THE ASSOCIATED GRADED ALGEBRAS OF BRAUER GRAPH ALGEBRAS I: FINITE REPRESENTATION TYPE

JING GUO AND YUMING LIU*

ABSTRACT. It is well known that Brauer graph algebras coincide with symmetric special biserial algebras and there has been a lot of work on Brauer graph algebras and their representation theory. Given a Brauer graph algebra A associated with a Brauer graph G, we denote by $gr(A)$ the graded algebra associated with the radical filtration of A. We give a criterion for $gr(A)$ to be representation-finite in terms of the graded degrees of vertices in G. Moreover, when $gr(A)$ is representation-finite, we give the precise relationship between the Auslander-Reiten quiver of A and the Auslander-Reiten quiver of $gr(A)$.

1. Introduction

In representation theory of finite dimensional algebras, the graded algebras associated with the radical filtration often play an important role. Let A be a finite dimensional algebra over a field, and let $gr(A)$ be the graded algebra associated with the radical filtration of A (see Definition 2.7). In [5], Cline, Parshall and Scott pointed out that homological properties of the homological dual $A^! = Ext^*(A/radA, A/radA)$ of A (often called the Yoneda algebra of A — another important graded algebra associated with A) are often reflected in terms of properties of the graded algebra $gr(A)$. In particular, they showed that for a quasi-hereditary algebra A, $gr(A)$ is sometimes isomorphic to the double dual $A^{!!}$ as graded algebras. However, if A has infinite global dimension (for example, if A is a non-semisimple self-injective algebra), then the homological dual $A^!$ is not finite dimensional and it is not known how to connect $A^!$ with the finite dimensional algebra $gr(A)$. Much less is known for both the ring theoretical properties and the representation theory of $gr(A)$ in this case.

Recently, Rickard and Rouquier studied the reconstruction problem for stable equivalence in [9] (a reformulation of this question is called the simple-image problem in [4]). When they tried to construct some self-injective algebra A, they found that they could only construct the associated graded algebra $gr(A)$, which is not self-injective any more in general. This gives another reason to study the relationship between A and $gr(A)$ under the assumption that A is an arbitrary finite dimensional algebra.

This paper is a first attempt towards this direction: we focus our study on the relationship between Brauer graph algebras and their associated graded algebras. Brauer graph algebras is a good class of algebras for our aim since, on the one hand they are closely related to the modular representation theory of finite groups, and on the other hand they coincide with symmetric special biserial algebras. A comprehensive survey on Brauer graph algebras can be found in [10].

To study the associated graded algebra $gr(A)$ of a Brauer graph algebra A, the first step should be to describe $gr(A)$ by quiver and relations. After this we will see that $gr(A)$ is a special biserial algebra for any Brauer graph algebra A. Using this point of view we can give a criterion for $gr(A)$ to be representation-finite in terms of the graded degrees of vertices in the associated Brauer graph G.

Definition. Let G be a Brauer graph. For each vertex v, we denote by $m(v)$ the multiplicity of v and by $val(v)$ the valency of v, with the convention that a loop is counted twice in $val(v)$. Moreover, if $val(v) = 1$, we denote by v' the unique vertex adjacent to v. For each vertex v in G, we define the graded degree $grd(v)$

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*Corresponding author.

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as follows. If G is the Brauer graph given by a single edge with both vertices v and v' of multiplicity 1, then $grd(v) = grd(v') = 1$; otherwise

$$grd(v) = \begin{cases} m(v)val(v), & \text{if } m(v)val(v) > 1, \\ grd(v'), & \text{if } m(v)val(v) = 1. \end{cases}$$

We use the name of graded degrees since this notion seems an appropriate tool to study the associated graded algebras of Brauer graph algebras. One of the main results in this paper is the following.

Theorem. *Let A be a Brauer graph algebra associated with a Brauer graph G. Then the graded algebra $gr(A)$ associated with the radical filtration of A is of finite representation type if and only if the following two conditions are satisfied:*
(1) *G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0;*
(2) *Denote by v_1 the exceptional vertex v_0 when $m_0 > 1$ or one of the vertices with maximal graded degree when $m_0 = 1$. For any vertex v_k in G, the walk $v_1 —— v_2 —— \cdots —— v_k$ from v_1 to v_k satisfies $grd(v_1) \geq grd(v_2) \geq \cdots \geq grd(v_k)$.*

As mentioned before, both a Brauer graph algebra A and its associated graded algebra $gr(A)$ are special biserial algebras. We can study the representation theory of A and $gr(A)$ by considering two closely related string algebras $\overline{A}$ and $\overline{gr(A)}$ (cf. Section 3). In particular, to describe the Auslander Reiten quivers of A and $gr(A)$, it is enough to describe the Auslander-Reiten quivers of $\overline{A}$ and $\overline{gr(A)}$. In case that $gr(A)$ is representation-finite we show a close connection between the Auslander-Reiten quiver of A and the Auslander-Reiten quiver of $gr(A)$. Roughly speaking, if $gr(A)$ is representation-finite, then the Auslander-Reiten quiver of $\overline{A}$ can be obtained from the Auslander-Reiten quiver of $\overline{gr(A)}$ by removing a special kind of mesh-closed subquivers called diamonds. The representation-infinite case for $gr(A)$ will be explored in future research.

This paper is organized as follows. In Section 2, we introduce the Brauer graph algebras and their associated graded algebras. In Section 3, for a Brauer graph algebra A and its associated graded algebra $gr(A)$, we study them from the special biserial algebra point of view. We define the string algebras $\overline{A}$ and $\overline{gr(A)}$ closely related to A and $gr(A)$ respectively, and prove some technical lemmas on the properties of the strings and bands in $\overline{A}$ and in $\overline{gr(A)}$. In Section 4, we present a criterion for $gr(A)$ to be representation-finite. In Section 5, we study the relationship between the Auslander-Reiten quiver of $gr(A)$ and the Auslander-Reiten quiver of A in case that $gr(A)$ is representation-finite.

2. Brauer graph algebras and their associated graded algebras

Throughout this paper, we fix an algebraically closed field k. All algebras will be finite dimensional algebras over k, all their modules will be finite dimensional left modules. For a module M, we denote by $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$ the socle and the radical of M, respectively; the length of the module M is denoted by $\ell(M)$, it means the number of composition factors in any composition series of M. We write a path p in a quiver from right to left and denote by $s(p)$ and $t(p)$ the start and the end of p, respectively; the length of a path is defined in an obvious way. By a simple cycle in a quiver we mean a cycle with no repeated arrows and no repeated vertices. Finally, we will abbreviate Auslander-Reiten quiver to AR-quiver, etc.

In this section, we introduce the Brauer graph algebras and their associated graded algebras.

2.1. **Brauer graph algebras.** We first recall the definition and general properties of Brauer graph algebras and then introduce the notion of graded degrees of vertices for any Brauer graph, which is directly related to algebraic structure of the corresponding Brauer graph algebra. For more details on Brauer graph algebras, we refer to [10], [7] and the references therein.

Definition 2.1. *A Brauer graph $G = (V(G), E(G))$ is a finite (undirected) connected graph with vertex set $V(G)$ and edge set $E(G)$ such that for each vertex v, there is a multiplicity $m(v) \in \mathbb{Z}_{>0}$ and a cyclic ordering of the edges incident to v. We will always display G as a graph with the edges incident to each vertex appearing in anticlockwise direction to reflect the cyclic ordering.*

A Brauer tree is a Brauer graph G such that $(V(G), E(G))$ is a tree and $m(v) = 1$ for all but at most one $v \in V(G)$. In this case we always choose a specified vertex v_0 (if $m(v) > 1$ then we choose $v_0 = v$), called the exceptional vertex, whose multiplicity will be denoted by m_0. A generalized Brauer tree is a Brauer graph G such that $(V(G), E(G))$ is a tree with at least two vertices with multiplicity greater than one.

In a Brauer graph $G = (V(G), E(G))$, we denote by $val(v)$ the valency or the ordinary degree of a vertex $v \in V(G)$; it is defined to be the number of edges in G incident to v, with the convention that a loop is counted twice in $val(v)$. An edge $i \in E(G)$ is said to be truncated at a vertex v if i is incident to v such that $m(v)val(v) = 1$.

Recall that any Brauer graph determines a finite dimensional basic symmetric k-algebra called Brauer graph algebra. Given a Brauer graph G and let A be the Brauer graph algebra associated with G. Then there is a quiver Q and an admissible ideal I such that $A \cong kQ/I$. The quiver $Q = (Q_0, Q_1)$ and the ideal I are constructed as follows.

If G is the Brauer graph given by a single edge with both vertices of multiplicity 1, then the quiver Q is given by one vertex and one loop and the corresponding Brauer graph algebra is isomorphic to $k[x]/(x^2)$. In the following description, we exclude this special case. The set Q_0 of vertices is given by the set of edges $E(G)$ of G, denoting the vertex in Q_0 corresponding to the edge i in $E(G)$ also by i. The set Q_1 of arrows is given by the cyclic ordering in G. Suppose that the cyclic ordering at a vertex v of G is given by $i_1 < i_2 < \cdots < i_n < i_1$; note that we might have $i_j = i_k$ for some $j \neq k$, if some of the edges are loops. We say that i_{j+1} is the successor of i_j for $1 \leq j \leq n-1$ and i_1 is the successor of i_n. Note that if v is a vertex at edge i with $val(v) = 1$ and if $m(v) > 1$ then $i < i$ and the successor of i is i, if $m(v) = 1$ then i does not have a successor. If i and j are two edges in $E(G)$ incident to a common vertex v and such that j is a successor of i in the cyclic ordering of the edges at v, then there is an arrow $i \to j$ in Q_1.

Since every arrow of Q starts and ends at an edge of G, there are at most two arrows starting and ending at every vertex of Q. Every vertex $v \in V(G)$ such that $m(v)val(v) \geq 2$, gives rise to an oriented cycle C_v in Q, which is unique up to cyclic permutation. We call C_v a special cycle at v. Note that if G contains no loops (in particular, if G is a generalized Brauer tree or a Brauer tree), then any special cycle is a simple cycle. Let C_v be such a special cycle at v. Then if C_v is a representative in its cyclic permutation class such that $t(C_v) = i = s(C_v)$, $i \in Q_0$, we say that C_v is a special i-cycle at v. If a special i-cycle at v has starting arrow α, then we denote this special i-cycle at v by $C_v(\alpha)$. Note that if $i \in E(G)$ is not a loop, then the special i-cycle at v is unique and we simply write C_v for this special i-cycle at v; if $i \in E(G)$ is a loop at v, then there are exactly two special i-cycles at v, an example of this kind is given in Example 2.3 (1).

We define the ideal I in the path algebra kQ generated by three types of relations. In the following we identify the set of edges $E(G)$ of a Brauer graph G with the set of vertices Q_0 of the corresponding quiver Q.

Relation of the first type:

$$C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')},$$

for any $i \in Q_0$ and for any special i-cycles $C_v(\alpha)$ and $C_{v'}(\alpha')$ at v and v' respectively such that both v and v' are not truncated.

Relation of the second type:

$$\alpha C_v(\alpha)^{m(v)},$$

for any $i \in Q_0$, any $v \in V(G)$ and where $C_v(\alpha)$ is a special i-cycle at v with starting arrow α.

Relation of the third type:

$$\beta\alpha,$$

for any $\alpha, \beta \in Q_1$ such that $\beta\alpha$ is not a subpath of any special cycle except if $\beta = \alpha$ is a loop associated with a vertex v of valency one and multiplicity $m(v) > 1$.

We note that I is an admissible ideal and the relations generating I do not constitute a minimal set of relations. Many of the relations, in particular many of the relations of the second type, are redundant.

The following theorem collects some general properties on Brauer graph algebras.

Theorem 2.2. (cf. [10, Subsection 2.5]) (1) *Given a Brauer graph G, the corresponding Brauer graph algebra $A \cong kQ/I$ is finite dimensional, basic, indecomposable and symmetric.*

(2) *Brauer graph algebras are special biserial and of tame representation type.*

(3) *A Brauer graph algebra is of finite representation type if and only if it is a Brauer tree algebra.*

Since Brauer graph algebras are special biserial (see the definition in Section 3), they are biserial and the composition factors of the maximal uniserial submodules of the indecomposable projective modules can be read from the Brauer graph. Let G be a Brauer graph and A the corresponding Brauer graph algebra. There is a one-to-one correspondence between the edges i in G and the simple A-modules S_i such that the projective cover P_i of S_i has the following structure. We have $P_i/rad(P_i) \cong soc(P_i) \cong S_i$, and $rad(P_i)/soc(P_i)$ is a direct sum of two (possibly zero) uniserial modules V_v and V_w corresponding to the two vertices incident to the edge i. Suppose that i is not truncated. Let $i, i_1, \cdots, i_{val(v)-1}$ be the successor sequence for i at v, and $i, j_1, \cdots, j_{val(w)-1}$ the successor sequence for i at w. Then V_v and V_w have composition series

$$S_{i_1}, \cdots, S_{i_{val(v)-1}}, S_i, S_{i_1}, \cdots, S_{i_{val(v)-1}}, \cdots, S_i, S_{i_1}, \cdots, S_{i_{val(v)-1}}$$

and

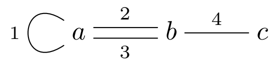
$$S_{j_1}, \cdots, S_{j_{val(w)-1}}, S_i, S_{j_1}, \cdots, S_{j_{val(w)-1}}, \cdots, S_i, S_{j_1}, \cdots, S_{j_{val(w)-1}}$$

respectively, such that, for $k = 1, \cdots, val(v) - 1$, the simple module S_{i_k} occurs precisely $m(v)$ times, and, for $l = 1, \cdots, val(w) - 1$, the simple module S_{j_l} occurs precisely $m(w)$ times. In the case where i is truncated, then P_i is itself uniserial. Suppose that i is not truncated at v and is truncated at w. Then P_i has composition series

$$S_i, S_{i_1}, \cdots, S_{i_{val(v)-1}}, S_i, S_{i_1}, \cdots, S_{i_{val(v)-1}}, \cdots, S_i, S_{i_1}, \cdots, S_{i_{val(v)-1}}, S_i$$

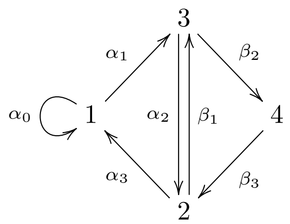
where, for $k = 1, \cdots, val(v) - 1$, the simple module S_{i_k} occurs precisely $m(v)$ times.

Example 2.3. (1) *Let G be the following Brauer graph*



with $m(a) = m(b) = m(c) = 1$. The cyclic ordering of the edges incident to vertex a is given by $1 < 1 < 3 < 2 < 1$, to vertex b is given by $2 < 3 < 4 < 2$ and to c is given by 4. We have $val(a) = 4$, $val(b) = 3$, $val(c) = 1$. Note that the edge 4 is a truncated edge since the vertex c is such that $m(c)val(c) = 1$.

The quiver Q of the corresponding Brauer graph algebra:



The special cycles in Q are the special 1-cycles at a corresponding to $\alpha_3\alpha_2\alpha_1\alpha_0$ and $\alpha_0\alpha_3\alpha_2\alpha_1$, the special 2-cycle at a given by $\alpha_2\alpha_1\alpha_0\alpha_3$, the special 2-cycle at b given by $\beta_3\beta_2\beta_1$, the special 3-cycle at a given by $\alpha_1\alpha_0\alpha_3\alpha_2$, the special 3-cycle at b given by $\beta_1\beta_3\beta_2$. Note that there are two distinct 1-cycles at the vertex a, namely $\alpha_3\alpha_2\alpha_1\alpha_0$ and $\alpha_0\alpha_3\alpha_2\alpha_1$.

Set of relations of the three types of the Brauer graph algebra:

The first type: $\alpha_3\alpha_2\alpha_1\alpha_0 - \alpha_0\alpha_3\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_0\alpha_3 - \beta_3\beta_2\beta_1, \alpha_1\alpha_0\alpha_3\alpha_2 - \beta_1\beta_3\beta_2$.

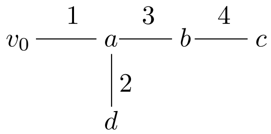
The second type: α^5, β^4. *(By abuse of notation, we write α^5 for $\alpha_0\alpha_3\alpha_2\alpha_1\alpha_0$, etc.)*

The third type: ${\alpha_0}^2, \alpha_1\alpha_3, \beta_1\alpha_2, \alpha_2\beta_1, \beta_2\alpha_1, \alpha_3\beta_3$.

The regular representation of the Brauer graph algebra is as follows:

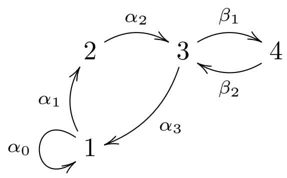
$$\begin{array}{ccc} & 1 & \\ 1 & & 3 \\ 3 & & 2 \\ 2 & & 1 \\ & 1 & \end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & \\ 1 & & 3 \\ 3 & & 4 \\ & 2 & \end{array} \oplus \begin{array}{ccc} & 3 & \\ 2 & & \\ 1 & & 4 \\ 1 & & 2 \\ & 3 & \end{array} \oplus \begin{array}{c} 4 \\ 2 \\ 3 \\ 4 \end{array}$$

(2) *Let G be the following Brauer tree*



with the multiplicity m_0 of the exceptional vertex v_0 is 2. The cyclic ordering of the edges incident to vertex v_0 is given by $1 < 1$, to vertex a is given by $1 < 2 < 3 < 1$, to vertex b is given by $3 < 4 < 3$, to c is given by 4 and to d is given by 2. We have $val(v_0) = 1$, $val(a) = 3$, $val(b) = 2$, $val(c) = 1$, $val(d) = 1$. The edges 4 and 2 are truncated edges.

