TRANSFER MAPS IN HOCHSCHILD (CO)HOMOLOGY AND APPLICATIONS TO STABLE AND DERIVED INVARIANTS AND TO THE AUSLANDER-REITEN CONJECTURE

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Abstract. Derived equivalences and stable equivalences of Morita type, and new (candidate) invariants thereof, between symmetric algebras will be investigated, using transfer maps as a tool. Close relationships will be established between the new invariants and the validity of the Auslander–Reiten conjecture, which states the invariance of the number of non-projective simple modules under stable equivalence. More precisely, the validity of this conjecture for a given pair of algebras, which are stably equivalent of Morita type, will be characterized in terms of data refining Hochschild homology (via Külshammer ideals) being invariant and also in terms of cyclic homology being invariant. Thus, validity of the Auslander–Reiten conjecture implies a whole set of ring theoretic and cohomological data to be invariant under stable equivalence of Morita type, and hence also under derived equivalence. We shall also prove that the Batalin-Vilkovisky algebra structure of Hochschild cohomology for symmetric algebras is preserved by derived equivalence. The main tools to be developed and used are transfer maps and their properties, in particular a crucial compatibility condition between transfer maps in Hochschild homology and Hochschild cohomology via the duality between them.

1. Introduction

Derived equivalences have been studied and used in representation theory of groups and of algebras since the pioneering work of Happel ([12, 13]), of Rickard ([28, 29, 30]) and of Keller ([15]). In particular, Broué's conjecture ([6]) has been the starting point of a major development in modular representation theory of finite groups, centreing around derived equivalences and their consequences. More generally, Broué ([7]) has introduced stable equivalences of Morita type, which are implied by derived equivalences between symmetric algebras. There are two key sets of problems about derived or stable equivalences. One is concerned with constructing derived or stable equivalences. The other one is about identifying and describing invariants, both of an abstract and of an explicit nature. It is this second complex of problems that we will address in this article.

While derived equivalences have been shown to preserve various cohomological invariants such as Hochschild (co)homology ([30]), K-theory([33, 8]) and cyclic homology([16]), much less is known for stable equivalences of Morita type. On the other hand, stable equivalences of Morita type are both more frequent and more explicit than derived equivalences. We propose to use transfer maps as a tool to set up and to investigate invariants under derived equivalences and under stable equivalences of Morita type. In the first part of this paper we will define the appropriate transfer maps and develop their basic properties, in particular a crucial compatibility condition (Theorem 2.10 and Corollary 2.12). In the second part we shall present some applications of the theory developed in the first part. We then will demonstrate the usefulness of the transfer maps in a general and abstract setup; we will work with symmetric algebras in full generality and with stable

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equivalences of Morita type in general, thus getting consequences for derived equivalences between group algebras, or blocks of group algebras, as well.

The consequences we will derive from properties of transfer maps are threefold. Firstly, we will discuss countable series of potential invariants, which all can be seen as refining Hochschild homology, but which are of a ring theoretical nature. While Hochschild homology in positive degree is known to be invariant under stable equivalences of Morita type ([23]), Hochschild homology in degree zero is an invariant if and only if the two algebras in question have the same number of non-projective simple modules, that is, when the so-called Auslander–Reiten conjecture ([2]) is valid in this situation ([24]). The series of candidate invariants that we are considering have been defined by Külshammer ([18, 19]) in terms of ideals of the centre of an algebra, and thus are closely related to degree zero Hochschild homology. We will show that any of these data is an invariant under stable equivalence of Morita type if and only if all others are so if and only if the Auslander–Reiten conjecture is valid in this case (Corollary 4.6 and Proposition 5.8). The main tool to relate these data for different algebras are, of course, transfer maps.

Secondly, we will look at another potential invariant, cyclic homology. Here, the transfer maps come in via Connes' operator. Cyclic homology is a derived invariant by a result of Keller ([16]). Surprisingly, it turns out that in odd degrees it is also invariant under stable equivalence of Morita type, while in even degrees it is so if and only if the Auslander–Reiten conjecture is valid for the given equivalence (Theorem 9.6).

Thus, our results provide various new interpretations of and a new approach to Auslander–Reiten conjecture (which at present appears to be far beyond reach), to be based on the equivalent versions of the conjecture contained in our results. To demonstrate feasibility of this approach, we give a sufficient criterion for the Auslander–Reiten conjecture to hold true. For two not necessarily symmetric algebras A and B over a field of positive characteristic, the conjecture follows from the existence of two stable equivalences of Morita type, one relating A and B and the other one relating their trivial extension algebras $\mathbb{T}(A)$ and $\mathbb{T}(B)$ (Corollary 8.2).

Thirdly, our results imply extensions and new proofs of various known results, especially on derived categories. We reprove some results of Alexander Zimmermann ([35, 36, 37]) concerning Külshammer ideals using transfer maps. We also prove that the Batalin-Vilkovisky algebra structure ([34][26]), in particular, the Gerstenhaber algebra structure, over Hochschild cohomology of symmetric algebras is preserved by a derived equivalence (Theorems 10.7 and 10.8).

Now we are going to describe the contents of this article in more detail. In Section 2, we recall transfer maps in Hochschild homology defined by Bouc and we introduce transfer maps in Hochschild cohomology for symmetric algebras. We prove the compatibility theorem between transfer maps in Hochschild homology and in Hochschild cohomology in this section. Section 3 studies when transfer maps preserve the product structure over Hochschild cohomology. P-power maps over zero-degree Hochschild homology groups are investigated in Section 4. Section 5 contains the stable version of Külshammer's ζ_n. P-power maps over the centre $Z(A)$, Külshammer's maps κ_n and their stable version are considered in the sixth section. Higher dimensional analogues are presented in the seventh section. Section 8 contains the trivial extension construction. Cyclic homology is studied in the ninth section and stable cyclic homology is introduced. We consider Batalin-Vilkovisky algebra structure in the last section.

2. Transfer maps

Let k be a field of arbitrary characteristic and let A be a finite dimensional k-algebra. In the sequel, $\otimes$ will denote $\otimes_k$. The bar resolution $\mathrm{Bar}_\bullet(A)$ is defined as follows: $\mathrm{Bar}_n(A) = A^{\otimes(n+2)}$ and for $n \geq 1$, the differential is $d' = \sum_{i=0}^{n}(-1)^i d'_i : \mathrm{Bar}_n(A) = A^{\otimes(n+2)} \to \mathrm{Bar}_{n-1}(A) = A^{\otimes(n+1)}$ where for $0 \leq i \leq n$, d'_i sends $a_0 \otimes \cdots \otimes a_{n+1}$ for $a_0, \cdots, a_{n+1} \in A$ to

$$a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}.$$

Then $(\mathrm{Bar}_\bullet(A), d')$ is a projective resolution of A as $A^e = A \otimes_k A^{op}$-modules. Let $(C_\bullet(A), d) = (A \otimes_{A^e} \mathrm{Bar}(A), Id_A \otimes d')$ be the Hochschild complex. Namely, for $n \geq 0$, $C_n(A) \cong A^{\otimes(n+1)}$ and for $n \geq 1$ the differential $d : C_n(A) \to C_{n-1}(A)$ sends $a_0 \otimes \cdots \otimes a_n$ with $a_0, \cdots, a_n \in A$ to

$$\sum_{i=0}^{n-1}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Next we recall the construction of transfer maps in Hochschild homology due to Bouc ([4]). Transfer maps in Hochschild homology and in cyclic homology also have been studied by Keller ([16]) and by Loday ([25]). Recall that for an algebra A, the Hochschild homology of degree zero is $HH_0(A) = A/K(A)$ where $K(A)$ is the subspace spanned by commutators.

Let A and B be two finite dimensional k-algebras and let M be an A-B-bimodule such that M is finitely generated and projective as a right B-module. There exist $x_i \in M$ and $\varphi_i \in Hom_B(M, B)$ with $1 \leq i \leq s$ such that for any $x \in M$, $x = \sum_i x_i \varphi_i(x)$. Then one can define a transfer map $t_M : HH_n(A) \to HH_n(B)$ for any $n \geq 0$. The construction on the level of Hochschild complexes is as follows: $t_M : C_n(A) \to C_n(B)$ sends $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to

$$\sum_{1 \leq i_0, \cdots, i_n \leq s} \varphi_{i_0}(a_0 x_{i_1}) \otimes \varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0}).$$

Bouc proved that this is a chain map and thus induces a morphism between Hochschild homology groups, written also by $t_M : HH_n(A) \to HH_n(B)$. In particular, the construction in degree zero

$$t_M : HH_0(A) = A/K(A) \to HH_0(B) = B/K(B)$$

is given by $a + K(A) \mapsto \sum_{i=1}^{s} \varphi_i(a x_i) + K(B)$.

We summarize basic properties of this transfer map in the following

Proposition 2.1. *([4, Section 3]) Let A, B and C be finite dimensional k-algebras.*

(1) If M is an A-B-bimodule and N is a B-C-bimodule such that M_B and N_C are finitely generated and projective, then $t_N \circ t_M = t_{M \otimes_B N} : HH_n(A) \to HH_n(C)$, for each $n \geq 0$.

(2) Let

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of A-B-bimodules which are finitely generated and projective as right B-modules. Then $t_M = t_L + t_N : HH_n(A) \to HH_n(B)$, for each $n \geq 0$.

(3) For a finitely generated projective A-B-bimodule P, the transfer map $t_P : HH_n(A) \to HH_n(B)$ is zero for each $n > 0$.

(4) Consider A as an A-A-bimodule by left and right multiplications, then $t_A : HH_n(A) \to HH_n(A)$ is the identity map for any $n \geq 0$.

Remark 2.2. *Given a bounded (cochain) complex $X^\bullet$ of A-B-bimodules whose terms are finitely generated and projective as right B-modules, one can also define a transfer map $t_{X^\bullet} : HH_n(A) \to HH_n(B)$ by $t_{X^\bullet} := \sum_i (-1)^i t_{X^i}$. Note that if $Y^\bullet$ is another bounded complex of A-B-bimodules whose terms are finitely generated and projective as right B-modules such that $X^\bullet$ and $Y^\bullet$ are quasi-isomorphic, then $t_{X^\bullet} = t_{Y^\bullet}$ ([4, Section 4]).*

We will define transfer maps in Hochschild cohomology for symmetric algebras. Our definition will turn out to coincide with the construction due to Linckelmann ([20]). We recall some basic facts about symmetric algebras and for details we refer to [20, Section 6]. A symmetric algebra is a finite dimensional k-algebra such that there is a symmetric non-degenerate associative bilinear form $(\ ,\ )_A : A \times A \to k$, or equivalently, $A \cong D(A) = A^* = \mathrm{Hom}_k(A, k)$ as bimodules. The image of the unit element of A under this isomorphism is called a symmetrizing form on A and is denoted by s. Note that the bilinear form can be given by $(a, a')_A = s(aa')$ for arbitrary $a, a' \in A$.

Now let B be another finite dimensional k-algebra and ${}_A M_B$ be an A-B-bimodule. Then $M^* = \mathrm{Hom}_k(M, k)$ is isomorphic to $\mathrm{Hom}_A(M, A)$ as B-A-bimodules. The isomorphism $\mathrm{Hom}_A(M, A) \cong M^*$ sends $f \in \mathrm{Hom}_A(M, A)$ to the composition $s \circ f$. The inverse map can be described as follows.

Let $\{u_i\}$ be a basis of A and let $\{v_i\}$ be the dual basis with respect to the bilinear form $(\ ,\ )_A$ or $s \in A^*$, that is, $(u_i, v_j) = \delta_{ij}$. Then the image of $\theta \in M^*$ under the inverse map is the map sending $x \in M$ to $\sum_i \theta(v_i x)u_i$.

Now suppose that ${}_AM$ is finitely generated and projective. Then we have isomorphisms of functors

$$\mathrm{Hom}_A(M, -) \cong \mathrm{Hom}_A(M, A) \otimes_A - \cong M^* \otimes_A -.$$

By adjointness, there is an adjoint pair $(M \otimes_B -, M^* \otimes_A -)$. One can compute the associate counit morphism $\varepsilon_M : M \otimes_B M^* \to A$. Let $x \in M$ and $\theta \in M^*$. Then $\varepsilon_M(x \otimes_A \theta) = \sum_i \theta(v_i x)u_i$.

Suppose now that B is also symmetric with $t \in B^*$ giving the symmetrizing form. If furthermore, M_B is finitely generated and projective, then M^* is finitely generated projective as left B-module and we have isomorphisms of functors

$$\mathrm{Hom}_B(M^*, -) \cong \mathrm{Hom}_B(M^*, B) \otimes_B - \cong M^{**} \otimes_B - \cong M \otimes_B -.$$

Hence there is another adjoint pair $(M^* \otimes_A -, M \otimes_B -)$. Its unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$ can be computed. Since M is finitely generated and projective as a right B-module, there exist $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \le i \le s$ such that for any $x \in M$, $x = \sum_i x_i\varphi_i(x)$. It follows that η_{M^*} sends $a \in A$ to $\sum_i ax_i \otimes_B t \circ \varphi_i$.

Remark 2.3. *We can give another realization of the above unit morphism η_{M^*} by considering adjoint pairs between the categories of right modules. Namely, since M is finitely generated and projective as a right B-module, there are isomorphisms of functors*

$$\mathrm{Hom}_B(M, -) \cong - \otimes_B \mathrm{Hom}_B(M, B) \cong - \otimes_B M^*.$$

Hence there is an adjoint pair $(- \otimes_A M, - \otimes_B M^*)$. A computation shows that the unit morphism $A \to M \otimes_B M^*$ of this adjoint pair coincides with the above η_{M^*}.

Now transfer maps in Hochschild cohomology can be defined. Recall that the Hochschild cohomology is the cohomology of the Hochschild complex $C^\bullet(A) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_\bullet(A), A)$. Then $C^n(A) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_n(A), A) = \mathrm{Hom}_k(A^{\otimes n}, A)$. Our goal is to define a chain map $t^M : C^\bullet(B) \to C^\bullet(A)$ for an A-B-bimodule ${}_AM_B$ which is finitely generated and projective as a left A-module and a right B-module for two symmetric algebras A and B. To this end, we need a chain map

$$\Theta_\bullet : \mathrm{Bar}_\bullet(A) \to M \otimes_B \mathrm{Bar}_\bullet(B) \otimes_B M^*$$

which lifts the unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$.

Proposition 2.4. *Let A and B be two finite dimensional k-algebras with B symmetric by a symmetrizing form $t \in B^*$. Let ${}_AM_B$ be an A-B-bimodule such that M_B is finitely generated and projective.*

Then for $n \ge 0$, the map

$$\Theta_n : \mathrm{Bar}_n(A) = A^{\otimes(n+2)} \to M \otimes_B \mathrm{Bar}_n(B) \otimes_B M^* = M \otimes B^{\otimes n} \otimes M^*$$

sending $a_0 \otimes \cdots \otimes a_{n+1}$ for $a_0, \cdots, a_{n+1} \in A$ to

$$\sum_{i_0,\cdots,i_n} a_0x_{i_1} \otimes \varphi_{i_1}(a_1x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_nx_{i_0}) \otimes t \circ \varphi_{i_0}a_{n+1}$$

commutes with the differential, where $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \le i \le s$ such that for any $x \in M$, $x = \sum_i x_i\varphi_i(x)$. Moreover, $\Theta_\bullet$ lifts the unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$.

Proof We first prove that $\Theta_\bullet$ lifts the unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$, that is, $\eta_{M^*}\mu_A = (Id_M \otimes \mu_B \otimes Id_{M^*})\Theta_0$ where $\mu_A : A \otimes_k A \to A$ and $\mu_B : B \otimes_k B \to B$ are the multiplication maps. For $a_0, a_1 \in A$, we have $(Id_M \otimes \mu_B \otimes Id_{M^*})\Theta_0(a_0 \otimes a_1) = (Id_M \otimes \mu_B \otimes Id_{M^*})(\sum_i a_0x_i \otimes t \circ \varphi_i a_1) = \sum_i a_0x_i \otimes_B t \circ \varphi_i a_1$, and on the other hand, $\eta_{M^*}\mu_A(a_0 \otimes a_1) = \eta_{M^*}(a_0a_1) = \sum_i a_0x_i \otimes_B t \circ \varphi_i a_1$ since η_{M^*} is an A-A-bimodule homomorphism.

Next, we prove that $\Theta_\bullet$ is a chain map. To do this, we need to prove that $\Theta_{n-1}d'_i = (Id_M \otimes d'_i \otimes Id_{M^*})\Theta_n$ for each $0 \le i \le n$. We shall only give the proof for $i = 0$ and for $i = n$, the other cases being similar.

