ALGEBRAIC MORSE THEORY VIA HOMOLOGICAL PERTURBATION LEMMA WITH TWO APPLICATIONS

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Abstract. As a generalization of the classical killing-contractible-complex lemma, we present algebraic Morse theory via homological perturbation lemma, in a form more general than existing presentations in the literature. Two-sided Anick resolutions due to E. Sköldberg are generalised to algebras given by quiver with relations and a minimal criterion is provided as well. Two applications of algebraic Morse theory are presented. It is shown that a Chinese algebras of rank $n \geq 1$ is homologically smooth and of global dimension $\frac{n(n+1)}{2}$, and the minimal two-sided projective resolution of a Koszul algebra introduced by N. Iyudu and S. Shkarin is constructed.

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Introduction

Discrete Morse theory has its origin in the work of K. S. Brown, R. Geoghegan [6]. In that paper, one encountered a cell complex with one vertex and infinitely many cells in each positive dimension. The authors, using ad hoc method, collapsed this cell complex to a quotient complex with only two cells in each positive dimension. K. S. Brown formalized and applied the collapsing method scheme to groups with a rewriting system [5]. Motivated by differential topology, R. Forman [16] rediscovered and developed this theory as a discrete version of the usual smooth Morse theory, hence the name "discrete Morse theory". Since then, this subject has received much attention in combinatorial and computational topology; see [17, 26, 27, 36] etc.

An algebraic version of discrete Morse theory has been developed by E. Sköldberg [38], D. Kozlov [25], M. Joellenbeck and V. Welker [24]. It has many applications in algebra, such as combinatorial commutative algebra [24], cohomology of Lie algebras [39, 29, 31, 32], Hochschild cohomology [34, 30], operad theory [13].

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E. Sköldberg [40] studied algebraic Morse theory from the viewpoint of homological perturbation lemma [15, 37, 4, 22, 11]. This paper pushes this idea further, generalising and complementing [38, 40] as well as [25, 24] and presents an application to Chinese algebras.

The layout of this paper is as follows.

The first section contains an introduction to Homological Perturbation Lemma (HPL) by providing some new proofs and observations which seem to be of independent interest. In the second section we use HPL to generalise the classical killing-contractible-complex lemma in homological algebra.

Algebraic Morse theory is presented in the third section and our definition of the key concept "Morse matching" is more general than those in [38, 25, 24].

One of the most important values of algebraic Morse theory is that it provides a method to construct two-sided Anick resolutions, compared with the original one-sided version in [1, 2]. This is particularly useful for Hochschild cohomology. In the fourth section, using the corresponding Gröbner-Shirshov basis theory [19], we furnish a construction of two-sided Anick resolutions for algebras given by quivers with relations, generalising [38, Section 3.2]. A similar resolution with different construction of differentials is provided by S. Chouhy and A. Solotar [9], which might be closely related to two-sided Anick resolutions.

The fifth section contains a criterion which gives a sufficient condition when the two-sided Anick resolution is minimal; see Theorem 5.4. We also provide a counter-example to a result that appeared in [24].

The sixth section contains an application of algebraic Morse theory to Chinese algebras [14, 7]. Using the Gröbner-Shirshov basis for Chinese algebras discovered by Y. Chen and J. Qiu [8], we prove that a Chinese algebra of rank $n \geq 1$ is homologically smooth, that is, it admits a finite length resolution by finitely generated projective bimodules, and its global dimension is $\frac{n(n+1)}{2}$; see Theorem 6.5.

In last section, we consider the quadratic algebra $A = k\langle x, y, z \mid x^2 + yx, xz, zy\rangle$ first studied by N. Iyudu and S. Shkarin [23] in their classification result for Hilbert series of Koszul algebras with three generators and three relations with the goal to answer two questions from the book [35]. In [12], V. Dotsenko and S. R. Chowdhury constructed the one-sided Anick resolution of A. We will use the algebraic Morse theory to construct the two-sided minimal resolution of A through its two-sided Anick resolution.

1. Homological Perturbation Lemma

In this section, we give an introduction to Homological Perturbation Lemma (HPL).

We first introduce some relevant notions which are apparently well known, although, up to our knowledge, some of them do not appear in the literature.

