tau(i)}$  is fully faithful,  $\underline{\text{mod}}\text{-}A_i$  is nonzero, and  $\underline{\text{mod}}\text{-}A_{\tau(i)}$  is indecomposable as a triangulated category, it follows from Proposition 2.6 that  $-\otimes_A e_i M_q e_{\tau(i)} : \underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\tau(i)}$  is a triangulated equivalence. Therefore  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A_i \rightarrow \underline{\text{mod}}\text{-}A_{\tau(i)}$  is a triangulated equivalence.  $\square$

#### 4. A VARIATION OF THE CONSTRUCTION IN PREVIOUS SECTION

There exist some examples of stable equivalences (cf. Subsection 6.1) which do not satisfies Assumptions 1 in last section, however if we modify some conditions, we may obtain a similar proposition, which will include these examples.

In this section, we make the following

**Assumption 2:** Let  $k$  be a field,  $A$  be a symmetric  $k$ -algebra,  $R$  be a non-semisimple symmetric subalgebra of  $A$  such that  $A_R$  is projective. Let  $B$  be another subalgebra of  $A$ , such that the following conditions hold:

(a')  $(\text{rad}R)B = B(\text{rad}R)$ ;

(b)  $B \otimes_k (R/\text{rad}R) \xrightarrow{\phi} A/(\text{rad}R)A$ ,  $b \otimes \bar{1} \mapsto \bar{b}$  is an isomorphism in  $\underline{\text{mod}}\text{-}R$ ;

(c)  $B$  has a periodic free  $B^e$ -resolution

$$0 \rightarrow B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \rightarrow 0;$$

(d) The image  $x$  of  $\delta_q(1)$  in  $(A \otimes_R A)^{m_{q-1}}$  satisfies  $rx = xr$  for all  $r \in R$ ;

(e) There exists a complex

$(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} (A \otimes_R A)^{m_{q-2}} \xrightarrow{d_{q-2}} \dots \rightarrow (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow 0;$   
 of  $A^e$ -modules such that the diagram

$$\begin{array}{ccccccccccc}
 (B \otimes_k B)^{m_{q-1}} & \xrightarrow{\delta_{q-1}} & (B \otimes_k B)^{m_{q-2}} & \xrightarrow{\delta_{q-2}} & \dots & \longrightarrow & (B \otimes_k B)^{m_1} & \xrightarrow{\delta_1} & (B \otimes_k B)^{m_0} & \xrightarrow{\delta_0} & B & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 (A \otimes_R A)^{m_{q-1}} & \xrightarrow{d_{q-1}} & (A \otimes_R A)^{m_{q-2}} & \xrightarrow{d_{q-2}} & \dots & \longrightarrow & (A \otimes_R A)^{m_1} & \xrightarrow{d_1} & (A \otimes_R A)^{m_0} & \xrightarrow{d_0} & A & \longrightarrow & 0
 \end{array}$$

is commutative, where the vertical morphisms are the obvious morphisms.

Note that the condition (a') is a generalization of (a) in Assumption 1, the conditions (b) and (c) are the same as in Assumption 1, and the conditions (d) and (e) are new. Clearly, if the triple  $(A, R, B)$  satisfies Assumption 1, then it also satisfies Assumption 2.

Similar to Lemma 3.4, there exist triangles  $M_1 \xrightarrow{i_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A \rightarrow$ ,  $M_2 \xrightarrow{i_2} (A \otimes_R A)^{m_1} \xrightarrow{f_1} M_1 \rightarrow$ ,  $\dots$ ,  $M_q \xrightarrow{i_q} (A \otimes_R A)^{m_{q-1}} \xrightarrow{f_{q-1}} M_{q-1} \rightarrow$  of  $\underline{\text{lrp}}(A)$  such that  $i_p f_p = d_p$  for  $1 \leq p \leq q-1$ . We have following proposition, which is an analogy of Theorem 3.5.

**Theorem 4.1.** *Let  $(A, R, B)$  be the triple that satisfies Assumption 2. If  $M_q$  is the  $A$ - $A$ -bimodule defined above, then  $-\otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  is a stable auto-equivalence of  $A$ .*

*Proof.* Since  $(\text{rad}R)B = B(\text{rad}R)$ ,  $T_A = A/(\text{rad}R)A$  and  $\Omega_A T = (\text{rad}R)A$  becomes  $B$ - $A$ -bimodules. The proof is similar to the proof of Theorem 3.5. The only difficulty is to show the diagrams in Figure 3 and Figure 6 are commutative.

*To show that the diagrams in Figure 3 are commutative.*

Since the image  $x$  of  $\delta_q(1)$  in  $(A \otimes_R A)^{m_{q-1}}$  satisfies  $rx = xr$  for all  $r \in R$ , we have  $\sum_l r b_{il} \otimes b'_{il} = \sum_l b_{il} \otimes b'_{il} r$  in  $A \otimes_R A$  for all  $r \in R$ . Therefore  $\sum_l \overline{r b_{il}} \otimes b'_{il} = \sum_l \overline{b_{il}} \otimes b'_{il} r$  in  $T \otimes_R A$  for all  $r \in R$ . Moreover, since  $\delta_q^i$  is a  $B^e$ -homomorphism,  $\sum_l b b_{il} \otimes b'_{il} = \sum_l b_{il} \otimes b'_{il} b$  in  $B \otimes_k B$  for all  $b \in B$ , and therefore  $\sum_l \overline{b b_{il}} \otimes b'_{il} = \sum_l \overline{b_{il}} \otimes b'_{il} b$  in  $T \otimes_R A$  for all  $b \in B$ . We have  $(h \otimes 1)(\phi \otimes 1)(\delta_q^i \otimes 1)(\overline{1}) = \sum_{l,j} \overline{\beta_j r_j b_{il}} \otimes b'_{il} = \sum_j \beta_j \cdot (\sum_l \overline{r_j b_{il}} \otimes b'_{il}) = \sum_j \beta_j \cdot (\sum_l \overline{b_{il}} \otimes b'_{il} r_j) = \sum_j \sum_l (\overline{\beta_j b_{il}} \otimes b'_{il}) \cdot r_j = \sum_j \sum_l (\overline{b_{il}} \otimes b'_{il} \beta_j) \cdot r_j = \sum_j \sum_l \overline{b_{il}} \otimes b'_{il} \beta_j r_j = (\phi \otimes 1)(\delta_q^i \otimes 1)h(\overline{1})$  and the diagram in Figure 3 commutes.

