SIMPLE-MINDED SYSTEMS IN STABLE MODULE CATEGORIES

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ABSTRACT. Simple-minded systems in stable module categories are defined by orthogonality and generating properties so that the images of the simple modules under a stable equivalence form such a system. Simple-minded systems are shown to be invariant under stable equivalences; thus the set of all simple-minded systems is an invariant of a stable module category. The simple-minded systems of several classes of algebras are described and connections to the Auslander-Reiten conjecture are pointed out.

1. INTRODUCTION

Three categories are usually associated with a finite dimensional algebra A: The module category $mod A$, which is an abelian category, the derived category $D^b(mod A)$, which is triangulated, and the stable category $\underline{mod}A$, which is also triangulated in case A is self-injective.

The abelian category $mod A$ is generated by the set of simple modules and, in a different sense, by each progenerator, that is, by a full set of indecomposable projective modules. Equivalences of module categories are described by Morita theory, in terms of images of projective modules or by progenerators. The triangulated category $D^b(mod A)$ is also generated by the set of simple modules and alternatively by each tilting complex. Equivalences of derived categories are described by Rickard's and Keller's versions of Morita theory, again in terms of images of projective modules. Rickard [21] has shown how to assign a tilting complex to a set of objects 'behaving like simple modules', thus allowing to switch between the two kinds of generators. Rouquier [23] formalised a concept of generators of triangulated categories and used it to define the dimension of a triangulated category.

The stable category $\underline{mod}A$ is generated by the set of simple modules, too. But the projective modules are not visible in this category and there is no substitute known for progenerators. An analogue of Morita theory for stable categories is missing. In particular, it is not known how to characterize equivalences of stable categories in terms of images of generators. In fact, it is not even known how to best define 'generators' of stable categories. This appears to be a major obstruction to solve a fundamental problem on stable categories, the Auslander-Reiten conjecture; this conjecture states that stable equivalences preserve the number of isomorphisms classes of non-projective simple modules.

The aim of this article is to suggest and to explore a new definition of generating sets of stable categories, which includes the set of (non-projective) simple modules as an example.

This suggestion is not the first one made. Pogorzaly [19, 20] introduced what he called maximal systems of orthogonal stable bricks. He showed that these generate the stable Grothendieck

group, and he used his concept to prove the Auslander-Reiten conjecture for self-injective special biserial algebras.

The main features of Pogorzaly's systems are mutual orthogonality and maximality. These properties are clearly invariant under stable equivalences, while it is a problem to show finiteness of the system in general. In contrast to this, the generating systems we are introducing here - the simple-minded systems - satisfy, in addition to the orthogonality properties, a generating condition that replaces maximality. Simple-minded systems always are finite (Proposition 2.7). Invariance under stable equivalences is not for free any more, but it is true; this is one of the main results we are going to prove (Theorem 3.2). The generating assumption we are using also provides a direct relation to the stable Grothendieck group.

We consider simple-minded systems for triangular algebras and one-point extensions and for Nakayama algebras, pointing out connections with the Auslander-Reiten conjecture. Moreover, we compare the new concept with that of Pogorzaly and we use the results of this comparison to define the concept of stable Loewy length.

2. Definition and basic properties

Let R be a commutative artin ring. Recall from [4] that an R-algebra A is called an artin algebra if A is finitely generated as a R-module. Important examples of artin algebras are finite dimensional algebras over a field.

Given an artin algebra A, we denote by mod$A$ the category of all finitely generated left A-modules. For an A-module X, we denote by $soc(X)$, $top(X)$, and $rad(X)$ its socle, top and radical, respectively. We denote by mod$_\mathcal{P}A$ the full subcategory of mod$A$ consisting of modules without direct summands isomorphic to a projective module. For an A-module X, there is a maximal summand (unique up to isomorphism) which has no nonzero projective summands. We call this summand the non-projective part of the module X.

The stable category $\underline{mod}A$ of A is defined as follows: The objects of $\underline{mod}A$ are the same as those of mod$A$, and the morphisms between two objects X and Y are given by the quotient R-module $\underline{\mathrm{Hom}}_A(X,Y) = \mathrm{Hom}_A(X,Y)/\mathcal{P}(X,Y)$, where $\mathcal{P}(X,Y)$ is the R-submodule of $\mathrm{Hom}_A(X,Y)$ consisting of those homomorphisms from X to Y which factor through a projective A-module.

Given two artin algebras A and B, we say that A and B are stably equivalent if their stable categories $\underline{mod}A$ and $\underline{mod}B$ are equivalent. The Auslander-Reiten translate $\tau = DTr$ over an artin algebra and the Heller functor (i.e. the syzygy functor) Ω over a self-injective algebra are typical examples of stable self-equivalences. For basic material on stable equivalence, we refer the reader to [2], [3], [4].

Let A be an artin algebra. In [3], Auslander and Reiten defined $e(A)$ to be the full additive subcategory of mod$A$ whose indecomposable objects are the indecomposable non-injective objects X in mod$A$, such that if $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an almost split sequence, then X or Y is projective. Using the notion of node introduced by Martinez-Villa (cf. [17]), the indecomposable objects of $e(A)$ consist of precisely the following three classes of modules: simple projective modules, nodes, and indecomposable non-simple non-injective projective modules. Clearly $e(A)$ has only a finite number of indecomposable modules. We denote by $e'(A)$ the full subcategory of $e(A)$ whose indecomposable objects are simple projective modules and nodes.

Let $\mathcal{C}$ be a class of A-modules. We denote by $\langle\mathcal{C}\rangle$ the full subcategory of mod$A$ consisting of modules which are direct summands of finite direct sums of objects in $\mathcal{C}$. For two subcategories $\mathcal{C}$ and $\mathcal{D}$ of mod$A$, we denote by $\langle\mathcal{C}\rangle * \langle\mathcal{D}\rangle$ the class of indecomposable A-modules Y such that there is a short exact sequence of the following form

$$(\dagger) \qquad 0 \longrightarrow X \longrightarrow Y \oplus P \longrightarrow Z \longrightarrow 0,$$

where $Z \in \langle \mathcal{D} \rangle$, $X \in \langle \mathcal{C} \cup e(A) \rangle$, and P is a projective A-module. We put $\langle \mathcal{C} \rangle_1 = \langle \mathcal{C} \rangle$ and we define inductively $\langle \mathcal{C} \rangle_n = \langle \langle \mathcal{C} \rangle_{n-1} * \langle \mathcal{C} \rangle \rangle$ for $n \geq 2$.

Definition 2.1. *Let A be an artin algebra. A class of objects $\mathcal{S}$ in $mod_{\mathcal{P}} A$ is called a* simple-minded system *(for short: s.m.s.) if the following two conditions are satisfied:*

(1) *(orthogonality condition) For any $S, T \in \mathcal{S}$,*

$$\underline{Hom}_A(S, T) = \left\{ \begin{array}{cc} 0 & S \neq T, \\ division\ ring, & S = T. \end{array} \right.$$

(2) *(generating condition) For any indecomposable non-projective A-module X, there exists some natural number n (depending on X) such that $X \in \langle \mathcal{S} \rangle_n$.*

Remark 2.2. (1) *The definition of a simple-minded system formally depends on the chosen algebra A. In Theorem 3.2 we will see that in fact a simple-minded system depends only on the equivalence class of the stable module category $\underline{mod}A$. Therefore, instead of talking of a simple-minded system over A we may then also talk of a simple-minded system in $\underline{mod}A$.*

(2) *By definition, there is no simple-minded system over a semisimple algebra. From now on, we assume that all algebras considered are non-semisimple.*

(3) *Let $A = B \times C$ be a direct product of artin algebras. Then it is easy to see that the simple-minded systems over A are exactly of the forms $\mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ is a simple-minded system over B and $\mathcal{S}_2$ is a simple-minded system over C.*

(4) *When the algebra A is self-injective, its stable module category is triangulated. In this setup, parallel and independent work of Rickard and Rouquier [22] is discussing the problem of reconstructing A from its stable module category. They are also using ([22], 3.2, hypothesis 1) the orthogonality and generating conditions satisfied by the simple modules. Moreover, they are adding a splitting field assumption and a condition using that there are no extensions between simple modules in negative degrees. To formulate the latter condition needs the triangulated structure.*

The following lemma is an easy consequence of our definition.

Lemma 2.3. *Let A be an artin algebra.*

(1) *Suppose that $\mathcal{S}$ is a simple-minded system over A. Then for any $X \in \mathcal{S}$, X is an indecomposable non-projective module. Moreover, the objects in $\mathcal{S}$ are pairwise non-isomorphic.*

(2) *Let $\mathcal{S}$ be a complete set of non-isomorphic simple non-projective A-modules. Then $\mathcal{S}$ is a simple-minded system over A.*

(3) *If $\mathcal{S}$ is a simple-minded system, then $\mathcal{S}$ generates the stable Grothendieck group $G_0^{st}(A)$ of A.*

Proof (1) is a direct consequence of the orthogonality condition. For (2), the orthogonality condition is clear. To prove the generating condition, we use the natural exact sequence $0 \longrightarrow rad(X) \longrightarrow X \longrightarrow top(X) \longrightarrow 0$ and induction on the Loewy length $ll(X)$ for an indecomposable module X in $mod_{\mathcal{P}} A$. (3) is an easy consequence of the definition of $G_0^{st}(A)$ (refer to Remark 2.4) and the generating condition on $\mathcal{S}$.