Definition 1.1. *(a) A homotopy retract datum (aka. HR datum)*

$$(L_*, b) \underset{p}{\overset{i}{\rightleftarrows}} (M_*, b) \qquad \circlearrowleft h \tag{1}$$

consists of the following

(i) two chain maps i and p between complexes (L_*, b) and (M_*, b), and

(ii) a homotopy h between ip and 1 (so $ip = 1 + bh + hb$).

(b) A HR datum is a strong quasi-isomorphism datum (aka. SQI datum) if moreover,

(iii) p and i are quasi-isomorphisms.

(c) A SQI datum is a homotopy equivalence datum (aka. HE datum) if moreover,

(iv) pi is homotopic to 1.

(d) A HE datum is a deformation retract datum (aka. DR datum), if moreover,

(v) $pi = 1$.

(e) A DR datum is a strong deformation retract datum (aka. SDR datum), if moreover,

(vi) $h^2 = 0, hi = 0, ph = 0$.

Remark 1.2.
(a) DR data and SDR data exist even from the beginning of HPL [15, 37, 4]. The name HE data appeared in the work of M. Crainic [11] with a different meaning, but which are exactly our SQI data.
(b) A DR datum can be always transferred to a SDR datum, as explained in [28].

We now introduce a condition which lies in the heart of HPL.

A perturbation δ of Datum (1) is a graded map on M_* of the same degree as b such that $b+\delta$ is a new differential. We call it *small*, if $1-\delta h$ is invertible. In this case, put

$$A=(1-\delta h)^{-1}\delta$$

and consider a new perturbed datum

$$(L_*, b_\infty) \underset{p_\infty}{\overset{i_\infty}{\rightleftarrows}} (M_*, b+\delta) \qquad \circlearrowleft h_\infty \tag{2}$$

with

$$i_\infty = i + hAi, p_\infty = p + pAh, h_\infty = h + hAh, b_\infty = b + pAi.$$

Remark 1.3. In a heuristic manner,

$$\begin{array}{rcccl} A &=& (1-\delta h)^{-1}\delta &=& \sum_{n=0}^{\infty}(\delta h)^n\delta = \sum_{n=0}^{\infty}\delta(h\delta)^n, \\ i_\infty &=& i+hAi &=& \sum_{n=0}^{\infty}(h\delta)^n i, \\ p_\infty &=& p+pAh &=& \sum_{n=0}^{\infty}p(\delta h)^n, \\ h_\infty &=& h+hAh &=& \sum_{n=0}^{\infty}h(\delta h)^n, \\ b_\infty &=& b+pAi &=& b+\sum_{n=0}^{\infty}p(\delta h)^n\delta i = b+\sum_{n=0}^{\infty}p\delta(h\delta)^n i. \end{array}$$

The following result summarizes known versions of HPL.

Theorem 1.4 (Homological Perturbation Lemma)**.** *Assume that δ is a small perturbation of Datum (1). If Datum (1) is a HR datum (resp. SQI datum, HE datum, SDR datum), then so is Datum (2).*

Remark 1.5.
(a) It is well known that Theorem 1.4 does NOT hold for DR data; see for example, [11, 2.3 Remarks (i)].
(b) The SDR version is the first version of HPL appeared in the literature [15, 37, 4]; the HR version is essentially contained in the proof of the SDR version, in particular, we have

$$i_\infty p_\infty = 1 + (b+\delta)h_\infty + h_\infty(b+\delta);$$

the HE version is firstly shown in [22]; the SQI version is contained in [11]. For a detailed historic account, see [21].

We will present a direct proof for the HE version compared with the original proof of [22], which used a mapping cylinder construction to deduce the HE version from the SDR version, and we will also provide a streamlined proof for the SQI version. In the course of proofs, we add some seemingly new observations.

In the proofs of this section, in order to facilitate the understanding of the reader, we often underline some terms which means that the terms will be changed in the next equality.

For the rest of this section, assume that we are given a HR datum (1) which is endowed with a small perturbation δ.

The following three equalities are crucial in the proof of the HR version of HPL.