*To show that the diagrams in Figure 6 are commutative.*

For  $r \in \text{rad}R \subseteq (\text{rad}R)A = \Omega_A T$ ,  $(\delta_q^i \otimes 1)(r) = \sum_l b_{il} \otimes b'_{il} r$ . There is a commutative diagram

$$\begin{array}{ccc}
 B \otimes_k (\text{rad}R)A & \xrightarrow{u} & A \otimes_k A \\
 \downarrow \nu \otimes 1 & & \downarrow p \\
 (\text{rad}R)A \otimes_R A & \xrightarrow{v} & A \otimes_R A
 \end{array}$$

in  $\text{mod}\text{-}A$ , where  $u, v, p$  are the obvious morphisms. Since  $\sum_l r b_{il} \otimes b'_{il} = \sum_l b_{il} \otimes b'_{il} r$  in  $A \otimes_R A$ ,  $(pu)(\sum_l b_{il} \otimes b'_{il} r) = \sum_l r b_{il} \otimes b'_{il} = v(\sum_l r b_{il} \otimes b'_{il})$ . Since  $v$  is injective and  $pu = v(\nu \otimes 1)$ ,  $(\nu \otimes 1)(\sum_l b_{il} \otimes b'_{il} r) = \sum_l r b_{il} \otimes b'_{il}$ . Then  $(h \otimes 1)(\nu \otimes 1)(\delta_q^i \otimes 1)(r) = (h \otimes 1)(\sum_l r b_{il} \otimes b'_{il}) = \sum_{l,j} \overline{x_j \gamma_j b_{il}} \otimes b'_{il} = \sum_j x_j \cdot (\sum_l \overline{\gamma_j b_{il}} \otimes b'_{il}) = \sum_j x_j \cdot (\sum_l \overline{b_{il}} \otimes b'_{il} \gamma_j) = \sum_j (\sum_l \overline{x_j b_{il}} \otimes b'_{il}) \cdot \gamma_j = \sum_j (\sum_l \overline{b_{il}} \otimes b'_{il} x_j) \cdot \gamma_j = \sum_{l,j} \overline{b_{il}} \otimes b'_{il} x_j \gamma_j = (\phi \otimes 1)(\delta_q^i \otimes 1)h(r)$ . So the diagram in Figure 6 is commutative.  $\square$

Recall that an  $A$ -module  $X$  is called a relatively  $R$ -projective module if  $X$  is isomorphic to a direct summand of  $X \otimes_R A_A$ . For  $A$ -modules  $X, Y$  with  $Y$  relatively  $R$ -projective, an  $A$ -homomorphism  $f : Y \rightarrow X$  is called a relatively  $R$ -projective cover of  $X$  if any  $A$ -homomorphism

$g : Z \rightarrow X$  with  $Z$  relatively  $R$ -projective factors through  $f$ . This is equivalent to the fact that  $f$  is a split epimorphism as an  $R$ -homomorphism.

**Proposition 4.2.** (Compare to [8, Proposition 6.5]) *Let  $\rho = - \otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  be the stable auto-equivalence of  $A$  in Theorem 4.1. If both  $A, R, B$  are elementary local  $k$ -algebras, then  $\rho(k)$  is isomorphic to  $\Omega_R^q(k)$  up to a summand of a relatively  $R$ -projective module. (Note that  $\Omega_R(X)$  denotes the kernel of some relatively  $R$ -projective cover of  ${}_A X$  and it is determined up to a summand of a relatively  $R$ -projective module.)*

*Proof.* Since  $R/\text{rad}R = k$ , we have an isomorphism  $\phi : B \rightarrow k \otimes_R A, b \mapsto 1 \otimes b$  in  $\underline{\text{mod}}\text{-}R$ , where the  $R$ -module structure of  $B$  is induced from the epimorphism  $R \rightarrow k$ . Applies the functors  $k \otimes_B -$  and  $k \otimes_A -$  to the complex  $0 \rightarrow B \xrightarrow{\delta_q} (B \otimes_k B)^{m_{q-1}} \xrightarrow{\delta_{q-1}} \dots \rightarrow (B \otimes_k B)^{m_1} \xrightarrow{\delta_1} (B \otimes_k B)^{m_0} \xrightarrow{\delta_0} B \rightarrow 0$  and the complex  $(A \otimes_R A)^{m_{q-1}} \xrightarrow{d_{q-1}} \dots \rightarrow (A \otimes_R A)^{m_1} \xrightarrow{d_1} (A \otimes_R A)^{m_0} \xrightarrow{d_0} A$  respectively, we get a commutative diagram in  $\underline{\text{mod}}\text{-}R$ :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & k & \xrightarrow{1 \otimes \delta_q} & B^{m_{q-1}} & \xrightarrow{1 \otimes \delta_{q-1}} & \dots & \longrightarrow & B^{m_1} & \xrightarrow{1 \otimes \delta_1} & B^{m_0} & \xrightarrow{1 \otimes \delta_0} & k & \longrightarrow & 0 . \\ & & & & \downarrow \phi^{m_{q-1}} & & & & \downarrow \phi^{m_1} & & \downarrow \phi^{m_0} & & \parallel & & \\ & & & & (k \otimes_R A)^{m_{q-1}} & \xrightarrow{1 \otimes d_{q-1}} & \dots & \longrightarrow & (k \otimes_R A)^{m_1} & \xrightarrow{1 \otimes d_1} & (k \otimes_R A)^{m_0} & \xrightarrow{1 \otimes d_0} & k & & \end{array}$$

Since the first row of the diagram is split exact as a complex of  $k$ -modules, it is also split exact as a complex of  $R$ -modules. Similar to the argument in Step 1.1 of the proof of Theorem 3.5, we have isomorphisms of split triangles

$$\begin{array}{ccccccc} L_{l+1} & \longrightarrow & B^{m_l} & \longrightarrow & L_l & \longrightarrow & \\ \downarrow & & \downarrow \phi^{m_l} & & \downarrow & & \\ k \otimes_A M_{l+1} & \xrightarrow{1 \otimes i_{l+1}} & (k \otimes_R A)^{m_l} & \xrightarrow{1 \otimes f_l} & k \otimes_A M_l & \longrightarrow & \end{array}$$

in  $\underline{\text{mod}}\text{-}R$  for  $0 \leq l \leq q-1$ , where  $L_0 = L_q = k, M_0 = A, f_0 = d_0$ . Therefore  $1 \otimes f_l : (k \otimes_R A)^{m_l} \rightarrow k \otimes_A M_l$  are split epimorphisms in  $\underline{\text{mod}}\text{-}R$  for  $0 \leq l \leq q-1$ .