The quiver Q of the corresponding Brauer tree algebra:



The special cycles in Q are the special 1-cycle at v_0 given by α_0, the special 1-cycle at a given by $\alpha_3\alpha_2\alpha_1$, the special 3-cycle at a given by $\alpha_2\alpha_1\alpha_3$, the special 3-cycle at b given by $\beta_2\beta_1$.

Set of relations of the three types of the Brauer tree algebra:

The first type: $\alpha_0{}^2 - \alpha_3\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_3 - \beta_2\beta_1$.

The second type: $\alpha_0{}^3, \alpha_1\alpha_3\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_3\alpha_2, \alpha_3\alpha_2\alpha_1\alpha_3, \beta^3$.

The third type: $\alpha_1\alpha_0, \alpha_0\alpha_3, \beta_1\alpha_2, \alpha_3\beta_2$.

The regular representation of the Brauer tree algebra is as follows:

$$\begin{array}{ccc} & 1 & \\ & & 2 \\ 1 & & \\ & & 3 \\ & 1 & \end{array} \oplus \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array} \oplus \begin{array}{ccc} & 3 & \\ 1 & & \\ & & 4 \\ 2 & & \\ & 3 & \end{array} \oplus \begin{array}{c} 4 \\ 3 \\ 4 \end{array}$$

We remark that in the Brauer tree case, the corresponding quiver Q consists of simple cycles, the simple cycles are in one-to-one correspondence with the vertices v of G that has either more than one edge incident to v or is an exceptional vertex with $m(v) > 1$ (we regard a loop as a simple cycle); moreover, any two simple cycles in Q meet in at most one vertex and every vertex in Q belongs to two simple cycles at most. The simple cycle of Q corresponding to the exceptional vertex v_0 is called the exceptional cycle (cf. [2, Section 3]).

The following is a key notion in later presentations.

Definition 2.4. *Let G be a Brauer graph. For each vertex v, we denote by $m(v)$ the multiplicity of v and by $val(v)$ the valency of v, with the convention that a loop is counted twice in $val(v)$. Moreover, if $val(v) = 1$, we denote by v' the unique vertex adjacent to v. For each vertex v in G, we define the graded degree $grd(v)$ as follows. If G is the Brauer graph given by a single edge with both vertices v and v' of multiplicity 1, then $grd(v) = grd(v') = 1$; otherwise*

$$grd(v) = \begin{cases} m(v)val(v), & \text{if } m(v)val(v) > 1, \\ grd(v'), & \text{if } m(v)val(v) = 1. \end{cases}$$

Example 2.5. *The graded degrees of vertices of the two Brauer graphs in Example 2.3 are as follows.*

(1) $grd(a) = m(a)val(a) = 4$, $grd(b) = m(b)val(b) = 3$, $grd(c) = grd(b) = 3$ *since* $m(c)val(c) = 1$.

(2) $grd(v_0) = m(v_0)val(v_0) = 2$, $grd(a) = m(a)val(a) = 3$, $grd(b) = m(b)val(b) = 2$, $grd(c) = grd(b) = 2$, $grd(d) = grd(a) = 3$.

Let G be a Brauer graph and $A = kQ/I$ the corresponding Brauer graph algebra. For any edge $v \overset{i}{\text{——}} w$ in G, we denote by S_i the corresponding simple A-module. Suppose that the projective cover P_i of S_i satisfies $rad(P_i) = U_v + U_w$ with $\ell(U_v) \leq \ell(U_w)$, where U_v and U_w are two uniserial modules with $U_v \cap U_w \cong S_i$. Note that if the edge i is a loop, then $v = w$ but $U_v \ncong U_w$. The following lemma shows that the graded degrees are directly related to algebraic structure of the corresponding Brauer graph algebra.

Lemma 2.6. *If* $U_v \cong S_i$, *then* $grd(v) = grd(w) = \ell(U_w)$; *if* $U_v \ncong S_i$, *then* $grd(v) = \ell(U_v)$ *and* $grd(w) = \ell(U_w)$.

Proof. If i is not truncated at both vertices v and w, then $U_v \ncong S_i$, and by the construction of U_v and U_w, we have $grd(v) = \ell(U_v)$ and $grd(w) = \ell(U_w)$. If i is truncated at v but not truncated at w, then $U_v \cong S_i$, and similarly we have $grd(v) = grd(w) = \ell(U_w)$. If i is truncated at both vertices v and w, then G is the Brauer graph given by a single edge which is truncated at both vertices and the conclusion clearly holds. □

2.2. **Graded algebra associated with the radical filtration of an algebra.** We recall the definition of graded algebra associated with the radical filtration of a finite dimensional algebra.

Definition 2.7. (see, for example [8, Subsection 1.6]) *Let A be a finite dimensional algebra. Denote by $\mathfrak{r}$ the radical $rad(A)$ of A. Then the graded algebra $gr(A)$ of A associated with the radical filtration is defined as follows. As a graded vector space,*

$$gr(A) = A/\mathfrak{r} \ \oplus\ \mathfrak{r}/\mathfrak{r}^2 \ \oplus\ \cdots \ \oplus\ \mathfrak{r}^t/\mathfrak{r}^{t+1} \ \oplus\ \cdots.$$

The multiplication of $gr(A)$ is given as follows. For any two homogeneous elements:

$$x + \mathfrak{r}^{m+1} \in \mathfrak{r}^m/\mathfrak{r}^{m+1}, \quad y + \mathfrak{r}^{n+1} \in \mathfrak{r}^n/\mathfrak{r}^{n+1},$$

we have

$$(x + \mathfrak{r}^{m+1}) \cdot (y + \mathfrak{r}^{n+1}) = xy + \mathfrak{r}^{m+n+1}.$$

By the above definition, the dimension of the algebra $gr(A)$ is equal to the dimension of A and the radical of $gr(A)$ is $rad(gr(A)) = \mathfrak{r}/\mathfrak{r}^2 \ \oplus\ \cdots \ \oplus\ \mathfrak{r}^t/\mathfrak{r}^{t+1} \ \oplus\ \cdots$. So the semisimple algebras associated with $gr(A)$ and A are isomorphic: $gr(A)/rad(gr(A)) \cong A/\mathfrak{r}$. It follows that A is a basic algebra if and only if $gr(A)$ is a basic algebra. In fact, we have the following general result.

Lemma 2.8. *If two algebras A and B are Morita equivalent, then their associated graded algebras $gr(A)$ and $gr(B)$ are Morita equivalent.*

Proof. Without loss of generality, we may assume that B is the basic algebra of A and has the form $B = eAe$, where e is an idempotent element in A with the property $AeA = A$. The radical of eAe is $e\mathfrak{r}e$, the radical square of eAe is $e\mathfrak{r}^2e$, and so on, where $\mathfrak{r}$ is the radical of A. Then $gr(eAe) = eAe/e\mathfrak{r}e \oplus e\mathfrak{r}e/e\mathfrak{r}^2e \oplus \cdots \oplus e\mathfrak{r}^te/e\mathfrak{r}^{t+1}e \oplus \cdots$. There is the following bijection for each t

$$e\mathfrak{r}^te/e\mathfrak{r}^{t+1}e \longrightarrow \overline{e}(\mathfrak{r}^t/\mathfrak{r}^{t+1})\overline{e},$$

$$\overline{eae} \longmapsto \overline{e} \cdot \overline{a} \cdot \overline{e},$$

where $\overline{e} = e + \mathfrak{r}, \overline{a} = a + \mathfrak{r}^{t+1}, \overline{eae} = eae + e\mathfrak{r}^{t+1}e$, and $\cdot$ is the multiplication in $gr(A)$. This induces an algebra isomorphism between $gr(eAe)$ and $\overline{e}gr(A)\overline{e}$. Since the A-module Ae is a projective generator, we have that the $gr(A)$-module $gr(A)\overline{e}$ is also a projective generator, and therefore the isomorphism $gr(eAe) \cong \overline{e}gr(A)\overline{e} \cong End_{gr(A)}(gr(A)\overline{e})^{op}$ implies that $gr(A)$ and $gr(eAe)$ are Morita equivalent. □

Although $gr(A)$ is a graded algebra, in this paper we study $gr(A)$ as an ungraded algebra. We will discuss concrete examples of associated graded algebras of Brauer graph algebras in next subsection.

2.3. **The associated graded algebras of Brauer graph algebras.** Now we describe the associated graded algebras of Brauer graph algebras by quivers and relations. Let A be a Brauer graph algebra associated with a Brauer graph G. Then we can assume that $A = kQ/I$ where Q and I are described as in Subsection 2.1.

Lemma 2.9. *Let $A = kQ/I$ be a Brauer graph algebra where the quiver Q and the admissible ideal I are described as in Subsection 2.1. The generating relations of the second and the third types in I are given by paths, and relation of the first type is of the form $\rho = p - q$, where p and q are two paths with $s(p) = t(p) = s(q) = t(q)$. For any relation $\rho = p - q$ of first type (suppose that the length of p is m and the length of q is n), we replace it by*

$$\rho' = \begin{cases} \rho, & m = n, \\ q, & m > n, \\ p, & m < n. \end{cases}$$

Then the associated graded algebra $gr(A)$ is isomorphic to kQ/I', where Q is the same quiver as above and I' is an admissible ideal whose generating relations are obtained from that of I by replacing each ρ by ρ'.

Proof. Since $gr(A) = A/\mathfrak{r} \oplus \mathfrak{r}/\mathfrak{r}^2 \oplus \cdots \oplus \mathfrak{r}^t/\mathfrak{r}^{t+1} \oplus \cdots$, where $\mathfrak{r}$ is the radical $rad(A)$ of A. We have

$$rad(gr(A)) = \mathfrak{r}/\mathfrak{r}^2 \oplus \cdots \oplus \mathfrak{r}^t/\mathfrak{r}^{t+1} \oplus \cdots,$$

$$rad^2(gr(A)) = \mathfrak{r}^2/\mathfrak{r}^3 \oplus \cdots \oplus \mathfrak{r}^t/\mathfrak{r}^{t+1} \oplus \cdots.$$

Therefore, $gr(A)/rad(gr(A)) \cong A/\mathfrak{r}$ and $rad(gr(A))/rad^2(gr(A)) \cong \mathfrak{r}/\mathfrak{r}^2$. Thus we can assume that $gr(A)$ has the form kQ/I'' with the same quiver Q and with some admissible ideal I''. We observe that the path relations and commutative relations of the form $p - q$ (where p and q have the same length) in I are also relations in I'', and that any commutative relation of the form $\rho = p - q$ (where p and q have the different length) in I gives a relation ρ' in I''.

We now consider an algebra $A' = kQ/I'$, where the generating relations of I' are obtained from that of I by replacing each ρ by ρ'. Since any generating relation in I' becomes zero in $gr(A)$ by definition, we have $I' \subset I''$. Since A and $gr(A)$ have the same dimension as vector spaces, in order to show $I'' = I'$, it is enough to show that the algebras A' and A have the same dimension. Since there is a bijection between indecomposable projective modules over A and A', it suffices to show that the dimensions of any indecomposable projective A-module P and its corresponding indecomposable projective A'-module P' are the same. There are three cases to be considered.

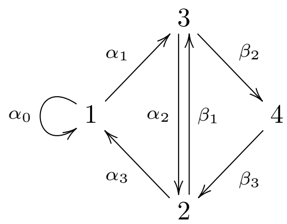
(1) P is a uniserial module. Then P' is a uniserial module with $\dim P' = \dim P$;
(2) $rad(P)/soc(P) = V_1 \oplus V_2$ with $\ell(V_1) = \ell(V_2)$, where V_1 and V_2 are two non-zero uniserial modules. Then $rad(P')/soc(P') = V_1' \oplus V_2'$ with $\ell(V_1') = \ell(V_2') = \ell(V_1)$, where V_1' and V_2' are two uniserial modules. Since both $\mathrm{soc}(P)$ and $\mathrm{soc}(P')$ are simple modules, we also have $\dim P' = \dim P$;
(3) $rad(P)/soc(P) = V_1 \oplus V_2$ with $0 < \ell(V_1) < \ell(V_2)$, where V_1 and V_2 are two uniserial modules. Then $rad(P') = V_1' \oplus V_2'$ with $\ell(V_1') = \ell(V_1)$ and $\ell(V_2') = \ell(V_2) + 1$, where V_1' and V_2' are two uniserial modules. So we still have $\dim P' = \dim P$.

Note that in the first two cases, the A'-module P' is projective-injective, and in the third case P' is projective but not injective. □

Remark 2.10. *There is a bijection between indecomposable projective modules over A and $gr(A)$. More precisely, if P is an indecomposable projective A-module, and if we identify $gr(A)$ with the algebra A' in the proof of Lemma 2.9, then the corresponding indecomposable projective $gr(A)$-module is just the module P'. As a result, any two Brauer graph algebras are isomorphic if and only if their associated graded algebras are isomorphic.*

Example 2.11. *According to Lemma 2.9, we can describe the associated graded algebras of the two Brauer graph algebras in Example 2.3. Note that in both examples, $gr(A)$ is not a symmetric algebra any more.*

(1) *$gr(A)$ is given by the quiver Q*



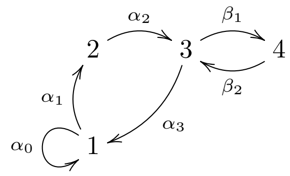
and relations

$$\alpha_3\alpha_2\alpha_1\alpha_0 - \alpha_0\alpha_3\alpha_2\alpha_1, \beta_3\beta_2\beta_1, \beta_1\beta_3\beta_2, \alpha_3\alpha_2\alpha_1\alpha_0\alpha_3, {\alpha_0}^2, \alpha_1\alpha_3, \beta_1\alpha_2, \alpha_2\beta_1, \beta_2\alpha_1, \alpha_3\beta_3.$$

The regular representation of $gr(A)$ is as follows:

$$\begin{array}{ccc} & 1 & \\ 1 & & 3 \\ 3 & & 2 \\ 2 & & 1 \\ & 1 & \end{array} \oplus \begin{array}{ccc} & 2 & \\ 1 & & \\ 1 & & 3 \\ 3 & & 4 \\ 2 & & \end{array} \oplus \begin{array}{ccc} & 3 & \\ 2 & & \\ 1 & & 4 \\ 1 & & 2 \\ 3 & & \end{array} \oplus \begin{array}{c} 4 \\ 2 \\ 3 \\ 4 \end{array}$$

(2) *$gr(A)$ is given by the quiver Q*



and relations

$${\alpha_0}^2, \beta_2\beta_1, \alpha_1\alpha_3\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_3\alpha_2, \alpha_3\alpha_2\alpha_1\alpha_3, \alpha_1\alpha_0, \alpha_0\alpha_3, \beta_1\alpha_2, \alpha_3\beta_2.$$

The regular representation of $gr(A)$ is as follows:

$$\begin{array}{ccc} & 1 & \\ & & 2 \\ 1 & & 3 \\ & & 1 \end{array} \oplus \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \end{array} \oplus \begin{array}{ccc} & 3 & \\ 1 & & \\ 2 & & 4 \\ & 3 & \end{array} \oplus \begin{array}{c} 4 \\ 3 \\ 4 \end{array}$$

We are grateful to the referee who suggests the following inspiring notations.

Definition 2.12. *Let $G = (V(G), E(G))$ be a Brauer graph with graded degree function grd and $A = kQ/I$ the corresponding Brauer graph algebra. We identify Q_0 with $E(G)$ by the natural bijection between them.*

(1) *We call an edge $v_1 \overset{i}{\text{---}} v_2$ in G with $grd(v_1) \neq grd(v_2)$ an unbalanced edge, and denote the endpoints of i by $v_L^{(i)}$, $v_S^{(i)}$ with $grd(v_L^{(i)}) > grd(v_S^{(i)})$. Whenever the context is clear we will omit the superscript (i).*

(2) *For any unbalanced edge $v_S \overset{i}{\text{---}} v_L$ in G, there is a relation of the first type $\rho_i = p_i - q_i$ in I, where $p_i = C_{v_S}^{m(v_S)}$, $q_i = C_{v_L}^{m(v_L)}$ are two paths with lengths $grd(v_S)$, $grd(v_L)$ respectively. Let e_i be the corresponding primitive idempotent in A. We define the following sets:*

$$\mathbb{W} = \{i \in Q_0 | rad(Ae_i)/soc(Ae_i) = V_1 \oplus V_2, V_1 \neq 0, V_2 \neq 0, \ell(V_1) \neq \ell(V_2)\} \subseteq Q_0, \tag{2.1}$$

$$\mathbb{P} = \bigcup_{i \in \mathbb{W}} \{r_i | r_i \text{ is the longer path between } p_i \text{ and } q_i\}. \tag{2.2}$$

Note that we can identify the set of unbalanced edges with $\mathbb{W}$ under the natural bijection between Q_0 and $E(G)$, and that $s(r_i) = t(r_i) = i$ for any r_i in $\mathbb{P}$.