$\square$

Remark 2.4. (1) *Let A be an artin algebra. Recall from [18] that the stable Grothendieck group $G_0^{st}(A)$ is by definition the cokernel of the Cartan map. In other words, there is the following short exact sequence*

$$K_0(A) \overset{C_A}{\to} G_0(A) \to G_0^{st}(A) \to 0,$$

where C_A is the Cartan matrix of A and where $K_0(A)$ (respectively, $G_0(A)$) is a free abelian group of finite rank generated by isomorphism classes of indecomposable projective modules (respectively, isomorphism classes of simple modules). For our purpose, we shall use the following equivalent definition of the stable Grothendieck group (cf. [17]): $G_0^{\mathrm{st}}(A)$ is the quotient group $\mathcal{L}_1/\mathcal{R}_1$, where $\mathcal{L}_1$ is the free group generated by the isomorphism classes $[X]$ of modules X in $\mathrm{mod}_\mathcal{P} A$, $\mathcal{R}_1$ is the subgroup generated by the following three classes of elements: (i) $[Y]-[X]-[Z]$, where $0 \longrightarrow X \oplus Q \longrightarrow Y \oplus P \longrightarrow Z \longrightarrow 0$ is an exact sequence such that Q, P are projective A-modules; (ii) $[X]+[Z]$, where $0 \longrightarrow X \oplus Q \longrightarrow P \longrightarrow Z \longrightarrow 0$ is an exact sequence such that Q, P are projective A-modules; (iii) $[Z]$, where $0 \longrightarrow Q \longrightarrow P \longrightarrow Z \longrightarrow 0$ is an exact sequence such that Q, P are projective A-modules. In particular, if A is a self-injective algebra, then $G_0^{\mathrm{st}}(A)$ is the quotient group $\mathcal{L}_1/\mathcal{R}_1$, where $\mathcal{L}_1$ is the free group generated by the isomorphism classes $[X]$ of modules X in $\mathrm{mod}_\mathcal{P} A$, $\mathcal{R}_1$ is the subgroup generated by the elements $[Y]-[X]-[Z]$ such that there is an exact sequence $0 \longrightarrow X \longrightarrow Y \oplus P \longrightarrow Z \longrightarrow 0$ with P projective.

(2) The statement (3) in Lemma 2.3 says that the simple-minded systems are just sets of generators of the stable Grothendieck group. For an artin algebra A of finite global dimension it can be easily proved that the stable Grothendieck group is trivial, while on the other hand, there still may exist nontrivial stable equivalence related to A and such information will be recorded in the simple-minded systems over A (see Theorem 3.2). This indicates that the simple-minded system is a finer notion than the stable Grothendieck group.

Before giving further properties of a simple-minded system, we prove the following lemma, to be used frequently.

Lemma 2.5. Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence of A-modules and M an A-module. If $\underline{Hom}_A(M,X) = \underline{Hom}_A(M,Z) = 0$, then $\underline{Hom}_A(M,Y) = 0$.

Proof Applying the functor $\mathrm{Hom}_A(-,M)$ to the above exact sequence and using the Auslander-Reiten formula, we get the following exact commutative diagram

$$
\begin{array}{ccccc}
Ext_A^1(Z,M) & \longrightarrow & Ext_A^1(Y,M) & \longrightarrow & Ext_A^1(X,M) \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
D\underline{Hom}_A(\tau^{-1}(M),Z) & \longrightarrow & D\underline{Hom}_A(\tau^{-1}(M),Y) & \longrightarrow & D\underline{Hom}_A(\tau^{-1}(M),X),
\end{array}
$$

where $D = \mathrm{Hom}_k(-,k)$ denotes the usual duality and $\tau^{-1} = TrD$ is the inverse of the Auslander-Reiten translation. When M runs through $\mathrm{mod}A$, $\tau^{-1}(M)$ runs through $\mathrm{mod}_\mathcal{P} A$. The lemma thus follows.

$\square$

Remark 2.6. For self-injective algebras, the same result holds true for $\underline{Hom}_A(-,M)$ and both results are special cases of [9, Lemma 1.4]. However, Lemma 2.5 does not hold for $\underline{Hom}_A(-,M)$ in general. For example, let A be a path algebra over a field given by the quiver
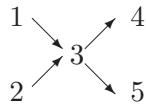
There is an exact sequence of A-modules $0 \longrightarrow 4 \longrightarrow \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \longrightarrow 3 \longrightarrow 0$. Here, $\underline{Hom}_A(4, \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}) = \underline{Hom}_A(3, \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}) = 0$, but $\underline{Hom}_A(\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}) \neq 0$.

The next proposition collects some elementary facts on a simple-minded system.

Proposition 2.7. *Let A be an artin algebra and let $\mathcal{S}$ be a simple-minded system over A. Then we have the following.*

(1) *$\mathcal{S}$ contains (up to isomorphism) any simple non-projective injective module.*

(2) *$\mathcal{S}$ contains (up to isomorphism) any node.*

(3) *Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two classes of objects in $\mathrm{mod}_{\mathcal{P}}A$ such that $\mathcal{S}_1 \subsetneqq \mathcal{S} \subsetneqq \mathcal{S}_2$. Then neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ is a simple-minded system.*

(4) *The number of objects in $\mathcal{S}$ is finite, that is, the cardinality $|\mathcal{S}| < \infty$.*

Proof (1) Let S be a simple non-projective injective module. Suppose that $\mathcal{S}$ does not contain S. Then S can be generated by an exact sequence of the following form

$$0 \longrightarrow X \longrightarrow S \oplus P \longrightarrow Z \longrightarrow 0,$$

where $X \in \mathrm{mod}A$, $Z \in \langle \mathcal{S} \rangle$ and P is a projective A-module. Moreover, we can assume that the morphism $S \longrightarrow Z$ in the above exact sequence is nonzero. But then $S \longrightarrow Z$ splits and therefore S is a direct summand of Z. This contradicts the assumption that $\mathcal{S}$ does not contain S and (1) follows.

(2) Similarly, suppose that S is a node and that $\mathcal{S}$ does not contain S. Let $S \in \langle \mathcal{S} \rangle_n \setminus \langle \mathcal{S} \rangle_{n-1}$. Then S can be generated by an exact sequence of the following form

$$0 \longrightarrow X \longrightarrow S \oplus P \longrightarrow Z \longrightarrow 0,$$

where $X \in \langle \mathcal{S} \rangle_{n-1}$, $Z \in \langle \mathcal{S} \rangle$ and P is a projective A-module. Clearly neither X nor Z contains a summand isomorphic to S. Since S is a node, we have an almost split sequence $0 \longrightarrow S \longrightarrow E \longrightarrow \tau^{-1}(S) \longrightarrow 0$ with E projective. Since X contains no summand isomorphic to S, any homomorphism $S \longrightarrow X$ must factor through the left almost split homomorphism $S \longrightarrow E$ and therefore $\underline{Hom}_A(S, X) = 0$. Similarly, $\underline{Hom}_A(S, Z) = 0$. It follows from Lemma 2.5 that $\underline{Hom}_A(S, S) = 0$. This contradiction shows that S is an object in $\mathcal{S}$.

(3) We only need to prove the statement for $\mathcal{S}_1$. Suppose that $\mathcal{S}_1$ is a simple-minded system and that $X \in \mathcal{S} \setminus \mathcal{S}_1$. Then X is generated from objects in $\langle \mathcal{S}_1 \cup e(A) \rangle$. By (2) and the definition of simple-minded systems, X is left orthogonal (in the stable category) to every object in $\langle \mathcal{S}_1 \cup e(A) \rangle$. It follows from Lemma 2.5 that $\underline{Hom}_A(X, X) = 0$. This is a contradiction and therefore $\mathcal{S}_1$ is not a simple-minded system.

(4) To generate all simple A-modules, we only need a finite number of objects in $\mathcal{S}$, say, $X_1, \cdots, X_n$. We can assume that $\{X_1, \cdots, X_n\}$ contains all nodes (otherwise, we just add the nodes into this set). We shall prove that $\mathcal{S} = \{X_1, \cdots, X_n\}$. Suppose that there exists $X \in \mathcal{S} \setminus \{X_1, \cdots, X_n\}$. Since $\underline{Hom}_A(X, X_i) = 0$ for all $1 \leq i \leq n$, by Lemma 2.5, we have that $\underline{Hom}_A(X, S) = 0$ for any simple module S. This is clearly a contradiction and therefore our conclusion follows.

$\square$

For self-injective algebras, any projective summand of X in the generating sequence (†) can be cancelled. Thus we get the following result.

Corollary 2.8. *Let A be a self-injective algebra and let $\mathcal{S}$ be a simple-minded system over A. For any indecomposable non-projective A-module X, there is a projective A-module P and a filtration*

$$X \oplus P = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$$

with the subquotients in $\langle \mathcal{S} \rangle$.

The above corollary suggests the following definition.

Definition 2.9. *Let A be a self-injective algebra and let $\mathcal{S}$ be a simple-minded system over A. For any indecomposable non-projective A-module X, we define $\mu(X)$ as the minimum integer m such that there is a projective A-module P and a filtration*

$$X \oplus P = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$$

with the subquotients in $\langle \mathcal{S} \rangle$. For a general A-module X, we define $\mu(X) = \max \mu(Y)$ where Y runs through all the indecomposable non-projective summands of X.