For every  $0 \leq l \leq q-1$  and for every  $R$ -module  $X_R$ , we have a commutative diagram

$$\begin{array}{ccc} \underline{\text{Hom}}_A(FX, (k \otimes_R A)^{m_l}) & \xrightarrow{\underline{\text{Hom}}_A(FX, 1 \otimes f_l)} & \underline{\text{Hom}}_A(FX, k \otimes_A M_l) , \\ \downarrow & & \downarrow \\ \underline{\text{Hom}}_R(X, (k \otimes_R A)^{m_l}_R) & \xrightarrow{\underline{\text{Hom}}_R(X, 1 \otimes f_l)} & \underline{\text{Hom}}_R(X, (k \otimes_A M_l)_R) \end{array}$$

where the vertical arrows are isomorphisms. Since  $1 \otimes f_l : (k \otimes_R A)^{m_l} \rightarrow k \otimes_A M_l$  is a split epimorphism in  $\underline{\text{mod}}\text{-}R$ ,  $\underline{\text{Hom}}_R(X, (k \otimes_R A)^{m_l}_R) \rightarrow \underline{\text{Hom}}_R(X, (k \otimes_A M_l)_R)$  is surjective, therefore  $\underline{\text{Hom}}_A(FX, (k \otimes_R A)^{m_l}) \rightarrow \underline{\text{Hom}}_A(FX, k \otimes_A M_l)$  is surjective. Then the morphism  $1 \otimes f_l : (k \otimes_R A)^{m_l} \rightarrow k \otimes_A M_l$  is a right  $F(\underline{\text{mod}}\text{-}R)$ -approximation. It follows that the  $A$ -homomorphism  $(1 \otimes f_l, \pi_l) : (k \otimes_R A)^{m_l} \oplus P_l \rightarrow k \otimes_A M_l$  is a relatively  $R$ -projective cover of  $k \otimes_A M_l$ , where  $\pi_l : P_l \rightarrow k \otimes_A M_l$  is the projective cover of  $k \otimes_A M_l$ . By the triangle  $k \otimes_A M_{l+1} \xrightarrow{1 \otimes i_{l+1}} (k \otimes_R A)^{m_l} \xrightarrow{1 \otimes f_l} k \otimes_A M_l \rightarrow$  in  $\underline{\text{mod}}\text{-}A$ , we see that  $k \otimes_A M_{l+1} \cong \Omega_R(k \otimes_A M_l)$ . Therefore  $\rho(k) = k \otimes_A M_q \cong \Omega_R(k \otimes_A M_{q-1}) \cong \dots \cong \Omega_R^q(k)$ .  $\square$

**Remark 4.3.** *Since the stable auto-equivalence in Theorem 3.5 is a special case of the stable auto-equivalence in Theorem 4.1, it also satisfies Proposition 4.2.*

## 5. ENDO-TRIVIAL MODULES OVER FINITE P-GROUPS

Let  $k$  be a field of characteristic  $p$  with  $p$  prime,  $P$  be a finite  $p$ -group and  $kP$  be its group algebra. A  $kP$ -module  $M$  is called endo-trivial if  $\text{End}_k(M) \cong k \oplus P$  for some projective module  $P$ . Two endo-trivial modules  $M, N$  are said to be equivalent if  $M \oplus P \cong N \oplus Q$  for some projective modules  $P, Q$ . The group  $T(P)$  has elements consisting of equivalence classes  $[M]$  of endo-trivial modules  $M$ . The operation is given by  $[M] + [N] = [M \otimes_k N]$ , see [4, Section 3].

Note that the stable auto-equivalences of Morita type of  $kP$  are precisely induced by endo-trivial modules. The next proposition shows that in most cases, our construction recovers all the stable auto-equivalences of  $kP$  corresponding to endo-trivial modules.

Let  $A = kP$  and  $R = kS, B = kL$  for some subgroups  $S, L$  of  $P$ . Suppose that the triple  $(A, R, B)$  satisfies Assumption 1 of Section 3. Let  $\rho_{S,L} := - \otimes_A M_q : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}A$  be the stable auto-equivalence of  $A$  in Theorem 3.5. Since  $\underline{\text{End}}_A(\rho_{S,L}(k)) \cong \underline{\text{End}}_A(k) \cong k$ , by [2, Theorem 1],  $\rho_{S,L}(k)$  is an endo-trivial module.

**Proposition 5.1.** *Let  $P$  be a finite  $p$ -group which is not generalized quaternion. Then there exist finitely many pairs  $(S_i, L_i)$  of subgroups of  $P$  such that the following conditions hold:*

- (1) *Each pair  $(S_i, L_i)$  gives a triple  $(A, kS_i, kL_i)$  satisfying Assumption 1;*
- (2)  *$T(P)$  is generated by  $[\Omega_{kP}(k)]$  and elements of the form  $[\rho_{S_i, L_i}(k)]$ , where  $\rho_{S_i, L_i}$  is the stable auto-equivalence of  $A = kP$  defined as above.*

In the following, for a subgroup  $H$  of a group  $G$ , we denote by  $N_G(H)$  and  $C_G(H)$  the normalizer and the centralizer of  $H$  in  $G$  respectively.

**Lemma 5.2.** *Let  $G$  be a group,  $H$  be a subgroup of  $G$  of order  $p$  with  $p$  prime. Then for every  $g \in G$ ,*

$$(6) \quad |HgH| = \begin{cases} p, & \text{if } g \in N_G(H); \\ p^2, & \text{otherwise.} \end{cases}$$

*Proof.* If  $g \notin N_G(H)$ , then  $g^{-1}Hg \neq H$ . Since  $|g^{-1}Hg| = |H| = p$ , we have  $|g^{-1}Hg \cap H| = 1$ . Therefore  $|HgH| = |g^{-1}HgH| = \frac{|g^{-1}Hg||H|}{|g^{-1}Hg \cap H|} = p^2$ .  $\square$

**Lemma 5.3.** *Let  $P$  be a finite  $p$ -group and  $H$  be a subgroup of  $P$  order  $p$ , then  $C_P(H) = N_P(H)$ .*

*Proof.* There is a group homomorphism  $\phi : N_P(H) \rightarrow \text{Aut}(H)$  such that  $\phi(g)(h) = ghg^{-1}$  for all  $g \in N_P(H)$  and  $h \in H$ . Moreover, the kernel of  $\phi$  is  $C_P(H)$ . Since  $\text{Aut}(H) \cong \text{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}^\times$ ,  $|\text{Aut}(H)| = p - 1$ . Therefore  $[N_P(H) : C_P(H)]$  divides  $p - 1$ . Since  $[N_P(H) : C_P(H)]$  is a power of  $p$ , it must equal to 1.  $\square$