(3) *If G is a Brauer tree and $v_S \overset{i}{\text{——}} v_L$ is an unbalanced edge, we write the subgraph of G by removing the edge i as follows: $G \setminus i = G_{i,L} \bigcup G_{i,S}$, where $G_{i,L}$ (resp. $G_{i,S}$) is the connected branch of $G \setminus i$ containing the vertex v_L (resp. v_S). Moreover, we denote the set of vertices in $G_{i,L}$ (resp. $G_{i,S}$) by $V(G_{i,L})$ (resp. $V(G_{i,S})$).*

In some cases, a Brauer graph algebra A is isomorphic to its associated graded algebra $gr(A)$. We give a description for this situation in terms of the graded degrees.

Proposition 2.13. *Let $A = kQ/I$ be a Brauer graph algebra associated with a Brauer graph G and $gr(A)$ the associated graded algebra of A. Then there is a natural bijection between the elements in $\mathbb{W}$ and the elements in $\mathbb{P}$, and both $\mathbb{W}$ and $\mathbb{P}$ are mapped bijectively to the set of unbalanced edges in G. Moreover, the following statements are equivalent.*

(1) *A is isomorphic to $gr(A)$ as algebras.*
(2) *The vertices in the Brauer graph G have the same graded degree.*
(3) *$\mathbb{W}$ (resp. $\mathbb{P}$) is an empty set.*

Proof. The results follow easily from Lemma 2.9, Lemma 2.6 and the fact that G is a connected graph. □

Remark 2.14. *In* [2], *Bogdanic introduced a notion of graded Brauer tree algebra. His definition of graded Brauer tree algebra means a Brauer tree algebra equipped with grading induced by a certain specific choice of gradings on the symmetric Nakayama algebra via a graded derived equivalence. In particular, the graded Brauer tree algebra is always representation-finite and symmetric. However, the associated graded algebra of a Brauer tree algebra is in general neither representation-finite nor symmetric.*

3. Special biserial algebras and string algebras

3.1. **From special biserial algebras to string algebras.** We recall some notions on special biserial algebra and string algebra. For more details, we refer to [6], [3] and [10].

Definition 3.1. *A finite dimensional k-algebra A is called special biserial if there is a quiver Q and an admissible ideal I in kQ such that A is Morita equivalent to kQ/I and such that kQ/I satisfies the following conditions:*
(1) *At every vertex v in Q there are at most two arrows starting at v and there are at most two arrows ending at v;*
(2) *For every arrow α in Q, there exists at most one arrow β such that $\beta\alpha \notin I$ and there exists at most one arrow γ such that $\alpha\gamma \notin I$.*

A special biserial algebra A is called a string algebra if the defining ideal I is generated by paths.

Given a special biserial algebra $A = kQ/I$, we can associate a string algebra $\bar{A}$ as follows. Set

$$L := \{i \in Q_0 \mid Ae_i \text{ is injective and not uniserial}\}, \quad S_0 := \bigoplus_{i \in L} soc(Ae_i).$$

Then S_0 is an ideal of A and the quotient algebra $\bar{A} = A/S_0$ is a string algebra (cf. [6, Section II.1.3]). Note that the operation $\overline{(\cdot)}$ preserves representation-finiteness and we can reconstruct the AR-quiver of A from the AR-quiver of $\overline{A}$ easily.

Suppose now that $A = kQ/I$ is a string algebra. For an arrow $\beta \in Q_1$, we denote by β^{-1} the formal inverse of β and set $s(\beta^{-1}) = t(\beta)$, $t(\beta^{-1}) = s(\beta)$, $(\beta^{-1})^{-1} = \beta$. For convenience, the formal inverse of an arrow will be called an inverse arrow. A word of length n is defined by a sequence $c_n \dots c_2 c_1$, where $c_i \in Q_1$ or $c_i^{-1} \in Q_1$, and where $t(c_i) = s(c_{i+1})$ for $1 \leq i \leq n-1$. We define

$$s(c_n \dots c_2 c_1) = s(c_1),\ t(c_n \dots c_2 c_1) = t(c_n), \text{ and } (c_n \dots c_2 c_1)^{-1} = c_1^{-1} c_2^{-1} \dots c_n^{-1}.$$

For every vertex v in Q, there is an empty word 1_v of length 0 such that $t(1_v) = s(1_v) = v$ and $1_v^{-1} = 1_v$. Suppose that a word $C := c_n \dots c_2 c_1$ satisfies $s(C) = t(C)$, we define a rotation of C as a word of the form $c_i \dots c_1 c_n \dots c_{i+1}$. The product of two words is defined by placing them next to each other, provided that the resulting sequence is a word.

A word C is called a string provided either $C = 1_v$ for some vertex v in Q or $C = c_n \dots c_2 c_1$ satisfying $c_{i+1} \neq c_i^{-1}$ for $1 \leq i \leq n-1$, and no subword (or its inverse) of C belongs to the ideal I. We say that a string $C = c_n \dots c_2 c_1$ with $n \geq 1$ is directed if all c_i are arrows, and C is inverse if all c_i are inverse arrows. A string C of positive length is called a band if all powers of C are strings and C is not a power of a string of smaller length.

On the set of words, we consider two equivalence relations. Firstly, $\sim$ denotes the relation which identifies C and C^{-1}; and secondly, we define $\sim_A$ to be the equivalence relation which identifies each word with its rotations and their inverses. Let $St(A)$ (or simply St) be a set of representatives of strings in A under $\sim$, and let $Ba(A)$ (or simply Ba) be the set of representatives of bands under $\sim_A$. In the following, we call a subword of a string a substring.

Example 3.2. *Let $A = kQ$ be the Kronecker algebra defined by the following quiver*

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

Then A is a string algebra and we can choose St and Ba as follows.

$$St = \{1_1, 1_2, \alpha, \beta, \alpha\beta^{-1}, \alpha^{-1}\beta, \beta\alpha^{-1}\beta, \alpha\beta^{-1}\alpha, \cdots\},$$
$$Ba = \{\alpha\beta^{-1}\} = \{\alpha^{-1}\beta\}.$$

We briefly recall the classification of indecomposable modules over a string algebra. Given a string algebra A, for each element C in $St(A)$, there is a unique indecomposable string A-module $M(C)$ up to isomorphism, and for each element b in $Ba(A)$, there are infinitely many non-isomorphic indecomposable band A-modules corresponding to b. Every module over a string algebra is defined either as a string module or as a band module. For the representation type of string algebras, there is the following theorem.

Theorem 3.3. ([6, Lemma II.8.1]) *A string algebra A is of finite representation type if and only if there is no band in A.*

3.2. **String algebras associated with Brauer graph algebras.** Since any Brauer graph algebra $A = kQ/I$ is special biserial, according to the description in Lemma 2.9, the associated graded algebra $gr(A) = kQ/I'$ is also special biserial. Thus we can reduce the study of A and $gr(A)$ to some string algebras $\overline{A}$ and $\overline{gr(A)}$. The string algebra $\overline{A}$ is defined by

$$\overline{A} = A/\bigoplus_{i \in L} soc(Ae_i), \tag{3.1}$$

where

$$L = \{i \in Q_0 | rad(Ae_i)/soc(Ae_i) = V_{i,1} \oplus V_{i,2}, V_{i,1} \neq 0, V_{i,2} \neq 0\}.$$

Recall that for each $i \in L$, there is a relation $\rho_i = p_i - q_i$ of the first type in I, where the length of p_i is $\ell(V_{i,1}) + 1$, the length of q_i is $\ell(V_{i,2}) + 1$. Therefore $\overline{A}$ can be described by the same quiver Q and an admissible ideal I_1 in kQ, where I_1 is generated by the ideal I and new relations $\{p_i, q_i \mid i \in L\}$. Similarly, the string algebra $\overline{gr(A)}$ is defined by

$$\overline{gr(A)} = gr(A)/\bigoplus_{i \in L'} soc(gr(A)e_i), \tag{3.2}$$

where

$$L' = \{i \in L | \ell(V_{i,1}) = \ell(V_{i,2})\}.$$

Note that for each $i \in L'$, there is a relation $\rho_i = p_i - q_i$ in I' such that p_i and q_i have the same length. Therefore $\overline{gr(A)}$ can be described by the same quiver Q and an admissible ideal I_2 in kQ, where I_2 is generated by the ideal I' and new relations $\{p_i, q_i \mid i \in L'\}$.

By the definitions of $\overline{A}$ and $\overline{gr(A)}$, we have that $\overline{A}$ is a quotient algebra of $\overline{gr(A)}$, that is, $\overline{A} \cong \overline{gr(A)}/I_3$, where the ideal I_3 is the k-vector space with basis given by the paths in the set $\mathbb{P}$ (this set is defined in (2.2)). Using notations from Subsection 2.3, we give displayed formulas of the ideals I, I', I_1, I_2, I_3 in kQ:

$$R_1 := \{\text{Relation of the first type in } I\}, \quad I_0 := \langle \text{Relation of the second type or the third type in } I \rangle;$$

$$I = I_0 + \langle R_1 \rangle;$$

$$I' = I_0 + \langle p_i - q_i \in R_1 \mid i \in Q_0, i \notin \mathbb{W}\rangle + \langle q_i \mid i \in \mathbb{W}, p_i - q_i \in R_1, q_i \text{ is shorter than } p_i\rangle;$$
$$I_1 = I_0 + \langle p_i, q_i \mid i \in Q_0, p_i - q_i \in R_1\rangle;$$
$$I_2 = I_0 + \langle p_i, q_i \mid i \in Q_0, i \notin \mathbb{W}, p_i - q_i \in R_1\rangle + \langle q_i \mid i \in \mathbb{W}, p_i - q_i \in R_1, q_i \text{ is shorter than } p_i\rangle;$$
$$I_3 = \langle r_i \in \mathbb{P} \mid i \in \mathbb{W}, p_i - q_i \in R_1\rangle = k\text{-vector space with basis } \{r_i \in \mathbb{P} \mid i \in \mathbb{W}\}.$$

We remind the reader that the four concerned algebras have the same quiver and the following displayed formulas:

$$A = kQ/I, \quad \overline{A} = kQ/I_1, \quad gr(A) = kQ/I', \quad \overline{gr(A)} = kQ/I_2.$$

Now we prove some general facts concerning the strings and bands in the string algebra $\overline{gr(A)} = kQ/I_2$.

Lemma 3.4. *Let $\overline{gr(A)} = kQ/I_2$. Suppose that the set $\mathbb{P}$ defined in* (2.2) *is non-empty. For any element C in $\mathbb{P}$ and a word β of length* 1*, if βC is a string in $\overline{gr(A)}$, then β is an inverse arrow. Similarly, if $C\beta$ is a string in $\overline{gr(A)}$, then β is an inverse arrow. In particular, if C is an element in $\mathbb{P}$, then C has no proper substring lying in $\mathbb{P}$.*

Proof. Any element C in $\mathbb{P}$ is a power of special cycle at some vertex in Q. If β is an arrow in Q, then βC is zero by relation of the second type or the third type, and so βC is not a string. Similarly, if β is an arrow in Q, then $C\beta$ is zero by relation of the second type or the third type, and so $C\beta$ is not a string. □

Lemma 3.5. *If C is a string in $\overline{gr(A)}$ and C is not a string in $\overline{A}$, then C or C^{-1} has a substring lying in $\mathbb{P}$.*

Proof. Since $\overline{A} \cong \overline{gr(A)}/I_3$, where the ideal I_3 is the k-vector space with basis given by the elements in $\mathbb{P}$, we have that the string C or its inverse C^{-1} has a substring lying in $\mathbb{P}$. □

3.3. **The case when A is a Brauer tree algebra.** In this subsection we assume that $A = kQ/I$ is a Brauer tree algebra associated with a Brauer tree G. Under this assumption A is of finite representation type, so is the string algebra $\overline{A} = kQ/I_1$, in particular, there is no band in $\overline{A}$.

Lemma 3.6. *Let $\overline{gr(A)} = kQ/I_2$ be defined in* (3.2) *such that $A = kQ/I$ is a Brauer tree algebra and $\mathbb{P}$ defined in* (2.2)*. If there is a band b in $\overline{gr(A)}$, then b has a substring lying in $\mathbb{P}$ (possibly after rotation or taking inverse of b).*

Proof. First note that a band b contains both arrow and inverse arrow. We can assume that b is a string in $\overline{A}$; otherwise, the claim follows from Lemma 3.5.

Since $\overline{A} = kQ/I_1$ is representation-finite, there exists some integer $m \geq 2$ chosen to be minimal such that b^m or its inverse has a substring lying in I_1, and this substring is not in I_2 since b^m is a string in $\overline{gr(A)} = kQ/I_2$. By Lemma 3.5, without loss of generality we may assume that this substring is an element of $\mathbb{P}$. We denote this element by e. If $m \geq 3$, by the minimality of m, then e has a substring b or b^{-1}, this contradicts the fact that b contains both arrow and inverse arrow. Thus we have $m = 2$ and e is a substring of b^2 or of its inverse. Since e is a directed string, it follows that e is a substring of b (possibly after rotation or taking inverse of b). □

A concrete example of a band in $\overline{gr(A)}$ which is also a string in $\overline{A}$ is given by $\alpha_1{\alpha_0}^{-1}\alpha_3\alpha_2$ in Example 2.11 (2), where the rotation of $\alpha_1{\alpha_0}^{-1}\alpha_3\alpha_2$ has a substring $\alpha_3\alpha_2\alpha_1 \in \mathbb{P}$. Clearly the existence of such a string implies that $\overline{gr(A)}$ is representation-infinite (although $\overline{A}$ is representation-finite).

Note that since $A = kQ/I$ is a Brauer tree algebra, the quiver Q consists of simple cycles such that any two simple cycles in Q meet in at most one vertex and every vertex in Q belongs to two simple cycles at most, so any path $\beta\alpha$ is a relation of the third type where α and β belong to distinct simple cycles. It follows that if $c_n \dots c_2c_1$ is a directed string in $\overline{gr(A)}$ or in $\overline{A}$, then all arrows c_i for $1 \leq i \leq n$ are in the same simple cycle of the quiver Q.

In the following, we define a special kind of strings in $\overline{gr(A)}$ which are closely related to walks in the Brauer tree G. We first introduce the notion of walk.

Definition 3.7. *Let v, w be two distinct vertices in a Brauer tree G. We define a walk from v to w to be a sequence $[v_1, a_1, v_2, a_2, v_3, \ldots, v_{k-1}, a_{k-1}, v_k]$ of vertices and edges, where $v_1 = v$, $v_k = w$, a_i is an edge incident to the vertices v_i and v_{i+1} for each $1 \le i \le k-1$ and all edges are pairwise distinct. We often simply write this walk by $[a_1, \ldots, a_{k-1}]$ and call it walk from edge a_1 to edge a_{k-1}. We define the length of a walk from v to w to be the number of edges in this walk; it will be denoted by $d_G(v, w)$.*

Lemma 3.8. *Let $\overline{gr(A)} = kQ/I_2$ such that $A = kQ/I$ is a Brauer tree algebra. For any $i, j \in Q_0$ where $i \neq j$, there exists a string $C = c_n \ldots c_2 c_1$ in $\overline{gr(A)}$ with $s(C) = i$ and $t(C) = j$ such that all $s(c_k)$ are pairwise distinct and $t(c_n)$ is different from $s(c_k)$ for each $1 \le k \le n$.*

Proof. The two vertices i and j in Q correspond to two distinct edges in the associated Brauer tree G which we denote by a_i and a_j, respectively. There is a walk $[a_{i_1}, \ldots, a_{i_k}]$ from a_i to a_j in G, where $a_{i_1} = a_i$, $a_{i_k} = a_j$, and all a_{i_l} are pairwise distinct for $1 \le l \le k$. We construct a string C in $\overline{gr(A)}$ with $s(C) = i$ and $t(C) = j$ as follows. First take a directed string C_1 with $s(C_1) = i_1$ and $t(C_1) = i_2$ such that C_1 lies in a simple cycle, next take an inverse string C_2 with $s(C_2) = i_2$ and $t(C_2) = i_3$ such that ${C_2}^{-1}$ lies in a simple cycle, then take a directed string C_3 with $s(C_3) = i_3$ and $t(C_3) = i_4$ such that C_3 lies in a simple cycle, and so on. Put $C = C_{k-1} \ldots C_2 C_1$. Then C is a string in $\overline{gr(A)}$ with $s(C) = i$ and $t(C) = j$ and satisfies the desired properties. Alternatively, we can construct C by first taking an inverse string, next taking a directed string, and so on. $\square$

Definition 3.9. *Let $\overline{gr(A)} = kQ/I_2$ such that $A = kQ/I$ is a Brauer tree algebra. Let $c_n \ldots c_1$ be a string in $\overline{gr(A)}$. We say that $c_n \ldots c_1$ is a simple string in $\overline{gr(A)}$ from $s(c_1)$ to $t(c_n)$ if all $s(c_k)$ are pairwise distinct and $t(c_n)$ is different from $s(c_k)$ for each $1 \le k \le n$.*

Note that the proof of Lemma 3.8 shows how we get two simple strings in $\overline{gr(A)}$ from any walk (of length ≥ 2) in G. For example, there are two simple strings from 1 to 4 in $\overline{gr(A)}$ of Example 2.11 (2), namely ${\beta_2}^{-1}\alpha_2\alpha_1$ and $\beta_1{\alpha_3}^{-1}$; both simple strings are constructed using the walk $[1, 3, 4]$ from edge 1 to edge 4 in G.