Proposition 2.10. *Let A be a self-injective algebra and let $\mathcal{S}$ be a simple-minded system over A. Then the values μ can take are bounded above as follows: For any A-module X, there is an inequality $\mu(X) \leq n_0 \cdot ll(A)$, where $ll(A)$ denotes the usual Loewy length of the regular module A and $n_0 = \mu(A/radA)$.*

Proof The proof proceeds by induction on the usual Loewy-length $ll(X)$ of X. Without loss of generality, we can assume that X is an indecomposable non-projective A-module. If $ll(X) = 1$, then X is a simple module and clearly $\mu(X) \leq n_0$. Now assume that $ll(X) = n > 1$. There is an exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0,$$

where $Y, Z \in \mathrm{mod}_{\mathcal{P}}A$, $ll(Y) = n - 1$ and Z is semisimple. By induction, there is a projective module P_1 such that $Y \oplus P_1$ has a $\langle \mathcal{S} \rangle$-filtration of length $\leq (n-1)n_0$, and there is a projective module P_2 such that $Z \oplus P_2$ has a $\langle \mathcal{S} \rangle$-filtration of length $\leq n_0$. It follows that the module $X \oplus P_1 \oplus P_2$ has a $\langle \mathcal{S} \rangle$-filtration of length $\leq (n-1)n_0 + n_0 = nn_0 \leq n_0 \cdot ll(A)$.

$\square$

3. Invariance under stable equivalences

In this section, we shall prove that the simple-minded systems are preserved by any stable equivalence. First of all, we recall some basic facts on functor categories due to Auslander and Reiten (cf. [2], [3]).

Let A be an artin algebra. We denote by $\mathrm{mod}(\mathrm{mod}A)$ the category of finitely presented contravariant functors F from $\mathrm{mod}A$ to abelian groups. By definition, $F \in \mathrm{mod}(\mathrm{mod}A)$ if and only if there is a morphism $f : X \longrightarrow Y$ in $\mathrm{mod}A$ such that F is the cokernel of the morphism

$$(-, f) : (-, X) \longrightarrow (-, Y),$$

where $\mathrm{Hom}_A(-, X) = (-, X)$ and $\mathrm{Hom}_A(-, f) = (-, f)$. Moreover, we denote by $\underline{mod}(\mathrm{mod}A)$ the full subcategory of $\mathrm{mod}(\mathrm{mod}A)$ whose objects are the functors which vanish on projective modules. The categories $\mathrm{mod}(\mathrm{mod}A)$ and $\underline{mod}(\mathrm{mod}A)$ have enough projective objects and enough injective objects. There is a natural functor $\underline{mod}A \longrightarrow \underline{mod}(\mathrm{mod}A)$ given by sending X to $(-, X)$, where $(-, X)(Y) = \underline{Hom}_A(Y, X)$, which induces an equivalence between $\underline{mod}A$ and the full subcategory of projective objects in $\underline{mod}(\mathrm{mod}A)$. In particular, we have that two artin algebras A and B are stably equivalent if and only if the categories $\underline{mod}(\mathrm{mod}A)$ and $\underline{mod}(\mathrm{mod}A)$ are equivalent. Notice also that the injective objects in $\underline{mod}(\mathrm{mod}A)$ are of the form $\mathrm{Ext}^1_A(-, X)$ with $X \in \mathrm{mod}A$.

The following lemma extends the result in [3, Lemma 3.4].

Lemma 3.1. *Let $\alpha : \underline{mod}A \longrightarrow \underline{mod}B$ be a stable equivalence and X be an indecomposable non-injective A-module. Denote also by α the induced equivalence: $\underline{mod}(\mathrm{mod}A) \longrightarrow \underline{mod}(\mathrm{mod}B)$. Then there is the following correspondence:*

$$\alpha(\mathrm{Ext}^1_A(-, X)) \simeq \begin{cases} \mathrm{Ext}^1_B(-, \alpha(X)) & \text{if } X \text{ is not in } e(A), \\ \mathrm{Ext}^1_B(-, Y) \text{ for some } Y \in e(B) & \text{if } X \text{ is in } e(A). \end{cases}$$

Moreover, if X is in $e'(A)$, then we also have $Y \in e'(B)$.

Proof Since X is an indecomposable non-injective A-module, the functor $Ext_A^1(-, X)$ is an indecomposable injective object in $\underline{mod}(modA)$. It follows that $\alpha(Ext_A^1(-, X))$ is an indecomposable injective object in $\underline{mod}(modB)$. If X is not in $e(A)$, by [3, Lemma 3.4], we have that $\alpha(Ext_A^1(-, X)) \simeq Ext_B^1(-, \alpha(X))$ with $\alpha(X)$ not in $e(B)$. If $X \in e(A)$, then $\alpha(Ext_A^1(-, X)) \simeq Ext_B^1(-, Y)$ for some indecomposable non-injective B-module Y since every indecomposable injective object in $\underline{mod}(modB)$ has this form. We claim that $Y \in e(B)$. Suppose that Y is not in $e(B)$. Again by [3, Lemma 3.4], we know that $X = \alpha^{-1}(Y)$ is not in $e(A)$. This contradiction shows that $Y \in e(B)$.

Now we suppose that $X \in e'(A)$. Then there is an almost split sequence

$$0 \longrightarrow X \xrightarrow{f} P \xrightarrow{g} Z \longrightarrow 0$$

with P a projective A-module. By [3, Proposition 2.1], we have an exact sequence

$$0 \longrightarrow \underline{(-, Z)} \longrightarrow Ext_A^1(-, X)$$

in $\underline{mod}(modA)$ such that $Ext_A^1(-, X)$ is an injective envelope of $\underline{(-, Z)}$. We consider the following almost split sequence

$$0 \longrightarrow Y \xrightarrow{f'} Q \xrightarrow{g'} \alpha(Z) \longrightarrow 0,$$

where $Y, Q \in modB$. We claim that Q is a projective B-module and therefore $Y \in e'(B)$. Otherwise, Q contains a non-projective direct summand and it follows from [4, Proposition 1.3] that P contains a non-projective direct summand, which is a contradiction! Again by [3, Proposition 2.1], there is an exact sequence

$$0 \longrightarrow \underline{(-, \alpha(Z))} \longrightarrow Ext_B^1(-, Y)$$

in $\underline{mod}(modB)$ such that $Ext_B^1(-, Y)$ is an injective envelope of $\underline{(-, \alpha(Z))}$. Since under the equivalence $\alpha : \underline{mod}(modA) \longrightarrow \underline{mod}(modB)$, the functor $\underline{(-, Z)}$ corresponds to $\underline{(-, \alpha(Z))}$, it follows that $\alpha(Ext_A^1(-, X)) \simeq Ext_B^1(-, Y)$.

$$\square$$

Theorem 3.2. *Let $\alpha : \underline{mod}A \longrightarrow \underline{mod}B$ be a stable equivalence and $\mathcal{S}$ be a simple-minded system over A. Then $\alpha(\mathcal{S})$ is a simple-minded system over B.*

Proof Obviously, $\alpha(\mathcal{S})$ is a class of objects in $mod_{\mathcal{P}}B$ and satisfies the orthogonality condition in $\underline{mod}B$. It remains to prove that $\langle \alpha(\mathcal{S}) \cup e(B) \rangle$ generates any module in $mod_{\mathcal{P}}B$. First we prove the following

Claim: Let $0 \longrightarrow X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \longrightarrow 0$ be an exact sequence in $modA$ which contains no split exact summands, where $X \in \langle mod_{\mathcal{P}}A \cup e(A) \rangle$, Y and Z are non-zero, $Z \in \langle \mathcal{S} \rangle$, $Y \in mod_{\mathcal{P}}A$ and P is a projective A-module. Then there is an exact sequence $0 \longrightarrow K \xrightarrow{f'} \alpha(Y) \oplus P' \xrightarrow{g'} \alpha(Z) \longrightarrow 0$ in $modB$ which contains no split exact summands, where $K \in \langle \alpha(X) \cup e(B) \rangle$ and P' is a projective B-module.

Proof of Claim: Since $0 \longrightarrow X \xrightarrow{f} Y \oplus P \xrightarrow{g} Z \longrightarrow 0$ is an exact sequence in $modA$ with no split exact summands, by [3, Proposition 2.1], we know that

$$\underline{(-, Y)} \xrightarrow{(-, g)} \underline{(-, Z)} \longrightarrow F \longrightarrow 0$$

is a minimal projective presentation of $F = Coker(-, g)$ in $\underline{mod}(modA)$, and that

$$0 \longrightarrow F \longrightarrow Ext_A^1(-, X) \xrightarrow{Ext_A^1(-, f)} Ext_A^1(-, Y)$$

is a minimal injective presentation of $F = Coker(-, g)$ in $\underline{mod}(modA)$.

Since $\alpha : \underline{mod}A \longrightarrow \underline{mod}B$ is a stable equivalence, we can choose $g'' : \alpha(Y) \longrightarrow \alpha(Z)$ such that $\alpha(\underline{g}) = \underline{g''}$ and choose $t : P' \longrightarrow \alpha(Z)$ such that $P' \longrightarrow \alpha(Z) \longrightarrow Cokerg''$ is a projective cover. Let $g' = (g'', t) : \alpha(Y) \oplus P' \longrightarrow \alpha(Z)$, and consider the exact sequence

$$0 \longrightarrow K \xrightarrow{f'} \alpha(Y) \oplus P' \xrightarrow{g'} \alpha(Z) \longrightarrow 0.$$

Clearly this sequence has no split exact summands. So, again by [3, Proposition 2.1], there is an exact sequence

$$(-, \alpha(Y)) \xrightarrow{(-, \alpha(g))} (-, \alpha(Z)) \longrightarrow G \longrightarrow Ext^1_B(-, K)$$

in $\underline{mod}(modB)$, where $G = Coker(-, \alpha(g)) = \alpha(F)$ and $G \longrightarrow Ext^1_B(-, K)$ is an injective envelope of G. It follows that $\alpha(Ext^1_A(-, X)) \simeq Ext^1_B(-, K)$. Write down $X = X_1 \oplus \cdots \oplus X_m \oplus X_{m+1} \oplus \cdots \oplus X_n$, where each X_i $(1 \le i \le n)$ is an indecomposable non-injective A-module and X_i is not in $e(A)$ for $1 \le i \le m$, $X_i \in e(A)$ for $m+1 \le i \le n$. By Lemma 3.1, $Ext^1_B(-, K) \simeq \alpha(Ext^1_A(-, X)) \simeq \bigoplus_{i=1}^n \alpha(Ext^1_A(-, X_i)) = Ext^1_B(-, \alpha(X_1)) \oplus \cdots \oplus Ext^1_B(-, \alpha(X_m)) \oplus \cdots \oplus Ext^1_B(-, K_n)$ for some K_i $(m+1 \le i \le n) \in e(B)$. Therefore $K \simeq \alpha(X_1) \oplus \cdots \oplus \alpha(X_m) \oplus K_{m+1} \oplus \cdots \oplus K_n \in \langle \alpha(X) \cup e(B) \rangle$. This finishes the proof of Claim.