Case $i=0$. For $a_0,\cdots,a_{n+1}\in A$, we have

$$\begin{aligned}
&\Theta_{n-1}d'_0(a_0\otimes\cdots\otimes a_{n+1})\\
=\ &\Theta_{n-1}(a_0a_1\otimes\cdots\otimes a_{n+1})\\
=\ &\sum_{i_2,\cdots,i_n,i_0} a_0a_1x_{i_2}\otimes\varphi_{i_2}(a_2x_{i_3})\otimes\cdots\otimes\varphi_{i_n}(a_nx_{i_0})\otimes t\circ\varphi_{i_0}a_{n+1}
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
&(Id_M\otimes d'_0\otimes Id_{M^*})\Theta_n(a_0\otimes\cdots\otimes a_{n+1})\\
=\ &(Id_M\otimes d'_0\otimes Id_{M^*})(\sum_{i_0,i_1,i_2,\cdots,i_n} a_0x_{i_1}\otimes\varphi_{i_1}(a_1x_{i_2})\otimes\cdots\otimes\varphi_{i_n}(a_nx_{i_0})\otimes t\circ\varphi_{i_0}a_{n+1})\\
=\ &\sum_{i_0,i_1,i_2,\cdots,i_n} a_0x_{i_1}\varphi_{i_1}(a_1x_{i_2})\otimes\varphi_{i_2}(a_2x_{i_3})\cdots\otimes\varphi_{i_n}(a_nx_{i_0})\otimes t\circ\varphi_{i_0}a_{n+1}\\
=\ &\sum_{i_0,i_2,\cdots,i_n}(\sum_{i_1} a_0x_{i_1}\varphi_{i_1}(a_1x_{i_2}))\otimes\varphi_{i_2}(a_2x_{i_3})\cdots\otimes\varphi_{i_n}(a_nx_{i_0})\otimes t\circ\varphi_{i_0}a_{n+1}\\
=\ &\sum_{i_2,\cdots,i_n,i_0} a_0a_1x_{i_2}\otimes\varphi_{i_2}(a_2x_{i_3})\otimes\cdots\otimes\varphi_{i_n}(a_nx_{i_0})\otimes t\circ\varphi_{i_0}a_{n+1}
\end{aligned}$$

where we use the equality

$$\sum_{i_1} x_{i_1}\varphi_{i_1}(a_1x_{i_2})=a_1x_{i_2}.$$

Case $i=n$. For $a_0,\cdots,a_{n+1}\in A$, we have

$$\begin{aligned}
&\Theta_{n-1}d'_n(a_0\otimes\cdots\otimes a_{n+1})\\
=\ &\Theta_{n-1}(a_0\otimes\cdots\otimes a_na_{n+1})\\
=\ &\sum_{i_0,i_1,\cdots,i_{n-1}} a_0x_{i_1}\otimes\varphi_{i_1}(a_1x_{i_2})\otimes\cdots\otimes\varphi_{i_{n-1}}(a_{n-1}x_{i_0})\otimes t\circ\varphi_{i_0}a_na_{n+1}
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
&(Id_M\otimes d'_n\otimes Id_{M^*})\Theta_n(a_0\otimes\cdots\otimes a_{n+1})\\
=\ &(Id_M\otimes d'_n\otimes Id_{M^*})(\sum_{i_0,\cdots,i_n} a_0x_{i_1}\otimes\varphi_{i_1}(a_1x_{i_2})\otimes\cdots\otimes\varphi_{i_n}(a_nx_{i_0})\otimes t\circ\varphi_{i_0}a_{n+1})\\
=\ &\sum_{i_0,\cdots,i_n} a_0x_{i_1}\otimes\varphi_{i_1}(a_1x_{i_2})\otimes\cdots\otimes\varphi_{i_{n-1}}(a_{n-1}x_{i_n})\otimes\varphi_{i_n}(a_nx_{i_0})t\circ\varphi_{i_0}a_{n+1}\\
=\ &\sum_{i_n,i_1,\cdots,i_{n-1},i_0} a_0x_{i_1}\otimes\varphi_{i_1}(a_1x_{i_2})\otimes\cdots\otimes\varphi_{i_{n-1}}(a_{n-1}x_{i_0})\otimes\varphi_{i_0}(a_nx_{i_n})t\circ\varphi_{i_n}a_{n+1}\\
=\ &\sum_{i_0,i_1,\cdots,i_{n-1}} a_0x_{i_1}\otimes\cdots\otimes\varphi_{i_{n-1}}(a_{n-1}x_{i_0})\otimes(\sum_{i_n}\varphi_{i_0}(a_nx_{i_n})t\circ\varphi_{i_n}a_{n+1}).
\end{aligned}$$

To finish the proof, we need to show that

$$t\circ\varphi_{i_0}a_na_{n+1}=\sum_{i_n}\varphi_{i_0}(a_nx_{i_n})t\circ\varphi_{i_n}a_{n+1}.$$

For $x\in M$, we have $t\circ\varphi_{i_0}a_na_{n+1}(x)=t(\varphi_{i_0}(a_na_{n+1}x))$ and on the other hand, we have

$$\sum_{i_n}\varphi_{i_0}(a_nx_{i_n})t\circ\varphi_{i_n}a_{n+1}(x)=\sum_{i_n}t\circ\varphi_{i_n}(a_{n+1}x\varphi_{i_0}(a_nx_{i_n}))=t(\sum_{i_n}\varphi_{i_n}(a_{n+1}x)\varphi_{i_0}(a_nx_{i_n}))$$
$$=t(\sum_{i_n}\varphi_{i_0}(a_nx_{i_n})\varphi_{i_n}(a_{n+1}x))=t(\varphi_{i_0}(a_n\sum_{i_n}x_{i_n}\varphi_{i_n}(a_{n+1}x)))=t(\varphi_{i_0}(a_na_{n+1}x)).$$

□

Now we can define transfer maps in Hochschild cohomology for symmetric algebras. Let A and B be two symmetric algebras and let ${}_AM_B$ be a bimodule such that ${}_AM$ and M_B are finitely generated and projective. Then for $f \in C^n(B) = \mathrm{Hom}_{B^e}(\mathrm{Bar}_n(B), B)$ with $n \geq 0$, we define $tr^M(f)$ to be the composition

$$\mathrm{Bar}_n(A) \stackrel{\Theta_n}{\to} M \otimes_B \mathrm{Bar}_n(B) \otimes_B M^* \stackrel{Id_M \otimes f \otimes Id_{M^*}}{\longrightarrow} M \otimes_B B \otimes_B M^* \stackrel{\varepsilon_M}{\to} A.$$

Proposition 2.5. (1) *For $f \in C^n(B) = \mathrm{Hom}_{B^e}(\mathrm{Bar}_n(B), B) \cong \mathrm{Hom}_k(B^{\otimes n}, B)$ with $n \geq 0$, the map $tr^M(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to*

$$\sum_{i_0, \cdots, i_n, j} (\varphi_{i_0}(v_j x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0})))_B u_j$$

where $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \leq i \leq s$ such that for any $x \in M$, $x = \sum_i x_i \varphi_i(x)$, where $(\ , \ )_B$ is the bilinear form over B and where $\{u_j\}, \{v_j\}$ are dual bases in A, that is, $(u_i, v_j)_A = \delta_{ij}$.

(2) *The map $tr^M : C^n(B) \to C^n(A)$ is a chain map and thus induces a transfer map $t^M : HH^n(B) \to HH^n(A)$ for $n \geq 0$. In particular, in degree zero, $t^M : Z(B) \to Z(A)$ is given by*

$$b \mapsto \sum_{i,j} (\varphi_i(v_j x_i), b)_B u_j.$$

Moreover, if we identify $Z(A)$ with $\mathrm{End}_{A^e}(A, A)$ ($Z(B)$ with $\mathrm{End}_{B^e}(B, B)$, respectively), then $t^M : Z(B) \to Z(A)$ coincides with the composition

$$A \stackrel{\eta_{M^*}}{\to} M \otimes_B B \otimes_B M^* \stackrel{Id_M \otimes f \otimes Id_{M^*}}{\longrightarrow} M \otimes_B B \otimes_B M^* \stackrel{\varepsilon_M}{\to} A.$$

Proof The proof of the first assertion is easy using the explicit construction of Θ_n and ε_M. The second assertion is also direct since $\Theta_\bullet$ is a chain map by Proposition 2.4.

□

Remark 2.6. *(1) Linckelmann introduced in* [20] *transfer maps for symmetric algebras as follows. Let A and B be two symmetric k-algebras . Let ${}_AM_B$ be an A-B-bimodule such that ${}_AM$ and M_B are finitely generated and projective. Let $\mathcal{P}_A$ (resp. $\mathcal{P}_B$) be a projective resolution of A (resp. of B) as bimodules. Suppose we are given $\zeta \in HH^n(B) \cong$* $\mathrm{Hom}_{K(B^e)}(\mathcal{P}_B, \mathcal{P}_B[n])$ *where $K(B^e)$ is the homotopy category of complexes of B^e-modules. Then we define $t^M(\zeta) \in HH^n(A) \cong$* $\mathrm{Hom}_{K(A^e)}(\mathcal{P}_A, \mathcal{P}_A[n])$ *to be the class in the homotopy category $K(A^e)$ of the composition*

$$\mathcal{P}_A \to M \otimes_B \mathcal{P}_B \otimes_B M^* \stackrel{Id_M \otimes \zeta \otimes Id_{M^*}}{\longrightarrow} M \otimes_B \mathcal{P}_B[n] \otimes_B M^* \to \mathcal{P}_A[n].$$

Here the first map in the composition lifts the unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$ and the third map lifts a translation of the counit morphism $\varepsilon_M : M \otimes_B M^* \to A$. It is obvious that our construction is a special case of Linckelmann's construction. We choose $\mathcal{P}_A$ (resp. $\mathcal{P}_B$) to be the Bar resolution $\mathrm{Bar}_\bullet(A)$ *(resp.* $\mathrm{Bar}_\bullet(B)$*) and we explicitly construct the first lift. Linckelmann's construction also works for a bounded complex $X^\bullet$ of A-B-bimodules whose terms are finitely generated and projective as left and right modules. But here we first define the case of modules, and point out some basic properties of transfer maps. Afterwards we will deal with the case of complexes in Remark 2.8.*

(2) As Linckelmann has pointed out in [20, Remark 2.10]*, the definition of the transfer map in Hochschild cohomology depends on the choice of the symmetrizing forms s on A and t on B in the following way: if $s' \in A^*$ and $t' \in B^*$ are some other symmetrizing forms, there are unique invertible elements $u \in Z(A)$ and $v \in Z(B)$ such that $s' = us$ and $t' = vt$. It follows that the corresponding transfer map $t^{M'}$ associated with s' and t' satisfies $t^{M'}([f]) = u^{-1}t^M(v[f])$ for any $[f] \in HH^n(B)$.*

As in Hochschild homology, the transfer maps in Hochschild cohomology satisfy the following properties proved in [20]. Here we shall give a different proof for them by using our explicit construction and some ideas from [4].

Proposition 2.7. *([20, Section 2]) Let A, B and C be (finite dimensional) symmetric k-algebras.*

(1) If M is an A-B-bimodule and N is a B-C-bimodule such that ${}_AM$, M_B, ${}_BN$ and N_C are finitely generated and projective as left and right modules, then $t^M \circ t^N = t^{M\otimes_B N} : HH^n(C) \to HH^n(A)$, for each $n \geq 0$.

(2) Let

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

be a short exact sequence of A-B-bimodules which are finitely generated and projective as left A-modules and as right B-modules. Then $t^M = t^L + t^N : HH^n(B) \to HH^n(A)$, for each $n \geq 0$.

(3) For a finitely generated projective A-B-bimodule P, the transfer map $t^P : HH^n(B) \to HH^n(A)$ is zero for each $n > 0$.

(4) Consider A as an A-A-bimodule by left and right multiplications. Then $t^A : HH^n(A) \to HH^n(A)$ is the identity map for any $n \geq 0$.

Proof (1) Since M is finitely generated and projective as a right B-module, there exist $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \leq i \leq s$ such that for any $x \in M$, $x = \sum_i x_i\varphi_i(x)$. Similarly, since N is finitely generated and projective as a right C-module, there exist $y_j \in N$ and $\psi_j \in Hom_C(N, C)$ with $1 \leq j \leq t$ such that for any $y \in N$, $y = \sum_j y_j\psi_j(y)$. Moreover, we choose the elements $x_i\otimes y_j \in M\otimes_B N$ and define $\theta_{i,j} \in Hom_C(M\otimes_B N, C)$ by $\theta_{i,j}(x\otimes y) = \psi_j(\varphi_i(x)y)$, where $1 \leq i \leq s$ and $1 \leq j \leq t$. Then for any $x \otimes y \in M \otimes_B N$, $\sum_{i,j}(x_i \otimes y_j)\theta_{i,j}(x \otimes y) = \sum_{i,j} x_i \otimes y_j\psi_j(\varphi_i(x)y) = \sum_i x_i \otimes \varphi_i(x)y = \sum_i x_i\varphi_i(x) \otimes y = x \otimes y$. Since A, B and C are symmetric k-algebras, we can choose bilinear forms $(\, , \,)_A$, $(\, , \,)_B$, and $(\, , \,)_C$, respectively. We choose dual bases $\{u_i\}, \{v_i\}$ in A, that is, $(u_i, v_j)_A = \delta_{ij}$. Similarly, we choose dual bases $\{p_j\}, \{q_j\}$ in B. To prove the statement, it suffices to prove that, for any $f \in \mathrm{Hom}_k(C^{\otimes n}, C)$, $tr^M \circ tr^N(f) = tr^{M\otimes_B N}(f)$.

For $f \in \mathrm{Hom}_k(C^{\otimes n}, C)$, we have $tr^N(f) \in \mathrm{Hom}_k(B^{\otimes n}, B)$ sends $b_1 \otimes \cdots \otimes b_n$ to

$$\sum_{j_0,\cdots,j_n,j} (\psi_{j_0}(q_jy_{j_1}), f(\psi_{j_1}(b_1y_{j_2}) \otimes \cdots \otimes \psi_{j_n}(b_ny_{j_0})))_C p_j.$$

Let $g = tr^N(f)$. Then we have $tr^M(g) = tr^M tr^N(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to

$$\sum_{i_0,\cdots,i_n,i} (\varphi_{i_0}(v_ix_{i_1}), g(\varphi_{i_1}(a_1x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_nx_{i_0})))_B u_i$$

$$= \sum_{i_0,\cdots,i_n,i} (\varphi_{i_0}(v_ix_{i_1}), \sum_{j_0,\cdots,j_n,j} (\psi_{j_0}(q_jy_{j_1}), f(\psi_{j_1}(\varphi_{i_1}(a_1x_{i_2})y_{j_2}) \otimes \cdots \otimes \psi_{j_n}(\varphi_{i_n}(a_nx_{i_0})y_{j_0})))_C p_j)_B u_i$$

$$= \sum_{i_0,\cdots,i_n,j_0,\cdots,j_n,i} \sum_j (\varphi_{i_0}(v_ix_{i_1}), (\psi_{j_0}(q_jy_{j_1}), f(\psi_{j_1}(\varphi_{i_1}(a_1x_{i_2})y_{j_2})\otimes\cdots\otimes\psi_{j_n}(\varphi_{i_n}(a_nx_{i_0})y_{j_0})))_C p_j)_B u_i$$

$$= \sum_{i_0,\cdots,i_n,j_0,\cdots,j_n,i} \sum_j (\psi_{j_0}(q_jy_{j_1}), f(\psi_{j_1}(\varphi_{i_1}(a_1x_{i_2})y_{j_2})\otimes\cdots\otimes\psi_{j_n}(\varphi_{i_n}(a_nx_{i_0})y_{j_0})))_C (\varphi_{i_0}(v_ix_{i_1}), p_j)_B u_i$$

$$= \sum_{i_0,\cdots,i_n,j_0,\cdots,j_n,i} (\sum_j \psi_{j_0}(q_jy_{j_1})(\varphi_{i_0}(v_ix_{i_1}), p_j)_B, f(\psi_{j_1}(\varphi_{i_1}(a_1x_{i_2})y_{j_2})\otimes\cdots\otimes\psi_{j_n}(\varphi_{i_n}(a_nx_{i_0})y_{j_0})))_C u_i.$$

On the other hand, we have $tr^{M\otimes_B N}(f)$ sends $a_1 \otimes \cdots \otimes a_n$ to

$$\sum_{i_0,\cdots,i_n,j_0,\cdots,j_n,i} (\theta_{i_0,j_0}(v_ix_{i_1} \otimes y_{j_1}), f(\theta_{i_1,j_1}(a_1x_{i_2} \otimes y_{j_2}) \otimes \cdots \otimes \theta_{i_n,j_n}(a_nx_{i_0} \otimes y_{j_0})))_C u_i$$

$$= \sum_{i_0,\cdots,i_n,j_0,\cdots,j_n,i} (\psi_{j_0}(\varphi_{i_0}(v_ix_{i_1})y_{j_1}), f(\psi_{j_1}(\varphi_{i_1}(a_1x_{i_2})y_{j_2}) \otimes \cdots \otimes \psi_{j_n}(\varphi_{i_n}(a_nx_{i_0})y_{j_0})))_C u_i.$$

It remains to prove that

$$\sum_j \psi_{j_0}(q_jy_{j_1})(\varphi_{i_0}(v_ix_{i_1}), p_j)_B = \psi_{j_0}(\varphi_{i_0}(v_ix_{i_1})y_{j_1}).$$

But this follows from the equality

$$\varphi_{i_0}(v_i x_{i_1}) = \sum_j (\varphi_{i_0}(v_i x_{i_1}), p_j)_B q_j.$$

(2) By assumption, both α and β are split as right B-module homomorphisms. So there exist right B-homomorphisms $\alpha' : M \to L$ and $\beta' : N \to M$ such that $\alpha'\alpha = 1_L, \beta\beta' = 1_N$ and $\alpha\alpha' + \beta'\beta = 1_M$. We choose $x_i \in L$ and $\varphi_i \in \mathrm{Hom}_B(L, B)$ with $1 \le i \le s$ such that for any $x \in L$, $x = \sum_i x_i\varphi_i(x)$. Similarly, we choose $y_j \in N$ and $\psi_j \in \mathrm{Hom}_B(N, B)$ with $1 \le j \le t$ such that for any $y \in N$, $y = \sum_j y_j\psi_j(y)$. We define $z_l \in M$ and $\pi_l \in \mathrm{Hom}_B(M, B)$ as follows

$$z_l = \begin{cases} \alpha(x_l) & 1 \le l \le s \\ \beta'(y_{l-s}) & s+1 \le l \le s+t, \end{cases} \qquad \pi_l = \begin{cases} \varphi_l\alpha' & 1 \le l \le s \\ \psi_{l-s}\beta & s+1 \le l \le s+t. \end{cases}$$

Then for any $z \in M$, $\sum_l z_l\pi_l(z) = \sum_i \alpha(x_i)(\varphi_i\alpha')(z) + \sum_j \beta'(y_j)(\psi_j\beta)(z) = \alpha(\sum_i x_i\varphi_i(\alpha'(z)) + \beta'(\sum_j y_j\psi_j(\beta(z)) = \alpha(\alpha'(z)) + \beta'(\beta(z)) = z$. As before, we choose bilinear forms $(\ ,\ )_A$ and $(\ ,\ )_B$, respectively, and let $\{u_i\}, \{v_i\}$ be dual bases in A. To prove the statement, it suffices to prove that, for any $f \in \mathrm{Hom}_k(B^{\otimes n}, B)$, $tr^L(f) + tr^N(f) = tr^M(f)$.