Conversely, for a simple string $C = c_n \ldots c_1$ in $\overline{gr(A)}$ with $s(c_1) = i, t(c_n) = j$, there is a (unique) walk in G such that C is one of the simple strings constructed from this walk as in the proof of Lemma 3.8. If $c_n \ldots c_1$ is a directed string or an inverse string, then $[i, j]$ is the desired walk. Otherwise, there exists $1 \le k_1 \le n-1$ such that $c_{k_1} \ldots c_1$ is a directed substring (resp. an inverse substring) and c_{k_1+1} is an inverse arrow (resp. an arrow), where $t(c_{k_1}) = i_1$, and we get a walk $[i, i_1]$ from the simple substring $c_{k_1} \ldots c_1$. For the simple substring $c_n \ldots c_{k_1+1}$, by an inductive argument we can get a walk $[i_1, \ldots, j]$. Putting them together we get the desired walk $[i, i_1, \ldots, j]$.

Lemma 3.10. *Let $\overline{gr(A)} = kQ/I_2$ such that $A = kQ/I$ is a Brauer tree algebra. If there is a string $C = c_n \ldots c_2 c_1$ satisfying $s(C) = t(C)$ in $\overline{gr(A)}$, then C has a substring C_1 such that $s(C_1) = t(C_1)$ and that C_1 or ${C_1}^{-1}$ is a directed string.*

Proof. If C contains a proper substring C' with positive length such that $s(C') = t(C')$, then it is enough to prove the statement for C'. Therefore, we can assume that C does not contain any proper substring C' with positive length such that $s(C') = t(C')$. We claim that under this assumption C is a directed or an inverse string and thus the statement is obviously true. Otherwise, the string C contains both arrow and inverse arrow, and without loss of generality we may assume that c_1 is an arrow. Then C has a substring $c_s \ldots c_1$ $(1 \le s < n)$ such that all c_i for $1 \le i \le s$ are arrows and c_{s+1} is an inverse arrow. Since any path $\beta\alpha$ is a relation of the third type where α and β belong to distinct simple cycles in Q, all arrows c_i for $1 \le i \le s$ lie in the same simple cycle of Q and the arrow c_{s+1}^{-1} lies in another simple cycle. By assumption, the string $C = c_n \ldots c_2 c_1$ contains some substring $c_n \ldots c_t$ $(s+1 < t \le n)$ satisfying the following conditions:

(1) $c_n \ldots c_t$ (or its inverse) is a directed string and all c_i for $t \le i \le n$ (or all ${c_i}^{-1}$ for $t \le i \le n$) lie in the same simple cycle of Q;

(2) The two substrings $c_s \ldots c_1$ and $c_n \ldots c_t$ do not intersect except at $s(C) = t(C)$, which is the start of $c_s \ldots c_1$ and the end of $c_n \ldots c_t$.

The existence of such a string C will produce an undirected cycle in the associated Brauer tree G, which is clearly wrong. $\square$

Lemma 3.11. *Let $\overline{gr(A)} = kQ/I_2$ such that $A = kQ/I$ is a Brauer tree algebra, and let $C = c_n \dots c_2 c_1$ be a string in $\overline{gr(A)}$. If C has no substring C_1 such that $s(C_1) = t(C_1)$ and that C_1 or ${C_1}^{-1}$ is directed, then all $s(c_k)$ are pairwise distinct and $t(c_n)$ is different from $s(c_k)$ for each $1 \leq k \leq n$, that is, C is a simple string.*

Proof. Suppose that $s(c_k) = s(c_l)$ for some $1 \leq k < l \leq n$. Then C contains a substring $C' := c_{l-1} \dots c_k$ such that $s(C') = t(C')$. By Lemma 3.10, C' contains a substring C_1 such that $s(C_1) = t(C_1)$ and that C_1 or ${C_1}^{-1}$ is directed, which is a contradiction to our assumption. It follows similarly as above that $t(c_n)$ is different from $s(c_k)$ for all k. □

4. Finite representation type

In this section, we continue to use the notations in last section: $A = kQ/I$ is a Brauer graph algebra associated with a Brauer graph G and $gr(A) = kQ/I'$ is the associated graded algebra. The related string algebras $\overline{A} = kQ/I_1$ and $\overline{gr(A)} = kQ/I_2$ are defined in (3.1) and (3.2), respectively. We will freely use the notation for the sets $\mathbb{W}$ and $\mathbb{P}$ as defined in (2.1) and (2.2), respectively. For a Brauer tree G, we denote the exceptional vertex by v_0 and its multiplicity by m_0.

Our aim in this section is to determine when $gr(A)$ is representation-finite. This is equivalent to determine when $\overline{gr(A)}$ is representation-finite. If there is no unbalanced edge in G, then by Proposition 2.13, $gr(A)$ is isomorphic to A, and so $gr(A)$ is representation-finite if and only if A is a Brauer tree algebra if and only if G is a Brauer tree. From now on we assume that G contains some unbalanced edges.

Lemma 4.1. *Let $\overline{gr(A)} = kQ/I_2$. Suppose that $v_1 \overset{i}{—} v_2$ is an edge in the associated Brauer graph G such that $C_{v_1}(\alpha_1)$ and $C_{v_2}(\alpha_2)$ are the special i-cycles at v_1 and v_2, respectively. If $C_{v_1}(\alpha_1)^{m(v_1)}$ is a string in $\overline{gr(A)}$ (that is, $C_{v_1}(\alpha_1)^{m(v_1)}$ is not in the ideal I_2), then $grd(v_1) > grd(v_2)$.*

Proof. By Lemma 2.9, our assumption shows that the indecomposable projective $\overline{gr(A)}$-module Q_i satisfies the following conditions: $rad(Q_i) = U_1 \oplus U_2$, $U_1 \neq 0$, $U_2 \neq 0$, $\ell(U_1) > \ell(U_2)+1$, $\ell(U_1) = grd(v_1)$, $\ell(U_2) = grd(v_2)-1$. Therefore $grd(v_1) > grd(v_2)$. □

Now let G be a Brauer tree. Recall from Definition 2.12 that for an unbalanced edge $v_S \overset{i}{—} v_L$ in G, we can write the subgraph of G by removing the edge i as follows: $G \setminus i = G_{i,L} \bigcup G_{i,S}$; $V(G_{i,L})$ (resp. $V(G_{i,S})$) denotes the set of vertices in $G_{i,L}$ (resp. $G_{i,S}$). Recall also from Definition 3.7 that for two distinct vertices v, w in G, the length of the walk from v to w is denoted by $d_G(v, w)$.

Lemma 4.2. *Suppose that G is a Brauer tree with an exceptional vertex v_0 and satisfies the following condition: for any unbalanced edge $v_S \overset{i}{—} v_L$, the set $V(G_{i,S})$ does not contain the exceptional vertex v_0. Then, for any unbalanced edge $v_S \overset{i}{—} v_L$ and for any adjacent vertices v and w in $V(G_{i,S})$ satisfying $d_G(v, v_S)+1 = d_G(w, v_S)$, we have $grd(v) \geq grd(w)$.*

Proof. Suppose that there is an unbalanced edge $v_S \overset{i}{—} v_L$ and that there are two adjacent vertices v and w in $V(G_{i,S})$ satisfying the following conditions: $d_G(v, v_S) + 1 = d_G(w, v_S)$ and $grd(v) < grd(w)$. We denote by j the edge incident to the vertices v and w, it is obvious that $j \neq i$. Then, by the assumption, j is an unbalanced edge and $V(G_{j,S})$ does not contain the exceptional vertex v_0. Moreover, we observe that $V(G_{j,L})$ is a subset of $V(G_{i,S})$ and therefore $V(G_{j,L})$ also does not contain the exceptional vertex v_0. This contradicts the fact that the subgraph $G \setminus j$ is a union of $G_{j,S}$ and $G_{j,L}$. □

We associate some properties of Brauer tree G with strings in $\overline{gr(A)}$ in the following two lemmas.

Lemma 4.3. *Let G be a Brauer tree and $\overline{gr(A)} = kQ/I_2$. Suppose that $C = c_n \dots c_l \dots c_1$ is a string in $\overline{gr(A)}$ satisfying $l < n$ and that $c_l \dots c_1$ or $c_1^{-1} \dots c_l^{-1}$ is an element of $\mathbb{P}$, where $s(c_1) = t(c_l) = t(c_n) = i$. We denote by $v_S \overset{i}{—} v_L$ the corresponding unbalanced edge in G. Then at least one of the following holds.*

(1) *The set $V(G_{i,S})$ contains the exceptional vertex v_0 with m_0 greater than 1.*
(2) *There are some adjacent vertices v, w in $V(G_{i,S})$, such that $d_G(v, v_S)+1 = d_G(w, v_S)$ and $grd(v) < grd(w)$.*

Proof. Without loss of generality, we may assume that $c_1^{-1}\dots c_l^{-1}$ is an element in $\mathbb{P}$. By Lemma 3.4, c_{l+1} is an arrow in Q.

If $n = l+1$, then c_{l+1} is a loop in Q and v_S is the exceptional vertex v_0 with m_0 greater than 1.

If $n > l+1$, without loss of generality, we may assume that $t(c_k) \neq i$ for each $l+1 \leq k \leq n-1$ and $t(c_n) = i$. Then $t(c_k)$ is a vertex in Q corresponding to an edge in $G_{i,S}$ for $l+1 \leq k \leq n-1$.

For the string $c_n \dots c_{l+1}$, since $t(c_n) = s(c_{l+1})$, by Lemma 3.10, $c_n \dots c_{l+1}$ or $c_{l+1}^{-1}\dots c_n^{-1}$ has a directed substring $c_{n_1}\dots c_{l_1}$ satisfying $s(c_{l_1}) = t(c_{n_1})$ such that $c_{n_1-1}\dots c_{l+1}$ or $c_{l+1}^{-1}\dots c_{n_1-1}^{-1}$ has no directed substring whose source and target are the same. There are two cases to be considered.

Case 1. If $l_1 = l+1$, then v_S is the exceptional vertex v_0 with m_0 greater than 1.

Case 2. If $l_1 > l+1$, for the string $c_{l_1-1}\dots c_{l+1}$, by Lemma 3.11, all $s(c_k)$ are pairwise distinct and $t(c_{l_1-1})$ is different from $s(c_k)$ for $l+1 \leq k \leq l_1-1$ and $j := s(c_{l_1}) \neq i$. In particular, $c_{l_1-1}\dots c_{l+1}$ is a simple string in $\overline{gr(A)}$ which gives rise to a walk from i to j in G. We denote by $v \overset{j}{\text{---}} w$ the edge in $G_{i,S}$ corresponding to j, where $d_G(v, v_S) + 1 = d_G(w, v_S)$. The above discussion shows that the special j-cycle C_w at w in G does not lie in the ideal I_2. Then w is the exceptional vertex v_0 with m_0 greater than 1 or $grd(v) < grd(w)$ by Lemma 4.1. $\square$

Lemma 4.4. *Let G be a Brauer tree and $\overline{gr(A)} = kQ/I_2$. Suppose that G satisfies the following condition:*

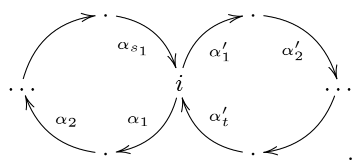
when $m_0 > 1$, there is an unbalanced edge $v_S \overset{i}{\text{---}} v_L$ such that $V(G_{i,S})$ contains the exceptional vertex v_0;
when $m_0 = 1$, there is an unbalanced edge $v_S \overset{i}{\text{---}} v_L$ and some adjacent vertices v, w in $V(G_{i,S})$, such that $d_G(v, v_S) + 1 = d_G(w, v_S)$ and $grd(v) < grd(w)$.

Then there is a band in $\overline{gr(A)}$.

Proof. In either case, we have an unbalanced edge $v_S \overset{i}{\text{---}} v_L$ with $s := grd(v_S)$ and $t := grd(v_L)$. Then we have that the vertex i in Q is the intersection of two simple cycles. We consider the two cases separately.

The first case. $m_0 > 1$, and $V(G_{i,S})$ contains the exceptional vertex v_0.

If $v_0 = v_S$, then $s = m_0 s_1$, where $s_1 = val(v_S)$. Therefore Q contains the following subquiver:



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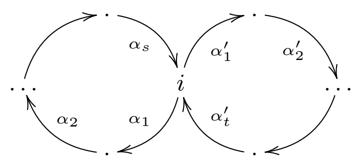
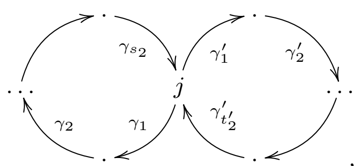
We have that $\alpha_1^{-1}\dots\alpha_{s_1}^{-1}\alpha'_t\dots\alpha'_1$ is a band in $\overline{gr(A)}$.

If $v_0 \neq v_S$, then Q contains the following subquiver



and $G_{i,S}$ contains some edge $v \overset{j}{\text{---}} w$ with $w = v_0$ and $d_G(v, v_S)+1 = d_G(w, v_S)$. Then Q contains the following subquiver



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where $grd(v) = s_2,\ grd(v_0) = m_0 t'_2,\ t'_2 = val(v_0)$, and $\gamma'_{t'_2} \dots \gamma'_1$ is not in I_2.

By Lemma 3.8, there exists a simple string $c_{k_1} \dots c_1$ satisfying $c_1 = \alpha_s^{-1}$ and $t(c_{k_1}) = j$. Note that the arrows $\gamma'_1, \dots, \gamma'_{t'_2}$ and their inverse arrows do not belong to the simple string $c_{k_1} \dots c_1$, and $c_{k_1} \dots c_1 \alpha'_t \dots \alpha'_1$ is a string.

(i) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \gamma_1^{-1}$), then $\gamma'_{t'_2} \dots \gamma'_1 c_{k_1} \dots c_1 \alpha'_t \dots \alpha'_1$ is also a string. By Lemma 3.8, there exists a simple string $c'_{k_2} \dots c'_1$ satisfying $c'_1 = \gamma_{s_2}^{-1}$ and $t(c'_{k_2}) = i$. Then

$$b := c'_{k_2} \dots c'_1 \gamma'_{t'_2} \dots \gamma'_1 c_{k_1} \dots c_1 \alpha'_t \dots \alpha'_1$$

is a band with source i in $\overline{gr(A)}$.

(ii) If c_{k_1} is an arrow (in other words, $c_{k_1} = \gamma_{s_2}$), then ${\gamma'_1}^{-1} \dots {\gamma'_{t'_2}}^{-1} c_{k_1} \dots c_1 \alpha'_t \dots \alpha'_1$ is also a string. In this situation we can similarly get a band in $\overline{gr(A)}$ as in (i).

The second case. $m_0 = 1$, and there are two adjacent vertices v and w in $V(G_{i,S})$ satisfying the following conditions: $d_G(v, v_S) + 1 = d_G(w, v_S)$ and $grd(v) < grd(w)$. Denote by j the edge in G incident to the vertices v and w and note that $j \neq i$. Then Q contains the following subquiver

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where $s_1 = grd(v)$, $t_1 = grd(w)$, $\alpha'_t \dots \alpha'_1$ and $\beta'_{t_1} \dots \beta'_1$ are not in I_2.

By Lemma 3.8, there exists a simple string $c_{k_1} \dots c_1$ satisfying $c_1 = \alpha_s^{-1}$ and $t(c_{k_1}) = j$. Note that the arrows $\beta'_1, \dots, \beta'_{t_1}$ and their inverse arrows do not belong to the simple string $c_{k_1} \dots c_1$, and $c_{k_1} \dots c_1 \alpha'_t \dots \alpha'_1$ is a string.

(i) If c_{k_1} is an inverse arrow (in other words, $c_{k_1} = \beta_1^{-1}$), then $\beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_1 \alpha'_t \cdots \alpha'_1$ is also a string. By Lemma 3.8, there exists a simple string $c'_{k_2} \dots c'_1$ satisfying $c'_1 = \beta_{s_1}^{-1}$ and $t(c'_{k_2}) = i$. Then

$$b := c'_{k_2} \dots c'_1 \beta'_{t_1} \dots \beta'_1 c_{k_1} \dots c_1 \alpha'_t \dots \alpha'_1$$

is a band with source i in $\overline{gr(A)}$.

(ii) If c_{k_1} is an arrow (in other words, $c_{k_1} = \beta_{s_1}$), then $(\beta'_1)^{-1} \dots (\beta'_{t_1})^{-1} c_{k_1} \dots c_1\ \alpha'_t \dots \alpha'_1$ is also a string. In this situation we can similarly get a band in $\overline{gr(A)}$ as in (i). ☐

We are now ready to give a description of the representation-finiteness for $gr(A)$ in terms of the graded degrees.