From the above claim, it is easy to see that $\alpha(\langle S \rangle_n) = \langle \alpha(S) \rangle_n$ for any natural number n. It follows that $\langle \alpha(S) \cup e(B) \rangle$ generates any module in $mod_{\mathcal{P}}B$.

$\square$

The theorem shows that simple-minded systems are stably invariant. As an application, we determine the simple-minded systems over the 4-dimensional weakly symmetric local k-algebra $A_t = k < x, y > /(x^2, y^2, xy - tyx)$, where k is an algebraically closed field and $t \ne 0$ is an element in k. If $chark = 2$ and $t = 1$, then A_t is isomorphic to the group algebra of the Klein 4-group.

Let S be the unique (up to isomorphism) simple A_t-module. The Auslander-Reiten quiver of A_t is known to have a component $\mathcal{C}$ containing S and a $\mathbb{P}_1(k)$-family of homogenous tubes. For any non-projective module X in the component $\mathcal{C}$, X is an image of S under some appropriate composition of the stable equivalence functors DTr and Ω. It follows that each non-projective A_t-module $X \in \mathcal{C}$ defines a simple-minded system over A_t. Conversely, every simple-minded system over A_t is of this form. In fact, if there is another kind of simple-minded system S over A_t, then by Proposition 2.7 (3), S should not contain any module in $\mathcal{C}$. So each $X \in S$ has even dimension. Since the unique indecomposable projective A_t-module is also even-dimensional, S can only generate even-dimensional modules, a contradiction!

The above argument works for all group algebras A of finite p-groups if the characteristic of the field k is the prime number p. Indeed, by a result of Carlson ([6]), an A-module M satisfies $\underline{Hom}_A(M, M) = k$ if and only if M is an endotrivial module. On the other hand, each endotrivial module M induces a stable self-equivalence of Morita type over A such that the unique simple module k is mapped to M. Therefore every endotrivial module defines a simple-minded system and these are all simple-minded systems over A. Theorem 3.2 implies:

Corollary 3.3. *The Auslander-Reiten conjecture holds true for two algebras (i.e. two algebras related by a stable equivalence have the same number of non-projective simple modules up to isomorphism) if one of the algebras is a group algebra of a finite p-group in characteristic p.*

This result is due to Linckelmann ([11, Theorem 3.4]).

Clearly, in the above examples, each simple-minded system can be obtained from the simple modules by applying a suitable stable self-equivalence. However, the following example shows that simple-minded systems over an artin algebra are in general not acted upon transitively by the group of stable self-equivalences. It may be interesting to determine all the artin algebras with transitive action of the stable self-equivalences on the simple-minded systems.

Example 3.4. *Let k be a field. Let A be a finite dimensional k-algebra with the following regular representation*

$$A \;=\; \begin{matrix} 1' \\ 2' \\ 1' \end{matrix} \oplus \begin{matrix} 2' \\ 1' \end{matrix} \quad \begin{matrix} 2' \\ 3' \\ 2' \end{matrix} \oplus \begin{matrix} 3' \\ 2' \\ 3' \end{matrix}$$

and let B be a finite dimensional k-algebra with the following regular representation

$$B \;=\; \begin{matrix} 1 \\ 2 \\ 3 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 3 \\ 1 \\ 2 \end{matrix} \oplus \begin{matrix} 3 \\ 1 \\ 2 \\ 3 \end{matrix} \,.$$

Both A and B are representation-finite and symmetric, and there is a stable equivalence of Morita type α between B and A such that $\alpha(1) = 1', \alpha(2) = \begin{smallmatrix} 3' \\ 2' \end{smallmatrix}, \alpha(3) = \begin{smallmatrix} 2' \\ 3' \end{smallmatrix}$ (cf. [15, Section 6]). By Theorem 3.2, $\{1', \begin{smallmatrix} 3' \\ 2' \end{smallmatrix}, \begin{smallmatrix} 2' \\ 3' \end{smallmatrix}\}$ is a simple-minded system. However, there is no stable self-equivalence β over A such that $\beta(\{1', 2', 3'\}) = \{1', \begin{smallmatrix} 3' \\ 2' \end{smallmatrix}, \begin{smallmatrix} 2' \\ 3' \end{smallmatrix}\}$. In fact, if β is such a stable self-equivalence then β must be a stable self-equivalence of Morita type (cf. [1]). Therefore the composition $\beta^{-1}\alpha$ is a stable equivalence of Morita type between B and A under which each simple B-module corresponds to a simple A-module. It follows from Linckelmann's theorem ([11, Theorem 2.1]) that B and A are Morita equivalent, which is clearly a contradiction.

The next example shows that simple-minded systems over an artin algebra may even fail to be acted upon transitively by arbitrary stable equivalences. This is in contrast with the situation for derived categories. Here, the main result of Rickard's work on derived equivalences for symmetric algebras in [21, Theorem 5.1] shows transitivity.

Example 3.5. *Let k be a field. Let A be a finite dimensional k-algebra with the following regular representation*

$$A \;=\; \begin{matrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{matrix} \oplus \begin{matrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \end{matrix} \,.$$

Suppose that B is another finite dimensional k-algebra such that there is a stable equivalence α between B and A. As in the above example, α is lifted to a stable equivalence of Morita type. Since A is an indecomposable representation-finite symmetric algebra, so is B (by [14]). It follows easily that A and B are Morita equivalent. Without loss of generality we can identify B and A and assume that α is a stable self-equivalence of Morita type over A. By Proposition 6.2, one can verify that there are precisely four simple-minded systems over A:

$$\mathcal{S}_1 = \{1, 2\}; \ \mathcal{S}_2 = \{1, \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}\}; \ \mathcal{S}_3 = \{\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}, 2\}; \ \mathcal{S}_4 = \{\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 1 \end{smallmatrix}\}.$$

Clearly α commutes with the syzygy functor Ω, and therefore α cannot map the simple-minded system $\mathcal{S}_1$ to $\mathcal{S}_2$.

4. Simple-minded systems and triangular algebras

In this section, we apply simple-minded systems to study triangular algebras and one-point extension algebras.

For simplicity, throughout this section, we consider (finite dimensional) quiver algebras of the form kQ/I, where k is a field, Q is a quiver and I is an admissible ideal in kQ. Recall that a quiver algebra kQ/I is said to be a triangular algebra if there is no oriented cycle in its quiver Q.

Proposition 4.1. *If $A = kQ/I$ is a triangular algebra, then A has only one simple-minded system $\mathcal{S} = \{$simple non-projective A-modules$\}$.*

Proof Clearly we can assume that the quiver Q contains no isolated vertices. Suppose that $\mathcal{S}$ is a simple-minded system. Then, by Proposition 2.7, $\mathcal{S}$ must contain all simple injective A-modules, say, $L_{11}, \cdots, L_{1i_1}$ (which correspond to the source vertices in the quiver Q of A). By the orthogonal condition, every other module in $\mathcal{S}$ has no composition factor isomorphic to L_{1j} ($1 \leq j \leq i_1$). Let $\{L_{21}, \cdots, L_{2i_2}\}$ be simple A-modules which correspond to such vertices v_L that v_L is not a sink vertex but is next to a source vertex in the quiver Q. Take such a simple A-module L which corresponds to a vertex v_L, that is, we are in the following situation:
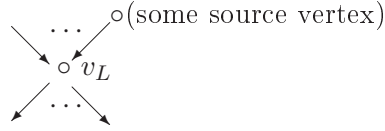
Then L can be generated by an exact sequence of the following form

$$0 \longrightarrow X \longrightarrow L \oplus P \longrightarrow Z \longrightarrow 0,$$

where $X \in \mathrm{mod}A$, $Z \in \langle \mathcal{S} \rangle$ and P is a projective A-module. Thus there exists some $S \in \mathcal{S}$ such that $L \subseteq soc(S)$. However, if L_{2j} ($1 \leq j \leq i_2$) is a composition factor of a module in $\mathcal{S}$, then L_{2j} must occur in its top. It follows that L is also contained in $top(S)$. This forces $L \simeq S$ by the indecomposability property. We thus proved that $\mathcal{S}$ contains the simple modules $L_{21}, \cdots, L_{2i_2}$. Observe that the modules other than the above two classes of simple modules in $\mathcal{S}$ have no composition factor isomorphic to such simple modules. Next we consider the simple non-projective A-modules $L_{31}, \cdots, L_{3i_3}$ which are "next" to simple modules $L_{21}, \cdots, L_{2i_2}$. Continuing by this way, we can prove that $\mathcal{S}$ contains all the simple non-projective A-modules and therefore $\mathcal{S} = \{$simple non-projective A-modules$\}$.