**Lemma 5.4.** *Let  $G$  be a finite group. If the trivial  $G$ -module  $k$  has a periodic free resolution of periodic  $n$ , then  $kG$  has a periodic free resolution as  $kG$ - $kG$ -bimodule of the same periodic.*

*Proof.* For  $X \in \text{mod}\text{-}kG$ , define a  $kG$ - $kG$ -bimodule structure on  $X \otimes_k kG$  by the formulas  $g \cdot (x \otimes \mu) = x \otimes g\mu$  and  $(x \otimes \mu) \cdot g = xg \otimes \mu g$ . It can be shown that the map  $X \mapsto X \otimes_k kG$  defines a functor  $\Phi$  from  $\text{mod}\text{-}kG$  to  $kG\text{-mod}\text{-}kG$ . Since the trivial  $G$ -module  $k$  has a periodic free resolution, there exists an exact sequence  $0 \rightarrow k \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow k \rightarrow 0$  of  $kG$ -modules, where  $F_0, \dots, F_{n-1}$  are free  $kG$ -modules. Let  $M = kG \otimes_k kG$  be the free  $kG$ - $kG$ -bimodule of rank 1. Then the map  $\Phi(kG) \rightarrow M, g \otimes h \mapsto hg^{-1} \otimes g$  is an isomorphism of  $kG$ - $kG$ -bimodules. So  $\Phi$  sends free  $kG$ -modules to free  $kG$ - $kG$ -bimodules. Applies the functor  $\Phi$  to the exact sequence  $0 \rightarrow k \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow k \rightarrow 0$ , we get an exact sequence  $0 \rightarrow \Phi(k) \rightarrow \Phi(F_{n-1}) \rightarrow \cdots \rightarrow \Phi(F_1) \rightarrow \Phi(F_0) \rightarrow \Phi(k) \rightarrow 0$  of  $kG$ - $kG$ -bimodules with  $\Phi(F_0), \dots, \Phi(F_{n-1})$  free. Note that  $\Phi(k) \cong kG$  as  $kG$ - $kG$ -bimodules.  $\square$

**Proof of Proposition 5.1.** *Case 1: Assume that  $P$  is a finite  $p$ -group having a maximal elementary abelian subgroup of rank 2.*

*Case 1.1:  $P$  is not semi-dihedral.*

By [4, Theorem 7.1],  $T(P)$  is a free abelian group generated by the classes of the modules  $\Omega_{kP}(k)$ ,  $N_2, \dots, N_r$ , where  $r$  is the number of conjugacy classes of connected components of the poset of all elementary abelian subgroups of  $P$  of rank at least 2 and the  $N_i$  are defined as follows. For  $2 \leq i \leq r$ , let  $S_i$  be the subgroups of  $P$  of order  $p$  in [4, Lemma 2.2(b)] with  $C_P(S_i) = S_i \times L_i$ , where  $L_i$  either cyclic or generalized quaternion. Let  $M_i = \Omega_{kP}^{-1}(k) \otimes_k \Omega_{P/S_i}(k)$ , where  $\Omega_{P/S_i}(k)$  denotes the kernel of a relatively  $kS_i$ -projective cover of the trivial  $kP$ -module  $k$ . Define

$$(7) \quad N_i = \begin{cases} \Gamma(M_i^{\otimes 2}), & \text{if } L_i \text{ is cyclic of order } \geq 3; \\ M_i, & \text{if } p = 2 \text{ and } L_i \text{ is cyclic of order } 2; \\ \Gamma(M_i^{\otimes 4}), & \text{if } p = 2 \text{ and } L_i \text{ is generalized quaternion,} \end{cases}$$

where  $\Gamma(M)$  denotes the sum of all the indecomposable summands of  $M$  having vertex  $P$ . Let  $A = kP$  and  $R_i = kS_i$ ,  $B_i = kL_i$  for  $2 \leq i \leq r$ . Note that  $R_i/\text{rad}R_i \cong k$ . Since  $L_i \leq C_P(S_i)$ , we have  $br = rb$  for any  $b \in B_i$  and  $r \in R_i$ . Let  $h_1, \dots, h_q$  be a complete set of double coset representatives for  $S_i$  in  $P$  which not belong to  $N_P(S_i)$ . Since  $P$  is a  $p$ -group and  $S_i$  is a subgroup of  $P$  of order  $p$ , by Lemma 5.3,  $N_P(S_i) = C_P(S_i)$ . Therefore  $P$  is a disjoint union of double cosets  $S_i g S_i = g S_i$  with  $g \in L_i$  and double cosets  $S_i h_n S_i$  with  $1 \leq n \leq q$ . By Lemma 5.2,  $|S_i h_n S_i| = p^2$  for  $1 \leq n \leq q$ , therefore the  $R_i$ - $R_i$ -subbimodule  $kS_i h_n S_i$  of  $A$  is isomorphic to  $R_i \otimes_k R_i$ . We have  $A/(\text{rad}R_i)A \cong (R_i/\text{rad}R_i) \otimes_{R_i} A = k \otimes_{R_i} A \cong \bigoplus_{g \in L_i} k \otimes_{R_i} k g S_i \oplus \bigoplus_{n=1}^q k \otimes_{R_i} k S_i h_n S_i \cong k^{|L_i|} \oplus R_i^q$  as  $R_i$ -modules. Moreover, the  $R_i$ -homomorphism  $\phi_i : B_i \otimes_{R_i} (R_i/\text{rad}R_i) \rightarrow A/(\text{rad}R_i)A$ ,  $b \otimes 1 \mapsto \bar{b}$  is isomorphic to the inclusion morphism  $k^{|L_i|} \rightarrow k^{|L_i|} \oplus R_i^q$ . Therefore  $\phi_i$  is an isomorphism in  $\text{mod-}R_i$ .