Theorem 4.5. *Let $A = kQ/I$ be a Brauer graph algebra associated with a Brauer graph G. Assume that G contains some unbalanced edges. Then the following three conditions are equivalent.*

(a) *The graded algebra $gr(A) = kQ/I'$ associated with the radical filtration of A is of finite representation type.*
(b) *The Brauer graph G satisfies the following combinatorial conditions:*
  (1) *G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0;*
  (2) *If $m_0 > 1$, then for any unbalanced edge $v_S \overset{i}{\text{———}} v_L$, the set $V(G_{i,S})$ does not contain the exceptional vertex v_0;*
  (3) *If $m_0 = 1$, then for any unbalanced edge $v_S \overset{i}{\text{———}} v_L$ and for any adjacent vertices v, w in $V(G_{i,S})$, the condition $d_G(v, v_S) + 1 = d_G(w, v_S)$ implies $grd(v) \geq grd(w)$.*
(c) *The Brauer graph G satisfies the following combinatorial conditions:*
  (1) *G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0;*
  (2) *Denote by v_1 the exceptional vertex v_0 when $m_0 > 1$ or one of the vertices with maximal graded degree when $m_0 = 1$. For any vertex v_k in G, the walk $[v_1, a_1, v_2, a_2, v_3, \dots, v_{k-1}, a_{k-1}, v_k]$ from v_1 to v_k satisfies $grd(v_1) \geq grd(v_2) \geq \cdots \geq grd(v_k)$.*

Proof. Since $gr(A) = kQ/I'$ is representation-finite if and only if so is the associated string algebra $\overline{gr(A)} = kQ/I_2$, we can replace $gr(A)$ by $\overline{gr(A)}$ in our statement, and in the following proof, a string (resp. a band) means a string (resp. a band) in the string algebra $\overline{gr(A)}$ unless otherwise specified. By Theorem 3.3, $\overline{gr(A)}$ is of finite representation type if and only if there is no band in $\overline{gr(A)}$.

$(a) \Longrightarrow (c)$ Suppose that $\overline{gr(A)}$ is of finite representation type, that is, there is no band in $\overline{gr(A)}$. Since $\overline{A}$ is a quotient algebra of $\overline{gr(A)}$ (cf. Subsection 3.2), we know that $\overline{A}$ and A are also of finite representation type. By Theorem 2.2, A is a Brauer tree algebra and G is a Brauer tree with an exceptional vertex v_0 of multiplicity m_0. This verifies the condition (1) in (c). In order to verify the condition (2) in (c), we suppose, on the contrary that, there exists a vertex v_k in G such that the walk $[v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k]$ from v_1 to v_k does not satisfy $grd(v_1) \geq grd(v_2) \geq \cdots \geq grd(v_k)$, where a_i is an edge incident to vertices v_i and v_{i+1} for each $1 \leq i \leq k-1$. In other words, there exists an unbalanced edge $v_i \overset{a_i}{\text{---}} v_{i+1}$ for some $1 \leq i \leq k-1$ with $grd(v_i) < grd(v_{i+1})$. There are two cases to be considered.

Case 1. If $m_0 > 1$, then $v_1 = v_0$. In this case, a_i is an unbalanced edge such that $V(G_{a_i,S})$ contains v_0.

Case 2. If $m_0 = 1$, then v_1 is one of the vertices with maximal graded degree. Observe that in this case we may find some unbalanced edge $w_1 \overset{i}{\text{---}} w_2$ with $grd(w_1) < grd(w_2)$ and some adjacent vertices w_3, w_4 in $V(G_{i,S})$, such that $d_G(w_3, w_1) + 1 = d_G(w_4, w_1)$ and $grd(w_3) < grd(w_4)$.

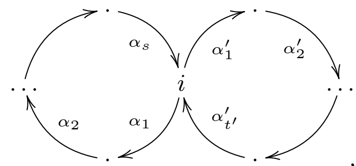
In either case the condition of Lemma 4.4 is satisfied and we can construct a band in $\overline{gr(A)}$, which is clearly a contradiction.

$(c) \Longrightarrow (b)$ The condition (1) in (b) clearly holds. To verify the conditions (2) and (3) in (b), We consider two cases by contradiction.

Case 1. $m_0 > 1$. Suppose that there is an unbalanced edge $w_1 \overset{a}{\text{---}} w_2$ with $grd(w_1) < grd(w_2)$ such that v_0 is in $V(G_{a,S})$. Since $v_1 = v_0$ in this case, and since there is a walk $[v_1, a_1, v_2, \ldots, w_1, a, w_2]$ from v_1 to w_2, we have $grd(w_1) \geq grd(w_2)$ by condition (2) in (c), which is clearly a contradiction. This verifies the condition (2) in (b).

Case 2. $m_0 = 1$. Suppose that there is an unbalanced edge $w_1 \overset{a}{\text{---}} w_2$ with $grd(w_1) < grd(w_2)$ such that there are adjacent vertices u and v in $V(G_{a,S})$ satisfying $d_G(u, w_1) + 1 = d_G(v, w_1)$ and $grd(u) < grd(v)$. Denote by a' the edge in G incident to the vertices u and v. Let $[v_1, a_1, v_2, \ldots, v]$ be the walk from v_1 to v satisfying $grd(v_1) \geq \cdots \geq grd(v)$. Since $grd(u) < grd(v)$, we must have $d_G(u, v_1) = d_G(v, v_1) + 1$. It follows that we have a walk $[v_1, a_1, v_2, \ldots, v, a', u, \ldots, w_1, a, w_2]$ from v_1 to w_2, and therefore $grd(w_1) \geq grd(w_2)$ by condition (2) in (c), which is again a contradiction. This verifies the condition (3) in (b).

$(b) \Longrightarrow (a)$ Suppose that $\overline{gr(A)}$ is of infinite representation type under the conditions (1), (2) and (3) in (b). Then we have a band b in $\overline{gr(A)}$. By Lemma 3.6, b has a substring lying in $\mathbb{P}$ (possibly after rotation or taking inverse of b). Without loss of generality we may assume that b has a substring lying in $\mathbb{P}$ and such a substring is given by $\alpha'_t \ldots \alpha'_1$ with $s(\alpha'_1) = t(\alpha'_t) = i$. Denote by $v_1 \overset{i}{\text{---}} v_2$ the corresponding edge in G, then $grd(v_1) \neq grd(v_2)$. Without loss of generality we assume $grd(v_1) < grd(v_2)$, where $grd(v_2) = t$. By the condition (2) in (b), when $m_0 > 1$, the vertex v_1 is not the exceptional vertex v_0. Thus regardless of $m_0 > 1$ or $m_0 = 1$, Q contains the following subquiver



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where $s = grd(v_1)$, $t' = val(v_2)$. Note that in this situation if $v_0 \neq v_2$, then $\alpha'_t \ldots \alpha'_1 = \alpha'_{t'} \ldots \alpha'_1$, and if $v_0 = v_2$, then $\alpha'_t \ldots \alpha'_1 = (\alpha'_{t'} \ldots \alpha'_1)^{m_0}$.

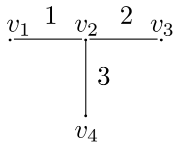
Since b contains both arrow and inverse arrow, we have $b \neq \alpha'_t \ldots \alpha'_1$. Up to a rotation of b, we can assume that $b = c_n \ldots c_1 \alpha'_t \ldots \alpha'_1$. By Lemma 3.4, c_1 is an inverse arrow and $c_1^{-1} = \alpha_s$. Since $t(c_n) = s(\alpha'_1) = i$, by Lemma 4.3, the set $V(G_{i,S})$ contains the exceptional vertex v_0 with m_0 greater than 1 or there are some adjacent vertices v_3, v_4 in $V(G_{i,S})$, such that $d_G(v_3, v_1) + 1 = d_G(v_4, v_1)$ and $grd(v_3) < grd(v_4)$. If the set $V(G_{i,S})$ contains the exceptional vertex v_0 with m_0 greater than 1, we get a contradiction to the condition (2) in (b). Otherwise, we further consider two subcases: if $m_0 = 1$, then we get a contradiction to the condition

(3) in (b); if $m_0 > 1$, then by the condition (2) in (b) and Lemma 4.2 we again get a contradiction. Hence, (b) implies that $\overline{gr(A)}$ is of finite representation type. $\square$

We note that our main result has an immediate consequence: if G is a Brauer tree with non-trivial multiplicity, then $gr(A)$ being representation-finite implies that the exceptional vertex has the maximal graded degree. The following corollary is easy to verify and we omit the proof.

Corollary 4.6. *Let G be a Brauer tree with n edges. Let A be a Brauer tree algebra associated with G and $gr(A)$ its associated graded algebra. Then we have the following.*

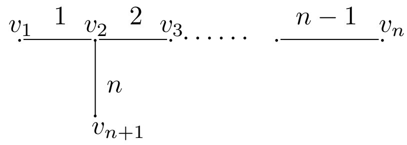
(1) *If $n = 1$, then $gr(A)$ is of finite representation type.*
(2) *If $n = 2$, then $gr(A)$ is of finite representation type.*
(3) *If $n = 3$, then $gr(A)$ is of infinite representation type if and only if the Brauer tree G is isomorphic to the following tree*



where $v_0 = v_1$, $m_0 = 2$.

Corollary 4.7. *For any pair (n, m) of positive integers with $n \geq 4$ and $m \geq 2$, there is a Brauer tree such that the associated graded algebra is of infinite representation type.*

Proof. Let G be the following Brauer tree



where $v_0 = v_n$ and $m_0 = m$. Let A be the Brauer tree algebra associated with G. If $m_0 \geq 2$, then the walk $[v_n, n-1, v_{n-1}, \ldots, v_3, 2, v_2]$ from v_n to v_2 satisfies $grd(v_3) < grd(v_2)$, where $grd(v_3) = 2$ and $grd(v_2) = 3$. By Theorem 4.5, the associated graded algebra $gr(A)$ is of infinite representation type for any $m_0 \geq 2$. $\square$

5. The Auslander-Reiten quivers

In this section, we assume that $A = kQ/I$ is a Brauer tree algebra associated with a Brauer tree G and that $gr(A)$ is its associated graded algebra. Let $\overline{A} = kQ/I_1$ and $\overline{gr(A)} = kQ/I_2$ be defined in (3.1) and (3.2), respectively. Throughout this section we assume that $\overline{gr(A)}$ is of finite representation type. The main result of this section is Theorem 5.13, which describes the relationship between the AR-quiver of $\overline{A}$ and the AR-quiver of $\overline{gr(A)}$.

According to the descriptions of string algebras $\overline{A}$ and $\overline{gr(A)}$ in Subsection 3.2, we have $\overline{A} \cong \overline{gr(A)}/I_3$, where the ideal I_3 is the k-vector space with basis given by the paths in the set $\mathbb{P}$ (see Equation (2.2)). If $\mathbb{P}$ is empty, then by Proposition 2.13, $A \cong gr(A)$ and $\overline{A} \cong \overline{gr(A)}$. Therefore $\overline{A}$ and $\overline{gr(A)}$ have the isomorphic AR-quiver. Throughout this section, we assume that $\mathbb{P}$ is not empty.

5.1. **The indecomposable $\overline{gr(A)}$-modules.** In this subsection, we count the number of non-isomorphic indecomposable $\overline{gr(A)}$-modules. The string algebra $\overline{gr(A)}$ is of finite representation type and there is a bijection between the isoclasses of indecomposable $\overline{gr(A)}$-modules and the strings in $St(\overline{gr(A)})$ (cf. [6, Section II.3]), so we can just count the strings in $St(\overline{gr(A)})$. For any string C in $\overline{gr(A)}$, we denote by $M(C)$ the indecomposable $\overline{gr(A)}$-module corresponding to the string C. For a detailed explanation of $M(C)$, we refer the reader to [3, p.160]. Note that C and C^{-1} define the same indecomposable $\overline{gr(A)}$-module.

For any string C in $\overline{gr(A)}$ that is not a string in $\overline{A}$, we have the following lemma. For the notion of unbalanced edge, we refer to Definition 2.12.

Lemma 5.1. *Let $\overline{gr(A)} = kQ/I_2$. Suppose that $C = c_n \dots c_l \dots c_1$ is a string in $\overline{gr(A)}$ satisfying $l < n$ and that $c_l \dots c_1$ or $c_1^{-1} \dots c_l^{-1}$ is an element of $\mathbb{P}$, where $s(c_1) = t(c_l) = i$. Denote by $v_S \overset{i}{-\!\!-\!\!-} v_L$ the corresponding unbalanced edge in G. Then $c_n \dots c_{l+1}$ is a simple substring of C such that $t(c_k)$ is in $G_{i,S}$ for each $l+1 \leq k \leq n$.*

Proof. To show that $t(c_k)$ is in $G_{i,S}$ for $l+1 \leq k \leq n$, it is equivalent to prove that $t(c_k) \neq i$ for $l+1 \leq k \leq n$. Suppose on the contrary that there exists $l+1 \leq m \leq n$ such that $t(c_m) = i$ and $t(c_k) \neq i$ for $l+1 \leq k \leq m-1$. For the substring $c_m \dots c_l \dots c_1$, by Lemma 4.3, the set $V(G_{i,S})$ contains the exceptional vertex v_0 with m_0 greater than 1, or there are some adjacent vertices v, w in $V(G_{i,S})$, such that $d_G(v, v_S) + 1 = d_G(w, v_S)$ and $grd(v) < grd(w)$. If the set $V(G_{i,S})$ contains the exceptional vertex v_0 with m_0 greater than 1, we get a contradiction to the condition (2) in Theorem 4.5 (b). Otherwise, we further consider two subcases: if $m_0 = 1$, then we get a contradiction to the condition (3) in Theorem 4.5 (b); if $m_0 > 1$, then by the condition (2) in Theorem 4.5 (b) and Lemma 4.2 we again get a contradiction.

It remains to show that $c_n \dots c_{l+1}$ is a simple string. It suffices to show that all $t(c_k)$ are pairwise distinct for $l \leq k \leq n$. Suppose that there exist k and t satisfying $l \leq t < k \leq n$ such that $t(c_k) = t(c_t) = s(c_{t+1})$ and that $t(c_m)$ is different from $t(c_s)$ for each $l \leq m < k$ and $l \leq s < m$. Repeating the similar proof as in the proof of Lemma 4.3, we get that the set $V(G_{i,S})$ contains the exceptional vertex v_0 with m_0 greater than 1 or there are some adjacent vertices v, w in $V(G_{i,S})$, such that $d_G(v, v_S) + 1 = d_G(w, v_S)$ and $grd(v) < grd(w)$. Again repeating the proof in the previous paragraph, we get a contradiction. □

Lemma 5.2. *If C is a string in $\overline{gr(A)}$, then C or C^{-1} has at most one substring lying in $\mathbb{P}$.*

Proof. Let C be a string in $\overline{gr(A)}$. By Lemma 3.5, either C is a string in $\overline{A}$ or C has a substring lying in $\mathbb{P}$ up to inverting C. Without loss of generality, we may assume that the string C has the form $c_n \dots c_l \dots c_1 c_0 \dots c_{-m}$ such that $l < n, 0 \leq m$ and that $c_l \dots c_1$ is an element of $\mathbb{P}$. By Lemma 3.4 and Lemma 5.1, $c_n \dots c_{l+1}$ is a simple substring of C such that c_{l+1} is an inverse arrow. Similarly, by Lemma 3.4 and a dual result of Lemma 5.1, $c_0 \dots c_{-m}$ is a simple string such that c_0 is an inverse arrow. Now our conclusion follows from the following general fact: $\mathbb{P}$ consists of pairwise distinct cycles (in the corresponding quiver Q) and every cycle in $\mathbb{P}$ is not a substring of any other cycle in $\mathbb{P}$. □

Remark 5.3. *Combining Lemma 5.2 and Lemma 3.6 we can get a necessary and sufficient condition for $gr(A)$ to be representation-finite in case that G is a Brauer tree: for any string C in $\overline{gr(A)}$, there is at most one subword of C whose underlying path is in $\mathbb{P}$. Note that for general Brauer graph G, the above condition is not sufficient for the representation-finiteness of $gr(A)$; for example, if A is a representation-infinite Brauer graph algebra and isomorphic to $gr(A)$, then $\overline{gr(A)}$ clearly satisfies the above condition since $\mathbb{P}$ is empty.*

In the following, we identify the set of unbalanced edges in G with the set $\mathbb{W}$ (see Definition 2.12). For an unbalanced edge $v_S \overset{i}{-\!\!-\!\!-} v_L$ in G, we define

$$n_i = \text{ the number of edges in } G_{i,S}. \tag{5.1}$$

Proposition 5.4. *The number of strings C in $St(\overline{gr(A)})$ that are not strings in $\overline{A}$ is $\sum\limits_{i \in \mathbb{W}} (n_i + 1)^2$.*

Proof. By Lemma 3.5 and Lemma 5.2, such a string C or C^{-1} has exactly one substring lying in $\mathbb{P}$. Since $C = C^{-1}$ in St, without loss of generality, we may assume that C has a substring $c_l \dots c_1$ lying in $\mathbb{P}$. Suppose that $s(c_l \dots c_1) = i$, and we denote by $v_S \overset{i}{-\!\!-\!\!-} v_L$ the corresponding unbalanced edge in G. Note that $c_l \dots c_1$ is a string in $\overline{gr(A)}$ and is not a string in $\overline{A}$. We count the number of strings C having the substring $c_l \dots c_1$. There are four cases to be considered.

Case 1. $C = c_l \dots c_1$.

Case 2. C has the form $c_n \dots c_{l+1} c_l \dots c_1$, where c_{l+1} is an inverse arrow. The number of strings C of this form is $n_i - 1$. In fact, by Lemma 5.1, the substring $c_n \dots c_{l+1}$ of C is a simple string starting at i and is uniquely determined by its ending vertex in Q, or equivalently, is uniquely determined by an edge in $G_{i,S}$. So the number of strings C of the form $c_n \dots c_l \dots c_1$ is n_i.

Case 3. C has the form $c_l \dots c_1 c_0 \dots c_{-n}$, where c_0 is an inverse arrow. By a similar result as Lemma 5.1, the substring $c_0 \dots c_{-n}$ of C is a simple string ending at i and is uniquely determined by its starting vertex in Q, or equivalently, is uniquely determined by an edge in $G_{i,S}$. Therefore the number of strings C of this form is again n_i.