For $f \in \mathrm{Hom}_k(B^{\otimes n}, B)$, the map $tr^L(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to

$$(i) \qquad \sum_{i_0,\cdots,i_n,i} (\varphi_{i_0}(v_i x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0})))_B u_i$$

and the map $tr^N(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to

$$(ii) \qquad \sum_{j_0,\cdots,j_n,i} (\psi_{j_0}(v_i y_{j_1}), f(\psi_{j_1}(a_1 y_{j_2}) \otimes \cdots \otimes \psi_{j_n}(a_n y_{j_0})))_B u_i.$$

On the other hand, $tr^M(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to

$$(iii) \qquad \sum_{l_0,\cdots,l_n,i} (\pi_{l_0}(v_i z_{l_1}), f(\pi_{l_1}(a_1 z_{l_2}) \otimes \cdots \otimes \pi_{l_n}(a_n z_{l_0})))_B u_i.$$

Observe that if $1 \le j, k \le s$, then $\pi_j(v_i z_k) = \varphi_j(\alpha'(v_i\alpha(x_k))) = \varphi_j(\alpha'(\alpha(v_i x_k))) = \varphi_j(v_i x_k)$. Similarly, if $s+1 \le j, k \le s+t$, then $\pi_j(v_i z_k) = \psi_{j-s}(\beta(v_i\beta'(y_{k-s}))) = \psi_{j-s}(v_i\beta(\beta'(y_{k-s}))) = \psi_{j-s}(v_i y_{k-s})$. However, if $1 \le k \le s$ and $s+1 \le j \le s+t$, then $\pi_j(v_i z_k) = \psi_{j-s}(\beta(v_i\alpha(v_i x_k))) = \psi_{j-s}(v_i\beta(\alpha(x_k))) = 0$. It follows that each term in (iii) occurs exactly once in (i) or (ii), and conversely, each term in (i) or (ii) is a term in (iii). Therefore, $(iii) = (i) + (ii)$.

(3) Without loss of generality we can assume that P is an indecomposable projective A-B-bimodule. Therefore $P \cong Ae \otimes_k fB$ for some idempotents $e \in A$ and $f \in B$. It follows that $t^P = t^{Ae} \circ t^{fB} : HH^n(B) \to HH^n(A)$ factors through $HH^n(k)$. However, $HH^n(k) = 0$ for each $n \ge 0$.

(4) For the regular A-A-bimodule A, we choose $x = 1, \varphi = 1_A$ so that $x\varphi(a) = a$ for any $a \in A$. We fix a bilinear form $(\ ,\ )_A$, and let $\{u_i\}, \{v_i\}$ be dual bases in A. Then for any $f \in \mathrm{Hom}_k(A^{\otimes n}, A)$, we have that $tr^A(f) \in \mathrm{Hom}_k(A^{\otimes n}, A)$ sends $a_1 \otimes \cdots \otimes a_n$ to

$$\sum_j (v_j, f(a_1 \otimes \cdots \otimes a_n)_A u_j = f(a_1 \otimes \cdots \otimes a_n).$$

□

Remark 2.8. *Given a bounded (cochain) complex $X^\bullet$ of A-B-bimodules whose terms are finitely generated and projective as left and right modules, one can also define a transfer map $t^{X^\bullet} : HH^n(B) \to HH^n(A)$ by $t^{X^\bullet} := \sum_i (-1)^i t^{X^i}$. Note that if $Y^\bullet$ is another bounded complex of A-B-bimodules whose terms are finitely generated and projective as left and right modules such that $X^\bullet$ and $Y^\bullet$ are quasi-isomorphic, then $t^{X^\bullet} = t^{Y^\bullet}$ (by the same argument as in* [4, Section 4]*).*

Let A be a symmetric algebra. Then there is a non-degenerate bilinear pairing between $HH^n(A)$ and $HH_n(A)$ for any $n \ge 0$ induced by the following isomorphism of complexes:

$$\mathrm{Hom}_k(C_\bullet(A), k) = \mathrm{Hom}_k(A \otimes_{A^e} \mathrm{Bar}_\bullet(A), k) \cong \mathrm{Hom}_{A^e}(\mathrm{Bar}_\bullet(A), A^*) \cong \mathrm{Hom}_{A^e}(\mathrm{Bar}_\bullet(A), A) = C^\bullet(A),$$

where the third isomorphism is induced by the isomorphism of bimodules $A^* \cong A$.

Lemma 2.9. *Let $f \in C^n(A) \cong \mathrm{Hom}_k(A^{\otimes n}, A)$. Then its image in $\mathrm{Hom}_k(A^{\otimes(n+1)}, k)$ under the inverse of the above isomorphism, denoted by $\Phi(f)$,*

$$\textit{sends } a_0 \otimes \cdots \otimes a_n \textit{ for } a_0, \cdots, a_n \in A \textit{ to } (a_0, f(a_1 \otimes \cdots \otimes a_n))_A.$$

Proof For $f \in C^n(A) \cong \mathrm{Hom}_k(A^{\otimes n}, A)$, its image in $\mathrm{Hom}_{A^e}(\mathrm{Bar}_\bullet(A), A)$ sends $a_0 \otimes \cdots \otimes a_n \otimes a_{n+1}$ to $a_0 f(a_1 \otimes \cdots \otimes a_n) a_{n+1}$. Its image in $\mathrm{Hom}_{A^e}(\mathrm{Bar}_\bullet(A), A^*)$ sends $a_0 \otimes \cdots \otimes a_n \otimes a_{n+1}$ to $a_0 f(a_1 \otimes \cdots \otimes a_n) a_{n+1} s$, where s is the symmetrizing form over A. Its image in $\mathrm{Hom}_k(A \otimes_{A^e} \mathrm{Bar}_\bullet(A), k)$ sends $a \otimes a_0 \otimes \cdots \otimes a_n \otimes a_{n+1}$ to $(a_0 f(a_1 \otimes \cdots \otimes a_n) a_{n+1} s)(a)$. Finally, its image in $\mathrm{Hom}_k(A^{\otimes(n+1)}, k)$ sends $a_0 \otimes \cdots \otimes a_n$ to $(a_0 f(a_1 \otimes \cdots \otimes a_n) s)(1) = s(a_0 f(a_1 \otimes \cdots \otimes a_n)) = (a_0, f(a_1 \otimes \cdots \otimes a_n))_A$. □

Let A and B be two symmetric algebras and let ${}_A M_B$ be a bimodule such that ${}_A M$ and M_B are finitely generated and projective. Given such a bimodule M, transfer maps are defined in Hochschild homology and also in Hochschild cohomology. It will be crucial to know whether they are compatible via the above duality. This question is answered by the following theorem.

Theorem 2.10. *Let A and B be two symmetric algebras and let ${}_A M_B$ be a bimodule such that ${}_A M$ and M_B are finitely generated and projective.*

Then there is a commutative diagram

$$\begin{array}{ccc} C^\bullet(B) & \xrightarrow{\cong} & \mathrm{Hom}_k(C_\bullet(B), k) \\ \downarrow{\scriptstyle tr^M} & & \downarrow{\scriptstyle -\circ tr_M} \\ C^\bullet(A) & \xrightarrow{\cong} & \mathrm{Hom}_k(C_\bullet(A), k) \end{array}$$

where the horizontal isomorphisms are the above duality and the rightmost map is induced by composing tr_M on the right-hand side.

Proof We compare the action of the two compositions on arbitrary elements.

Let $a_0, \cdots, a_n \in A$ and let $f \in C^n(B) = \mathrm{Hom}_k(B^{\otimes n}, B)$. Then

$$\begin{aligned} & \Phi(f) tr_M(a_0 \otimes \cdots \otimes a_n) \\ = \ & \Phi(f)\Big(\sum_{i_0, \cdots, i_n} \varphi_{i_0}(a_0 x_{i_1}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0})\Big) \\ = \ & \sum_{i_0, \cdots, i_n} (\varphi_{i_0}(a_0 x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0})))_B \end{aligned}$$

and on the other hand,

$$\begin{aligned} & \Phi(tr^M(f))(a_0 \otimes \cdots \otimes a_n) \\ = \ & (a_0, tr^M(f)(a_1 \otimes \cdots \otimes a_n))_A \\ = \ & (a_0, \sum_{i_0, \cdots, i_n, j} (\varphi_{i_0}(v_j x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0}))_B u_j)_A \\ = \ & \sum_{i_0, \cdots, i_n, j} (a_0, u_j)_A (\varphi_{i_0}(v_j x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0}))_B \\ = \ & \sum_{i_0, \cdots, i_n} (\varphi_{i_0}((\sum_j (a_0, u_j)_A) v_j x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0}))_B \\ = \ & \sum_{i_0, \cdots, i_n} (\varphi_{i_0}((a_0 x_{i_1}), f(\varphi_{i_1}(a_1 x_{i_2}) \otimes \cdots \otimes \varphi_{i_n}(a_n x_{i_0}))_B, \end{aligned}$$

where $\{u_j\}, \{v_j\}$ are dual bases in A.

□

Remark 2.11. *An analogous result holds true when replacing the above M by a bounded complex $X^\bullet$ of A-B-bimodules whose terms are finitely generated and projective as left and right modules.*

Corollary 2.12. *Let A and B be two symmetric algebras and let ${}_AM_B$ be a bimodule such that ${}_AM$ and M_B are finitely generated and projective.*

Then the transfer maps $t^M : Z(B) \to Z(A)$ and $t_M : A/K(A) \to B/K(B)$ satisfy the following compatibility property:

$$(t_M(\overline{a}), b)_B = (\overline{a}, t^M(b))_A, \qquad \text{where } \overline{a} \in A/K(A), b \in Z(B).$$

Proof In degree zero case, both $C_0(B)$ and $C^0(B)$ identify with B and the duality $\Phi : C^0(B) \cong B \to Hom_k(B, k) \cong Hom_k(C_0(B), k)$ (cf. Lemma 2.9) maps any $b \in B$ to $(-, b)_B \in Hom_k(B, k)$. Now the conclusion follows easily from the degree zero case of Theorem 2.10.

□

Let $Z^{pr}(A)$ be the projective centre of A, that is, the set of A-A-bimodule homomorphisms from A to itself which factor through projective bimodules. The projective centre is an ideal of the centre $Z(A)$ and we denote the stable centre by $Z^{st}(A) = Z(A)/Z^{pr}(A)$.

Lemma 2.13. *Let A and B be two symmetric algebras and let ${}_AM_B$ be a bimodule such that ${}_AM$ and M_B are finitely generated and projective. Then $t^M(Z^{pr}(B)) \subseteq Z^{pr}(A)$ and hence there is an induced map $t^M_{st} : Z^{st}(B) \to Z^{st}(A)$.*

Proof Note that $t^M : Z(B) \to Z(A)$ coincides with the composition

$$A \overset{\eta_{M^*}}{\to} M \otimes_B B \otimes_B M^* \overset{Id_M \otimes f \otimes Id_{M^*}}{\longrightarrow} M \otimes_B B \otimes_B M^* \overset{\varepsilon_M}{\to} A.$$

Therefore our conclusion follows from the fact that $M \otimes_B P \otimes_B M^*$ is a projective A-A-bimodule for any projective B-B-bimodule P.

□

Next we recall the definition of stable Hochschild homology of degree zero and its basic properties (cf. [24]).

Definition 2.14. *Let A be a finite dimensional k-algebra over a field k with the decomposition ${}_AA = \bigoplus_{i=1}^r Ae_i$, where Ae_i $(1 \leq i \leq r)$ are indecomposable projective A-modules. The stable Hochschild homology group $HH_0^{\mathrm{st}}(A)$ of degree zero is defined to be a subgroup of the 0-degree Hochschild homology group $HH_0(A) = A/K(A)$, namely*

$$HH_0^{\mathrm{st}}(A) = \{a \in A \mid \text{ the trace of the map } Ae_i \to Ae_i (b \mapsto ab) \text{ vanishes for any } 1 \leq i \leq r\}/K(A).$$

The main property of this group is its invariance under derived equivalences and stable equivalences of Morita type.

Theorem 2.15.
1. [24, Corollary 4.5] *Let A and B be two derived equivalent finite dimensional algebras over an algebraically closed field. Then* $\dim HH_0^{\mathrm{st}}(A) = \dim HH_0^{\mathrm{st}}(B)$.
2. [24, Theorem 4.7] *Let A and B be two finite dimensional algebras over an algebraically closed field which are stably equivalent of Morita type. Then* $\dim HH_0^{\mathrm{st}}(A) = \dim HH_0^{\mathrm{st}}(B)$.
3. [24, Proposition 4.13] *If A is symmetric, then $HH_0^{st}(A) = Z^{pr}(A)^\perp/K(A)$.*
4. [24, Proof of Theorem 4.7] *Let M be an A-B-bimodule such that M induces a stable equivalence of Morita type between A and B. Then $t_M(HH_0^{st}(A)) \subseteq HH_0^{st}(B)$ and we denote its restriction on $HH_0^{st}(A)$ by $t^{st}_M : HH_0^{st}(A) \to HH_0^{st}(B)$.*

Now let A and B be two symmetric algebras which are stably equivalent of Morita type (defined by the bimodule ${}_AM_B$). We have defined two stable transfer maps $t^M_{st} : Z^{st}(B) \to Z^{st}(A)$ and $t^{st}_M : HH_0^{st}(A) \to HH_0^{st}(B)$. By Theorem 2.15 (3), the symmetrizing form $s \in A^*$ induces a non-degenerate bilinear pairing $(\ ,\ )_A : Z^{st}(A) \times HH_0^{st}(A) \to k$, and the symmetrizing form $t \in B^*$ induces a non-degenerate bilinear pairing $(\ ,\ )_B : Z^{st}(B) \times HH_0^{st}(B) \to k$. As in Corollary 2.12, t^M_{st} and t^{st}_M are compatible via $(\ ,\ )_A$ and $(\ ,\ )_B$.

Proposition 2.16. *Let M be an A-B-bimodule such that M induces a stable equivalence of Morita type between two symmetric algebras A and B. Then*

$$(t_M^{st}(\overline{a}), [b])_B = (\overline{a}, t_{st}^M([b]))_A$$

where $\overline{a} = a + K(A) \in HH_0^{st}(A)$ and $[b] = b + Z^{pr}(B) \in Z^{st}(B)$ for $a \in A$ and $b \in Z(B)$.

Proof This is an easy consequence of Corollary 2.12 and the previous discussion.

□

3. Transfer maps and product structure of Hochschild cohomology

In this section, we investigate the problem when transfer maps in Hochschild cohomology preserve the product structure of the Hochschild cohomology algebras. We concentrate on the case where the bimodule (bounded complex of bimodules, respectively) is given by a stable equivalence of Morita type (a derived equivalence, respectively).

Recall first the definition of a stable equivalence of Morita type.

Definition 3.1. *([7]) Let A and B be two finite dimensional k-algebras. We say that A and B are stably equivalent of Morita type, if there are two bimodules ${}_AM_B$ and ${}_BN_A$ which are projective as left modules and as right modules and such that we have bimodule isomorphisms:*

$${}_AM \otimes_B N_A \cong {}_AA_A \oplus {}_AP_A, \quad {}_BN \otimes_A M_B \cong {}_BB_B \oplus {}_BQ_B$$

where ${}_AP_A$ and ${}_BQ_B$ are projective bimodules.

From now on, when talking about a stable equivalence of Morita type between algebras A and B, we shall always assume that A and B have no semisimple direct summands (as algebras). By [9, Corollary 3.1] and [22], we may and will assume that both $(N\otimes_A -, M\otimes_B -)$ and $(M\otimes_B -, N\otimes_A -)$ are pairs of adjoint functors. In particular, we can identify N with $\mathrm{Hom}_A(M, A)$ as B-A-bimodules. The bimodule isomorphism ${}_AM \otimes_B N_A \cong {}_AA_A \oplus {}_AP_A$, defines a projection $p : M \otimes_B N \to A$ and an injection $i : A \to M \otimes_B N$. Then we have $p \circ i = 1_A$. Note that i can be chosen as the unit morphism η_N of the adjoint pair $(N \otimes_A -, M \otimes_B -)$ (see [9]), but in general, for a fixed choice of i, one cannot choose p as the counit morphism ε_M of the adjoint pair $(M \otimes_B -, N \otimes_A -)$.

Remark 3.2. *Note that in general the composition $\varepsilon_M\eta_N$ is an invertible element in the centre $Z(A)$. Indeed, $\varepsilon_M\eta_N$ is an element in the centre $Z(A)$ since both ε_M and η_N are A-A-bimodule homomorphisms. It is also invertible by the following argument. Without loss of generality, we may assume that A is an indecomposable (non-simple) algebra and therefore the centre $Z(A)$ is a local algebra. Both ε_M and η_N induce invertible elements in the stable centre $Z^{st}(A) = Z(A)/Z^{pr}(A)$ and $Z^{pr}(A) \subseteq radZ(A)$ in this case. It follows that $\varepsilon_M\eta_N$ is also invertible in $Z(A)$.*

Now suppose that A and B are symmetric k-algebras. One can identify N with $M^* = Hom_k(M, k)$. Under a suitable choice of symmetrizing forms over A and B, i can be chosen as the unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$ of the adjoint pair $(M^* \otimes_A -, M \otimes_B -)$, and p can be chosen as the counit morphism $\varepsilon_M : M \otimes_B M^* \to A$ of the adjoint pair $(M \otimes_B -, M^* \otimes_A -)$. Explicit computations for η_{M^*} and ε_M have been given in Section 2. Note that η_{M^*} depends on the choice of the symmetrizing form t on B and that ε_M depends on the choice of the symmetrizing form s on A. Since the composition $\varepsilon_M\eta_{M^*}$ is an invertible element, say u, in the centre $Z(A)$, we can choose another symmetrizing form $s' \in A^*$ such that $s' = us$. With respect to s' and t, the composition $\varepsilon_M\eta_{M^*}$ is the identity element $1 \in A$.