Let  $k$  denotes the trivial  $L_i$ -module. When  $L_i$  is cyclic, then  $\Omega_{kL_i}^2(k) \cong k$ . Moreover, when  $L_i$  is cyclic of order 2, then  $\Omega_{kL_i}(k) \cong k$ . When  $L_i$  is generalized quaternion, by [6, Proposition 3.16],  $\Omega_{kL_i}^4(k) \cong k$ . Since  $B_i = kL_i$  is local, the periodic projective resolution of  $k$  is also a periodic free resolution. By Lemma 5.4,  $B_i$  has a periodic free resolution as a  $B_i$ - $B_i$ -bimodule of periodic  $n_i$ , where

$$(8) \quad n_i = \begin{cases} 2, & \text{if } L_i \text{ is cyclic of order } \geq 3; \\ 1, & \text{if } p = 2 \text{ and } L_i \text{ is cyclic of order } 2; \\ 4, & \text{if } p = 2 \text{ and } L_i \text{ is generalized quaternion.} \end{cases}$$

Therefore the triple  $(A, R_i, B_i)$  satisfies Assumption 1 in Section 3. By Proposition 4.2 and Remark 4.3,  $\rho_{S_i, L_i}(k) \cong \Omega_{P/S_i}^{n_i}(k)$ . Since  $\Omega_{P/S_i}(k)^{\otimes n_i} \oplus V \cong \Omega_{P/S_i}^{n_i}(k) \oplus W$  for some relatively  $kS_i$ -projective modules  $V, W$ ,

$$(9) \quad N_i = \begin{cases} \Gamma(\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k)), & \text{if } L_i \text{ is cyclic of order } \geq 3, \\ & \text{or } p = 2 \text{ and } L_i \text{ is generalized quaternion;} \\ \Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k), & \text{if } p = 2 \text{ and } L_i \text{ is cyclic of order } 2. \end{cases}$$

When  $L_i$  is cyclic of order  $\geq 3$ , or when  $p = 2$  and  $L_i$  is generalized quaternion, since both  $\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k)$  and  $\Gamma(\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k))$  are endo-trivial modules,  $\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k) \cong \Gamma(\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k)) \oplus V$  for some projective  $kP$ -module  $V$ . Therefore  $[\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k)] = [\Gamma(\Omega_{kP}^{-n_i}(k) \otimes_k \rho_{S_i, L_i}(k))]$  in  $T(P)$ . So  $T(P)$  is generated by  $[\Omega_{kP}(k)]$  and  $[\rho_{S_i, L_i}(k)]$  for  $2 \leq i \leq r$ .

*Case 1.2:  $P$  is semi-dihedral.*

The semi-dihedral of order  $2^n$  ( $n \geq 4$ ) is given by  $SD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = x^{2^{n-2}-1} \rangle$ . Let  $S = \langle y \rangle$  be a subgroup of  $P = SD_{2^n}$ . Then  $C_P(S) = S \times S'$ , where  $S' = \langle x^{2^{n-2}} \rangle$ . Let  $A = kP$ ,  $R = kS$ ,  $B = kS'$ . Similar to Case 1.1, the triple  $(A, R, B)$  satisfies Assumption 1. Since  $B$  has a free resolution of periodic 1 as a  $B$ - $B$ -bimodule, by Proposition 4.2 and Remark 4.3,  $\rho_{S, S'}(k) \cong \Omega_{P/S}(k)$ , which is exactly the module  $L$  defined in [3, Section 7]. By [3, Theorem 7.1],



$T(P)$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ , generated by  $[\Omega_{kP}(k)]$  and  $[\Omega_{kP}(L)]$ , where the element  $[\Omega_{kP}(L)]$  has order 2. Therefore  $[\Omega_{kP}(k)]$  together with  $[\rho_{S,L}(k)]$  generates  $T(P)$ .

*Case 2: Assume that  $P$  is a finite  $p$ -group which do not have a maximal elementary abelian subgroup of rank 2.*

Since  $P$  is not generalized quaternion, either  $P$  is cyclic or every maximal elementary abelian subgroup of  $P$  has rank at least 3 (cf. [4, Introduction]). By [7, Corollary 8.8] and [5, Corollary 1.3],  $T(P)$  is generated by  $[\Omega_{kP}(k)]$ . So the conclusion also holds in this case.  $\square$

**Remark 5.5.** *An example of  $p$ -group which has a maximal elementary abelian subgroup of rank 2 and which is not semi-dihedral is the dihedral group  $D_8 = \langle x, y \mid x^4 = y^2 = 1, yxy = x^{-1} \rangle$  of order 8, where  $E = \{1, x^2, y, x^2y\}$  is a maximal elementary abelian subgroup of  $Q_8$  of rank 2. An example of  $p$ -group whose maximal elementary abelian subgroups have rank at least 3 is  $D_8 * D_8 = (D_8 \times D_8) / \langle (x^2, x^2) \rangle$ , see [4, Section 6].*

**Remark 5.6.** *For every positive integer  $n \geq 2$ , the generalized quaternion group  $Q_{4n}$  of order  $4n$  is defined by the presentation  $\langle x, y \mid x^{2n} = 1, y^2 = x^n, yxy^{-1} = x^{-1} \rangle$ . When  $n = 2$  it is the usual quaternion group. The generalized quaternion group  $Q_{4n}$  is a  $p$ -group if and only if  $n$  is a power of 2. The reason why we exclude generalized quaternion groups in Proposition 5.1 is that the endo-trivial module  $L$  constructed in [3, Section 6] may not be a relative syzygy of the trivial  $kP$ -module.*

## 6. EXAMPLES IN NON-LOCAL CASE

6.1. In this subsection, let  $G$  be a finite group and  $N, H$  be subgroups of  $G$  such that  $N_G(N) = N \rtimes H$  and  $|NgN| = |N|^2$  for any  $g \in G - N_G(N)$ . Let  $k$  be a field whose characteristic divides  $|N|$ , and let  $A = kG, R = kN, B = kH$ . Assume that the trivial  $kH$ -module  $k$  has a periodic free resolution.

**Proposition 6.1.** *The triple  $(A, R, B)$  as above satisfies Assumption 2 of Section 4, so it defines a stable auto-equivalence of  $A$  by Theorem 4.1.*

*Proof.* Since  $N$  is a subgroup of  $G$ ,  $A_R$  is projective. We need to check that the triple  $(A, R, B)$  satisfies the assumptions (a') to (e) at the beginning of Section 4.

Suppose the semidirect product  $N \rtimes H$  is defined by the group homomorphism  $\eta : H \rightarrow \text{Aut}(N)$ . For any  $\sum_{n \in N} \lambda_n n \in \text{rad}R$  and  $h \in H$ , the group automorphism  $\eta(h) : N \rightarrow N$  induces an automorphism  $\eta_h$  of  $R$ , and  $h(\sum_{n \in N} \lambda_n n) = \sum_{n \in N} \lambda_n \eta(h)(n)h = \eta_h(\sum_{n \in N} \lambda_n n)h$ . Since  $\eta_h(\text{rad}R) = \text{rad}R$ ,  $\eta_h(\sum_{n \in N} \lambda_n n) \in \text{rad}R$ . Therefore  $B(\text{rad}R) \subseteq (\text{rad}R)B$ . Similarly, it can be shown that  $(\text{rad}R)B \subseteq B(\text{rad}R)$ . So the assumption (a') holds.