Case 4. C has the form $c_n \dots c_{l+1} c_l \dots c_1 c_0 \dots c_{-m}$, where both c_0 and c_{l+1} are inverse arrows. By a similar consideration, we know that the number of strings C of this form is $(n_i)^2$.

Hence the number of strings having the substring $c_l \dots c_1$ is $(n_i)^2 + 2(n_i) + 1 = (n_i + 1)^2$. $\square$

Since each indecomposable $\overline{gr(A)}$-module either can be identified as an indecomposable $\overline{A}$-module or is of the form $M(C)$, where $M(C)$ is an indecomposable $\overline{gr(A)}$-module corresponding to a string C in $St(\overline{gr(A)})$ which is not a string in $\overline{A}$, we have the following direct consequence.

Corollary 5.5. *The number of non-isomorphic indecomposable $\overline{gr(A)}$-modules is $N_{\overline{A}} + \sum_{i \in \mathbb{W}} (n_i + 1)^2$, where $N_{\overline{A}}$ is the number of non-isomorphic indecomposable $\overline{A}$-modules, and n_i is defined in (5.1).*

By [1, Section X.3], we have that the Brauer tree algebra A is stably equivalent to a symmetric Nakayama algebra $A_n^{nm_0}$, where n is the number of edges in Brauer tree G, m_0 is the multiplicity of the exceptional vertex v_0 in Brauer tree G and $nm_0 + 1$ is the Loewy length of regular $A_n^{nm_0}$-module. It follows from the stable equivalence that the number of non-isomorphic indecomposable A-modules is $(nm_0 + 1)n$, and by the definition of $\overline{A}$, the number of non-isomorphic indecomposable $\overline{A}$-modules is $(nm_0 + 1)n - n + m$, where m is the number of non-isomorphic indecomposable uniserial projective A-modules. So we have the following result.

Corollary 5.6. *The number of non-isomorphic indecomposable $\overline{gr(A)}$-modules is $n^2 m_0 + m + \sum_{i \in \mathbb{W}} (n_i + 1)^2$, where n is the number of edges in the Brauer tree G, and m is the number of non-isomorphic indecomposable uniserial projective A-modules, and n_i is defined in (5.1).*

5.2. **The AR-quiver of $\overline{gr(A)}$.** In this subsection, we study the AR-quiver of $\overline{gr(A)}$. For general Auslander-Reiten theory we refer the reader to [1]. The irreducible maps and the AR-sequences over a string algebra are described in [3], and there is a brief introduction to the results in [6]. We denote by τ the AR-translation. Let us first recall from [6, Section II.5] some definitions and notations about strings, which are used to determine irreducible maps between indecomposable modules over a string algebra.

We say that a string C starts on a peak (resp. starts in a deep) provided there is no arrow β such that $C\beta$ (resp. $C\beta^{-1}$) is a string. Dually, a string C ends on a peak (resp. ends in a deep) provided there is no arrow β such that $\beta^{-1}C$ (resp. βC) is a string.

Let C be a string, if string C, not starting on a peak (resp. not ending on a peak), say $C\beta$ (resp. $\beta^{-1}C$) is a string for some arrow β, then there is a unique (if exists) directed string D such that $C\beta D^{-1}$ (resp. $D\beta^{-1}C$) is a string starting in a deep (resp. ending in a deep). When D is either a directed string or $1_{s(\beta)}$, we denote $C\beta D^{-1}$ (resp. $D\beta^{-1}C$) by C_h (resp. by ${}_hC$). If string C, not starting in a deep (resp. not ending in a deep), say $C\beta^{-1}$ (resp. βC) is a string for some arrow β, then there is a unique (if exists) directed string D such that $C\beta^{-1}D$ (resp. $D^{-1}\beta C$) is a string starting on a peak (resp. ending on a peak). When D is either a directed string or $1_{t(\beta)}$, we denote $C\beta^{-1}D$ (resp. $D^{-1}\beta C$) by C_c (resp. by ${}_cC$).

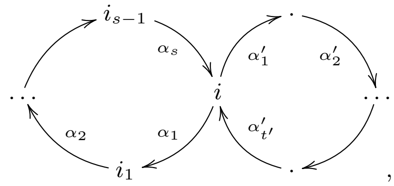
Proposition 5.7. ([6, Section II.5, II.6]) *The canonical embeddings $M(C) \longrightarrow M(C_h)$, $M(C) \longrightarrow M({}_hC)$, and the canonical projections $M(C_c) \longrightarrow M(C)$, $M({}_cC) \longrightarrow M(C)$ are irreducible maps. All irreducible maps ending at string modules are of these forms.*

Lemma 5.8. *Let $\mathscr{M}$ be the set of all isoclasses of indecomposable $\overline{gr(A)}$-module. For an unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G, let $r_i \in \mathbb{P}$. Then*

$$\sum_{N \in \mathscr{M}} dim_k Irr(N, M(r_i)) = 1, \sum_{N \in \mathscr{M}} dim_k Irr(M(r_i), N) = 1,$$

where $M(r_i)$ is the indecomposable $\overline{gr(A)}$-module corresponding to the string r_i, $Irr(N, M(r_i))$ and $Irr(M(r_i), N)$ are the k-vector spaces of irreducible morphisms from N and to N respectively.

Proof. For an unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G with $s := grd(v_S)$ and $t := grd(v_L)$, the quiver Q contains the following subquiver



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where $t = t'$ when $v_0 \neq v_L$ and $t = t'm_0$ when $v_0 = v_L$. By the definition of r_i, we have $r_i = (\alpha'_{t'} \dots \alpha'_1)^m$, where $m = 1$ when $v_0 \neq v_L$ and $m = m_0$ when $v_0 = v_L$. Then $M(r_i) \cong \overline{gr(A)}e_i/\overline{gr(A)}\alpha_1$, where e_i is the primitive idempotent corresponding to i. By [6, II.6.2], there exists an AR-sequence

$$0 \longrightarrow M(D) \longrightarrow M(r_i\alpha_1^{-1}D) \longrightarrow M(r_i) \longrightarrow 0,$$

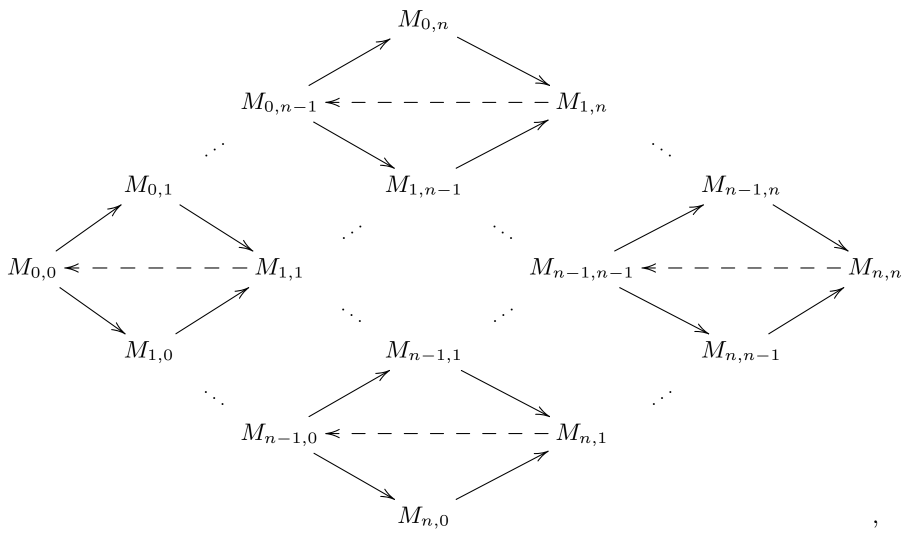
where D is a string such that $r_i\alpha_1^{-1}D = (r_i)_c$. Therefore $\sum\limits_{N \in \mathscr{M}} dim_k Irr(N, M(r_i)) = 1$.

When the vertex i_{s-1} in Q belongs to two simple cycles in Q (resp. i_{s-1} belongs to one simple cycle in Q), there exists a directed string D (resp. a string D of length 0) such that $M(D) \cong \overline{gr(A)}e_{i_{s-1}}/\overline{gr(A)}\alpha_s$, where $e_{i_{s-1}}$ is the primitive idempotent corresponding to i_{s-1}. Again by [6, II.6.2], there exists an AR-sequence

$$0 \longrightarrow M(r_i) \longrightarrow M(D\alpha_s^{-1}r_i) \longrightarrow M(D) \longrightarrow 0.$$

Therefore $\sum\limits_{N \in \mathscr{M}} dim_k Irr(M(r_i), N) = 1$. $\square$

Let Λ be a string algebra and Γ_Λ the AR-quiver of Λ. We define a diamond of length n in Γ_Λ to be a connected subquiver of the following form



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where the dashed arrows indicate the AR-translations, $n \geq 1$, all modules in the subquiver are pairwise non-isomorphic, and there exists the following AR-sequence for each pair (p, q) satisfying $0 \leq p, q \leq n-1$:

$$0 \longrightarrow M_{p,q} \longrightarrow M_{p,q+1} \oplus M_{p+1,q} \longrightarrow M_{p+1,q+1} \longrightarrow 0.$$

We remark that according to the terminology in [11, Section XVIII.2.13], a diamond in Γ_Λ is a mesh-closed subquiver. It would be interesting to know whether the full subcategory of Λ-mod consisting of all modules in a diamond has some good categorical properties.

Lemma 5.9. *Let $r_i \in \mathbb{P}$. If C is a simple string in $\overline{gr(A)}$ such that r_iC is a string in $\overline{gr(A)}$, then there are an irreducible map $M(r_iC) \longrightarrow M(r_iC')$ and an AR-sequence as follows:*

$$0 \longrightarrow M(r_iC) \longrightarrow M({}_h(r_iC)) \oplus M(r_iC') \longrightarrow M({}_h(r_iC')) \longrightarrow 0,$$

where C' is either a simple string in $\overline{gr(A)}$ or 1_i. In particular, we have $\tau(M({}_h(r_iC'))) = M(r_iC)$.

Proof. For an unbalanced edge $v_S \overset{i}{\text{———}} v_L$ in G with $s := grd(v_S) < grd(v_L) =: t$, the quiver Q contains the following subquiver

$$\begin{tikzcd}
 & i_{s-1} \arrow[dr, bend left, "\alpha_s"] & & \cdot \arrow[dr, bend left, "\alpha'_2"] & \\
\cdots \arrow[ur, bend left] & & i \arrow[dl, bend left, "\alpha_1"] \arrow[ur, bend left, "\alpha'_1"] & & \cdots \arrow[dl, bend left] \\
 & i_1 \arrow[ul, bend left, "\alpha_2"] & & \cdot \arrow[ul, bend left, "\alpha'_{t'}"] &
\end{tikzcd},$$

where $t = t'$ when $v_0 \neq v_L$ and $t = t'm_0$ when $v_0 = v_L$. Note that $\alpha_s \dots \alpha_1$ is in the ideal I_2. By the definition of r_i, we have $r_i = (\alpha'_{t'} \dots \alpha'_1)^m$, where $m = 1$ when $v_0 \neq v_L$ and $m = m_0$ when $v_0 = v_L$.

Since C is a simple string in $\overline{gr(A)}$ such that r_iC is a string in $\overline{gr(A)}$, by Lemma 3.4, $C = \alpha_1^{-1}c_1$, where c_1 is either a simple string in $\overline{gr(A)}$ or 1_{i_1}. Note that the string r_iC does not end on a peak. There are two cases as follows.

If r_iC does not start on a peak, by [6, II.6.2], then there is the following AR-sequence:

$$0 \longrightarrow M(r_iC) \longrightarrow M({}_h(r_iC)) \oplus M(r_iC') \longrightarrow M({}_h(r_iC')) \longrightarrow 0,$$

where $C' = C_h$ is a simple string in $\overline{gr(A)}$.

If r_iC starts on a peak, by [6, II.6.2], then there is the following AR-sequence:

$$0 \longrightarrow M(r_iC) \longrightarrow M({}_h(r_iC)) \oplus M(r_iC') \longrightarrow M({}_h(r_iC')) \longrightarrow 0,$$

where $(C')_c = C$. In this case C' is either a simple string in $\overline{gr(A)}$ or 1_i.

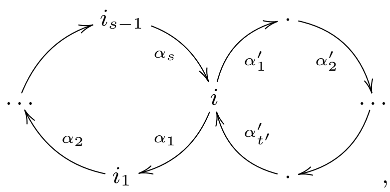
In both cases, we have an AR-sequence of the form $0 \to M(r_iC) \to M({}_h(r_iC)) \oplus M(r_iC') \to M({}_h(r_iC')) \to 0$, where C' is either a simple string in $\overline{gr(A)}$ or 1_i. ☐

Proposition 5.10. *Let n_i be defined in (5.1) and $r_i \in \mathbb{P}$. Then the indecomposable $\overline{gr(A)}$-modules corresponding to the strings having the substring r_i form the following diamond $\mathcal{D}_i$ of length n_i in the AR-quiver of $\overline{gr(A)}$:*

$$\begin{tikzcd}[column sep=small]
 & & & & M(D_{0,n_i}) \arrow[dr] & & & & \\
 & & & M(D_{0,n_{i-1}}) \arrow[ur] \arrow[dr] & & M(D_{1,n}) \arrow[ll, dashed] & & & \\
 & & \iddots & & & & \ddots & & \\
 & M(D_{0,1}) \arrow[dr] & & \iddots & M(D_{1,n_{i-1}}) \arrow[ur] & \ddots & & M(D_{n_{i-1},n}) \arrow[dr] & \\
M(D_{0,0}) \arrow[ur] \arrow[dr] & & M(D_{1,1}) \arrow[ll, dashed] & & & & M(D_{n_{i-1},n_{i-1}}) \arrow[ur] \arrow[dr] & & M(D_{n_i,n_i}) \arrow[ll, dashed] \\
 & M(D_{1,0}) \arrow[ur] & & \ddots & M(D_{n_{i-1},1}) \arrow[dr] & \iddots & & M(D_{n_i,n_{i-1}}) \arrow[ur] & \\
 & & \ddots & & & & \iddots & & \\
 & & & M(D_{n_{i-1},0}) \arrow[ur] \arrow[dr] & & M(D_{n_i,1}) \arrow[ll, dashed] & & & \\
 & & & & M(D_{n_i,0}) \arrow[ur] & & & &
\end{tikzcd},$$

where $M(D_{p,q})$ is a string module corresponding to some properly defined string $d_qr_ic_p$ for $0 \leq p, q \leq n_i$, and $c_{n_i} = d_0 = 1_i$. In particular, $M(D_{0,0})$ (resp. $M(D_{n_i,n_i})$) is the projective cover (resp. injective hull) of the simple $\overline{gr(A)}$-module S_i, $M(D_{n_i,0}) = M(r_i)$, and $M(D_{0,n_i}) = M(d_{n_i}r_ic_0)$.

Proof. Recall that $\overline{gr(A)} = kQ/I_2$. For an unbalanced edge $v_S \overset{i}{\text{——}} v_L$ in G with $s := grd(v_S) < grd(v_L) =: t$, the quiver Q contains the following subquiver



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where $t = t'$ when $v_0 \neq v_L$ and $t = t'm_0$ when $v_0 = v_L$. Note that $\alpha_s \ldots \alpha_1$ is in the ideal I_2. By the definition of r_i, we have $r_i = (\alpha'_{t'} \ldots \alpha'_1)^m$, where $m = 1$ when $v_0 \neq v_L$ and $m = m_0$ when $v_0 = v_L$. By Proposition 5.4, there are $(n_i + 1)^2$ indecomposable $\overline{gr(A)}$-modules corresponding to the strings having the substring r_i. We are going to show that these modules form a diamond of length n_i in the AR-quiver Γ of $\overline{gr(A)}$. The proof will be divided into several steps. First we recall that a path $x_0 \to x_1 \to \cdots \to x_n$ in Γ is said to be sectional provided $x_{i-1} \neq \tau(x_{i+1})$ for all $1 \leq i < n$.

(1) There is a sectional path of length n_i in Γ, starting at the projective module $\overline{gr(A)}e_i$ and ending at $M(r_i)$, where e_i is the primitive idempotent corresponding to i. Every module on this sectional path is of the form $M(r_i c)$, where c is either 1_i or a simple string ending at i and starting at a vertex in Q which corresponds to an edge in the subgraph $G_{i,S}$ of the Brauer tree G.