$\square$

Corollary 4.2. *If $\alpha : \underline{modB} \longrightarrow \underline{modA}$ is a stable equivalence such that A is a triangular algebra, then α maps each non-projective simple B-module to a non-projective simple A-module, and therefore A and B have the same number of non-projective simple modules.*

Proof Since $\mathcal{S} = \{$simple non-projective B-modules$\}$ is a simple-minded system over B, $\alpha(\mathcal{S})$ is a simple-minded system. But $\mathcal{S}' = \{$simple non-projective A-modules$\}$ is the only simple-minded system. It follows that $\alpha(\mathcal{S}) = \mathcal{S}'$, and A and B have the same number of non-projective simple modules.

$\square$

The next result is a special case of the result in [8, Theorem 4.3]. Note that here we allow the algebras have nodes. Given two finite dimensional algebras A and B. Recall that A and B are said to be stably equivalent of Morita type if there are two bimodules $_AM_B$ and $_BN_A$ which are projective as left modules and as right modules such that we have bimodule isomorphisms:

$$_AM \otimes_B N_A \simeq {}_AA_A \oplus {}_AP_A, \quad _BN \otimes_A M_B \simeq {}_BB_B \oplus {}_BQ_B$$

where $_AP_A$ and $_BQ_B$ are projective bimodules.

Proposition 4.3. *Let A and B be algebras over a field k. Suppose that A and B have no semisimple summands and that their maximal semsimple quotient algebras are separable. If two bimodules $_AM_B$ and $_BN_A$ define a stable equivalence of Morita type between A and B such that B is a triangular algebra, then A and B are Morita equivalent.*

To prove Proposition 4.3, we need the following lemma which is a generalization of Linckelmann's result in [11, Theorem 2.1(ii)] for self-injective algebras.

Lemma 4.4. *Let A and B be two indecomposable nonsimple algebras over a field k, whose maximal semisimple quotient algebras are separable. If two indecomposable bimodules $_AM_B$ and $_BN_A$ define a stable equivalence of Morita type between A and B, then $N \otimes_A S$ is an indecomposable B-module for each simple A-module S.*

Proof Note that under the assumption of this lemma, by [8] we can assume that both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs. In particular, $N \otimes_A -$ and $M \otimes_B -$ map projective (injective, respectively) modules to projective (injective, respectively) modules, and P and Q are projective-injective bimodules. We first state two simple facts.

Fact 1: For any indecomposable non-(projective-injective) A-module X, $N \otimes_A X$ is a non-(projective-injective) B-module; Similarly, we have the result for $M \otimes_B -$.

Otherwise, $M \otimes_B N \otimes_A X \simeq X \oplus P \otimes_A X$ is a projective-injective A-module and so is X. This will be a contradiction!

Fact 2: For any indecomposable non-(projective-injective) A-module X, suppose that $N \otimes_A X \simeq Y \oplus E$ with Y an indecomposable non-(projective-injective) B-module. Then E is a projective-injective B-module.

Otherwise, let $E = Z \oplus E'$ with Z an indecomposable non-(projective-injective) B-module. Then $M \otimes_B N \otimes_A X \simeq X \oplus P \otimes_A X \simeq M \otimes_B Y \oplus M \otimes_B Z \oplus M \otimes_B E'$. The left hand side of this equality contains only one indecomposable non-(projective-injective) summand but the right hand side contains at least two indecomposable non-(projective-injective) summands. A contradiction!

Now let $S = Ae/radAe$ be a simple A-module. We want to show that $N \otimes_A S$ is an indecomposable B-module. There are two cases to be considered.

Case 1. $S = Ae/radAe$ is isomorphic to $soc(e'A)$ as left A-module for some indecomposable projective-injective right A-module (or equivalently, an indecomposable projective-injective A^{op}-module) $e'A$. (Note that $soc(e'A)$ is an ideal in A since $e'A$ is projective-injective, see [7, Section 9.2]). Then, for any indecomposable projective-injective B-module Bf, $Bf \otimes_k e'A$ is a projective-injective $B \otimes_k A^{op}$-module and $soc(Bf \otimes_k e'A) \simeq soc(Bf) \otimes_k soc(e'A)$ by the separability assumption. Since as a $B \otimes_k A^{op}$-module, N has no projective summands, $soc(Bf \otimes_k e'A)N = 0$. On the other hand, $soc(Bf \otimes_k e'A)N = soc(Bf)Nsoc(e'A) \simeq soc(Bf)(N \otimes_A soc(e'A)) \simeq soc(Bf)(N \otimes_A S) = 0$. This implies that the B-module $N \otimes_A S$ contains no projective-injective summands and therefore $N \otimes_A S$ is indecomposable by Fact 1 and 2.

Case 2. $S = Ae/radAe$ is not isomorphic to $soc(e'A)$ as left A-module for any indecomposable projective-injective right A-module $e'A$. Since P is a projective bimodule, we have a decomposition of the following form: $P = \bigoplus_{i,j} Ae_i \otimes_k e_jA$ where the e_i's and the e_j's are some primitive idempotents in A. Since P is also an injective bimodule, each Ae_i and each e_jA are also injective modules. By [8, Theorem 4.3], for any such idempotent e_j, Ae_j is projective-injective and $Ae_j/radAe_j$ is isomorphic to the socle of some projective-injective A-module. Therefore the idempotent e is different from any one of the above e_j. It follows that $e_jA \otimes_A (Ae/radAe) = 0$

for each e_j and $P \otimes_A (Ae/radAe) = 0$. This implies that $M \otimes_B N \otimes_A (Ae/radAe) \simeq (Ae/radAe)$ and therefore $N \otimes_A (Ae/radAe)$ must be an indecomposable B-module.

$\square$

Proof of Proposition 4.3. By [14], we can assume that both A and B are indecomposable k-algebras, and that ${}_A M_B$ and ${}_B N_A$ are indecomposable bimodules. For any simple A-module S, Lemma 4.4 implies that $N \otimes_A S$ is an indecomposable B-module. We want to show that $N \otimes_A S$ is simple. There are two cases to be considered.

Case 1. S is non-projective. In this case, Corollary 4.2 implies that $N \otimes_A S$ is simple.

Case 2. S is simple projective. In this case, by [13, Lemma 3.1], $N \otimes_A S$ must contain a simple projective summand and therefore is also simple.

We have proved that $N \otimes_A -$ maps each simple A-module to a simple B-module. By the generalization of Linckelmann's theorem (see [12, Theorem 1.1]), the functor $N \otimes_A - : \mathrm{mod}A \longrightarrow \mathrm{mod}B$ gives a Morita equivalence.

$\square$

Finally, we prove a general fact on simple-minded systems of one-point extension algebras.

Proposition 4.5. *Let B be a finite dimensional algebra over a field k and let $A = \begin{pmatrix} B & M \\ 0 & k \end{pmatrix}$ be a one-point extension algebra of B by a B-module M.*

(1) If $\mathcal{S}$ is a simple-minded system over B, then $\mathcal{S}' = \mathcal{S} \cup \{L\}$ is a simple-minded system over A, where L is the simple injective A-module with projective cover $\begin{pmatrix} M \\ k \end{pmatrix}$.

(2) Each simple-minded system has the form $\mathcal{S}' = \mathcal{S} \cup \{L\}$ where $\mathcal{S}$ is a simple-minded system over B and L is as above.

Proof (1) There is a canonical algebra epimorphism $A \longrightarrow B$ given by $\begin{pmatrix} b & m \\ 0 & x \end{pmatrix} \mapsto b$. So every B-module is automatically an A-module by this map. In particular, B is a projective A-module and the embedding functor ${}_A B \otimes_B - : \mathrm{mod}B \longrightarrow \mathrm{mod}A$ induces a functor ${}_A B \otimes_B - : \underline{\mathrm{mod}}B \longrightarrow \underline{\mathrm{mod}}A$. Note that ${}_A B \otimes_B - : \underline{\mathrm{mod}}B \longrightarrow \underline{\mathrm{mod}}A$ is a fully faithful functor, and that $e(A) \supseteq e(B)$.

Clearly we have $\underline{Hom}_A(X, Y) = 0$ for any $X, Y \in \mathcal{S}$. Since every A-module in $\mathcal{S}$ has no composition factor isomorphic to L, we also have $\underline{Hom}_A(X, L) = \underline{Hom}_A(L, X) = 0$ for any $X \in \mathcal{S}$. This proves the orthogonality condition for $\mathcal{S}'$. Now let Y be any indecomposable non-projective A-module. Notice that if L is a composition factor of Y, then L must occur in the top of Y. We consider two cases.

Case 1. Y has no composition factor isomorphic to L. In this case Y is a B-module and can be generated by $\langle \mathcal{S} \cup e(B) \rangle$. Since $B \otimes_B e(B) \subset e(A)$, we know that Y is generated by $\langle \mathcal{S}' \cup e(A) \rangle$.

Case 2. Y contains a composition factor isomorphic to L. We have an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow L^m \longrightarrow 0,$$

where m is a natural number and X contains no composition factor isomorphic to L. It is readily seen that this case is reduced to Case 1.