For a stable equivalence of Morita type, by a result of the second author ([22, Corollary 2.4]), if B is symmetric, then so is A. More precisely, one can deduce from the bilinear form on B given by $t \in B^*$ a bilinear form on A which makes it a symmetric algebra. We shall show that, with respect to this pair of bilinear forms, the composition $\varepsilon_M\eta_{M^*}$ is the identity element $1 \in A$.

Lemma 3.3. *Let A and B be two symmetric algebras which are stably equivalent of Morita type given by ${}_AM_B$ and ${}_BN_A$. Suppose that $t \in B^*$ gives the bilinear form on B.*
Then the induced bilinear form on A can be described as follows:

$$s : A \to k, \quad a \mapsto \sum t(\varphi_i(ax_i))$$

where $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_B(M, B)$ with $1 \le i \le s$ such that for any $x \in M$, $x = \sum_i x_i\varphi_i(x)$.

Proof The proof follows exactly that of [22, Corollary 2.4]. Since M_B is projective, we have by [1, Proposition 20.11], $M \otimes_B D(B) \cong D(\mathrm{Hom}_B(M, B))$ as A-B-bimodules. As ${}_BN$ is projective, by [32, Lemma 3.59], we have A-A-bimodule isomorphisms

$$\begin{aligned} M \otimes_B B \otimes_B N &\cong M \otimes_B D(B) \otimes_B N \\ &\cong D(\mathrm{Hom}_B(M, B)) \otimes_B N \\ &\cong D(\mathrm{Hom}_B(N, \mathrm{Hom}_B(M, B))) \\ &\cong D(\mathrm{Hom}_B(N, B) \otimes_B \mathrm{Hom}_B(M, B))) \\ &\cong D(M \otimes_B N). \end{aligned}$$

We have thus $M \otimes_B N \cong D(M \otimes_B N)$. If we compose it with the injection $j : A \to M \otimes_B N$ and the surjection: $D(M \otimes_B N) \to D(A), f \mapsto f \circ j$, one obtains an A-A-bimodule homomorphism

$$\sigma : A \to M \otimes_B N \cong D(M \otimes_B N) \to D(A).$$

We claim that σ is an isomorphism so that σ defines a symmetrizing bilinear form for the algebra A. In fact, from the isomorphisms $M \otimes_B N \cong A \oplus P$ and $M \otimes_B N \cong D(M \otimes_B N)$, we get an isomorphism $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} : A \oplus P \cong D(A) \oplus D(P)$ with $h_1 = \sigma$. Suppose that the inverse of h is given by $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} : D(A) \oplus D(P) \cong A \oplus P$. Then we have that $g_1h_1 + g_2h_3 = 1_A \in \mathrm{End}_{A^e}(A)$. Since A has no projective A^e-summand and g_2h_3 factors through a projective A^e-module, $g_2h_3 \in \mathrm{radEnd}_{A^e}(A)$ and therefore $g_1h_1 = 1_A - g_2h_3$ is an isomorphism. It follows that $\sigma = h_1$ is an isomorphism.

Now the element $s \in D(A)$ is just the image of $1 \in A$ under σ and can be computed explicitly.

□

Proposition 3.4. *Let A and B be two symmetric algebras which are stably equivalent of Morita type given by ${}_AM_B$ and ${}_BN_A = M^*$. Suppose that $t \in B^*$ gives the bilinear form over B and that the bilinear form on A is induced from that of B as in Lemma 3.3.*
Then $\varepsilon_M\eta_{M^*} = 1_A$.

Proof By the explicit computations of η_{M^*} and ε_M in Section 2 and by Lemma 3.3, we have

$$\varepsilon_M(\eta_{M^*}(1)) = \varepsilon_M(\sum_i x_i \otimes t \circ \varphi_i) = \sum_{ij} t \circ \varphi_i(v_jx_i)u_j = \sum_j s(v_j)u_j = 1.$$

The conclusion follows from the fact that both η_{M^*} and ε_M are A-A-bimodule homomorphisms.

□

Now we can state the result that transfer maps preserve product structure once the above choice of bilinear form over A has been made.

Theorem 3.5. *Let A and B be two symmetric algebras which are related by a stable equivalence of Morita type that is given by ${}_AM_B$ and ${}_BN_A = M^*$. Suppose that the bilinear form on A is induced from that of B as in Lemma 3.3.*
Then the transfer map $t^M : HH^*_{st}(B) = HH^*(B)/Z^{pr}(B) \to HH^*_{st}(A)$ is an isomorphism of algebras.

This result is indicated without proof in [20, Remark 2.13]. We give a proof here based on a result of Pogorzaly ([27]). He proved that a stable equivalence of Morita type between self-injective algebras induces an isomorphism of stable Hochschild cohomology algebras $HH^*_{st}(A) =$

$HH^*(A)/Z^{pr}(A)$. He identified $HH^n_{st}(A) \cong \underline{\mathrm{Hom}}_{A^e}(\Omega^n_{A^e}(A), A)$ and the isomorphism he constructed sends $f : \Omega^n_{B^e}(B) \to B$ to the composition in the stable category A^e-$\underline{\mathrm{mod}}$

$$\Omega^n_{A^e}(A) \cong M \otimes_B \Omega^n_{B^e}(B) \otimes_B N \overset{Id_M \otimes f \otimes Id_N}{\longrightarrow} M \otimes_B B \otimes_B N \cong A,$$

where the first isomorphism in the above composition is induced from the injection $i : A \to M \otimes_B N$, and the last isomorphism is induced from the projection $p : M \otimes_B N \to A$. Note that if the algebras are symmetric and ${}_BN_A = M^*$, then we can choose i as the unit morphism $\eta_{M^*} : A \to M \otimes_B M^*$ and p as the counit morphism $\varepsilon_M : M \otimes_B M^* \to A$. To prove Theorem 3.5, it suffices to prove that the above map can be deduced by our transfer map.

Proof If $g : \mathrm{Bar}_n(B) \to B$ $(n > 0)$ is a map of B^e-modules such that the composition with the differential $d' : \mathrm{Bar}_{n+1}(B) \to \mathrm{Bar}_n(B)$ is zero, then we get a map of B^e-modules $f : \Omega^n_{B^e}(B) \to B$; and if g is equal to a composition of a map $\mathrm{Bar}_{n-1}(B) \to B$ with the differential $d' : \mathrm{Bar}_n(B) \to \mathrm{Bar}_{n-1}(B)$, then the corresponding f factors through a projective B^e-module. When B is a self-injective algebra, this gives an isomorphism $HH^n_{st}(B) \cong \underline{\mathrm{Hom}}_{B^e}(\Omega^n_{B^e}(B), B)$ $(n > 0)$. Since $\Theta : \mathrm{Bar}_\bullet(A) \to M \otimes_B \mathrm{Bar}_\bullet(B) \otimes_B N$ is a chain map, we have a commutative diagram:

$$\begin{array}{ccccccc}
\Omega^n_{A^e}(A) & \xrightarrow{\Psi} & M \otimes_B \Omega^n_{B^e}(B) \otimes_B N & \xrightarrow{Id_M \otimes f \otimes Id_N} & M \otimes_B B \otimes_B N & \xrightarrow{\varepsilon_M} & A \\
\downarrow & & \downarrow & & \| & & \| \\
\mathrm{Bar}_n(A) & \xrightarrow{\Theta_n} & M \otimes_B \mathrm{Bar}_n(B) \otimes_B N & \xrightarrow{Id_M \otimes g \otimes Id_N} & M \otimes_B B \otimes_B N & \xrightarrow{\varepsilon_M} & A
\end{array}$$

where the first two vertical morphisms are natural injections, where Ψ is induced from Θ_n and thus induced from η_{M^*}. It is easy to see that Ψ is an isomorphism in the stable category of A^e-modules. Since ε_M is an isomorphism in the stable category, the composition of all morphisms in the upper sequence is exactly the image of f under Pogorzaly's isomorphism. We are done.

□

Now we look at derived equivalences between symmetric algebras. Let A and B be two finite dimensional symmetric algebras which are derived equivalent given by a two-sided tilting (cochain) complex ${}_AX^\bullet_B$. Note that ${}_AX^\bullet_B$ can be chosen as a bounded complex whose terms are finitely generated bimodules and projective as left and right modules. In the following, we always assume that a two-sided tilting complex has this form. It is well known that if B is symmetric, so is A. Similarly, we have the following result.

Proposition 3.6. *Let A and B be two symmetric algebras, related by a derived equivalence given by a two-sided tilting complex $X^\bullet$ whose terms are bimodules that are projective as left A-modules and right B-modules.*
Then there is a choice of symmetrizing forms s for A and t for B such that the transfer map $t^{X^\bullet} : HH^*(B) \to HH^*(A)$ is an isomorphism between Hochschild cohomology algebras.

This result has been indicated without proof in [20, Remark 2.13]. We give a proof here based on the next lemma. Before we state this lemma, we recall the following result proved by Rickard (cf. [17, Theorem 9.2.8]): If A and B are two symmetric algebras, related by a derived equivalence given by $X^\bullet$ a two-sided tilting complex whose terms are bimodules that are projective as left A-modules and right B-modules, then $X^\bullet \otimes_B X^{\bullet *} \cong A$ in the chain homotopy category $K^b(A \otimes_k A^{op})$ and $X^{\bullet *} \otimes_A X^\bullet \cong B$ in the chain homotopy category $K^b(B \otimes_k B^{op})$. Therefore the functors $X^\bullet \otimes_B -$ and $X^{\bullet *} \otimes_A -$ are quasi-inverse equivalences between the chain homotopy categories $K^b(A)$ and $K^b(B)$ (where $X^{\bullet *}$ is the dual complex of $X^\bullet$ over k). As before, we denote by $\varepsilon_{X^\bullet} : X^\bullet \otimes_B X^{\bullet *} \to A$ the counit morphism of the adjoint pair $(X^\bullet \otimes_B -, X^{\bullet *} \otimes_A -)$ and by $\eta_{X^{\bullet *}} : A \to X^\bullet \otimes_B X^{\bullet *}$ the unit morphism of the adjoint pair $(X^{\bullet *} \otimes_A -, X^\bullet \otimes_B -)$.

Lemma 3.7. *Let A and B be two symmetric algebras. Fix a symmetrizing form s for A and a symmetrizing form t for B. Suppose that A and B are related by a derived equivalence given by a two-sided tilting complex $X^\bullet$, whose terms are bimodules that are projective as left A-modules and*

right B-modules.
Then, in degree zero the counit morphism $\varepsilon_{X^\bullet} : X^\bullet \otimes_B X^{\bullet *} \to A$ is given by $\varepsilon_{X^\bullet} = \sum_i (-1)^i \varepsilon_{X^i}$, and in degree zero the unit morphism $\eta_{X^{\bullet *}} : A \to X^\bullet \otimes_B X^{\bullet *}$ is given by $\eta_{X^{\bullet *}} = \sum_i \eta_{X^{i*}}$ (where ε_{X^i} is the counit morphism of the adjoint pair $(X^i \otimes_B -, X^{i^*} \otimes_A -)$ and $\eta_{X^{i*}}$ is the unit morphism of the adjoint pair $(X^{i^*} \otimes_A -, X^i \otimes_B -)$). Moreover, we can choose the symmetrizing form s for A such that the composition $\varepsilon_{X^\bullet}\eta_{X^{\bullet *}}$ is the identity element in the centre $Z(A)$.

Proof Given two (cochain) complexes $U^\bullet := (\cdots \to U^m \overset{d^m_{U^\bullet}}{\to} U^{m+1} \to \cdots)$ and $V^\bullet := (\cdots \to V^m \overset{d^m_{V^\bullet}}{\to} V^{m+1} \to \cdots)$ in A-mod, we set the complex

$$Hom^\bullet_A(U^\bullet, V^\bullet) := (\cdots \to Hom^m_A(U^\bullet, V^\bullet) \overset{d^m}{\to} Hom^{m+1}_A(U^\bullet, V^\bullet) \to \cdots)$$

where

$$Hom^m_A(U^\bullet, V^\bullet) := \prod_{j-i=m} Hom_A(U^i, V^j) \qquad \textit{and}$$

$$d^m : Hom_A(U^i, V^j) \to Hom_A(U^{i-1}, V^j) \times Hom_A(U^i, V^{j+1})$$

$$\alpha \mapsto (\alpha d^{i-1}_{U^\bullet}, (-1)^i d^j_{V^\bullet}\alpha).$$

In particular, the dual complex of $X^\bullet$ over k is given by $X^{\bullet *}$, and $d^m : (X^{-m})^* \to (X^{-m-1})^*$ $(\alpha \mapsto \alpha d^{-m-1}_{U^\bullet})$. Assume now that $W^\bullet := (\cdots \to W^m \overset{d^m_{W^\bullet}}{\to} W^{m+1} \to \cdots)$ is a (cochain) complex in A^{op}-mod. Define the complex

$$W^\bullet \otimes_A U^\bullet := (\cdots \to (W^\bullet \otimes_A U^\bullet)^m \overset{d^m}{\to} (W^\bullet \otimes_A U^\bullet)^{m+1} \to \cdots)$$

where

$$(W^\bullet \otimes_A U^\bullet)^m := \bigoplus_{i+j=m} W^i \otimes_A U^j \qquad \textit{and}$$

$$d^m : W^i \otimes_A U^j \to (W^{i+1} \otimes_A U^j) \oplus (W^i \otimes_A U^{j+1})$$

$$w \otimes u \mapsto (d^i_{W^\bullet}(w) \otimes u, (-1)^i w \otimes d^j_{U^\bullet}(u)).$$

Now let $X^\bullet$ be a two-sided tilting complex whose terms are bimodules that are projective as left A-modules and right B-modules. Then the adjunction isomorphism $\phi : Hom^\bullet_A(X^\bullet \otimes_B Y^\bullet, Z^\bullet) \cong Hom^\bullet_A(Y^\bullet, X^{\bullet *} \otimes_A Z^\bullet)$ can be realized as follows. For any $i, j, k \in \mathbb{Z}$, there is a natural adjunction isomorphism

$$\phi^{i,j,k} : Hom_A(X^i \otimes_B Y^j, Z^k) \to Hom_A(Y^j, X^{i^*} \otimes_A Z^k).$$

We just put

$$\phi = \bigoplus_{i,j,k} (-1)^{ij} \phi^{i,j,k} \ .$$

From this, the counit morphism $\varepsilon_{X^\bullet}$ and the unit morphism $\eta_{X^{\bullet *}}$ can be computed easily. Since the functors $X^\bullet \otimes_B -$ and $X^{\bullet *} \otimes_A -$ are quasi-inverse equivalences, the last statement is clear.

□

Proof of Proposition 3.6 We choose the symmetrizing forms s for A and t for B such that the composition $\varepsilon_{X^\bullet}\eta_{X^{\bullet *}}$ is the identity element in the centre $Z(A)$. It is well known that derived equivalences preserve the Hochschild cohomology algebras (see, for example, [17, Corollary 6.3.7]). We shall prove that our transfer map coincides with the isomorphism given in [17, Corollary 6.3.7]. First we recall the isomorphism between $HH^*(B)$ and $HH^*(A)$. Given a representative $f : \mathrm{Bar}_n(B) \to B$ of an element in $HH^n(B)$, it corresponds to a morphism $f : \mathrm{Bar}_\bullet(B) \to B[n]$ in $K^b(B)$. Tensoring from the left by $X^\bullet$ and from the right by $X^{\bullet *}$, we get a morphism $Id_{X^\bullet} \otimes f \otimes Id_{X^{\bullet *}} : X^\bullet \otimes_B \mathrm{Bar}_\bullet(B) \otimes_B X^{\bullet *} \to X^\bullet \otimes_B X^{\bullet *}[n]$ in $K^b(A)$. Then the composition morphism in $K^b(A)$

$$(\varepsilon_{X^\bullet}[n])(Id_{X^\bullet} \otimes f \otimes Id_{X^{\bullet *}})\Delta : \mathrm{Bar}_\bullet(A) \to X^\bullet \otimes_B \mathrm{Bar}_\bullet(B) \otimes_B X^{\bullet *} \to X^\bullet \otimes_B X^{\bullet *}[n] \to A[n]$$

represents the image of f in $HH^*(A)$, where $\Delta : \mathrm{Bar}_\bullet(A) \to X^\bullet \otimes_B \mathrm{Bar}_\bullet(B) \otimes_B X^{\bullet *}$ lifts the unit morphism $\eta_{X^{\bullet *}} : A \to X^\bullet \otimes_B X^{\bullet *}$. Note that Δ is unique up to homotopy. Since $\eta_{X^{\bullet *}} = \sum_i \eta_{X^{i*}}$, we know that Δ can be taken as $\sum_i \Theta_{n,i}$ in degree $-n$, where for each i,

$$\Theta_{n,i} : \mathrm{Bar}_n(A) \to X^i \otimes_B \mathrm{Bar}_n(B) \otimes_B X^{i^*}$$

is the morphism which lifts the unit morphism $\eta_{X^{i*}} : A \to X^i \otimes_B X^{i^*}$ as given in Proposition 2.4. On the other hand, the image of f under our transfer map is given by $t^{X^\bullet}(f) = \sum_i (-1)^i t^{X^i}(f)$, where $t^{X^i}(f)$ is the following composition (which is the transfer map defined by the A-B-bimodule X^i)

$$\mathrm{Bar}_n(A) \overset{\Theta_{n,i}}{\to} X^i \otimes_B \mathrm{Bar}_n(B) \otimes_B X^{i^*} \overset{Id_{X^i} \otimes f \otimes Id_{X^{i*}}}{\longrightarrow} X^i \otimes_B B \otimes_B X^{i^*} \overset{\varepsilon_{X^i}}{\to} A.$$

It follows that the image of f given in [17, Corollary 6.3.7] coincides with $t^{X^\bullet}(f) = \sum_i (-1)^i t^{X^i}(f)$. □

4. P-power maps over the (stable) Hochschild homology of degree zero

Throughout this section, k denotes a field of characteristic $p > 0$. Let A be a finite dimensional k-algebra. There exist p-power maps over $HH_0(A) = A/K(A)$ defined as follows: for any $n \geq 0$,

$$\mu_A^{p^n} : HH_0(A) \to HH_0(A), \quad a + K(A) \mapsto a^{p^n} + K(A).$$

These maps are semi-linear. Moreover, $\mathrm{Ker}(\mu_A^{p^n}) = T_n(A)/K(A)$, where

$$T_n(A) := \{a \in A | a^{p^n} \in K(A)\}$$

and $\mathrm{Im}(\mu_A^{p^n}) = P_n(A)/K(A)$, where

$$P_n(A) := \{a^{p^n} : a \in A\} + K(A).$$

Note that if $n = 0$, $T_0(A) = K(A)$ and $P_0(A) = A$. We thank the referee for pointing out the fact that the abelian group $P_n(A)$ is not a vector space unless the ground field k is perfect.