The  $R$ -homomorphism  $\phi$  is given by  $kH \otimes_k (kN/\text{rad}kN) \rightarrow (kN/\text{rad}kN) \otimes_{kN} kG, h \otimes \bar{n} \mapsto \bar{1} \otimes hn$ . We have  $(kN/\text{rad}kN) \otimes_{kN} kG \cong (kN/\text{rad}kN) \otimes_{kN} kN_G(N) \oplus (\oplus_{i=1}^t (kN/\text{rad}kN) \otimes_{kN} kNg_iN)$  as  $R$ -modules, where each  $g_i$  belongs to  $G - N_G(N)$  such that  $G - N_G(N)$  is a disjoint union of all  $Ng_iN$ s. Since  $|Ng_iN| = |N|^2$ ,  $kNg_iN \cong R \otimes_k R$  as  $R^e$ -modules, so each  $(kN/\text{rad}kN) \otimes_{kN} kNg_iN$  is a projective  $R$ -module. Moreover, the image of  $\phi$  is  $(kN/\text{rad}kN) \otimes_{kN} kN_G(N)$ . Since  $(kN/\text{rad}kN) \otimes_{kN} kN_G(N) \cong \oplus_{h \in H} (kN/\text{rad}kN) \otimes_{kN} kNh$ ,  $\dim_k((kN/\text{rad}kN) \otimes_{kN} kN_G(N)) = |H| \dim_k(kN/\text{rad}kN) = \dim_k(kH \otimes_k (kN/\text{rad}kN))$ , so  $\phi$  induces an  $R$ -isomorphism from  $kH \otimes_k (kN/\text{rad}kN)$  to  $(kN/\text{rad}kN) \otimes_{kN} kN_G(N)$ . Therefore  $\phi$  is an isomorphism in  $\underline{\text{mod}}\text{-}R$  and the assumption (b) holds.

Since the trivial  $kH$ -module  $k$  has a periodic free resolution, by Lemma 5.4 the  $kH$ - $kH$ -bimodule  $kH$  also has a periodic free resolution. Then the assumption (c) holds. Assume the periodic free resolution of the trivial  $kH$ -module  $k$  is given by the exact sequence  $0 \rightarrow k \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow k \rightarrow 0$ , where  $F_0, \dots, F_{n-1}$  are free  $kG$ -modules. Then the exact sequence  $0 \rightarrow \Phi(k) \rightarrow \Phi(F_{n-1}) \rightarrow \cdots \rightarrow \Phi(F_1) \rightarrow \Phi(F_0) \rightarrow \Phi(k) \rightarrow 0$  gives a periodic free resolution of the  $kH$ - $kH$ -bimodule  $kH$ , where  $\Phi = - \otimes_k kH$  is the functor defined in the proof of Lemma 5.4.

Let  $f : kH \rightarrow kH$ ,  $1 \mapsto \sum_{h \in H} \lambda_h h$  be a morphism in  $\text{mod-}kH$ , then  $\Phi(f)$  is isomorphic to the  $kH$ - $kH$ -homomorphism  $\tilde{f} : kH \otimes_k kH \rightarrow kH \otimes_k kH$ ,  $1 \otimes 1 \mapsto \sum_{h \in H} \lambda_h h^{-1} \otimes h$ , by the isomorphism  $\Phi(kH) \rightarrow kH \otimes_k kH$ ,  $g \otimes h \mapsto hg^{-1} \otimes g$ . Since for any  $n \in N$ ,  $(\sum_{h \in H} \lambda_h h^{-1} \otimes h)n = \sum_{h \in H} \lambda_h h^{-1} \otimes \eta(h)(n)h = \sum_{h \in H} \lambda_h h^{-1} \eta(h)(n) \otimes h = \sum_{h \in H} \lambda_h \eta(h^{-1})(\eta(h)(n))h^{-1} \otimes h = n(\sum_{h \in H} \lambda_h h^{-1} \otimes h)$  in  $kG \otimes_{kN} kG$ , there is a  $kG$ - $kG$ -homomorphism  $\alpha : kG \otimes_{kN} kG \rightarrow kG \otimes_{kN} kG$  such the diagram

$$\begin{array}{ccc} kH \otimes_k kH & \xrightarrow{\tilde{f}} & kH \otimes_k kH \\ \downarrow & & \downarrow \\ kG \otimes_{kN} kG & \xrightarrow{\alpha} & kG \otimes_{kN} kG \end{array}$$

commutes, where the vertical morphisms are the obvious one. Moreover, for any  $kH$ -homomorphism  $g : kH \rightarrow k$ ,  $1 \mapsto \lambda$ ,  $\Phi(f)$  is isomorphic to the  $kH$ - $kH$ -homomorphism  $\tilde{g} : kH \otimes_k kH \rightarrow kH$ ,  $1 \otimes 1 \mapsto \lambda$ . Therefore there is a  $kG$ - $kG$ -homomorphism  $\beta : kG \otimes_{kN} kG \rightarrow kG$  such the diagram

$$\begin{array}{ccc} kH \otimes_k kH & \xrightarrow{\tilde{g}} & kH \\ \downarrow & & \parallel \\ kG \otimes_{kN} kG & \xrightarrow{\beta} & kG \end{array}$$

commutes. Since each  $F_i$  is a free  $kH$ -module, the assumption (e) holds.

Each  $kH$ -homomorphism  $u : k \rightarrow kH$  maps 1 to some  $\lambda(\sum_{h \in H} h)$ , where  $\lambda \in k$ . Then  $\Phi(u)$  is isomorphic to the  $kH$ - $kH$ -homomorphism  $\tilde{u} : kH \rightarrow kH \otimes_k kH$ ,  $1 \mapsto \lambda(\sum_{h \in H} h^{-1} \otimes h)$ . Since for every  $n \in N$ ,  $(h^{-1} \otimes h)n = h^{-1} \otimes \eta(h)(n)h = h^{-1} \eta(h)(n) \otimes h = \eta(h^{-1})(\eta(h)(n))h^{-1} \otimes h = n(h^{-1} \otimes h)$  in  $kG \otimes_{kN} kG$ , the image  $x$  of  $\tilde{u}(1)$  in  $kG \otimes_{kN} kG$  satisfies  $rx = xr$  for every  $r \in R = kN$ . Therefore the assumption (d) holds.  $\square$

Suppose the trivial  $kH$ -module  $k$  has a periodic free resolution of periodic  $n$ , then by Lemma 5.4,  $B = kH$  also has a periodic free resolution of periodic  $n$ . Let  $\rho$  be the stable auto-equivalence of  $A = kG$  in Theorem 4.1 with respect to this periodic free resolution of  $B$ . Similar to Proposition 4.2, we have following proposition.