(2) Suppose that $\mathcal{S}'$ is a simple-minded system. Then, by Proposition 2.7, $\mathcal{S}'$ must contain the simple injective module L. So $\mathcal{S}' = \mathcal{S} \cup \{L\}$ with $\mathcal{S}$ a class of objects in $\mathrm{mod}_{\mathcal{P}}A$. For any $X \in \mathcal{S}$, X contains no composition factor isomorphic to L: otherwise, $L \in top(X)$ and $\underline{Hom}_A(X, L) \neq 0$. A contradiction! Therefore $\mathcal{S} \subseteq \mathrm{mod}_{\mathcal{P}}B$. We shall prove that $\mathcal{S}$ is a

simple-minded system over B. Obviously, $\mathcal{S}$ satisfies the orthogonality condition in $\underline{mod}B$ since $\underline{mod}B$ is a full subcategory of $\underline{mod}A$. Now let Y be an indecomposable B-module in $\mathrm{mod}_{\mathcal{P}}B$. Then without loss of generality we can assume that the last exact sequence in $\mathrm{mod}A$ which generates Y has the following form:

$$0 \longrightarrow X \longrightarrow Y \oplus P \longrightarrow Z \longrightarrow 0,$$

where $X \in \mathrm{mod}A$, $Z \in \langle \mathcal{S} \rangle$ and P is a projective B-module. It follows that X is a B-module and therefore all the exact sequences involved in generating Y lie in $\mathrm{mod}B$. So Y is generated by $\langle \mathcal{S} \cup e(B) \rangle$, and $\mathcal{S}$ is a simple-minded system over B.

$\square$

Remark 4.6. (1) *Using the above proposition, we get a simple proof of Proposition 4.1 as follows: without loss of generality we assume that A is an indecomposable algebra. Therefore A can be obtained by a finite number of one-point extensions from a single point and the conclusion follows immediately from Proposition 4.5.*

(2) *The result in Proposition 4.5 can not be generalized to triangular matrix algebras, i.e. algebras of the form $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are arbitrary algebras. For example, let k be an algebraically closed field. Let $A = B = k[x]/(x^3)$ be two finite dimensional algebras over k and let $M = k[x]/(x^3)$ be the natural A-B-bimodule. Consider the triangular matrix algebra $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \simeq k(\alpha \circlearrowleft 1 \xleftarrow{\beta} 2 \circlearrowright \gamma)/(\alpha^3, \gamma^3, \alpha\beta - \beta\gamma)$. Clearly $(x^2)/(x^3)$ is a simple-minded system which corresponds to the simple Λ-module 1, and $(x)/(x^3)$ is a simple-minded system over B which corresponds to the Λ-module $\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$. But $\{1, \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\}$ is not a simple-minded system over Λ since $\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$ is not self-orthogonal.*

5. SIMPLE-MINDED SYSTEMS AND SELF-INJECTIVE ALGEBRAS

In this section, we shall compare the simple-minded systems with Pogorzaly's maximal systems of stable orthogonal bricks over a self-injective algebra. We simplify Pogorzaly's definition and drop one condition used by him to exclude a few trivial cases; in this way we arrive at 'weakly simple-minded system'. For representation finite self-injective algebras we show these to coincide with the simple-minded systems defined before. Thus, for these algebras Pogarzaly's concept essentially coincides with ours. Once this has been achieved, we introduce the notion of stable Loewy length for modules in a stable category.

Let A be a self-injective algebra over an algebraically closed field k. Recall from [20, 19] that an indecomposable A-module X in $\underline{mod}A$ is said to be a stable A-brick if its stable endomorphism ring $\underline{End}_A(X)$ is isomorphic to k. A family $\{X_i\}_{i \in I}$ of stable A-bricks is said to be a maximal system of orthogonal stable A-bricks if the following conditions are satisfied:

(1) $\tau(X_i) \not\cong X_i$ for any $i \in I$;

(2) $\underline{Hom}_A(X_i, X_j) = 0$ for any $i \neq j$;

(3) For any nonzero object $X \in \underline{mod}A$, there exists some $i \in I$ such that $\underline{Hom}_A(X, X_i) \neq 0$ and there exists some $j \in I$ such that $\underline{Hom}_A(X_j, X) \neq 0$.

Note that the above definition can be simplified. Indeed, one half of the assumption in condition (3) is enough: the two conditions "For any nonzero object $X \in \underline{mod}A$, there exists some $i \in I$ such that $\underline{Hom}_A(X, X_i) \neq 0$" and "For any nonzero object $X \in \underline{mod}A$, there exists some $j \in I$ such that $\underline{Hom}_A(X_j, X) \neq 0$" are equivalent. This can be seen from a general fact on

stable categories over a self-injective algebra proved in [19]. The general fact was presented in the proof of [19, Proposition 1]. For convenience of the reader, we include the proof here.

Proposition 5.1. ([19, Proof of Proposition 1]) *Let A be a self-injective artin algebra. Let M be an indecomposable non-projective A-module and X be any indecomposable A-module. If there is a nonzero homomorphism $\underline{f} : X \longrightarrow M$ in $\underline{mod}A$, then there is a nonzero homomorphism $\underline{h} : \tau^{-1}\Omega(M) \longrightarrow X$ such that $\underline{fh} \neq 0$ in $\underline{mod}A$.*

Proof If $X \simeq M$, then there is a nonsplit exact sequence in mod$A$: $0 \longrightarrow \Omega(M) \longrightarrow P(M) \longrightarrow M \longrightarrow 0$, where $P(M) \longrightarrow M$ is a projective cover of M. It follows from the Auslander-Reiten formula that $\underline{Hom}_A(\tau^{-1}\Omega(M), M) \simeq Ext_A^1(M, \Omega(M)) \neq 0$.

Assume now that $X \not\simeq M$. Consider the following exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \longrightarrow & P(M) & \xrightarrow{l} & M & \longrightarrow & 0 \\
 & & {\scriptstyle j}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle 1}\downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X \oplus P(M) & \xrightarrow{(f,l)} & M & \longrightarrow & 0,
\end{array}
$$

where $f : X \longrightarrow M$ is a representative of $\underline{f}$ in mod$A$, $l : P(M) \longrightarrow M$ is a projective cover, i is a canonical embedding and j is induced from i. Applying the snake lemma we get the following exact sequence of A-modules:

$$0 \longrightarrow \Omega(M) \xrightarrow{j} Y \xrightarrow{s} X \longrightarrow 0.$$

Note that j is not a split monomorphism since otherwise s is a split epimorphism and therefore $f(X) = 0$, a contradiction! By the property of almost split sequence, we get the following exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \longrightarrow & Z & \xrightarrow{r} & \tau^{-1}\Omega(M) & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\downarrow & & {\scriptstyle t}\downarrow & & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & \Omega(M) & \xrightarrow{j} & Y & \xrightarrow{s} & X & \longrightarrow & 0,
\end{array}
$$

where the first row is an almost split sequence, h is induced from t. We first note that $h \neq 0$ since otherwise r will be a split epimorphism, and this is clearly a contradiction! Next we show that $\underline{h} \neq 0$. Suppose that this is not the case. We have the following exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \xrightarrow{r'} & Z & \xrightarrow{r} & \tau^{-1}\Omega(M) & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\downarrow & & {\scriptstyle t}\downarrow & {\scriptstyle P}\nearrow\!\!\searrow & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & \Omega(M) & \xrightarrow{j} & Y & \xrightarrow{s} & X & \longrightarrow & 0,
\end{array}
$$

where h factors through the projective cover P of X, the homomorphism $P \longrightarrow Y$ is induced from the property of projective modules, and we denote the composition map $\tau^{-1}\Omega(M) \longrightarrow P \longrightarrow Y$ by u. We have that $Im(t-ur) \subseteq j(\Omega(M))$ and that $(t-ur)r'(\Omega(M)) = tr'(\Omega(M)) = j(\Omega(M)) \simeq \Omega(M)$. Therefore r' is a split monomorphism which is clearly a contradiction. Finally, let us prove that $\underline{fh} \neq 0$. Indeed, if fh factors through $P(M)$ then $fh = lh'$ for some $h' : \tau^{-1}\Omega(M) \longrightarrow P(M)$, and consequently h factors through Y by the following pullback diagram

$$\begin{array}{ccc}
\tau^{-1}\Omega(M) & & \\
& \searrow \quad \xrightarrow{\ h\ } & \\
h' \downarrow & Y \xrightarrow{\ s\ } X & \\
& \downarrow \qquad \downarrow f & \\
P(M) \xrightarrow{\ l\ } & M. &
\end{array}$$

Hence there is $h_1 : \tau^{-1}\Omega(M) \longrightarrow Y$ with $h = sh_1$. Thus $st = sh_1 r$ and $Im(t - h_1 r) \subseteq Im(j)$. Then, as before, a contradiction can be deduced. This shows that $\underline{fh} \neq 0$.

<div align="right">□</div>

Corollary 5.2. *In the definition of maximal system of orthogonal bricks, the two conditions "For any nonzero object $X \in \underline{mod}A$, there exists some $i \in I$ such that $\underline{Hom}_A(X, X_i) \neq 0$" and "For any nonzero object $X \in \underline{mod}A$, there exists some $j \in I$ such that $\underline{Hom}_A(X_j, X) \neq 0$" are equivalent.*

Proof It suffices to prove it for X indecomposable. Suppose that the condition "For any nonzero object $X \in \underline{mod}A$, there exists some $i \in I$ such that $\underline{Hom}_A(X, X_i) \neq 0$" is satisfied. By Proposition 5.1, $\underline{Hom}_A(X_i, \Omega^{-1}\tau(X)) \simeq \underline{Hom}_A(\tau^{-1}\Omega(X_i), X) \neq 0$. When X runs through the nonzero objects in $\underline{mod}A$, so does $\Omega^{-1}\tau(X)$. Therefore we have proved the second condition. The proof of the other direction is similar.

<div align="right">□</div>

In order to compare simple-minded systems with Pogorzaly's maximal systems of orthogonal bricks over a self-injective algebra, we introduce the following definition (note that our definition here applies in any artin algebra).