Lemma 4.1. *(*[24, Lemma 7.1]*) Let A and B be two finite dimensional k-algebras. Given an A-B bimodule ${}_AM_B$ which is finitely generated and projective as right B-module, there is a commutative diagram:*

$$\begin{array}{ccc} HH_0(A) & \xrightarrow{t_M} & HH_0(B) \\ \downarrow{\scriptstyle \mu_A^{p^n}} & & \downarrow{\scriptstyle \mu_B^{p^n}} \\ HH_0(A) & \xrightarrow{t_M} & HH_0(B). \end{array}$$

Corollary 4.2. *Suppose that two finite dimensional k-algebras A and B are related by a derived equivalence that is given by a bounded two-sided tilting complex ${}_AX_B^\bullet$ whose terms are finitely generated and projective on either side. Then there is a commutative diagram for each $n \geq 0$,*

$$\begin{array}{ccc} HH_0(A) & \overset{t_{X^\bullet}}{\underset{\sim}{\longrightarrow}} & HH_0(B) \\ \downarrow{\scriptstyle \mu_A^{p^n}} & & \downarrow{\scriptstyle \mu_B^{p^n}} \\ HH_0(A) & \overset{t_{X^\bullet}}{\underset{\sim}{\longrightarrow}} & HH_0(B). \end{array}$$

Hence, for each $n \geq 0$,

1. $\mathrm{dim} T_n(A)/K(A) = \mathrm{dim} T_n(B)/K(B)$*;*
2. *If k is a perfect field, then we have* $\mathrm{dim} P_n(A)/K(A) = \mathrm{dim} P_n(B)/K(B)$*.*

Proof By Proposition 2.1 and Remark 2.2, the transfer map $t_{X^\bullet} : HH_0(A) \to HH_0(B)$ is an isomorphism. The commutativity of the diagram follows from the previous lemma, as $t_{X^\bullet} = \sum_i (-1)^i t_{X^i}$.

□

Remark 4.3. *The assertion (1) was first proved by C. Bessenrodt, T. Holm and A. Zimmermann (*[3, Corollary 1.2]*), under the stronger assumption that the ground field k is perfect.*

Proposition 4.4. *Let k be an algebraically closed field of characteristic $p > 0$ and let A be a finite dimensional k-algebra.*

1. *For $n \geq 0$, there is an inclusion $\mu_A^{p^n}(HH_0^{\mathrm{st}}(A)) \subseteq HH_0^{\mathrm{st}}(A)$ and thus an induced map*
$$\mu_{A,st}^{p^n} := \mu_A^{p^n}|_{HH_0^{st}(A)} : HH_0^{st}(A) \to HH_0^{st}(A).$$
2. *For $n \geq 0$, $\mathrm{Ker}(\mu_{A,st}^{p^n}) = T_n(A)/K(A)$.*
3. *For $n \geq 0$, we define $P_n^{st}(A)/K(A) := \mathrm{Im}(\mu_{A,st}^{p^n}) \subseteq HH_0^{st}(A)$.*
4. *$\mathrm{dim}P_n(A)/K(A) = \mathrm{dim}P_n^{st}(A)/K(A) + \mathrm{rank}_p C_A$, where C_A is the Cartan matrix of A.*

Proof The first assertion is [24, Corollary 7.2]. The second follows from [24, Lemma 7.3] which says that $T_n(A)/K(A) \subseteq HH_0^{st}(A)$ for each $n \geq 0$. For the last statement,
$$\mathrm{dim}P_n(A)/K(A) - \mathrm{dim}P_n^{st}(A)/K(A) = \mathrm{dim}HH_0(A) - \mathrm{dim}HH_0^{st}(A) = \mathrm{rank}_p C_A,$$
where the last equality is [24, Theorem 4.4].

□

Proposition 4.5. *(*[24, Proposition 7.4]*) Let k be an algebraically closed field of characteristic $p > 0$ and let A and B be two finite dimensional k-algebras. Let ${}_AM_B$ be an A-B bimodule, which is finitely generated and projective as right B-module.*
Then we have a commutative diagram
$$\begin{array}{ccc} HH_0^{\mathrm{st}}(A) & \xrightarrow{t_M^{st}} & HH_0^{\mathrm{st}}(B) \\ {\scriptstyle \mu_{A,st}^{p^n}}\downarrow & & {\scriptstyle \mu_{B,st}^{p^n}}\downarrow \\ HH_0^{\mathrm{st}}(A) & \xrightarrow{t_M^{st}} & HH_0^{\mathrm{st}}(B). \end{array}$$

Corollary 4.6. *Let k be an algebraically closed field of characteristic $p > 0$ and let A and B be two finite dimensional algebras related by a stable equivalence of Morita type that is given by $({}_AM_B, {}_BN_A)$.*
Then the following diagram is commutative:
$$\begin{array}{ccc} HH_0^{st}(A) & \xrightarrow[\sim]{t_M^{st}} & HH_0^{st}(B) \\ {\scriptstyle \mu_{A,st}^{p^n}}\downarrow & & \downarrow{\scriptstyle \mu_{B,st}^{p^n}} \\ HH_0^{st}(A) & \xrightarrow[\sim]{t_M^{st}} & HH_0^{st}(B). \end{array}$$

Therefore,

1. *for any $n \geq 0$, $\mathrm{dim}P_n^{st}(A)/K(A) = \mathrm{dim}P_n^{st}(B)/K(B)$;*
2. *for any $n \geq 0$, $\mathrm{dim}T_n(A)/K(A) = \mathrm{dim}T_n(B)/K(B)$;*
3. *under the condition that A and B have no semisimple summands, the Auslander–Reiten conjecture holds for this stable equivalence of Morita type if and only if $\mathrm{dim}P_n(A)/K(A) = \mathrm{dim}P_n(B)/K(B)$ for some $n \geq 0$.*

Proof The commutativity of the above diagram was proved in the previous proposition and the assertions (1) and (2) thus follow. The last assertion is implied by (4) of Proposition 4.4 and [24, Proposition 5.1].

□

Remark 4.7. *The maps $\mu_{A,st}^{p^n}$ are invariant under derived equivalences by Proposition 4.5 applied to a two-sided tilting complex.*

5. Külshammer's maps ζ_n and their stable versions

Throughout this section, k denotes a perfect field of characteristic $p > 0$. Let A be a symmetric k-algebra. Then there is a non-degenerate associative symmetric bilinear form $(\ , \ )_A : A \times A \to k$. Since $Z(A) = K(A)^{\perp}$ ([19, (35)]), we have an induced bilinear form, denoted also by $(\ , \ )_A : Z(A) \times A/K(A) \to k$. Using the p-power maps over $A/K(A)$, for $n \geq 0$, Külshammer introduced a map $\zeta_n : Z(A) \to Z(A)$ by the defining equation

$$(a, a'^{p^n} + K(A))_A = (\zeta_n(a), a' + K(A))_A^{p^n}$$

for any $a \in Z(A)$ and $a' \in A$. It is easy to see ([19, (46) and (47)]) that $\mathrm{Ker}(\zeta_n) = P_n^{\perp}(A)$ and $\mathrm{Im}(\zeta_n) = T_n^{\perp}(A)$. One can prove ([19, (36)]) that $T_n^{\perp}(A)$ form a decreasing sequence of ideals in $Z(A)$ and their intersection is $R(A) := Z(A) \cap \mathrm{Soc}(A)$ ([19, (37)]), the so-called Reynolds ideal. The ideals $T_n^{\perp}(A)$ are called generalised Reynolds ideals or Külshammer ideals. Note that by [5, Lemma 4.1 (iii)], $Z^{pr}(A) \subseteq R(A) \subseteq T_n^{\perp}(A)$ for any $n \geq 0$. (The statement of this result in [5, Lemma 4.1 (iii)] is assuming that k is algebraically closed, but it is easy to verify that the proof doesn't use this assumption.)

Proposition 5.1. *Let A and B be two symmetric k-algebras and let ${}_AM_B$ be a bimodule which is finitely generated and projective on either side.*
Then the following diagram is commutative:

$$\begin{array}{ccc} Z(B) & \xrightarrow{t^M} & Z(A) \\ {\scriptstyle \zeta_n^B}\downarrow & & {\scriptstyle \zeta_n^A}\downarrow \\ Z(B) & \xrightarrow{t^M} & Z(A). \end{array}$$

Proof For any $a \in A$ and $b \in Z(B)$, write $\overline{a} = a + K(A) \in A/K(A)$. Then

$$\begin{aligned} (t^M(\zeta_n^B(b)), \overline{a})_A^{p^n} &= (\zeta_n^B(b), t_M(\overline{a}))_B^{p^n} \\ &= (b, t_M(\overline{a})^{p^n})_B \\ &= (b, t_M(\overline{a}^{p^n}))_B \\ &= (t^M(b), \overline{a}^{p^n})_A \\ &= (\zeta_n^A(t^M(b)), \overline{a})_A^{p^n}, \end{aligned}$$

where the first and the forth equality use Corollary 2.12 and the third one uses Lemma 4.1. Since a is arbitrary, this implies $t^M(\zeta_n^B(b)) = \zeta_n^A(t^M(b))$.

□

Now we can give a new proof of the following theorem of Zimmermann.

Theorem 5.2. [35, Theorem 1] *Let A and B be symmetric k-algebras. Let A and B be derived equivalent given by a bounded two sided tilting complex $X^\bullet$ whose terms are finitely generated and projective on either side.*
Then symmetrizing forms s for A and t for B can be chosen in such a way that the following diagram is commutative:

$$\begin{array}{ccc} Z(B) & \xrightarrow[\sim]{t^{X^\bullet}} & Z(A) \\ {\scriptstyle \zeta_n^B}\downarrow & & \downarrow{\scriptstyle \zeta_n^A} \\ Z(B) & \xrightarrow[\sim]{t^{X^\bullet}} & Z(A). \end{array}$$

Thus

1. *for $n \geq 0$, $t^{X^\bullet}(T_n^{\perp}(B)) = T_n^{\perp}(A)$, that is, $T_n^{\perp}(B)$ and $T_n^{\perp}(A)$ are isomorphic as ideals via the algebra isomorphism $t^{X^\bullet} : Z(B) \to Z(A)$;*
2. *for $n \geq 0$, $t^{X^\bullet}(P_n^{\perp}(B)) = P_n^{\perp}(A)$.*

Proof The isomorphism is a consequence of Proposition 3.6. Since $t^{X^\bullet} = \sum_i (-1)^i t^{X^i}$, the commutativity of the diagram follows from the preceding proposition.

□

Now we consider the stable version of ζ_n^A.

Lemma 5.3. (1) *Let A be a finite dimensional algebra. Fix $n \geq 0$. If $a + K(A) \in HH_0^{st}(A) \cap (P_n(A)/K(A))$, then there exists $b + K(A) \in HH_0^{st}(A)$ such that $a - b^{p^n} \in K(A)$.*

(2) *Let A be a symmetric algebra. Then the inverse image of $Z^{pr}(A)$ under $\zeta_n^A : Z(A) \to Z(A)$ is $P_n^\perp(A) + Z^{pr}(A)$.*

Proof (1) Let $\{e_1, \cdots, e_n\}$ be a complete list of representatives of primitive orthogonal idempotents. Let $t_{Ae_i} : HH_0(A) \to HH_0(k) = k$ be the transfer maps in Hochschild homology. If $a + K(A) \in HH_0^{st}(A) \cap P_n(A)/K(A)$, then there exists $b + K(A) \in HH_0(A)$ such that $a - b^{p^n} \in K(A)$ and $t_{Ae_i}(b^{p^n} + K(A)) = 0$ for all i. By [24, Lemma 7.1], $t_{Ae_i}(b + K(A))^{p^n} = 0$ for all i, and therefore $t_{Ae_i}(b + K(A)) = 0$ for all i since k is perfect. This implies $b + K(A) \in HH_0^{st}(A)$.

(2) Let $a \in Z(A)$. Then $\zeta_n(a) \in Z^{pr}(A)$ if and only if a is orthogonal to the subspace generated by $\{b^{p^n} + K(A) \in HH_0(A) : b + K(A) \in HH_0^{st}(A) = Z^{pr}(A)^\perp / K(A)\}$. By (1), this subspace is just $(Z^{pr}(A)^\perp \cap P_n(A))/K(A)$ and thus $a \in P_n^\perp(A) + Z^{pr}(A)$.

□

Proposition 5.4. *Suppose that A is a symmetric k-algebra.*

(1) *For $n \geq 0$, we have $\zeta_n^A(Z^{pr}(A)) \subseteq Z^{pr}(A)$ and thus an induced map: $\zeta_n^{A,st} : Z^{st}(A) \to Z^{st}(A)$.*

(2) *For $n \geq 0$, $\mathrm{Im}(\zeta_n^{A,st}) = T_n^\perp(A)/Z^{pr}(A)$.*

(3) *For $n \geq 0$, $\mathrm{Ker}(\zeta_n^{A,st}) \simeq P_n^\perp(A)$.*

Proof Let $a' + K(A) \in HH_0^{st}(A)$. Then $a'^{p^n} + K(A) \in HH_0^{st}(A) = Z^{pr}(A)^\perp / K(A)$ by Proposition 4.4 (1). For $a \in Z^{pr}(A)$, $(\zeta_n(a), a' + K(A)) = (a, (a')^{p^n} + K(A))^{p^{-n}} = 0$ and (1) follows.

The assertion (2) is obvious, since $Z^{pr}(A) \subseteq T_n^\perp(A)$ for any $n \geq 0$.

By (2), we have

$$\mathrm{dimKer}(\zeta_n^{A,st}) = \mathrm{dim}Z^{st}(A) - \mathrm{dimIm}(\zeta_n^{A,st}) = \mathrm{dim}Z(A)/T_n^\perp(A) = \mathrm{dim}P_n^\perp(A).$$

On the other hand, by Lemma 5.3 (2), we have

$$\mathrm{Ker}(\zeta_n^{A,st}) = (P_n^\perp(A) + Z^{pr}(A))/Z^{pr}(A) \cong P_n^\perp(A)/(P_n^\perp(A) \cap Z^{pr}(A)).$$

So $Z^{pr}(A) \cap P_n^\perp(A) = \{0\}$ and (3) follows.

□

Remark 5.5. *The above proof shows that for a symmetric algebra A defined over a perfect field of characteristic $p > 0$, $Z^{pr}(A) \cap P_n^\perp(A) = \{0\}$ for $n \geq 0$.*

Remark 5.6. *Let A be a symmetric algebra. Since $HH_0^{st}(A) = Z^{pr}(A)^\perp / K(A)$, we have an induced non-degenerate bilinear pairing $Z^{st}(A) \times HH_0^{st}(A) \to k$. The map $\zeta_n^{A,st}$ satisfies*

$$(\zeta_n^{A,st}([a]), \overline{a'})^{p^n} = ([a], \mu_A^{p^n}(\overline{a'}))$$

where $[a] = a + Z^{pr}(A) \in Z^{st}(A)$ and $\overline{a'} = a' + K(A) \in HH_0^{st}(A)$ for $a \in Z(A)$ and $a' \in A$. This equation can also be used to define $\zeta_n^{A,st}$.

Corollary 5.7. *Let A and B be two symmetric k-algebras and let ${}_AM_B$ be a bimodule which is finitely generated and projective on either side.*

Then there is a commutative diagram

$$\begin{array}{ccc} Z^{st}(B) & \xrightarrow{t_{st}^M} & Z^{st}(A) \\ {\scriptstyle \zeta_n^{B,st}}\downarrow & & \downarrow{\scriptstyle \zeta_n^{A,st}} \\ Z^{st}(B) & \xrightarrow{t_{st}^M} & Z^{st}(A). \end{array}$$

Proof This follows from Proposition 5.1 and Proposition 5.4 (1).

□

Proposition 5.8. *Let A and B be two finite dimensional symmetric algebras which are are related by a stable equivalence of Morita type given by $({}_AM_B, {}_BN_A)$. Suppose that the bilinear form of A is induced from that of B.*
Then there is a commutative diagram

$$\begin{array}{ccc} Z^{st}(B) & \xrightarrow[\sim]{t_{st}^M} & Z^{st}(A) \\ {\scriptstyle \zeta_n^{B,st}}\downarrow & & \downarrow{\scriptstyle \zeta_n^{A,st}} \\ Z^{st}(B) & \xrightarrow[\sim]{t_{st}^M} & Z^{st}(A). \end{array}$$

Thus

(1) *for $n \geq 0$, $\mathrm{dim}P_n^{\perp}(A) = \mathrm{dim}P_n^{\perp}(B)$;*
(2) *for $n \geq 0$, $T_n^{\perp}(B)/Z^{pr}(B)$ and $T_n^{\perp}(A)/Z^{pr}(A)$ are isomorphic as ideals via the algebra isomorphism $t_{st}^M : Z^{st}(B) \to Z^{st}(A)$.*
(3) *under the condition that A and B have no semisimple summands, the Auslander–Reiten conjecture holds for this stable equivalence of Morita type if and only if $\mathrm{dim}T_n^{\perp}(A) = \mathrm{dim}T_n^{\perp}(B)$ for some $n \geq 0$.*

Proof By [22], B is also symmetric. The proof easily follows from Theorem 3.5 and Corollary 5.7.