**Proposition 6.2.** *For the trivial  $kG$ -module  $k$ ,  $\rho(k) \cong \Omega_{G/N}^n(k)$ , where  $\Omega_{G/N}(M)$  denotes the kernel of some relatively  $kN$ -projective cover of  $M$ .*

*Proof.* Consider  $B = kH$  as a module over  $R = kN$ , where each  $n \in N$  acts trivially on  $B$ . Let  $\psi : B \rightarrow k \otimes_R A$ ,  $h \mapsto 1 \otimes h$  be a  $k$ -linear homomorphism, where  $k$  denotes the trivial  $R$ -module. Since for any  $h \in H$  and  $n \in N$ ,  $(1 \otimes h)n = 1 \otimes hn = 1 \otimes \eta(h)(n)h = 1 \otimes h$  in  $k \otimes_R A$ ,  $\psi$  is also an  $R$ -homomorphism. Since  $k \otimes_R A \cong k \otimes_{kN} kN_G(N) \oplus (\bigoplus_{i=1}^t k \otimes_{kN} kNg_iN)$  as  $R$ -modules, where each  $g_i$  belongs to  $G - N_G(N)$  such that  $G - N_G(N)$  is a disjoint union of all  $Ng_iNs$ ,  $\psi$  is an isomorphism in  $\text{mod-}R$ . The rest of the proof is similar to that of Proposition 4.2.  $\square$

**Example 6.3.** *Let  $k$  be a field of characteristic 2 which contains cubic roots of unity,  $G = S_4$  be the symmetric group on 4 letters, and  $A = kG$ . Let  $e_1 = 1 + (123) + (132)$ ,  $e_2 = 1 + \omega(123) + \omega^2(132)$ ,  $e_3 = 1 + \omega^2(123) + \omega(132)$  be three idempotents of  $A$ , where  $\omega \in k$  is a cubic root of unity. Then  $1 = e_1 + e_2 + e_3$  is a decomposition of 1 into primitive orthogonal idempotents. The basic algebra of  $A$  is  $\Lambda = fAf$ , where  $f = e_1 + e_2$ . It can be shown that  $\Lambda$  is given by the quiver*

$$\delta \circlearrowleft 1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2 \circlearrowright \gamma$$

with relations  $\alpha\beta = \delta^2 = \gamma\alpha = \gamma\beta = 0$  and  $\alpha\delta\beta = \gamma^2$ .

(i) Let  $S = \langle(12)\rangle$  be a subgroup of  $G$ , then  $N_G(S) = C_G(S) = S \times L$ , where  $L = \langle(34)\rangle$ . By Lemma 5.2,  $|SgS| = |S|^2$  for any  $g \in G - N_G(S)$ . Let  $R = kS$ ,  $B = kL$ . Since the trivial

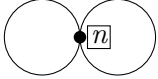
$B$ -module  $k$  satisfies  $\Omega_B(k) \cong k$ , by Proposition 6.1, the triple  $(A, R, B)$  defines a stable auto-equivalence  $\rho$  of  $A$ . Moreover,  $\rho$  is induced by the functor  $-\otimes_A K$ , where  $K$  is the kernel of the  $A^e$ -homomorphism  $A \otimes_R A \rightarrow A$ , which is given by multiplication. Since  $\Lambda$  is Morita equivalent to  $A$ , the stable auto-equivalence  $\rho$  induces a stable auto-equivalence  $\mu$  of  $\Lambda$ . It can be shown that  $\mu(1) = \begin{smallmatrix} 1 & & \\ & 1 & 2 \\ & & 2 \end{smallmatrix}$  and  $\mu(2) = \begin{smallmatrix} 1 & 2 & \\ & 1 & 2 \\ & & 2 \end{smallmatrix}$ .

(ii) Let  $N = \{(1), (12), (34), (12)(34)\}$  be a subgroup of  $G$ , then  $N_G(N) = \{(1), (12), (34), (12)(34), (13)(24), (1324), (14)(23), (1423)\} = N \rtimes H$ , where  $H = \langle (13)(24) \rangle$ . A calculation shows that  $G = N_G(N) \cup N(13)N$ , where  $|N(13)N| = |N|^2$ . Let  $R' = kN$ ,  $B' = kH$ . Since the trivial  $B'$ -module  $k$  satisfies  $\Omega_{B'}(k) \cong k$ , by Proposition 6.1, the triple  $(A, R', B')$  defines a stable auto-equivalence  $\rho'$  of  $A$ . Moreover,  $\rho'$  is induced by the functor  $-\otimes_A K'$ , where  $K'$  is the kernel of the  $A^e$ -homomorphism  $A \otimes_{R'} A \rightarrow A$ , which is given by multiplication. Let  $\mu'$  be the stable auto-equivalence of  $\Lambda$  induced by  $\rho'$ . It can be shown that  $\mu'(1) = \begin{smallmatrix} 1 & 2 & \\ & 1 & 2 & 1 \\ & & 2 & & 1 \end{smallmatrix}$  and  $\mu'(2) = \begin{smallmatrix} 1 & 2 & 1 \\ & 1 & 2 & 1 \\ & & 2 & & 1 \end{smallmatrix}$ .

(iii) Let  $P = \langle (1324) \rangle$  be a subgroup of  $G$ , then  $N_G(P) = \{(1), (12), (34), (12)(34), (13)(24), (1324), (14)(23), (1423)\} = P \rtimes Q$ , where  $Q = \langle (12) \rangle$ . We have  $G = N_G(P) \cup P(13)P$ , where  $|P(13)P| = |P|^2$ . Let  $R'' = kP$ ,  $B'' = kQ$ . Similar to case (2) above, the triple  $(A, R'', B'')$  defines a stable auto-equivalence  $\rho''$  of  $A$ , which is induced by the functor  $-\otimes_A K''$ , where  $K''$  is the kernel of the  $A^e$ -homomorphism  $A \otimes_{R''} A \rightarrow A$ . Let  $\mu''$  be the stable auto-equivalence of  $\Lambda$  induced by  $\rho''$ , then  $\mu''(1) = \begin{smallmatrix} 1 & 2 & \\ & 1 & 2 \end{smallmatrix}$  and  $\mu''(2) = \begin{smallmatrix} 1 & 2 & \\ & 1 & 2 \end{smallmatrix}$ .

$\Omega_\Lambda^{-2}(2) = \begin{smallmatrix} 1 & 2 & \\ & 1 & 2 & 1 \\ & & 2 & & 1 \end{smallmatrix}$ , which is same as Case (ii).