In fact, we have $\overline{gr(A)}e_i \cong M(r_i c_0)$ with $c_0 = \alpha_1^{-1} \ldots \alpha_{s-1}^{-1} =: \alpha_1^{-1} c'_0$. By Lemma 5.9, there exist a unique string of the form $r_i c_1$, an irreducible map $M(r_i c_0) \longrightarrow M(r_i c_1)$ and an AR-sequence:

$$0 \longrightarrow M(r_i c_0) \longrightarrow M({}_h(r_i c_0)) \oplus M(r_i c_1) \longrightarrow M({}_h(r_i c_1)) \longrightarrow 0,$$

where c_1 is either 1_i or a simple string ending at i. If $c_1 \neq 1_i$, then we can apply Lemma 5.9 repeatedly to get a unique path $M(r_i c_0) \longrightarrow M(r_i c_1) \longrightarrow \cdots$ in Γ, where every c_p is either a simple string ending at i or 1_i and $\tau(M({}_h(r_i c_p))) = M(r_i c_{p-1})$ for $p \geq 1$. Moreover, if $c_p = 1_i$ for some p, then we stop the above procedure. We claim that all the modules appearing in the above path are pairwise distinct. Otherwise, suppose that $M(r_i c_p) = M(r_i c_{p'})$ with $p < p'$, then by the construction of AR-sequences, we have $M(r_i c_{p-1}) = M(r_i c_{p'-1})$, $M(r_i c_{p-2}) = M(r_i c_{p'-2})$, and so on. Without loss of generality, we may assume that $M(r_i c_0) = M(r_i c_q)$ for some $q > 0$. It follows that $M(r_i c_{q-1}) = \tau(M({}_h(r_i c_q))) = \tau(M({}_h(r_i c_0)))$. On the other hand, by the following AR-sequence:

$$0 \longrightarrow M(C) \longrightarrow M({}_hC) \oplus M(r_i c_0) \longrightarrow M({}_h(r_i c_0)) \longrightarrow 0,$$

where $C\alpha'_1 = r_i$, we can see that $\tau(M({}_h(r_i c_0)))$ is a string module $M(C)$, where C contains no substring of the form r_i or r_i^{-1}. Therefore, $M(r_i c_{q-1}) = \tau(M({}_h(r_i c_0)))$ is a contradiction.

Since $\tau(M({}_h(r_i c_p))) = M(r_i c_{p-1})$ for $p \geq 1$, the path $M(r_i c_0) \longrightarrow M(r_i c_1) \longrightarrow \cdots$ is sectional in Γ. Moreover, since $\overline{gr(A)}$ is representation-finite, the above path must have finite length and end at $M(r_i)$. We claim that every module of the form $M(r_i c)$ appears in the above sectional path and therefore this path is of length n_i, where c is either 1_i or a simple string ending at i.

Otherwise, suppose that $M(r_i c)$ does not appear in the above path, then we can use Lemma 5.9 to get a unique sectional path from $M(r_i c)$ to $M(r_i)$ such that every module in this path is of the form $M(r_i c')$. By the uniqueness, we know that $M(r_i c)$ must appear in the path $M(r_i c_0) \longrightarrow M(r_i c_1) \longrightarrow \cdots$, which is a contradiction. Hence we get a sectional path

$$({}_0S) \quad M(r_i c_0) \longrightarrow M(r_i c_1) \longrightarrow \cdots \longrightarrow M(r_i c_{n_i})$$

in Γ, where $c_0 = \alpha_1^{-1} \ldots \alpha_{s-1}^{-1}$, $c_{n_i} = 1_i$, $c_q = \alpha_1^{-1} c'_q$ is a simple string ending at i and c'_q is a simple string or 1_{i_1} for each $0 \leq q \leq n_i - 1$. We rename the modules along the sectional path by $M(D_{0,0}), M(D_{1,0}), \cdots, M(D_{n_i,0})$, respectively. Note that if we denote 1_i by d_0, then every string $D_{p,0}$ has the form $d_0 r_i c_p$ for $0 \leq p \leq n_i$.

(2) As a consequence of (1), for every module $M(D_{p,0}) = M(r_i c_p)$, where $0 \leq p \leq n_i - 1$, we have the following AR-sequence:

$$0 \longrightarrow M(r_i c_p) \longrightarrow M({}_h(r_i c_p)) \oplus M(r_i c_{p+1}) \longrightarrow M({}_h(r_i c_{p+1})) \longrightarrow 0,$$

where the string ${}_h(r_ic_p)$ has the form $d_1r_ic_p$ for $0 \le p \le n_i$ and d_1 is a simple string starting at i. Therefore we also have a sectional path of length n_i in Γ as follows:

$$({}_1S) \quad M(d_1r_ic_0) \longrightarrow M(d_1r_ic_1) \longrightarrow \cdots \longrightarrow M(d_1r_ic_{n_i}),$$

where $c_0, \cdots, c_{n_i}$ and d_1 are defined as in (1). We rename the modules along this sectional path by $M(D_{0,1})$, $M(D_{1,1})$, $\cdots$, $M(D_{n_i,1})$, respectively. Note that for every pair $(p, 0)$, where $0 \le p \le n_i - 1$, there exists the following AR-sequence:

$$0 \longrightarrow M(D_{p,0}) \longrightarrow M(D_{p+1,0}) \oplus M(D_{p,1}) \longrightarrow M(D_{p+1,1}) \longrightarrow 0.$$

(3) By similar arguments as in (1) and (2), for every $q \ge 0$, as long as $d_q \ne \alpha_2^{-1} \ldots \alpha_s^{-1}$, we can construct a sectional path of length n_i in Γ as follows:

$$({}_{q+1}S) \quad M(d_{q+1}r_ic_0) \longrightarrow M(d_{q+1}r_ic_1) \longrightarrow \cdots \longrightarrow M(d_{q+1}r_ic_{n_i}),$$

where $c_0, \cdots, c_{n_i}$ are defined as in (1) and d_{q+1} is a simple string starting at i. We rename the modules along the sectional path $({}_{q+1}S)$ by $M(D_{0,q+1})$, $M(D_{1,q+1})$, $\cdots$, $M(D_{n_i,q+1})$, respectively. Note that for every pair (p, q), where $0 \le p, q \le n_i - 1$, there exists the following AR-sequence:

$$0 \longrightarrow M(D_{p,q}) \longrightarrow M(D_{p+1,q}) \oplus M(D_{p,q+1}) \longrightarrow M(D_{p+1,q+1}) \longrightarrow 0.$$

(4) We claim that the sequence $d_0, d_1, \cdots$ is a finite sequence and stops at $d_{n_i} = \alpha_2^{-1} \ldots \alpha_s^{-1}$. Indeed, if we start with the module $M(r_i) = M(d_0r_i)$ and use the same method as in (1), we can construct a sectional path of length n_i in Γ:

$$M(d_0r_i) \longrightarrow M(d_1r_i) \longrightarrow \cdots \longrightarrow M(d_{n_i}r_i).$$

Note that $M(d_{n_i}r_i)$ is the injective module corresponding to i, and that every string d_q has the form $d'_q\alpha_s^{-1}$ with d'_q a simple string or $1_{i_{s-1}}$ for each $0 < q \le n_i$. Moreover, for each $0 < q < n_i$, there exists the following AR-sequence:

$$0 \longrightarrow M(D_{n_i,q}) \longrightarrow M(D_{n_i,q+1}) \oplus M(d'_q) \longrightarrow M(d'_{q+1}) \longrightarrow 0.$$

(5) Summarizing the above discussion, we know that $M(D_{p,q})$ are exactly the modules corresponding to the strings having the substring r_i for $0 \le p, q \le n_i$ and they form a diamond of length n_i in Γ, which we denote by $\mathcal{D}_i$. $\square$

Definition 5.11. *Let i be an unbalanced edge and $r_i \in \mathbb{P}$. We call the diamond $\mathcal{D}_i$ described in Proposition 5.10 is the diamond associated with i. Note that the indecomposable $\overline{gr(A)}$-modules $M(D_{p,q})$ $(0 \le p, q \le n_i)$ in $\mathcal{D}_i$ correspond precisely to the strings having the substring r_i.*

Proposition 5.10 together with Corollary 5.5 implies that the indecomposable $\overline{gr(A)}$-modules that do not belong to the union of all diamonds $\mathcal{D}_i$ in the AR-quiver of $\overline{gr(A)}$ can be identified as indecomposable $\overline{A}$-modules. In order to describe the relationship between the AR-quiver of $\overline{gr(A)}$ and the AR-quiver of $\overline{A}$, we next construct the AR-sequences around the diamond $\mathcal{D}_i$ associated with an unbalanced edge i.

We keep all the notations in Proposition 5.10 and in its proof. When $c_1 \ne 1_i$, for the string $c'_0 = \alpha_2^{-1} \ldots \alpha_{s-1}^{-1}$, since c'_0 does not end on a peak and it starts on a peak if and only if $D_{0,0} = r_i\alpha_1^{-1}c'_0$ starts on a peak. By [6, II.6.2], there exists the following AR-sequence:

$$0 \longrightarrow M(c'_0) \longrightarrow M(D_{0,0}) \oplus M(c'_1) \longrightarrow M(D_{1,0}) \longrightarrow 0.$$

We can repeat the above discussion for $M(c'_0)$ and get the following AR-sequence for each $1 \le p \le n_i - 2$:

$$0 \longrightarrow M(c'_p) \longrightarrow M(D_{p,0}) \oplus M(c'_{p+1}) \longrightarrow M(D_{p+1,0}) \longrightarrow 0.$$

Moreover, by the proofs of Lemma 5.8 and Proposition 5.10, we have the following AR-sequences:

$$0 \longrightarrow M(c'_{n_i-1}) \longrightarrow M(D_{n_i-1,0}) \longrightarrow M(D_{n_i,0}) \longrightarrow 0;$$

$$0 \longrightarrow M(C) \longrightarrow M(d'_1\alpha_s^{-1}C) \oplus M(D_{0,0}) \longrightarrow M(D_{0,1}) \longrightarrow 0.$$

For the string $d'_1\alpha_s^{-1}C$, it does not start on a peak. It is easy to see that $d'_1\alpha_s^{-1}C$ ends on a peak if and only if $d'_1\alpha_s^{-1}$ ends on a peak. Again by [6, II.6.2], we have the following AR-sequence:

$$0 \longrightarrow M(d'_1\alpha_s^{-1}C) \longrightarrow M(D_{0,1}) \oplus M(d'_2\alpha_s^{-1}C) \longrightarrow M(D_{0,2}) \longrightarrow 0,$$

where $M(D_{0,2}) = M(d'_2\alpha_s^{-1}r_i\alpha_1^{-1}c'_0)$ and $d'_2\alpha_s^{-1}r_i\alpha_1^{-1}c'_0 = (d'_2\alpha_s^{-1}C)_h$.

By the similar discussion as above, for each $2 \leq q \leq n_i - 1$, there exists the following AR-sequence:

$$0 \longrightarrow M(d'_q\alpha_s^{-1}C) \longrightarrow M(D_{0,q}) \oplus M(d'_{q+1}\alpha_s^{-1}C) \longrightarrow M(D_{0,q+1}) \longrightarrow 0,$$

where $M(D_{0,q+1}) = M(d'_{q+1}\alpha_s^{-1}r_i\alpha_1^{-1}c'_0)$ and $d'_{q+1}\alpha_s^{-1}r_i\alpha_1^{-1}c'_0 = (d'_{q+1}\alpha_s^{-1}C)_h$.

For the string C, we define a string e such that $C = \alpha'_{t'}e$. Consider the string $\alpha_2^{-1}\dots\alpha_s^{-1}C = d'_{n_i}\alpha_s^{-1}C$, we have that it ends on a peak and it does not start on a peak. Note that $\alpha_2^{-1}\dots\alpha_s^{-1}C = \alpha_2^{-1}\dots\alpha_s^{-1}\alpha'_{t'}e = {}_c(e)$ and $(d'_{n_i}\alpha_s^{-1}C)_h = D_{0,n_i}$. Then there exists an AR-sequence as follows:

$$0 \longrightarrow M(d'_{n_i}\alpha_s^{-1}C) \longrightarrow M(D_{0,n_i}) \oplus M(e) \longrightarrow M(e_h) \longrightarrow 0,$$

where $e_h = e\alpha'_1\alpha_1^{-1}\dots\alpha_{s-1}^{-1} = e\alpha'_1\alpha_1^{-1}c'_0$ and $c'_0 = \alpha_2^{-1}\dots\alpha_{s-1}^{-1} = d'_{n_i}$.

For the string $D_{0,n_i} = d'_{n_i}\alpha_s^{-1}r_i\alpha_1^{-1}c'_0$, we have that it starts on a peak if and only if $\alpha_1^{-1}c'_0$ starts on a peak. Note that the string D_{0,n_i} ends on a peak and $d'_{n_i}\alpha_s^{-1}r_i\alpha_1^{-1}c'_0 = {}_c(e\alpha'_1\alpha_1^{-1}c'_0)$. By [6, II.6.2], there exists an AR-sequence as follows:

$$0 \longrightarrow M(D_{0,n_i}) \longrightarrow M(D_{1,n_i}) \oplus M(e\alpha'_1\alpha_1^{-1}c'_0) \longrightarrow M(e\alpha'_1\alpha_1^{-1}c'_1) \longrightarrow 0.$$

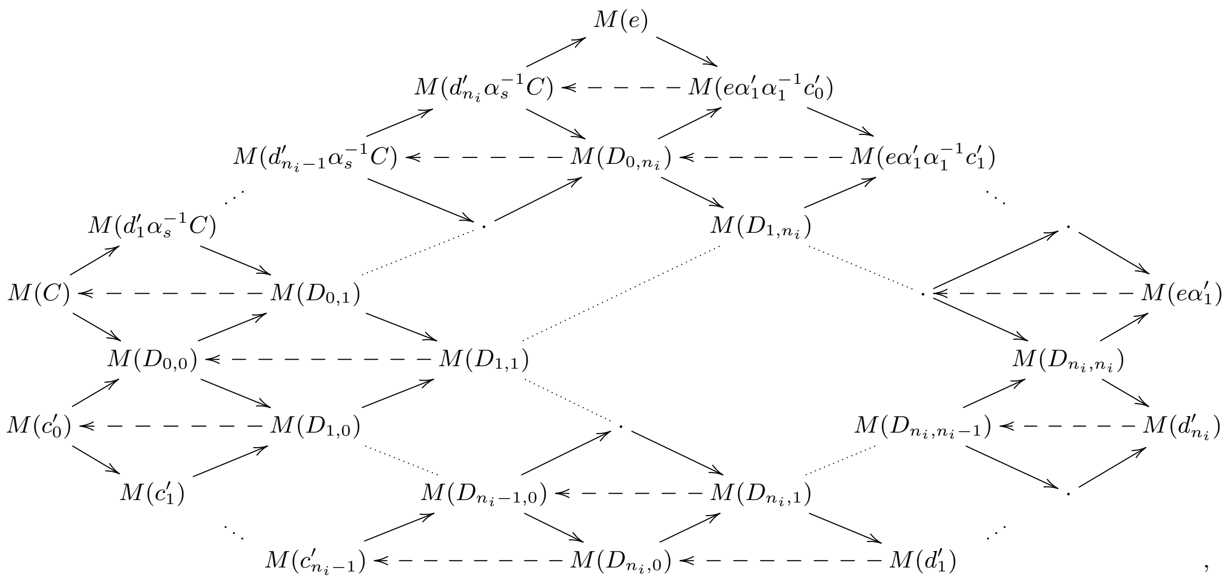
Similarly, for each $1 \leq p \leq n_i - 2$, there exists an AR-sequence as follows:

$$0 \longrightarrow M(D_{p,n_i}) \longrightarrow M(D_{p+1,n_i}) \oplus M(e\alpha'_1\alpha_1^{-1}c'_p) \longrightarrow M(e_0\alpha'_1\alpha_1^{-1}c'_{p+1}) \longrightarrow 0.$$

Finally, for the string $D_{n_i-1,n_i} = d'_{n_i}\alpha_s^{-1}r_i\alpha_1^{-1}c'_{n_i-1}$, we have that it starts on a peak and ends on a peak. Note that $d'_{n_i}\alpha_s^{-1}r_i\alpha_1^{-1}c'_{n_i-1} = (d'_{n_i}\alpha_s^{-1}C\alpha'_1)_c = (d'_{n_i}\alpha_s^{-1}\alpha'_t e\alpha'_1)_c = {}_c(e\alpha'_1)_c$. By [6, II.6.2], there exists an AR-sequence as follows:

$$0 \longrightarrow M(D_{n_i-1,n_i}) \longrightarrow M(D_{n_i,n_i}) \oplus M(e\alpha'_1\alpha_1^{-1}c'_{n_i}) \longrightarrow M(e\alpha'_1) \longrightarrow 0.$$

The above discussion shows that there is the following subquiver (which we denote by $\widetilde{\mathcal{D}_i}$) of the AR-quiver of $\overline{gr(A)}$:



,

where the modules $M(c'_0)$ and $M(d'_{n_i})$ are identified.

From the above discussions, we have the following result.

Corollary 5.12. *Let n_i be defined in (5.1) and $r_i \in \mathbb{P}$. Then $\tau^{n_i}(I_i) \cong P_i$, where P_i is the projective cover of simple $\overline{gr(A)}$-module S_i corresponding to i, I_i is the injective hull of S_i. Moreover, we have the following AR-sequence:*

$$0 \longrightarrow M(d'_{n_i}\alpha_s^{-1}C) \longrightarrow M(D_{0,n_i}) \oplus M(e) \longrightarrow M(e_h) \longrightarrow 0,$$

where the modules $M(d'_{n_i}\alpha_s^{-1}C)$, $M(e_h)$, $M(D_{0,n_i})$, $M(e)$ are described as before. Note that the module $M(e_h)$ (resp. $M(d'_{n_i}\alpha_s^{-1}C)$) viewed as a $\overline{A}$-module is the projective cover (resp. the injective hull) of the simple $\overline{A}$-module corresponding to i.

We are now ready to prove the main result of this section.