Definition 5.3. *Let A be an artin algebra. A class of objects $\mathcal{S}$ in $mod_{\mathcal{P}}A$ is called a* weakly simple-minded system *if the following two conditions are satisfied:*

(1) *(orthogonality condition) For any $S, T \in \mathcal{S}$,*

$$\underline{Hom}_A(S, T) = \left\{ \begin{array}{cc} 0 & S \neq T, \\ division\ ring, & S = T. \end{array} \right.$$

(2) *(weak generating condition) For any indecomposable non-projective A-module X, there exists some $S \in \mathcal{S}$ (depends on X) such that $\underline{Hom}_A(X, S) \neq 0$.*

Remark 5.4. *According to Remark 2.6, for general artin algebras, the weak generating condition in Definition 5.3 is not symmetric, that is, "$\underline{Hom}_A(X, S) \neq 0$" cannot be replaced by "$\underline{Hom}_A(S, X) \neq 0$".*

It is easy to see that every simple-minded system is a weakly simple-minded system. The reason is as follows: Let A be an artin algebra and let $\mathcal{S}$ be a simple-minded system. To show that $\mathcal{S}$ is a weakly simple-minded system, we only need to prove the weak generating condition. Let $0 \neq X \in mod_{\mathcal{P}}A$. Suppose that $\underline{Hom}_A(X, T) = 0$ for all $T \in \mathcal{S}$. Then we have that $\underline{Hom}_A(X, S) = 0$ for any simple module S (cf. the proof of Lemma 2.5). This is clearly a contradiction and therefore $\mathcal{S}$ satisfies the weak generating condition. Thus the question arises: Is every weakly simple-minded system also a simple-minded system?

At least for representation-finite self-injective finite dimensional algebras, we can prove that the above question has a positive answer. First we need a lemma. Let A be a finite dimensional algebra over a field k and let $\mathcal{S} = \{M_1, \cdots, M_n\}$ be a weakly simple-minded system. Let X

be an A-module in $\text{mod}_{\mathcal{P}}A$. Suppose that $\dim_k \underline{Hom}_A(X, M_i) = d_i$ for $1 \leq i \leq n$. Following [19], we will say that $\bigoplus_{i=1}^n M_i^{d_i}$ is an s-top of X with respect to $\mathcal{S}$. Of course, s-top(X) is well-defined for X. We consider the following exact sequence in $\text{mod}A$:

$$(*) \qquad 0 \longrightarrow X_1 \overset{h=(h',h'')}{\longrightarrow} X \oplus P \overset{(f,g)}{\longrightarrow} \text{s-top}(X) \longrightarrow 0,$$

where $f : X \longrightarrow \text{s-top}(X)$ is such a morphism that the coordinates of $\underline{f}$ form a basis of the nonzero k-space $\underline{Hom}_A(X, \text{s-top}(X))$ and $g : P \longrightarrow \text{s-top}(X)$ is such a morphism that $P \longrightarrow \text{s-top}(X) \longrightarrow Coker(f)$ is a projective cover.

Lemma 5.5. *Let X be an A-module in $\text{mod}_{\mathcal{P}}A$. Up to isomorphism, the non-projective part of the module X_1 in the above sequence $(*)$ is independent of the choice of the homomorphism $f : X \longrightarrow s\text{-}top(X)$.*

Proof First we note that if we replace $g : P \longrightarrow \text{s-top}(X)$ in the above sequence $(*)$ by the projective cover $g' : Q \longrightarrow \text{s-top}(X)$, then $\ker(f, g')$ and X_1 have isomorphic non-projective part. Now we choose another homomorphism $f' : X \longrightarrow \text{s-top}(X)$ such that the coordinates of $\underline{f'}$ still form a k-basis of $\underline{Hom}_A(X, \text{s-top}(X))$. There clearly is an A-module isomorphism $\alpha : \text{s-top}(X) \longrightarrow \text{s-top}(X)$ such that $\alpha f - f'$ factors through the projective cover $q_2 : P' \longrightarrow \text{s-top}(X)$. More precisely, there is a homomorphism $q_1 : X \longrightarrow P'$ such that $\alpha f - f' = q_2 q_1$. Hence we get the following exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y_1 & \longrightarrow & X \oplus P' & \xrightarrow{(f,\alpha^{-1}q_2)} & \text{s-top}(X) & \longrightarrow & 0 \\
& & d\downarrow & & \left(\begin{smallmatrix} 1 & 0 \\ q_1 & 1 \end{smallmatrix}\right)\downarrow & & \alpha\downarrow & & \\
0 & \longrightarrow & Y_1' & \longrightarrow & X \oplus P' & \xrightarrow{(f',q_2)} & \text{s-top}(X) & \longrightarrow & 0,
\end{array}
$$

where d is induced from the isomorphism $\left(\begin{smallmatrix} 1 & 0 \\ q_1 & 1 \end{smallmatrix}\right)$. It follows that d is an isomorphism. In particular, Y_1 and Y_1' have isomorphic non-projective parts.

$\square$

Clearly, if A is a self-injective algebra, X_1 contains no projective summands. However, in general X_1 may contain projective summands (although by our assumption, X contains no projective summands). According to [19], we define the s-radical of X with respect to $\mathcal{S}$ to be the non-projective part of X_1 in the above sequence $(*)$. This is well-defined up to isomorphism, and we shall denote it by s-rad(X). Moreover, we denote s-rad(s-rad$^{i-1}(X)$) by s-rad$^i(X)$.

Theorem 5.6. *Let A be a representation-finite self-injective finite dimensional algebra over a field k and let $\mathcal{S} = \{M_1, \cdots, M_n\}$ be a weakly simple-minded system. Then $\mathcal{S}$ even is a simple-minded system.*

Proof We only need to prove the generating condition. Let X be an indecomposable non-projective A-module. Suppose that $\dim_k \underline{Hom}_A(X, M_i) = d_i$ for $1 \leq i \leq n$. As before, we consider the following exact sequence in $\text{mod}A$:

$$(*) \qquad 0 \longrightarrow X_1 \overset{h=(h',h'')}{\longrightarrow} X \oplus P \overset{(f,g)}{\longrightarrow} \text{s-top}(X) \longrightarrow 0,$$

where $f : X \longrightarrow \text{s-top}(X)$ is such a morphism that the coordinates of $\underline{f}$ form a basis of the nonzero k-space $\underline{Hom}_A(X, \text{s-top}(X))$ and $g : P \longrightarrow \text{s-top}(X)$ is such a morphism that $P \longrightarrow \text{s-top}(X) \longrightarrow Coker(f)$ is a projective cover. Let $M = \bigoplus_{i=1}^n M_i$. Since $\underline{mod}A$ is a triangulated category (with translation functor $\Omega^{-1} : \underline{mod}A \longrightarrow \underline{mod}A$) and the above exact sequence induces a triangle

$$X_1 \overset{h}{\longrightarrow} X \overset{f}{\longrightarrow} \text{s-top}(X) \overset{e}{\longrightarrow} \Omega^{-1}(X_1)$$

in $\underline{mod}A$, after applying the contravariant cohomological functor $\underline{Hom}_A(-, M)$ to the above triangle, we get the following long exact sequence of k-spaces

$$\cdots \longrightarrow {}_{(}\Omega^{-1}(X), M) \overset{(\Omega^{-1}(h), M)}{\Longrightarrow} {}_{(}\Omega^{-1}(X_1), M) \overset{(e, M)}{\Longrightarrow} {}_{(}\text{s-top}(X), M) \overset{(f, M)}{\Longrightarrow} {}_{(}X, M) \longrightarrow \cdots.$$

We claim that (f, M) is an isomorphism. Indeed, the spaces ${}_{(}\text{s-top}(X), M)$ and ${}_{(}X, M)$ have the same k-dimension $\Sigma_{i=1}^n d_i$ and the canonical basis elements of ${}_{(}\text{s-top}(X), M)$ map to the coordinates of $\underline{f}$ which form a basis of ${}_{(}X, M)$. It follows that $(\Omega^{-1}(h), M)$ is an epimorphism and that $\dim_k \underline{Hom}_A(\Omega^{-1}(X), M) \geq \dim_k \underline{Hom}_A(\Omega^{-1}(X_1), M)$. We can assume that $X_1 \neq 0$ since otherwise $X \simeq \text{s-top}(X) \in \langle \mathcal{S} \rangle$ and we are done. Note also that X_1 contains no projective summand. For any indecomposable summand of X_1 (we still denote it by X_1), we can similarly take an exact sequence as $(*)$ in $\text{mod}A$:

$$0 \longrightarrow X_2 \overset{h_1}{\longrightarrow} X_1 \oplus P_1 \overset{(f_1, g_1)}{\longrightarrow} \text{s-top}(X_1) \longrightarrow 0.$$

From this we also deduce a canonical epimorphism $(\Omega^{-1}(h_1), M) : {}_{(}\Omega^{-1}(X_1), M) \longrightarrow {}_{(}\Omega^{-1}(X_2), M)$ and get an inequality $\dim_k \underline{Hom}_A(\Omega^{-1}(X_1), M) \geq \dim_k \underline{Hom}_A(\Omega^{-1}(X_2), M)$. Continuing in this way, we obtain a sequence of epimorphisms between k-spaces:

$$ {}_{(}\Omega^{-1}(X), M) \overset{(\Omega^{-1}(h), M)}{\Longrightarrow} {}_{(}\Omega^{-1}(X_1), M) \overset{(\Omega^{-1}(h_1), M)}{\Longrightarrow} {}_{(}\Omega^{-1}(X_2), M) \overset{(\Omega^{-1}(h_2), M)}{\Longrightarrow} \cdots.$$

The above sequence is induced from the following sequence

$$\cdots \longrightarrow X_3 \overset{h_2}{\longrightarrow} X_2 \overset{h_1}{\longrightarrow} X_1 \overset{h}{\longrightarrow} X$$

in $\underline{mod}A$ and the latter one is again induced from the following sequence

$$(**) \qquad \cdots \longrightarrow X_3 \overset{h_2'}{\longrightarrow} X_2 \overset{h_1'}{\longrightarrow} X_1 \overset{h'}{\longrightarrow} X$$

in $\text{mod}A$. To finish our proof, it suffices to prove the following conclusion: there exists some natural number m such that $X_m = 0$ (and consequently $X_i = 0$ for all $i \geq m$).