6.2. In this subsection, we consider a class of non-local Brauer graph algebras and construct stable auto-equivalences over them. In general, such stable auto-equivalences are not induced by derived auto-equivalences.

**Example 6.4.** Let  $A$  be the Brauer graph algebra given by the Brauer graph , where  $n \geq 1$ . Then  $A$  is given by the quiver

$$\alpha \circlearrowleft 1 \begin{matrix} \xrightarrow{\gamma} \\ \xleftarrow{\delta} \end{matrix} 2 \circlearrowright \beta$$

with relations  $(\alpha\delta\beta\gamma)^n = (\delta\beta\gamma\alpha)^n$ ,  $(\beta\gamma\alpha\delta)^n = (\gamma\alpha\delta\beta)^n$ ,  $\alpha^2 = \delta\gamma = \beta^2 = \gamma\delta = 0$ . Let  $R = k[\alpha] \times k[\beta]$ ,  $B = k[x]$  be two subalgebras of  $A$ , where  $x = (\delta\beta\gamma\alpha)^{n-1}\delta\beta\gamma + (\gamma\alpha\delta\beta)^{n-1}\gamma\alpha\delta$ . The triple  $(A, R, B)$  satisfies Assumption 1 in Section 3.

(1) If  $\text{char}(k) = 2$ , then  $B$  has a periodic free  $B^e$ -resolution  $0 \rightarrow B \rightarrow B \otimes_k B \xrightarrow{\mu} B \rightarrow 0$  of period 1, where  $\mu$  is the map given by multiplication. According to Theorem 3.5, the functor  $-\otimes_A K$  induces a stable auto-equivalence of  $A$ , where  $K$  is the kernel of the  $A^e$ -homomorphism  $A \otimes_R A \rightarrow A$  given by multiplication. Let  $S_i$  be the simple  $A$ -module which corresponds to the vertex  $i$ . A calculation shows that  $S_1 \otimes_A K \cong \text{rad}(e_1 A / \alpha A)$  and  $S_2 \otimes_A K \cong \text{rad}(e_2 A / \beta A)$ . Note that neither  $S_1 \otimes_A K$  nor  $S_2 \otimes_A K$  belongs to the  $\Omega_A$ -orbit of any simple  $A$ -module.

When  $n = 2$ , we have  $e_1A = \begin{matrix} 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{matrix}$  and  $e_2A = \begin{matrix} 2 \\ 2 & 1 \\ 1 & 2 \\ 2 & 2 \\ 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$ . Let  $X = S_1 \otimes_A K \cong \text{rad}(e_1A/\alpha A)$ ,

then  $X$  is the uniserial  $A$ -module  $\begin{matrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{matrix}$ . Let  $\Lambda = \text{End}_A(A \oplus S_1)$  and  $\Gamma = \text{End}_A(A \oplus X)$ . By the

construction in [11, Corollary 1.2], there is a stable equivalence of Morita type between  $\Lambda$  and  $\Gamma$ . The Cartan matrix  $C_\Lambda$  of  $\Lambda$  is given by

$$C_\Lambda = \begin{pmatrix} 8 & 8 & 1 \\ 8 & 8 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and the Cartan matrix  $C_\Gamma$  of  $\Gamma$  is given by

$$C_\Gamma = \begin{pmatrix} 8 & 8 & 3 \\ 8 & 8 & 4 \\ 3 & 4 & 2 \end{pmatrix}.$$

A calculation shows that  $C_\Lambda$  is congruent to

$$M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

over integers and  $C_\Gamma$  is congruent to

$$N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 8 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

over integers. If a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is congruent to  $N$  over integers, then it can be shown that  $a_{11}$  is even. Therefore the matrices  $M$  and  $N$  are not congruent over integers. So the matrices  $C_\Lambda$  and  $C_\Gamma$  are also not congruent over integers, which implies that  $\Lambda$  and  $\Gamma$  are not derived equivalent. According to [10, Proposition 6.1], the stable auto-equivalence of  $A$  induced by the functor  $- \otimes_A K$  cannot be lifted to a derived auto-equivalence.

(2) If  $k$  is a field of arbitrary characteristic, then  $B$  has a periodic free  $B^e$ -resolution  $0 \rightarrow B \rightarrow B \otimes_k B \xrightarrow{f} B \otimes_k B \xrightarrow{\mu} B \rightarrow 0$  of period 2, where  $f(1 \otimes 1) = 1 \otimes x - x \otimes 1$  and  $\mu$  is the map given by multiplication. According to Theorem 3.5, the functor  $- \otimes_A K'$  induces a stable auto-equivalence of  $A$ , where  $K'$  is given by the short exact sequence  $0 \rightarrow K' \rightarrow (A \otimes_R A) \oplus P \xrightarrow{(h_1, h_2)} K \rightarrow 0$  of  $A^e$ -modules. Here  $K$  is the kernel of the  $A^e$ -homomorphism  $A \otimes_R A \rightarrow A$

given by multiplication,  $h_1(1 \otimes 1) = 1 \otimes x - x \otimes 1$ , and  $h_2 : P \rightarrow K$  is the projective cover of  $K$  as an  $A^e$ -module. A calculation shows that  $S_1 \otimes_A K'$  (resp.  $S_2 \otimes_A K'$ ) is isomorphic to the  $A$ -module  $X_1$  (resp.  $X_2$ ) in  $\underline{\text{mod}}\text{-}A$ , where  $X_1$  (resp.  $X_2$ ) is given by the short exact sequence  $0 \rightarrow X_1 \rightarrow (e_1A/\alpha A) \oplus e_2A \xrightarrow{(u_1, u_2)} \text{rad}(e_1A/\alpha A) \rightarrow 0$  (resp. the short exact sequence  $0 \rightarrow X_2 \rightarrow (e_2A/\beta A) \oplus e_1A \xrightarrow{(v_1, v_2)} \text{rad}(e_2A/\beta A) \rightarrow 0$ ), where  $u_1(\bar{e}_1) = \overline{(\delta\beta\gamma\alpha)^{n-1}\delta\beta\gamma}$  (resp.  $v_1(\bar{e}_2) = \overline{(\gamma\alpha\delta\beta)^{n-1}\gamma\alpha\delta}$ ) and  $u_2 : e_2A \rightarrow \text{rad}(e_1A/\alpha A)$  (resp.  $v_2 : e_1A \rightarrow \text{rad}(e_2A/\beta A)$ ) is the projective cover of  $\text{rad}(e_1A/\alpha A)$  (resp.  $\text{rad}(e_2A/\beta A)$ ). Note that neither  $X_1$  nor  $X_2$  belongs to the  $\Omega_A$ -orbit of any simple  $A$ -module.

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