□

Remark 5.9. (1) *The above claim (2) generalises* [24, Proposition 7.10] *which proved an equality of dimensions*

$$\mathrm{dim}T_n^{\perp}(B)/Z^{pr}(B) = \mathrm{dim}T_n^{\perp}(A)/Z^{pr}(A).$$

(2) *For $n = 0$, $T_0^{\perp}(A) = Z(A)$. Thus claim (3) generalises* [24, Corollary 1.2] *in case of positive characteristic.*
(3) *The stable version ζ_n^{st} is invariant also under derived equivalences, by Corollary 5.7.*

6. P-power maps over the (stable) centre and Külshammer's maps κ_n

Throughout this section, k denotes a perfect field of characteristic $p > 0$. Let A be a symmetric k-algebra. In [18], using the p-power maps over $Z(A)$, for $n \geq 0$, Külshammer introduced a map $\kappa_n^A : HH_0(A) \to HH_0(A)$ by the defining equation

$$(a^{p^n}, \overline{a'}) = (a, \kappa_n^A(\overline{a'}))^{p^n}$$

for any $a \in Z(A)$ and $a' \in A$. This construction is in some sense dual to that of the maps ζ_n and we can also define a stable version of κ_n^A. In contrast to the case of ζ_n, however, we cannot give a satisfactory description of their kernels and cokernels.

Let A be a finite dimensional k-algebra. Since the centre $Z(A)$ is a commutative algebra, the p-power maps preserve the multiplication in $Z(A)$ and they are denoted by $\mu_{p^n}^A : Z(A) \to Z(A)$, $a \mapsto a^{p^n}$ for $n \geq 0$. Note that $\mathrm{Ker}\mu_{p^n}^A = T_n(Z(A)) := \{a \in Z(A) : a^{p^n} = 0\}$ and $\mathrm{Im}\mu_{p^n}^A = P_n(Z(A)) := \{a^{p^n} : a \in Z(A)\}$. Since derived equivalences preserve the centres as algebras, these maps are invariant under derived equivalences. Since $Z^{pr}(A)$ is an ideal of $Z(A)$, $\mu^A_{p^n}(Z^{pr}(A)) \subseteq Z^{pr}(A)$. So there is an induced map $\mu^{A,st}_{p^n} : Z^{st}(A) \to Z^{st}(A)$. Note that if $n = 0$, then $T_0(Z(A)) = 0$ and $P_0(Z(A)) = Z(A)$. Proposition 3.5 gives the following commutative diagram:

$$\begin{array}{ccc} Z^{st}(B) & \xrightarrow[\sim]{t^M} & Z^{st}(A) \\ \Big\downarrow{\scriptstyle \mu^{B,st}_{p^n}} & & \Big\downarrow{\scriptstyle \mu^{A,st}_{p^n}} \\ Z^{st}(B) & \xrightarrow[\sim]{t^M} & Z^{st}(A). \end{array}$$

Lemma 6.1. *Let A be a Frobenius algebra. Then $Z^{pr}(A) \subseteq \mathrm{Soc}(A)$. As a consequence, if A has no semi-simple summands, then $Z^{pr}(A)^2 = 0$ and $Z^{pr}(A) \subseteq T_n(Z(A))$ for $n \geq 1$.*

Proof The proof imitates that of [24, Proposition 4.11]. Let $\{a_i\}$ be a basis of A and let $\{b_i\}$ be a dual basis with respect to the bilinear form over A, that is, $(a_i, b_j) = \delta_{ij}$. Then by [24, Proposition 1.3 (1)], $Z^{pr}(A)$ is the image of the map $\tau : A \to A, x \mapsto \sum b_i x a_i$. We shall prove that $\mathrm{Im}(\tau) \subseteq \mathrm{Soc}(A)$.

We carefully choose a basis $\{a_i\}$ and its dual basis $\{b_i\}$ in A, as follows. Suppose that

$$A/J(A) \simeq M_{u_1}(k) \times \cdots M_{u_r}(k).$$

Write E^t_{ij} the matrix in $M_{u_t}(k)$ whose entry at the position (i, j) is 1 and is zero elsewhere. Then take $a_1 = e_1, a_2 = e_2, \cdots, a_m = e_m \in A$ such that their images in $A/J(A)$ correspond to the matrices E^t_{ii} for $1 \leq i \leq u_t$ and $1 \leq t \leq r$ and take $a_{m+1}, \cdots, a_n$ such that their images in $A/J(A)$ correspond to E^t_{ij} for $1 \leq i \neq j \leq u_t$ and $1 \leq t \leq r$. Then $\{a_1 \cdots, a_n\}$ are linearly independent in A and their images in $A/J(A)$ form a basis of the vector space $A/J(A)$. Note that for $m+1 \leq u \leq n$, $a_u \in e_i A e_j$ for some $1 \leq i \neq j \leq m$. Moreover let $a_{n+1}, \cdots, a_s \in J(A)$ such that their images in $J(A)/J^2(A)$ is a basis of the vector space $J(A)/J^2(A)$, let $a_{s+1}, \cdots, a_t \in J^2(A)$ such that their images in $J^2(A)/J^3(A)$ is a basis of the vector space $J^2(A)/J^3(A)$, and so on. Let $b_1, b_2, \cdots$ be the dual basis. Then $\{b_1, \cdots, b_n\}$ is a basis of $J(A)^{\perp} = \mathrm{Soc}(A)$, $\{b_1, \cdots, b_s\}$ is a basis of $J^2(A)^{\perp}$, $\{b_1, \cdots, b_t\}$ is a basis of $J^3(A)^{\perp}$, etc. Now we can prove the first assertion. Since for any $x \in A$,

$$J(A)x = 0 \Longleftrightarrow x \in \mathrm{Soc}(A) \Longleftrightarrow xJ(A) = 0,$$

we need to prove that $J(A) \cdot \mathrm{Im}(\tau) = 0$. It is easy to see that $J(A) \cdot J^n(A)^{\perp} \cdot J^{n-1}(A) = 0$ for $n \geq 1$. Let $y \in J(A)$ and $x \in A$. For $1 \leq i \leq n$, we get $y b_i x a_i \in J(A) \cdot \mathrm{Soc}(A) \cdot A \cdot A = 0$; for $n + 1 \leq i \leq s$, we get $y b_i x a_i \in J(A) \cdot J^2(A)^{\perp} \cdot A \cdot J(A) = 0$; for $s + 1 \leq i \leq t$, we get $y a_i x b_i \in J(A) \cdot J^3(A)^{\perp} \cdot A \cdot J^2(A) = 0$, etc. This proves that $\mathrm{Im}(\tau) \subseteq \mathrm{Soc}(A)$.

Now if A has no semi-simple direct summand, then $Z^{pr}(A)^2 \subseteq (\mathrm{Soc}(A))^2 \subseteq \mathrm{Soc}(A)J(A) = 0$ and thus for $n \geq 1$, $Z^{pr}(A) \subseteq T_n(Z(A))$.

□

As a consequence, we have

Corollary 6.2. *Let A be a finite dimensional algebra without semi-simple direct summands. Then for $n \geq 1$,*

(1) $T_n(Z(A))/Z^{pr}(A) \subseteq \mathrm{Ker}\mu^{A,st}_{p^n} \subseteq T_{n+1}(Z(A))/Z^{pr}(A)$;
(2) $P_n(Z(A)) \twoheadrightarrow \mathrm{Im}\mu^{A,st}_{p^n} \twoheadrightarrow P_{n+1}(Z(A))$.

The inclusions in (1) can be strict, as illustrated by the following

Example 6.3. *Let $A = k[X]/(X^{p^m+1})$ with $m \geq 1$. Then $Z^{pr}(A) = (X^{p^m})/(X^{p^m+1})$ and $Z^{st}(A) = Z(A)/Z^{pr}(A) \cong k[X]/(X^{p^m})$. Hence, for $1 \leq n \leq m$, $T_n(Z(A))/Z^{pr}(A) =< \overline{X^i}, p^{m-n} < i \leq p^m - 1 >$ and $Ker\mu^{A,st}_{p^n} =< \overline{X^i}, p^{m-n} \leq i \leq p^m - 1 >$.*

Now we consider the stable version of κ_n. Recall that $\mathrm{Ker}(\kappa^A_n) = P^{\perp}_n(Z(A))/K(A)$ and $\mathrm{Im}(\kappa^A_n) = T^{\perp}_n(Z(A))/K(A)$ ([19, (52)(53)]).

Proposition 6.4. *([36, Proposition 2.3]) Let A and B be symmetric k-algebras. Suppose that A and B are derived equivalent, and the equivalence is given by a bounded two sided tilting complex $X^\bullet$ whose terms are finitely generated and projective on either side.*
Then there exists an isomorphism $t_{X^\bullet} : HH_0(A) \xrightarrow{\sim} HH_0(B)$ such that

$$\begin{array}{ccc} HH_0(A) & \xrightarrow[\sim]{t_{X^\bullet}} & HH_0(B) \\ {\scriptstyle \kappa_n^A}\downarrow & & \downarrow{\scriptstyle \kappa_n^B} \\ HH_0(A) & \xrightarrow[\sim]{t_{X^\bullet}} & HH_0(B). \end{array}$$

Proof For any $b \in Z(B)$ and $a \in A$, write $\overline{a} = a + K(A) \in HH_0(A)$. We have

$$\begin{aligned} (b, \kappa_n^B t_{X^\bullet}(\overline{a}))_B^{p^n} &= (b^{p^n}, t_{X^\bullet}(\overline{a}))_B \\ &= (t^{X^\bullet}(b^{p^n}), \overline{a})_A \\ &= (t^{X^\bullet}(b)^{p^n}, \overline{a})_A \\ &= (t^{X^\bullet}(b), \kappa_n^A(\overline{a}))_A^{p^n} \\ &= (b, t_{X^\bullet}\kappa_n^A(\overline{a}))_B^{p^n}, \end{aligned}$$

where the second and the fifth equality use Corollary 2.12 and the third one uses Proposition 3.6. We have thus $\kappa_n^B t_{X^\bullet}(\overline{a}) = t_{X^\bullet}\kappa_n^A(\overline{a})$, as $b \in Z(B)$ is arbitrary.

□

Now we consider the stable version of κ_n^A.

Lemma 6.5. *Suppose that A is a symmetric algebra. For $n \geq 0$, we have $\kappa_n(HH_0^{st}(A)) \subseteq HH_0^{st}(A)$ and thus an induced map $\kappa_n^{A,st} : HH_0^{st}(A) \to HH_0^{st}(A)$.*

Proof For $\overline{a'} = a' + K(A) \in HH_0^{st}(A) = Z^{pr}(A)^\perp / K(A)$ and $a \in Z^{pr}(A)$, since $a^{p^n} \in Z^{pr}(A)$, we have

$$(a, \kappa_n^A(\overline{a'}))_A^{p^n} = (a^{p^n}, \overline{a'})_A = 0.$$

Thus $\kappa_n^A(\overline{a'}) \in HH_0^{st}(A)$.

□

Remark 6.6. *Let A be a symmetric algebra. Since we have an induced non-degenerate bilinear pairing $Z^{st}(A) \times HH_0^{st}(A) \to k$, the map $\kappa_n^{A,st}$ satisfies*

$$([a], \kappa_n^{A,st}(\overline{a'}))^{p^n} = ([a]^{p^n}, \overline{a'})$$

where $[a] = a + Z^{pr}(A) \in Z^{st}(A)$ and $\overline{a'} = a' + K(A) \in HH_0^{st}(A)$ for $a \in Z(A)$ and $a' \in A$. This equation can also be used to define $\kappa_n^{A,st}$.

Although we don't have a good description of the kernel and the cokernel of $\kappa_n^{A,st}$, we record the following

Proposition 6.7. *Let A and B be two finite dimensional symmetric algebras which are stably equivalent of Morita type, where the equivalence is given by $({}_AM_B, {}_BN_A)$. Suppose that the bilinear form of A is induced from that of B. Then the following diagram is commutative:*

$$\begin{array}{ccc} HH_0^{st}(A) & \xrightarrow[\sim]{t_M^{st}} & HH_0^{st}(B) \\ {\scriptstyle \kappa_n^{A,st}}\downarrow & & \downarrow{\scriptstyle \kappa_n^{B,st}} \\ HH_0^{st}(A) & \xrightarrow[\sim]{t_M^{st}} & HH_0^{st}(B). \end{array}$$

Proof This is obvious.

□

Remark 6.8. *The new maps κ_n^{st} are invariant under derived equivalences between symmetric algebras, by an analogue of Proposition 6.7.*

7. Higher dimensional analogue

Zimmermann introduced higher dimensional analogues of the map κ_n in [36]. Let us recall his construction.

Let A be a symmetric algebra defined over a perfect field of characteristic $p > 0$. We have a non-degenerate bilinear paring (cf. Lemma 2.9) for each $m \geq 0$

$$(\ , \ )_m : HH^m(A) \times HH_m(A) \to k$$

which generalizes the form in degree zero $(\ , \ ) : Z(A) \times A/K(A) \to k$. Since there exists a cup product on the Hochschild cohomology, one defines $\kappa_n^{(m),A} : HH_{p^n m}(A) \to HH_m(A)$ by the equation

$$(f^{p^n}, x)_{p^n m} = (f, \kappa_n^{(m)}(x))_m^{p^n}$$

for $f \in HH^m(A)$ and $x \in HH_{p^n m}(A)$. Obviously $\kappa_n^{(0),A} = \kappa_n^A$.

Theorem 7.1. *Let A and B be two symmetric k-algebras*

(1) *(*[36, Theorem 1]*) Suppose that A and B are related by a derived equivalence that is given by a bounded two sided tilting complex ${}_A X_B^\bullet$ whose terms are finitely generated and projective on either side.*
Then for each $m \geq 1$, there is a commutative diagram

$$\begin{array}{ccc} HH_{p^n m}(A) & \xrightarrow[\sim]{t_{X^\bullet}} & HH_{p^n m}(B) \\ \downarrow{\scriptstyle \kappa_n^{(m),A}} & & \downarrow{\scriptstyle \kappa_n^{(m),B}} \\ HH_m(A) & \xrightarrow[\sim]{t_{X^\bullet}} & HH_m(B). \end{array}$$

(2) *Suppose that A and B are related by a stable equivalence of Morita type that is given by $({}_A M_B, {}_B N_A)$.*
Then for each $m \geq 1$, there is a commutative diagram

$$\begin{array}{ccc} HH_{p^n m}(A) & \xrightarrow[\sim]{t_M} & HH_{p^n m}(B) \\ \downarrow{\scriptstyle \kappa_n^{(m),A}} & & \downarrow{\scriptstyle \kappa_n^{(m),B}} \\ HH_m(A) & \xrightarrow[\sim]{t_M} & HH_m(B). \end{array}$$

Proof The proof can be obtained by imitating the proofs of Proposition 6.4 and of Proposition 6.7 using Proposition 3.6 or Theorem 3.5.

□

Remark 7.2. *One can also obtain a proof of* (2) *in the above result by imitating the original proof of Zimmermann.*

8. Trivial extensions

Let A be a finite-dimensional algebra defined over a perfect field k of characteristic $p > 0$. We denote by A^* the k-linear dual $\mathrm{Hom}_k(A, k)$ which becomes an A-A-bimodule by setting $(afa')(b) = f(a'ba)$ for all $a, a', b \in A$ and $f \in A^*$.

The trivial extension $\mathbb{T}(A) := A \oplus A^*$ is the k-algebra defined by the multiplication $(a, f)(b, g) := (ab, ag + fb)$ for all $a, b \in A$ and $f, g \in A^*$. The trivial extension $\mathbb{T}(A)$ is a symmetric algebra, with respect to the bilinear form $((a, f), (b, g)) = f(b) + g(a)$.

Let $[A, A^*]$ denotes the commutator subspace of A^*, that is, the subspace generated by $af - fa$ for arbitrary $a \in A$ and $f \in A^*$. If V is a subspace of A, then $Ann_{A^*}(V) := \{f \in A^* | f(V) = 0\}$.

The following long proposition is collecting properties of the trivial extension algebra.

Proposition 8.1. (1) $Z(\mathbb{T}(A)) = Z(A) \oplus Ann_{A^*}(K(A))$;
(2) $K(\mathbb{T}(A)) = K(A) \oplus [A, A^*]$;
(3) $\mathbb{T}(A)/K(\mathbb{T}(A)) = A/K(A) \oplus A^*/[A, A^*]$;
(4) *for* $n \geq 1$, $T_n(\mathbb{T}(A)) = T_n(A) \oplus A^*$;
(5) *for* $n \geq 1$, $P_n(\mathbb{T}(A)) = P_n(A) \oplus [A, A^*]$;
(6) *for* $n \geq 1$, $T_n(\mathbb{T}(A))/K(\mathbb{T}(A)) = T_n(A)/K(A) \oplus A^*/[A, A^*]$;
(7) *for* $n \geq 1$, $P_n(\mathbb{T}(A))/K(\mathbb{T}(A)) = P_n(A)/K(A) \oplus 0$;
(8) *for* $n \geq 1$, $T_n^{\perp}(\mathbb{T}(A)) = 0 \oplus Ann_{A^*}(T_n(A))$;
(9) *for* $n \geq 1$, $P_n^{\perp}(\mathbb{T}(A)) = Z(A) \oplus Ann_{A^*}(P_n(A))$;
(10) *for* $n \geq 1$, $T_n(Z(\mathbb{T}(A))) = T_n(Z(A)) \oplus Ann_{A^*}(K(A))$;
(11) *for* $n \geq 1$, $P_n(Z(\mathbb{T}(A))) = P_n(Z(A)) \oplus 0$;
(12) *for* $n \geq 1$, $T_n^{\perp}(Z(\mathbb{T}(A))) = K(A) \oplus Ann_{A^*}(T_n(Z(A)))$;
(13) *for* $n \geq 1$, $P_n^{\perp}(Z(\mathbb{T}(A))) = A \oplus Ann_{A^*}(P_n(Z(A)))$;
(14) *for* $n \geq 1$, $T_n^{\perp}(Z(\mathbb{T}(A)))/K(\mathbb{T}(A)) = 0 \oplus Ann_{A^*}(T_n(Z(A)))/[A, A^*]$;
(15) *for* $n \geq 1$, $P_n^{\perp}(Z(\mathbb{T}(A)))/K(\mathbb{T}(A)) = A/K(A) \oplus Ann_{A^*}(P_n(Z(A)))/[A, A^*]$.