Theorem 5.13. *Let Γ be the AR-quiver of $\overline{gr(A)}$. For each unbalanced edge i, let n_i be defined in (5.1) and $\mathcal{D}_i$ the diamond associated with i. Then the AR-quiver of $\overline{A}$ can be obtained from Γ by applying the following operations for all unbalanced edges i:*

(1) *Remove the diamond $\mathcal{D}_i$ and the related arrows to $\mathcal{D}_i$;*
(2) *Add the following $2n_i$ irreducible maps:*

$$M(d'_q\alpha_s^{-1}C) \longrightarrow M(d'_q), M(c'_p) \longrightarrow M(e\alpha'_1\alpha_1^{-1}c'_p),$$

where $1 \le q \le n_i$, $0 \le p \le n_i - 1$;

(3) *Add the following $2n_i$ AR-translations:*

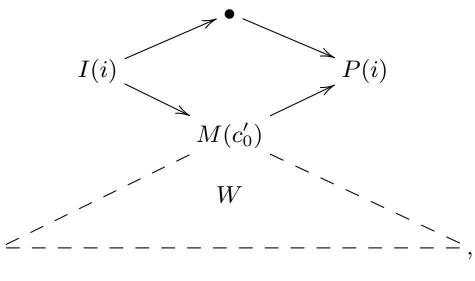
$$M(d'_{q-1}\alpha_s^{-1}C) \dashleftarrow M(d'_q), M(C) \dashleftarrow M(d'_1),$$

$$M(c'_{p-1}) \dashleftarrow M(e\alpha'_1\alpha_1^{-1}c'_p), M(c'_{n_i-2}) \dashleftarrow M(e\alpha'_1),$$

where $2 \le q \le n_i$, $0 \le p \le n_i - 1$;

(4) *Remove the AR-translation $M(d'_{n_i}\alpha_s^{-1}C) \dashleftarrow M(e_h)$.*

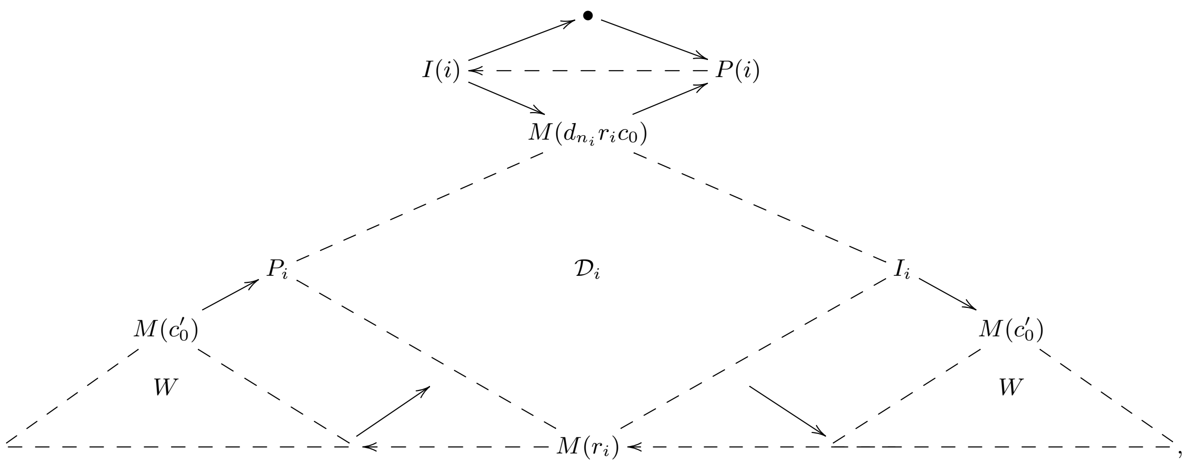
Remark 5.14. *By inverting the process in Theorem 5.13, we can construct Γ from the AR-quiver Γ_1 of $\overline{A}$ by inserting the diamonds $\mathcal{D}_i$'s. The inserting process of $\mathcal{D}_i$ can be illustrated as follows. First we look for the injective hull $I(i)$ and the projective cover $P(i)$ of the simple $\overline{A}$-module $S(i)$ in Γ_1, which are easy to find, since they correspond to an "empty mesh" in Γ_1:*



where the triangular part W denotes the wing with peak $M(c'_0)$ in Γ_1. Here a wing with peak $M_{0,n}$ (where $n \ge 0$) in the AR-quiver means a mesh-closed subquiver of the following form (possibly after removing some uniserial projective-injective modules):

$$\begin{tikzcd}[column sep=small]
 & & & & M_{0,n} \arrow[dr] & & & & \\
 & & & M_{0,n-1} \arrow[ur] \arrow[dr] & & M_{1,n} \arrow[ll, dashed] & & & \\
 & M_{0,1} \arrow[dr] & \iddots & & M_{1,n-1} \arrow[ur] & & \ddots & M_{n-1,n} \arrow[dr] & \\
M_{0,0} \arrow[ur] & & M_{1,1} \arrow[ll, dashed] & \iddots & \cdots & \ddots & M_{n-1,n-1} \arrow[ur] & & M_{n,n}. \arrow[ll, dashed]
\end{tikzcd}$$

Note that the bottom line in W lies on the one of the mouths of Γ_1. Next we insert the diamond $\mathcal{D}_i$ into the above empty mesh and get the following diagram (viewed as a subquiver of the AR-quiver Γ of $\overline{gr(A)}$):



where the two triangular parts W are identified in Γ, and P_i (resp. I_i) is the projective cover (resp. injective hull) of the simple $\overline{gr(A)}$-module S_i. Note that the bottom vertex $M(r_i)$ of $\mathcal{D}_i$ together with the bottom line in W lies on the one of the mouths of Γ. Note also that the empty mesh between $I(i)$ and $P(i)$ in Γ_1 becomes a real mesh in Γ.

Proof of Theorem 5.13. We denote by Γ_0 the translation quiver obtained from Γ after the operations (1)—(4), and by Γ_1 the AR-quiver of $\overline{A}$. Then Proposition 5.10 together with Corollary 5.5 implies that the vertices of Γ_0 are in one-to-one correspondence with the vertices of Γ_1. Since both $\overline{gr(A)}$ and $\overline{A}$ are string algebras, for any vertex X in Γ or in Γ_1, there are at most two arrows from X and at most two arrows to X. We are going to show that Γ_0 and Γ_1 are isomorphic as translation quivers, where in both cases the translation is given by the AR-translation τ.

First we note that in an AR-quiver, vertices correspond to indecomposable modules, arrows correspond to irreducible maps, and meshes (with dashed arrow inside) correspond to AR-sequences. Observe from the quiver $\widetilde{\mathcal{D}_i}$ before Corollary 5.12 that, if X is a module in diamond $\mathcal{D}_i$ and Y is a module in a distinct diamond $\mathcal{D}_j$, then X and Y can not appear simultaneously in an AR-sequence. For any two indecomposable $\overline{A}$-modules X and Y, by the relationship between strings and modules, an arrow $X \longrightarrow Y$ in Γ_0 corresponds to an arrow $X \longrightarrow Y$ in Γ_1, and vice versa; if exactly one of X and Y belongs to some diamond $\mathcal{D}_i$, then an arrow $X \longrightarrow Y$ in Γ would be destroyed when passing to Γ_0, but there will be some new arrow in Γ_0 (one of the maps in the operation (2)) and this new arrow corresponds to an arrow in Γ_1. It is easy to check that an arrow in Γ_1 is either coming from an arrow in Γ or coming from an operation in (2), and so Γ_0 and Γ_1 are isomorphic as ordinary quivers. Similarly, a mesh in which all vertices do not belong to any diamond $\mathcal{D}_i$ in Γ_0 corresponds to a mesh in Γ_1, and vice versa; if some of vertices of a mesh in Γ belong to some diamond $\mathcal{D}_i$, then it will be destroyed when passing to Γ_0, but there will be some new mesh in Γ_0 (one of the AR-sequences implicated in the operation (3)) and this new mesh corresponds to a mesh in Γ_1.

Moreover, by Corollary 5.12, for each unbalanced edge i, if we view the two $\overline{gr(A)}$-modules $M(e_h)$ and $M(d'_{n_i}\alpha_s^{-1}C)$ as $\overline{A}$-modules, where the $\overline{gr(A)}$-modules $M(e_h)$ and $M(d'_{n_i}\alpha_s^{-1}C)$ are neither projective nor injective, then they are the projective cover and the injective hull of the simple $\overline{A}$-module corresponding to i, respectively. So in order to pass from Γ to Γ_1, we should remove the dashed arrow from $M(e_h)$ to $M(d'_{n_i}\alpha_s^{-1}C)$ in Γ (the AR-translation in the operation (4)). Now it is easy to check that there is a bijection between the meshes in Γ_0 and the meshes in Γ_1, and therefore they are isomorphic as translation quivers. $\square$

Corollary 5.15. *The $\overline{gr(A)}$-module $M(r_i)$ in the diamond $\mathcal{D}_i$ satisfies $\tau^{n_i+1}(M(r_i)) \cong M(r_i)$.*

Proof. First of all, $\tau(M(r_i))$ is the module $M(c'_{n_i-1})$ appearing in the quiver $\widetilde{\mathcal{D}_i}$ before Corollary 5.12, and there is only one irreducible map from $\tau(M(r_i))$ in the AR-quiver of $\overline{gr(A)}$. We next show that there is no projective module in the τ-orbit of $M(r_i)$.

Otherwise, suppose that there is a projective $\overline{gr(A)}$-module P in the τ-orbit of $M(r_i)$. Then P is not uniserial; otherwise, P is also injective and therefore projective-injective, this contradicts the fact that P lies in a τ-orbit. Moreover, P must have the following form: $rad(P) = U_1 \oplus U_2$, where U_1 and U_2 are two non-zero uniserial modules with the same length. It follows that there are two irreducible maps to P and also two irreducible maps from P in the AR-quiver of $\overline{gr(A)}$. Both P and $\tau(M(r_i))$ can be identified as indecomposable $\overline{A}$-modules, according to the relationship between AR-quiver of $\overline{gr(A)}$ and the AR-quiver Γ_1 of $\overline{A}$ (see Theorem 5.13), P and $\tau(M(r_i))$ lie in the same τ-orbit in Γ_1, there are also two irreducible maps from P and only one irreducible map from $\tau(M(r_i))$. This contradicts the shape of the stable AR-quiver of the Brauer tree algebra A, it is of the form $\mathbb{Z}A_{nm_0}/\tau^n$, where n is the number of non-isomorphic simple A-modules, m_0 is the multiplicity of the exceptional vertex of the associated Brauer tree, and A_{nm_0} is the Dynkin quiver of type A_{nm_0}; in particular, P has two direct predecessors.

Therefore there is no projective module in the τ-orbit of $M(r_i)$ in the AR-quiver of $\overline{gr(A)}$. Since $\overline{gr(A)}$ is of finite representation type, there exists some natural number m such that $\tau^m(M(r_i)) \cong M(r_i)$. Now it is easy to see from the quiver $\widetilde{\mathcal{D}_i}$ before Corollary 5.12 that $m = n_i + 1$, note that the modules $M(c_0')$ and $M(d_n')$ are identified in this quiver. $\square$

Finally, we give an example to illustrate the above results.

Example 5.16. *Let G be the following Brauer tree*

$$\begin{tikzcd} \underset{v_0}{\cdot} \arrow[r, dash, "1"] & \cdot \arrow[r, dash, "2"] & \cdot \end{tikzcd} ,$$

where the multiplicity m_0 of the exceptional vertex v_0 is 3. Then the Brauer tree algebra $A = kQ/I$ associated with the Brauer tree G is given by the following quiver Q

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha_0"] \arrow[r, bend left, "\beta_0"] & 2 \arrow[l, bend left, "\beta_1"] \end{tikzcd}$$

with relations

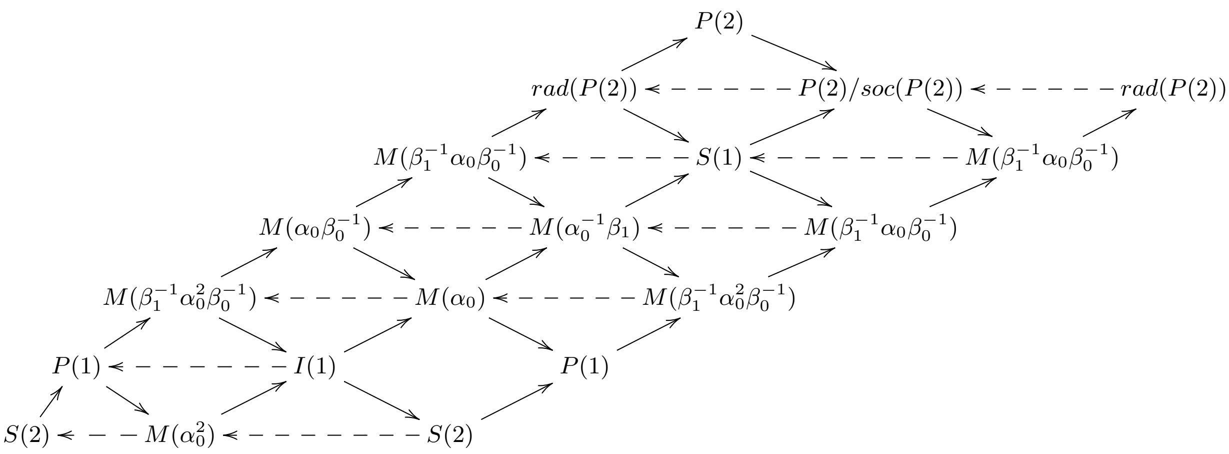
$$\alpha_0\beta_1, \beta_0\alpha_0, \alpha_0^4, \beta_0\beta_1\beta_0, \beta_1\beta_0\beta_1, \alpha_0^3 - \beta_1\beta_0.$$

By Lemma 2.9, the associated graded algebra $gr(A)$ can be described as the same quiver Q with relations

$$\alpha_0\beta_1, \beta_0\alpha_0, \alpha_0^4, \beta_0\beta_1\beta_0, \beta_1\beta_0\beta_1, \beta_1\beta_0.$$

By the definitions of $\overline{A}$ and $\overline{gr(A)}$, we have $\overline{A} = A/soc(Ae_1)$ and $\overline{gr(A)} = gr(A)$.

Let $S(i)$ be the simple $\overline{A}$-module corresponding to i and $P(i)$ (resp. $I(i)$) is the projective cover (resp. injective hull) of $S(i)$ for $i = 1, 2$. The AR-quiver of $\overline{A}$ is the following:



For the algebra $\overline{gr(A)}$, by Proposition 5.10, its AR-quiver contains the diamond $\mathcal{D}_1$ associated with 1:

$$\begin{tikzcd}
 & M(\beta_1^{-1}\alpha_0^3\beta_0^{-1}) \arrow[dr] & \\
M(\alpha_0^3\beta_0^{-1}) \arrow[ur] \arrow[dr] & & M(\beta_1^{-1}\alpha_0^3) \arrow[ll, dashed] \\
 & M(\alpha_0^3) \arrow[ur] &
\end{tikzcd}$$

where $M(\alpha_0^3\beta_0^{-1})$ is the projective cover of simple $\overline{gr(A)}$-module S_1 corresponding to 1, $M(\beta_1^{-1}\alpha_0^3)$ is the injective hull of S_1.

We have known that any $\overline{A}$-module is also a $\overline{gr(A)}$-module. Using the above notations, by Theorem 5.13, the AR-quiver of $\overline{gr(A)}$ can be obtained from that of $\overline{A}$ by inserting the diamond $\mathcal{D}_1$:

$$\begin{tikzcd}
 & & & & & & & P(2) \arrow[dr] & & & \\
 & & & & & & rad(P(2)) \arrow[ur] \arrow[dr] & & P_2/soc(P(2)) \arrow[ll, dashed] \arrow[dr] & & rad(P(2)) \arrow[ll, dashed] \\
 & & & & & M(\beta_1^{-1}\alpha_0\beta_0^{-1}) \arrow[ur] \arrow[dr] & & S(1) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & M(\beta_1^{-1}\alpha_0\beta_0^{-1}) \arrow[ll, dashed] \arrow[ur] & \\
 & & & & M(\alpha_0\beta_0^{-1}) \arrow[ur] \arrow[dr] & & M(\alpha_0^{-1}\beta_1) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & M(\alpha_0\beta_0^{-1}) \arrow[ll, dashed] \arrow[ur] & & \\
 & & & M(\beta_1^{-1}\alpha_0^2\beta_0^{-1}) \arrow[ur] \arrow[dr] & & M(\alpha_0) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & M(\beta_1^{-1}\alpha_0^2\beta_0^{-1}) \arrow[ll, dashed] \arrow[ur] & & & \\
 & & P(1) \arrow[ur] \arrow[dr] & & I(1) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & P(1) \arrow[ll, dashed] \arrow[ur] & & & & \\
 & M(\beta_1^{-1}\alpha_0^3\beta_0^{-1}) \arrow[ur] \arrow[dr] & & M(\alpha_0^2) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & M(\beta_1^{-1}\alpha_0^3\beta_0^{-1}) \arrow[ll, dashed] \arrow[ur] & & & & & \\
M(\alpha_0^3\beta_0^{-1}) \arrow[ur] \arrow[dr] & & M(\beta_1^{-1}\alpha_0^3) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & M(\alpha_0^3\beta_0^{-1}) \arrow[ur] & & & & & & \\
S(2) \arrow[u] & M(\alpha_0^3) \arrow[l, dashed] \arrow[ur] & & S(2) \arrow[ll, dashed] \arrow[ur] & & & & & & &
\end{tikzcd}$$

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Jing Guo: School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P. R. China

E-mail address: `gjws@mail.ustc.edu.cn`

Yuming Liu: School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P. R. China

E-mail address: `ymliu@bnu.edu.cn`