By our assumption, all the modules in the above sequence $(**)$ are indecomposable. We claim that all homomorphisms in $(**)$ are non-isomorphisms. In fact, if in the original sequence $(*)$ the s-radical X_1 contains an indecomposable summand X_1' such that $h' : X_1' \longrightarrow X$ is an isomorphism, then the inequality $\dim_k \underline{Hom}_A(\Omega^{-1}(X), M) \geq \dim_k \underline{Hom}_A(\Omega^{-1}(X_1), M)$ implies that X_1 can not contain any other summands, and therefore X_1 must be isomorphic to X. This would lead to the absurd conclusion that the sequence $(*)$ splits and that $X \simeq X \oplus \text{s-top}(X)$. Similarly, one can show that all $h_i'(i \geq 1)$ are non-isomorphisms. Since A is representation-finite and the modules in $\text{mod}A$ have bounded length, by [4, Corollary 1.3], for some large m ($m \leq 2^b$, where b denotes the least upper bound of the lengths of the indecomposable modules in $\text{mod}A$), the composition $h'h_1'\cdots h_m'$ is zero in $\text{mod}A$. It follows that the composition $(\Omega^{-1}(h_m), M) \cdots (\Omega^{-1}(h_1), M) (\Omega^{-1}(h), M)$ is zero. Since all $(\Omega^{-1}(h_i), M)$ are epimorphisms, we know that $\underline{Hom}_A(\Omega^{-1}(X_m), M) = 0$. By the weak generating condition, we know that $\Omega^{-1}(X_m) = 0$. It follows that $X_m = 0$ since $\Omega^{-1} : \underline{mod}A \longrightarrow \underline{mod}A$ is an equivalence.

$\square$

Remark 5.7. *Suppose that A is any (not necessarily representation-finite) self-injective algebra over a field k. The above proof implies that for any indecomposable non-projective A-module X, s-rad(X) cannot contain a direct summand isomorphic to X. Indeed, if this is the case, we can take all X_i equal to X in the above proof, and finally we get that $X = 0$, which is a contradiction. Moreover, it is easy to see that all the modules in $\{s\text{-}rad^i(X)|i = 0, 1, 2, \cdots\}$ are pairwise disjoint, i.e. do not have isomorphic direct summands.*

It is well-known that the Loewy length is a very useful concept in the module category $\text{mod}A$. It would be interesting to generalize this notion to the stable module category $\underline{mod}A$. Here,

we replace the simple modules by a simple-minded system. Indeed, Lemma 5.5 supplies a way to define the stable Loewy length of an object in $\underline{mod}A$.

Definition 5.8. *Let A be a finite dimensional algebra over a field k and let $\mathcal{S}$ be a simple-minded system over A. For any indecomposable non-projective A-module X in $mod_{\mathcal{P}}A$, we define the stable Loewy length of X with respect to $\mathcal{S}$ (which we denote by s-ll(X)) to be the least number m such that $s\text{-}rad^m(X) = 0$. If there is no such m, then we define $s\text{-}ll(X) = \infty$. For a general module $X \in modA$, we define $s\text{-}ll(X)$ to be the stable Loewy length of its non-projective part.*

Corollary 5.9. *Let A be a representation-finite self-injective finite dimensional algebra over a field k and let $\mathcal{S}$ be any simple-minded system. Then the stable Loewy length satisfies the inequality $s\text{-}ll(X) \leq 2^b$ for any $X \in modA$, where b denotes the maximum of the lengths of the indecomposable modules in modA.*

Proof This is an easy consequence of the proof of Theorem 5.6.

$\square$

Example 5.10. *Consider the algebra A in Example 3.5. Both $\mathcal{S}_1 = \{1, 2\}$ and $\mathcal{S}_2 = \{1, \begin{smallmatrix} & 2 \\ & 1 \\ & 2 \end{smallmatrix}\}$ are simple-minded systems over A. For any indecomposable non-projective A-module X, the stable Loewy length s-ll(X) with respect to $\mathcal{S}_1$ is equal to the usual Loewy length ll(X). However, the stable Loewy length s-ll(X) with respect to $\mathcal{S}_2$ is usually different from ll(X). For example, the stable Loewy length of the A-module $\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 2 \end{smallmatrix}$ with respect to $\mathcal{S}_2$ is equal to 2 while its usual Loewy length is 4.*

6. Nakayama algebras

One motivation to define simple-minded systems is to explore the potential use of this concept for the Auslander-Reiten conjecture. This conjecture says that two stably equivalent artin algebras have the same number of non-isomorphic non-projective simple modules. Based on the observation in Theorem 3.2, we pose the following question.

Question 6.1. *Is the cardinality of each simple-minded system over an artin algebra A equal to the number of non-isomorphic non-projective simple A-modules?*

A positive answer to this question implies the Auslander-Reiten conjecture. Actually, Pogorzaly [20] used an analogous idea in his setup to prove the conjecture for self-injective special biserial algebras. We think that besides the relationship with Auslander-Reiten conjecture, Question 6.1 is interesting in itself. Proposition 4.1 shows that the answer is positive for triangular algebras. The next proposition answers this question for Nakayama algebras.

Proposition 6.2. *Let A be a Nakayama algebra and let $\mathcal{S}$ be a simple-minded system. Then the cardinality of $\mathcal{S}$ is equal to the number of non-isomorphic non-projective simple A-modules. Moreover, if we assume that $\mathcal{S} = \{M_1, \cdots, M_n\}$ and that $\{S_1, \cdots, S_n\}$ is a complete set of non-isomorphic non-projective simple A-modules, then both the set of tops $top(M_1), \cdots, top(M_n)$ and the set of socles $soc(M_1), \cdots, soc(M_n)$ coincide, up to ordering, with the set of simple modules $S_1, \cdots, S_n$.*

Proof First we remind the reader that every indecomposable module over a Nakayama algebra is uniserial. Let S be any non-projective simple A-module. Then there exists some $M_i \in \mathcal{S}$ such that $S \simeq soc(M_i)$ by the weak generating condition. This shows that each non-projective

simple A-module occurs as a socle of some $M_i \in \mathcal{S}$. On the other hand, any two different M_i and M_j must have non-isomorphic socles. Indeed, if M_i and M_j satisfy $soc(M_i) \simeq soc(M_j)$, then there is a monomorphism from one module to another module, say, $M_i \hookrightarrow M_j$. But clearly in this case this homomorphism does not factor through a projective module and therefore $M_i \simeq M_j$ by the orthogonality condition. We have proved that the cardinality of $\mathcal{S}$ is equal to the number of non-isomorphic non-projective simple A-modules and that the socle series $soc(M_1), \cdots, soc(M_n)$ is a rearrangement of $S_1, \cdots, S_n$. To prove the statement for top series, it suffices to show that any two different M_i and M_j must have non-isomorphic tops. In fact, if M_i and M_j satisfy $top(M_i) \simeq top(M_j)$, then there is an epimorphism from one module to another module, say, $M_i \twoheadrightarrow M_j$. But clearly this homomorphism does not factor through a projective module and therefore $M_i \simeq M_j$ by the orthogonality condition.

$\square$

We now give an example to illustrate the above proposition.

Example 6.3. *We consider the Nakayama algebra B in Example 3.4. First we display the Auslander-Reiten quiver of B as follows:*
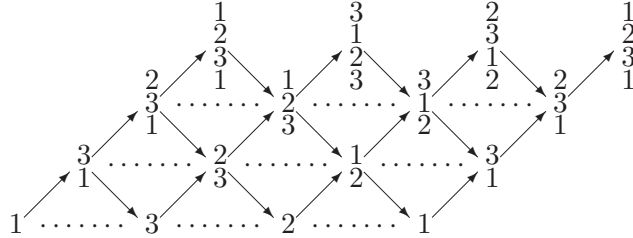
where the dotted lines indicate the Auslander-Reiten translation, and the same vertices are identified. Clearly each indecomposable non-projective B-module is self-orthogonal in $\underline{mod}B$. Using Proposition 6.2, it is not hard to verify that there are precisely five simple-minded systems over B:

$$\mathcal{S}_1 = \{1, 2, 3\};\ \mathcal{S}_2 = \{1, \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}\};\ \mathcal{S}_3 = \{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}, 3\};\ \mathcal{S}_4 = \{\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, 2, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\};\ \mathcal{S}_5 = \{\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}\}.$$

On the other hand, since the algebra A in Example 3.4 is stably equivalent to B, there are also five simple-minded systems over A. However, if we consider the following quotient algebra (which is still a Nakayama algebra but not self-injective) of B

$$B' = \begin{smallmatrix} & & 2 & & 3 \\ 1 & & 3 & & 1 \\ 2 & \oplus & 1 & \oplus & 2 \\ 3 & & 2 & & 3 \end{smallmatrix},$$

then there are two simple-minded systems over B':

$$\mathcal{S}_1' = \{1, 2, 3\};\ \mathcal{S}_2' = \{1, \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}\}.$$

This reflects the fact that there are many more (non-trivial) stable equivalences related to B than related to B'. However, if we consider the number of orbits of the simple-minded systems under stable self-equivalences, then in both cases, the number is 2.

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