Proof Most assertions have been proved in [3] and the rest are easy to verify.

□

Corollary 8.2. *Let A and B be two indecomposable finite dimensional algebras defined over an algebraically closed field of characteristic $p > 0$. Suppose that A and B are stably equivalent of Morita type and that A is symmetric.*
Then the condition that $\mathbb{T}(A)$ and $\mathbb{T}(B)$ are also stably equivalent of Morita type implies that Auslander–Reiten conjecture holds for A and B, that is, they have the same number of isoclasses of non-projective simple modules.

Proof Since A is symmetric, for any $n \geq 0$,

$$\dim Ann_{A^*}(P_n(A)) = \dim A/P_n(A) = \dim T_n(A)/K(A).$$

By Corollary 4.6 (2), for any $n \geq 0$, $\dim T_n(A)/K(A) = \dim T_n(B)/K(B)$. On the other hand, Corollary 5.8 (1) implies that for any $n \geq 0$, $\dim P_n^{\perp}(\mathbb{T}(A)) = \dim P_n^{\perp}(\mathbb{T}(B))$. By Proposition 8.1 (9), we obtain that $\dim Z(A) = \dim Z(B)$ and this implies the Auslander–Reiten conjecture by Theorem 1.1 of [24].

□

This motivates the following

Question 8.3. *Let A and B be two indecomposable, non-simple finite dimensional algebras which are stably equivalent of Morita type. Are their trivial extensions algebras $\mathbb{T}(A)$ and $\mathbb{T}(B)$ also stably equivalent of Morita type?*

Although we feel that the answer should be negative, we do not know of a counter-example.

9. Stable cyclic homology

In this section we study the invariance of cyclic homology under stable equivalences of Morita type. The case of derived equivalences was considered by Keller in [16] using transfer maps. We shall adopt the same approach, but we work in the setup of ordinary algebras. For basic notions about cyclic homology, we refer the reader to [25] or a very readable brief introduction [14].

Let k be a commutative ring with unit. We shall write $\otimes$ instead of $\otimes_k$. In order to agree with the usual notations in cyclic homology, we shall modify our notations. We shall use $R, T, \cdots$ to denote k-algebras. Given a k-algebra R, the cyclic homology group $HC_n(R)$ is defined to be the homology of the total complex of the following double complex $CC_{\bullet\bullet}(R)$:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_3(R) & \xleftarrow{id-t} & C_3(R) & \xleftarrow{N} & C_3(R) & \xleftarrow{id-t} & C_3(R) & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_2(R) & \xleftarrow{id-t} & C_2(R) & \xleftarrow{N} & C_2(R) & \xleftarrow{id-t} & C_2(R) & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_1(R) & \xleftarrow{id-t} & C_1(R) & \xleftarrow{N} & C_1(R) & \xleftarrow{id-t} & C_1(R) & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_0(R) & \xleftarrow{id-t} & C_0(R) & \xleftarrow{N} & C_0(R) & \xleftarrow{id-t} & C_0(R) & \xleftarrow{N} & \cdots
\end{array}$$

We recall the construction of the above double complex. Let $(C'_\bullet(R), b')$ be the Bar complex. Namely, for $n \geq 0$, $C'_n(R) = R^{\otimes(n+2)}$ and for $n \geq 0$ the differential $b' : C'_n(R) \to C'_{n-1}(R)$ sends $x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1}$ with $x_0, \cdots, x_{n+1} \in R$ to

$$\sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}.$$

Note that here we use b' to denote the differential. Let $(C_\bullet(R), b)$ be the Hochschild complex. Namely, for $n \geq 0$, $C_n(R) = C'_{n-1}(R) = R^{\otimes(n+1)}$ and for $n \geq 1$ the differential $b : C_n(R) \to C_{n-1}(R)$ sends $x_0 \otimes x_1 \otimes \cdots \otimes x_n$ with $x_0, \cdots, x_n \in R$ to

$$\sum_{i=0}^{n-1} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n + (-1)^n x_n x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}.$$

Note that here we use b to denote the differential. For $n \geq 0$, consider the endomorphism t of $C_n(R) = R^{\otimes(n+1)}$ defined for $x_0, \cdots, x_n \in R$ by

$$t(x_0 \otimes \cdots \otimes x_n) = (-1)^n x_n \otimes x_0 \otimes \cdots \otimes x_{n-1}.$$

In the diagram, we denote by N the associated norm map

$$N = id + t + \cdots + t^n : C_n(R) \to C_n(R).$$

One can verify that $b(id - t) = (id - t)b'$ and $Nb = b'N$. Thus $CC_{\bullet\bullet}(R)$ is a double complex. We define, for $n \geq 0$, another map $s : C_n(R) \to C_{n+1}(R)$ by

$$s(x_0 \otimes \cdots \otimes x_n) = 1 \otimes x_0 \otimes \cdots \otimes x_n.$$

Now the Connes' operator $B : C_n(R) \to C_{n+1}(R)$ is defined to be

$$B = (id - t)sN : C_n(R) \to C_{n+1}(R).$$

Since it can be shown that $B^2 = Bb + bB = 0$, B induces a map $HH_n(R) \to HH_{n+1}(R)$, still denoted by B.

The cyclic homology is defined to be $HC_n(R) := H_n(Tot(CC_{\bullet\bullet}(R)))$ for $n \geq 0$. It is easy to see that the bicomplex formed by the leftmost two columns, denoted by $CC^{\{2\}}_{\bullet\bullet}$, has a total complex which is quasi-isomorphic to the Hochschild complex $(C_\bullet(R), b)$. One has a short exact sequence of bicomplexes,

$$0 \to CC^{\{2\}}_{\bullet\bullet}(R) \xrightarrow{I} CC_{\bullet\bullet}(R) \xrightarrow{S} CC_{\bullet\bullet}(R)[2, 0] \to 0$$

where for the bicomplex $CC_{\bullet\bullet}(R)$, $(CC_{\bullet\bullet}(R)[2,0])_{pq} = CC_{p-2,q}(R)$. We have thus a long exact sequence, which is called Connes' long exact sequence,

$$\cdots \xrightarrow{\partial} HH_n(R) \xrightarrow{I} HC_n(R) \xrightarrow{S} HC_{n-2}(R) \xrightarrow{\partial} HH_{n-1}(R) \xrightarrow{I} HC_{n-1}(R) \xrightarrow{S} \cdots .$$

The degree -2 map S is called the periodicity map and we use ∂ to denote the connecting morphism instead of the usual B to avoid possible confusion. As a consequence of this exact sequence, one has $HH_0(R) \cong HC_0(R)$ and

$$HC_n(k) = \begin{cases} k & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Now let R and T be two k-algebras. Let M be an R-T-bimodule, which is finitely generated and projective as right T-module. Then one can also define a transfer map $t_M : HC_n(R) \to HC_n(T)$ for $n \geq 0$ as in the case of Hochschild homology. The construction was given in [16], and also indicated in [4, Section 4.4]. The construction goes as follows. We begin with the transfer maps for the Hochschild complex $t_M : C_\bullet(R) \to C_\bullet(T)$ and prove that it commutes with the operators b', b and t (see Section 2). This provides a morphism from the bicomplex $CC_{\bullet\bullet}(R)$ to $CC_{\bullet\bullet}(T)$ which induces the desired map $t_M : HC_n(R) \to HC_n(T)$.

We include the proof of the following

Lemma 9.1. *With the assumptions above, for $n \geq 0$, there are commutative diagrams*

$$\begin{array}{ccc} C_n(R) & \xrightarrow{b} & C_{n-1}(R) \\ {\scriptstyle t_M}\downarrow & & \downarrow{\scriptstyle t_M} \\ C_n(T) & \xrightarrow{b} & C_{n-1}(T), \end{array} \qquad \begin{array}{ccc} C_n(R) & \xrightarrow{t} & C_n(R) \\ {\scriptstyle t_M}\downarrow & & \downarrow{\scriptstyle t_M} \\ C_n(T) & \xrightarrow{t} & C_n(T), \end{array} \qquad \begin{array}{ccc} C'_n(R) & \xrightarrow{b'} & C'_{n-1}(R) \\ {\scriptstyle t_M}\downarrow & & \downarrow{\scriptstyle t_M} \\ C'_n(T) & \xrightarrow{b'} & C'_{n-1}(T). \end{array}$$

As a consequence, there is a morphism of bicomplexes $t_M : CC_{\bullet\bullet}(R) \to CC_{\bullet\bullet}(T)$ and thus $t_M : HC_n(R) \to HC_n(T)$.

Proof We prove the commutativity of the first diagram, the proofs of the second and the third being similar. Since M_T is finitely generated and projective, there are $x_i \in M$ and $\varphi_i \in \mathrm{Hom}_T(M,T)$ such that for each $x \in M$, we have $x = \sum_i x_i\varphi_i(x)$. For $a_0, \cdots, a_n \in R$, we have

$$\begin{aligned} t_M b(a_0 \otimes \cdots \otimes a_n) &= \textstyle\sum_{i=0}^{n-1}(-1)^i t_M(a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n) \\ &\quad +(-1)^n t_M(a_na_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) \\ &= \textstyle\sum_{i=0}^{n-1}(-1)^i \sum_{s_0,\cdots,s_i,s_{i+2},\cdots,s_n} \varphi_{s_0}(a_0x_{s_1}) \otimes \cdots \otimes \varphi_{s_{i-1}}(a_{i-1}x_{s_i}) \\ &\qquad \otimes\varphi_{s_i}(a_ia_{i+1}x_{s_{i+2}}) \otimes \varphi_{s_{i+2}}(a_{i+2}x_{s_{i+3}}) \otimes \cdots \otimes \varphi_{s_n}(a_nx_{s_0}) \\ &\quad +(-1)^n \textstyle\sum_{s_n,s_1,\cdots,s_{n-1}} \varphi_{s_n}(a_na_0x_{s_1}) \otimes \cdots \otimes \varphi_{s_{n-1}}(a_{n-1}x_{s_n}) \end{aligned}$$

and

$$\begin{aligned} bt_M(a_0 \otimes \cdots \otimes a_n) &= \textstyle\sum_{t_0,\cdots,t_n} b(\varphi_{t_0}(a_0x_{t_1}) \otimes \cdots \otimes \varphi_{t_n}(a_nx_{t_0})) \\ &= \textstyle\sum_{t_0,\cdots,t_n}\sum_{i=0}^{n-1}(-1)^i\varphi_{t_0}(a_0x_{t_1}) \otimes \cdots \otimes \varphi_{t_{i-1}}(a_{i-1}x_{t_i}) \\ &\qquad \otimes\varphi_{t_i}(a_ix_{t_{i+1}})\varphi_{t_{i+1}}(a_{i+1}x_{t_{i+2}}) \otimes \varphi_{t_{i+2}}(a_{i+2}x_{t_{i+3}}) \otimes \cdots \otimes \varphi_{t_n}(a_nx_{t_0}) \\ &\quad +\textstyle\sum_{t_0,\cdots,t_n}(-1)^n\varphi_{t_n}(a_nx_{t_0})\varphi_{t_0}(a_0x_{t_1}) \otimes \cdots \otimes \varphi_{t_{n-1}}(a_{n-1}x_{t_n}) \\ &= \textstyle\sum_{i=0}^{n-1}(-1)^i \sum_{t_0,\cdots,t_i,t_{i+2},\cdots,t_n} \varphi_{t_0}(a_0x_{t_1}) \otimes \cdots \otimes \varphi_{t_{i-1}}(a_{i-1}x_{t_i}) \\ &\qquad \otimes(\textstyle\sum_{t_{i+1}} \varphi_{t_i}(a_ix_{t_{i+1}})\varphi_{t_{i+1}}(a_{i+1}x_{t_{i+2}})) \otimes \cdots \otimes \varphi_{t_n}(a_nx_{t_0}) \\ &\quad +(-1)^n \textstyle\sum_{t_1,\cdots,t_n}(\sum_{t_0} \varphi_{t_n}(a_nx_{t_0})\varphi_{t_0}(a_0x_{t_1})) \otimes \cdots \otimes \varphi_{t_{n-1}}(a_{n-1}x_{t_n}) \\ &= \textstyle\sum_{i=0}^{n-1}(-1)^i \sum_{t_0,\cdots,t_i,t_{i+2},\cdots,t_n} \varphi_{t_0}(a_0x_{t_1}) \otimes \cdots \otimes \varphi_{t_{i-1}}(a_{i-1}x_{t_i}) \\ &\qquad \otimes\varphi_{t_i}(a_ia_{i+1}x_{t_{i+2}}) \otimes \varphi_{t_{i+2}}(a_{i+2}x_{t_{i+3}}) \otimes \cdots \otimes \varphi_{t_n}(a_nx_{t_0}) \\ &\quad +(-1)^n \textstyle\sum_{t_1,\cdots,t_n} \varphi_{t_n}(a_na_0x_{t_1})) \otimes \varphi_{t_1}(a_1x_{t_2}) \otimes \cdots \otimes \varphi_{t_{n-1}}(a_{n-1}x_{t_n}) \end{aligned}$$

where the last equality uses two other equalities, which are true by definition,

$$\sum_{t_{i+1}} \varphi_{t_i}(a_ix_{t_{i+1}})\varphi_{t_{i+1}}(a_{i+1}x_{t_{i+2}}) = \varphi_{t_i}(\sum_{t_{i+1}} a_ix_{t_{i+1}}\varphi_{t_{i+1}}(a_{i+1}x_{t_{i+2}})) = \varphi_{t_i}(a_ia_{i+1}x_{t_{i+2}})$$

and

$$\sum_{t_0}\varphi_{t_n}(a_nx_{t_0})\varphi_{t_0}(a_0x_{t_1}) = \varphi_{t_n}(\sum_{t_0} a_nx_{t_0}\varphi_{t_0}(a_0x_{t_1})) = \varphi_{t_n}(a_na_0x_{t_1}).$$

We have thus

$$t_Mb(a_0\otimes\cdots\otimes a_n) = bt_M(a_0\otimes\cdots\otimes a_n).$$

□

Corollary 9.2. *With the assumptions above, there is a commutative diagram of exact sequences of bicomplexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CC^{\{2\}}_{\bullet\bullet}(R) & \xrightarrow{I} & CC_{\bullet\bullet}(R) & \xrightarrow{S} & CC_{\bullet\bullet}(R)[2,0] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \\ 0 & \longrightarrow & CC^{\{2\}}_{\bullet\bullet}(T) & \xrightarrow{I} & CC_{\bullet\bullet}(T) & \xrightarrow{S} & CC_{\bullet\bullet}(T)[2,0] & \longrightarrow & 0 \end{array}$$

and hence there is the following exact commutative diagram

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{S} & HC_{n-1}(R) & \xrightarrow{\partial} & HH_n(R) & \xrightarrow{I} & HC_n(R) & \xrightarrow{S} & HC_{n-2}(R) & \xrightarrow{\partial} & HH_{n-1}(R) & \xrightarrow{I} & HC_{n-1}(R) & \xrightarrow{S} & \cdots \\ & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} & & \\ \cdots & \xrightarrow{S} & HC_{n-1}(T) & \xrightarrow{\partial} & HH_n(T) & \xrightarrow{I} & HC_n(T) & \xrightarrow{S} & HC_{n-2}(T) & \xrightarrow{\partial} & HH_{n-1}(T) & \xrightarrow{I} & HC_{n-1}(T) & \xrightarrow{S} & \cdots \end{array}$$

Corollary 9.3. *With the assumptions above, there is a commutative diagram for any $n \geq 0$,*

$$\begin{array}{ccc} HH_n(R) & \xrightarrow{B} & HH_{n+1}(R) \\ \downarrow{\scriptstyle t_M} & & \downarrow{\scriptstyle t_M} \\ HH_n(T) & \xrightarrow{B} & HH_{n+1}(T). \end{array}$$

Now we list some properties of the new transfer maps on cyclic homology.

Proposition 9.4. *Let R, T and U be k-algebras.*

(1) *If M is an R-T-bimodule and N is a T-U-bimodule such that M_T and N_U are finitely generated and projective, then there is an equality $t_N \circ t_M = t_{M\otimes_T N} : HC_n(R) \to HC_n(U)$, for each $n \geq 0$.*

(2) *Let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of R-T-bimodules which are finitely generated and projective as right T-modules. Then $t_M = t_L + t_N : HC_n(R) \to HC_n(T)$, for each $n \geq 0$.

(3) *Let k be an algebraically closed field and let R and T be finite dimensional k-algebras. For a finitely generated projective R-T-bimodule P, the transfer map $t_P : HC_n(R) \to HC_n(T)$ is zero for n odd.*

(4) *Consider R as an R-R-bimodule by left and right multiplications, then $t_R : HC_n(R) \to HC_n(R)$ is the identity map for any $n \geq 0$.*

Proof The assertions (1), (2) and (4) follow from the corresponding statements for transfer maps between Hochschild homology groups. Let us prove the assertion (3). Recall that $HC_n(k) = k$ if n is even and $HC_n(k) = 0$ if n is odd. Since k is an algebraically closed field, one can assume that (without loss of generality) $P = Re \otimes fT$ for certain idempotents $e \in R$ and $f \in T$. By (1), $t_P = t_{Re} \circ t_{fT} : HC_n(R) \to HC_n(k) \to HC_n(T)$. The assertion (3) thus follows.

□

Remark 9.5. *As in the case of Hochschild homology, for a bounded (cochain) complex $X^\bullet$ of R-T-bimodules whose terms are finitely generated and projective as right T-modules, one can define a transfer map $t_{X^\bullet} : HC_n(R) \to HC_n(T)$ by $t_{X^\bullet} := \sum_i (-1)^i t_{X^i}$. Note that if $Y^\bullet$ is another bounded complex of R-T-bimodules whose terms are finitely generated and projective as right T-modules such that $X^\bullet$ and $Y^\bullet$ are quasi-isomorphic, then $t_{X^\bullet} = t_{Y^\bullet}$.*

Theorem 9.6. *Let k be an algebraically closed field of arbitrary characteristic and let R and T be two finite dimensional k-algebras which are stably equivalent of Morita type.*

(1) *For $n > 0$ odd,* $\dim HC_n(R) = \dim HC_n(T)$*;*
(2) *Suppose that R and T have no semi-simple direct summands. Then for any $n \geq 0$ even, the following statements are equivalent*
 (i) *Auslander–Reiten conjecture holds for this stable equivalence of Morita type;*
 (ii) $\dim HC_n(R) = \dim HC_n(T)$.

Proof Suppose that two bimodules M and N define a stable equivalence of Morita type between R and T by $M \otimes_T N \simeq R \oplus P$, $N \otimes_R M \simeq T \oplus Q$. Since $t_R = 1_{HC_n(R)}$ and $t_T = 1_{HC_n(T)}$, the transfer maps $t_M : HC_n(R) \to HC_n(T)$ and $t_N : HC_n(T) \to HC_n(R)$ are mutually inverse group isomorphisms for all $n > 0$ by Proposition 9.4. This proves the first statement.

For the second statement, one uses the long exact sequence connecting Hochschild homology and cyclic homology. By Corollary 9.2, for $n \geq 2$ even, we have a commutative diagram

$$\begin{array}{ccccccccccccc}
\cdots \to & HC_{n-1}(R) & \to & HH_n(R) & \to & HC_n(R) & \to & HC_{n-2}(R) & \to & HH_{n-1}(R) & \to & HC_{n-1}(R) & \to \cdots \\
& \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow\cong & \\
\cdots \to & HC_{n-1}(T) & \to & HH_n(T) & \to & HC_n(T) & \to & HC_{n-2}(T) & \to & HH_{n-1}(T) & \to & HC_{n-1}(T) & \to \cdots
\end{array}$$

where the second and the third isomorphisms deduces from [24, Remark 3.3] and where the second (and hence the forth) follows from (1). We have thus by 5-lemma that $HC_n(R) \cong HC_n(T)$ if and only if $HC_{n-2}(R) \cong HC_{n-2}(T)$, but $HC_0(R) \cong HH_0(R)$ and by Theorem 1.1 of [24], $HH_0(R) \cong HH_0(T)$ is equivalent to the Auslander–Reiten conjecture. We are done.

□

By the previous theorem, one can define a stable version of cyclic homology for a finite dimensional algebra over a field.

Definition 9.7. *Let R be a finite dimensional algebra over a field k. The stable cyclic homology of R, in degree n, is defined to be*

$$HC_n^{st}(R) = \bigcap_P \mathrm{Ker}(t_P : HC_n(R) \to HC_n(k))$$

where P runs through the set of isomorphism classes of finite dimensional left projective R-modules (which here are considered as R-k-bimodules).

Obviously $HC_n^{st}(R)$ differs from $HC_n(R)$ only when n is even. In Connes' long exact sequence one can replace $HC_n(R)$ by $HC_n^{st}(R)$ and $HH_0(R)$ by $HH_0^{st}(R)$. This is the content of the following

Proposition 9.8. *There is a long exact sequence*

$$\cdots \xrightarrow{\partial} HH_n^{st}(R) \xrightarrow{I} HC_n^{st}(R) \xrightarrow{S} HC_{n-2}^{st}(R) \xrightarrow{\partial} HH_{n-1}^{st}(R) \xrightarrow{I} HC_{n-1}^{st}(R) \xrightarrow{S} \cdots.$$

If M is an R-T-bimodule such that M_T is finitely generated and projective, then the following diagram commutes

$$\begin{array}{ccccccccccccc}
\cdots \xrightarrow{S} & HC_{n-1}^{st}(R) & \xrightarrow{\partial} & HH_n^{st}(R) & \xrightarrow{I} & HC_n^{st}(R) & \xrightarrow{S} & HC_{n-2}^{st}(R) & \xrightarrow{\partial} & HH_{n-1}^{st}(R) & \xrightarrow{I} & HC_{n-1}^{st}(R) & \xrightarrow{S} \cdots \\
& t_M\downarrow & & t_M\downarrow & & t_M\downarrow & & t_M\downarrow & & t_M\downarrow & & t_M\downarrow & \\
\cdots \xrightarrow{S} & HC_{n-1}^{st}(T) & \xrightarrow{\partial} & HH_n^{st}(T) & \xrightarrow{I} & HC_n^{st}(T) & \xrightarrow{S} & HC_{n-2}^{st}(T) & \xrightarrow{\partial} & HH_{n-1}^{st}(T) & \xrightarrow{I} & HC_{n-1}^{st}(T) & \xrightarrow{S} \cdots
\end{array}$$

Proof We have the following commutative diagram with exact columns and whose lower two rows are also exact,

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & HH_n^{st}(R) & \xrightarrow{I} & HC_n^{st}(R) & \xrightarrow{S} & HC_{n-2}^{st}(R) & \xrightarrow{\partial} & HH_{n-1}^{st}(R) & \xrightarrow{I} & HC_{n-1}^{st}(R) & \xrightarrow{S} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\partial} & HH_n(R) & \xrightarrow{I} & HC_n(R) & \xrightarrow{S} & HC_{n-2}(R) & \xrightarrow{\partial} & HH_{n-1}(R) & \xrightarrow{I} & HC_{n-1}(R) & \xrightarrow{S} & \cdots \\
 & & {\scriptstyle\sum t_P}\downarrow & & {\scriptstyle\sum t_P}\downarrow & & {\scriptstyle\sum t_P}\downarrow & & {\scriptstyle\sum t_P}\downarrow & & {\scriptstyle\sum t_P}\downarrow & & \\
\cdots & \xrightarrow{\partial} & \oplus_P HH_n(k) & \xrightarrow{I} & \oplus_P HC_n(k) & \xrightarrow{S} & \oplus_P HC_{n-2}(k) & \xrightarrow{\partial} & \oplus_P HH_{n-1}(k) & \xrightarrow{I} & \oplus_P HC_{n-1}(k) & \xrightarrow{S} & \cdots
\end{array}$$

where P runs through the set of isoclasses of finite dimensional left projective R-modules. We need to show that the first row is exact as well. This can be done by diagram chasing, using the facts that $HC_n(k) = \begin{cases} k & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$ and that $HH_n(k) = \begin{cases} k & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$

□

Theorem 9.9. *The stable cyclic homology is invariant under derived equivalences and stable equivalences of Morita type.*

Proof The invariance of the stable cyclic homology under stable equivalences of Morita type is easy by Definition 9.7 and Proposition 9.4 (refer to [24, Theorem 4.7]). The case of derived equivalence is a consequence of the commutative diagram in Proposition 9.8 (we only need to replace t_M by $t_{X^\bullet}$, where $X^\bullet$ is a two-sided tilting complex; note that $t_{X^\bullet}$ gives isomorphisms between stable Hochschild homology groups) and the fact that $HC_0^{st} \cong HH_0^{st}$.

□

Remark 9.10. *As in the case of ordinary cyclic homology, transfer maps can also be defined over negative cyclic homology and periodic cyclic homology. We leave the details to the reader.*

To conclude this Section, we add some comments on transfer maps in cyclic cohomology and its variants.

Let k be a commutative ring with unit and let R be a k-algebra. The cyclic cohomology is defined to be the cohomology of the total complex of $CC^{\bullet\bullet}(R) := \mathrm{Hom}_k(CC_{\bullet\bullet}(R), k)$. If we have another k-algebra T and an R-T-bimodule M such that M_T is finitely generated and projective, then we have the transfer map defined as above $t_M : CC_{\bullet\bullet}(R) \to CC_{\bullet\bullet}(T)$ which in turn induces a map $t^M : CC^{\bullet\bullet}(T) \to CC^{\bullet\bullet}(R)$. This new chain map induces a map in cyclic cohomology, denoted also by $t^M : HC^n(T) \to HH^n(R)$ and called transfer map in cyclic cohomology. Obviously, these new maps have some properties like those of Proposition 9.4. Evidently, we can also define transfer maps over negative cyclic cohomology and periodic cyclic cohomology. Now let R be a finite dimensional algebra over a field. For $n \geq 0$, the stable cyclic cohomology $HC^n_{st}(R)$ is defined to be the quotient of $HC^n(R)$ by the sum of the images of $t^P : HC^n(k) \to HC^n(R)$ where P runs through the set of isoclasses of (finitely generated) projective R-modules. We can also define stable versions of negative cyclic cohomology and periodic cyclic cohomology. We have also similar results as Proposition 9.8 and Theorem 9.9.

10. Batalin–Vilkovisky vs Gerstenhaber

Recall the definition of Gerstenhaber algebras and Batalin–Vilkovisky algebras. Let k be a commutative ring with unit.

Definition 10.1. *A Gerstenhaber algebra is a graded k-module $A = \oplus_{i\in\mathbb{Z}} A^i$ equipped with two linear maps: a cup product*

$$\cup : A^i \otimes A^j \to A^{i+j}, x \otimes y \mapsto x \cup y$$

and a Lie bracket of degree -1

$$[\ ,\ ] : A^i \otimes A^j \to A^{i+j-1}, x \otimes y \mapsto [x, y]$$

such that

(a) *the cup product* $\cup$ *makes* A *into a graded commutative algebra;*
(b) *the Lie bracket* $[-,-]$ *gives* A *a structure of graded Lie algebra of degree* -1. *This means that for homogeneous elements* $a, b, c \in A$

$$[a, b] = -(-1)^{(|a|-1)(|b|-1)}[b, a]$$

and

$$(-1)^{(|a|-1)(|c|-1)}[[a, b], c] + (-1)^{(|b|-1)(|a|-1)}[[b, c], a] + (-1)^{(|c|-1)(|b|-1)}[[c, a], b] = 0$$

where $|a|$ *denotes the degree of* a*;*
(c) *the cup product and the Lie bracket satisfy the Poisson rule. This means that for any* $c \in A^{|c|}$ *the adjunction map* $[-, c] : A^i \to A^{i+|c|-1}, a \mapsto [a, c]$ *is a* $(|c| - 1)$*-derivation, i.e. for homogeneous* $a, b \in A$,

$$[a \cup b, c] = [a, c] \cup b + (-1)^{|a|(|c|-1)} a \cup [b, c].$$

Definition 10.2. *A Batalin–Vilkovisky (BV) algebra is a Gerstenhaber algebra* A *together with a degree* -1 *operator* $\Delta : A^\bullet \to A^{\bullet-1}$ *satisfying* $\Delta \circ \Delta = 0$ *and*

$$[a, b] = -(-1)^{(|a|-1)|b|}(\Delta(a \cup b) - (\Delta a) \cup b - (-1)^{|a|} a \cup (\Delta b))$$

for $a, b \in A$.

The first examples of Gerstenhaber algebras are Hochschild cohomology algebra of rings first discovered by Gerstenhaber in [11]. We recall his construction. Let R be a k-algebra. Let $f \in C^n(R) = \mathrm{Hom}_k(R^{\otimes n}, R)$ and $g \in C^m(R)$ with $n, m \geq 0$. If $n, m \geq 1$, then for $1 \leq i \leq n$, define

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n+m-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});$$

if $n \geq 1$ and $m = 0$, then $g \in A$ and for $1 \leq i \leq n$, define

$$(f \circ_i g)(a_1 \otimes \cdots \otimes a_{n-1}) = f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g \otimes a_i \otimes \cdots \otimes a_{n-1});$$

for any other case, define $f \circ_i g$ to be zero. Now denote

$$f \circ g = \sum_{i=1}^{n} (-1)^{(n-1)(i-1)} f \circ_i g$$

and

$$[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f.$$

The above $[\ ,\ ]$ is just the Gerstenhaber Lie bracket over the Hochschild cohomology algebra. This Lie bracket with the usual cup product makes the Hochschild cohomology algebra into a Gerstenhaber algebra.

Tradler noticed that the Hochschild cohomology algebra of a symmetric algebra is a BV algebra in [34]. This fact has been reproved by many authors ([26, 10] etc). For a symmetric algebra R, he showed that the operator Δ over Hochschild cohomology corresponds to the Connes' B operator on Hochschild homology via the duality between Hochschild cohomology and homology. Our main result in this section is that a derived equivalence between symmetric algebras preserves the structure of BV algebras of the Hochschild cohomology algebras.

Theorem 10.3. *Let* k *be a field. Let* R *and* T *be two symmetric algebras which are derived equivalent, by an equivalence given by a two-sided tilting complex* ${}_R X^\bullet_T$ *whose terms are bimodules that are projective as left* R*-modules and right* T*-modules. Then the transfer map* $t^{X^\bullet} : HH^*(T) \to HH^*(R)$ *is an isomorphism of BV-algebras.*

Proof We need to show that transfer maps in Hochschild cohomology commute with the operator Δ. By the compatibility theorem 2.10, we only need to show that transfer maps in Hochschild homology commute with Connes' B operator, but this is Corollary 9.3.

□

As a consequence, we obtain a special case of a theorem of Keller ([16]) which says that a derived equivalence preserves the structure of Gerstenhaber algebra structure over the Hochschild cohomology algebras.

Corollary 10.4. *With the above assumptions, the transfer map $t^{X^\bullet} : HH^*(T) \to HH^*(R)$ is an isomorphism of Gerstenhaber algebras.*

Menichi ([26, corollary 1.7]) also proved that the negative cyclic cohomology of a symmetric algebra is a graded Lie algebra of degree -2. His construction is as follows. Let R be a k-algebra. Then we have the long exact sequence

$$\cdots \xrightarrow{S} HC_{-}^{n+1}(R) \xrightarrow{\partial} HH^n(R) \xrightarrow{I} HC_{-}^{n}(R) \xrightarrow{S} HC_{-}^{n+2}(R) \xrightarrow{\partial} \cdots .$$

Let $a \in HC_{-}^{n}(R)$ and $b \in HC_{-}^{m}(R)$. Then the Lie bracket on $HC_{-}^{*}(R)$ is defined as follows

$$[a, b] = (-1)^{|a|} I(\partial a \cup \partial b).$$

Theorem 10.5. *Let R and T be two symmetric algebras which are related by a derived equivalence that is given by ${}_RX^\bullet_T$ a two-sided tilting complex whose terms are bimodules that are projective as left R-modules and right T-modules. Then the transfer map $t^{X^\bullet} : HC_{-}^{*}(T) \to HC_{-}^{*}(R)$ is an isomorphism of graded Lie algebras.*

Proof This follows from Proposition 3.6 and the analogue of Corollary 9.2 for negative cyclic cohomology.

□

Now we consider the invariance of the above structures under a stable equivalence of Morita type. Let k be a field and let R be a symmetric k-algebra. Then by Corollary 9.3, transfer maps commute with the operator $\Delta : HH^{n+1}(R) \to HH^n(R)$. Hence there is an induced map $\Delta_{st} : HH_{st}^{n+1}(R) \to HH_{st}^{n}(R)$ for $n \geq 0$. This means that the stable Hochschild cohomology algebra of a symmetric algebra is still a BV algebra. This also proves that the projective centre of a symmetric algebra is a Lie ideal for the Gerstenhaber Lie bracket. We have thus proved the following

Lemma 10.6. *Let k be a field and let R be a symmetric k-algebra. Then HH_{st}^{*} is a BV algebra with the Δ-operator induced from that of HH^* and as a consequence, the projective centre is a Lie ideal for the Gerstenhaber Lie algebra structure over the Hochschild cohomology algebra.*

Now for stable negative cyclic cohomology, we have a long exact sequence

$$\cdots \xrightarrow{S} HC_{-,st}^{n+1}(R) \xrightarrow{\partial} HH_{st}^{n}(R) \xrightarrow{I} HC_{-,st}^{n}(R) \xrightarrow{S} HC_{-,st}^{n+2}(R) \xrightarrow{\partial} \cdots .$$

Let $a \in HC_{-,st}^{n}(R)$ and $b \in HC_{-,st}^{m}(R)$. Then by [26, Lemma 7.2], we define a Lie bracket on $HC_{-,st}^{*}(R)$ as follows

$$[a, b] = (-1)^{|a|} I(\partial a \cup \partial b).$$

Combining Theorem 10.3, Theorem 10.5 and Lemma 10.6, we easily obtain the following

Theorem 10.7. *Let k be an algebraically closed field. Let R and T be two symmetric algebras which are related by a stable equivalence of Morita type that is given by ${}_RM_T$ and ${}_TN_R$, which are projective as R-modules and as T-modules. Suppose that the bilinear form on R is induced from that of T. Then the transfer map $t^M : HH_{st}^{*}(T) \to HH_{st}^{*}(R)$ is an isomorphism of BV-algebras and $t^M : HC_{-,st}^{*}(T) \to HC_{-,st}^{*}(R)$ is an isomorphism of graded Lie algebras.*

Since stable Hochschild cohomology is invariant under a derived equivalence, similarly as in the preceding result, we have the following

Theorem 10.8. *Let k be an algebraically closed field. Let R and T be two symmetric algebras which are derived equivalent, where the equivalence is given by ${}_RX_T^\bullet$ a tilting complex whose terms are bimodules that are projective as left R-modules and right T-modules. Then the transfer map $t^{X^\bullet}: HH_{st}^*(T) \to HH_{st}^*(R)$ is an isomorphism of BV-algebras and $t^{X^\bullet}: HC_{-,st}^*(T) \to HC_{-,st}^*(R)$ is an isomorphism of graded Lie algebras.*

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