Lemma 1.6 (see for example [11])**.** *We have the following equalities:*

$$\delta hA = Ah\delta = A-\delta, \tag{3}$$

$$(1-\delta h)^{-1} = 1+Ah, (1-h\delta)^{-1} = 1+hA, \tag{4}$$

$$AipA + Ab + bA = 0. \tag{5}$$

From now on we assume both the HR version and the SDR version of HPL. We are going to deduce the SQI and the HE version of HPL from these assumptions. The following interesting observations are useful in the proof of the SQI and the HE version of HPL.

Lemma 1.7. *The maps* $p_\infty i - 1 : (L_*, b) \longrightarrow (L_*, b_\infty)$ *and* $pi_\infty - 1 : (L_*, b_\infty) \longrightarrow (L_*, b)$ *are chain maps, that is,*

$$b_\infty(p_\infty i - 1) = (p_\infty i - 1)b \tag{6}$$

$$(pi_\infty - 1)b_\infty = b(pi_\infty - 1). \tag{7}$$

Proof Left to the reader.

□

Lemma 1.8. *The chain maps*

$$i \circ (pi_\infty - 1) : (L_*, b_\infty) \longrightarrow (M_*, b),$$
$$(p_\infty i - 1) \circ p : (M_*, b) \longrightarrow (L_*, b_\infty),$$
$$(pi_\infty - 1) \circ p_\infty : (M_*, b + \delta) \longrightarrow (L_*, b),$$

and

$$i_\infty \circ (p_\infty i - 1) : (L_*, b) \longrightarrow (M_*, b + \delta)$$

are all null-homotopic.

Proof In fact, we have the following equalities:

$$i(pi_\infty - 1) = hi_\infty b_\infty + bhi_\infty, \tag{8}$$

$$(p_\infty i - 1)p = b_\infty p_\infty h + p_\infty hb, \tag{9}$$

$$(pi_\infty - 1)p_\infty = ph_\infty(b + \delta) + bph_\infty, \tag{10}$$

$$i_\infty(p_\infty i - 1) = (b + \delta)h_\infty i + h_\infty ib. \tag{11}$$

whose proofs can be verified directly.

□

Lemma 1.9. *Let* $h' = ph_\infty i$ *and* $h'' = p_\infty hi_\infty$. *The following equalities hold:*

$$pi - 1 = bh' + h'b - (pi_\infty - 1)(p_\infty i - 1), \tag{12}$$

and

$$p_\infty i_\infty - 1 = b_\infty h'' + h''b_\infty - (p_\infty i - 1)(pi_\infty - 1). \tag{13}$$

Proof We only prove Equality (13), while that of Equality (12) can be done similarly.
For Equality (13),

$$\begin{aligned}
p_\infty i_\infty - 1 - b_\infty h'' - h''b_\infty &= p_\infty i_\infty - 1 - \underline{b_\infty p_\infty} hi_\infty - p_\infty h\underline{i_\infty b_\infty} \\
&= p_\infty i_\infty - 1 - \overline{p_\infty(\underline{b} + \delta)}\underline{h}i_\infty - p_\infty \underline{h}(\underline{b} + \delta)i_\infty \\
&= \underline{p_\infty i_\infty} - 1 - p_\infty ipi_\infty + \underline{p_\infty i_\infty} - \underline{p_\infty}\delta hi_\infty - p_\infty h\delta \underline{i_\infty} \\
&= -p_\infty ipi_\infty + 2p_\infty i_\infty - 1 - p\underline{(1 + Ah)}\delta hi_\infty - p_\infty h\underline{\delta(1 + hA)}i \\
&\overset{(3)}{=} -p_\infty ipi_\infty + \underline{2p_\infty i_\infty} - 1 - \underline{pAhi_\infty} - \underline{p_\infty hAi} \\
&= -p_\infty ipi_\infty - 1 + pi_\infty + p_\infty i \\
&= -(p_\infty i - 1)(pi_\infty - 1),
\end{aligned}$$

where the second equality follows from the fact that p_∞ and i_∞ are chain maps (by the HR version) .

□

We now prove the SQI and the HE version of Theorem 1.4.

Proof of the SQI version of Theorem 1.4

Since a SQI datum (1) is of course a HR datum, the perturbed datum (2) is at least a HR datum. So it suffices to prove that p_∞ and i_∞ are quasi-isomorphisms.

As $i_\infty p_\infty$ is homotopic to 1, it follows that p_∞ is injective in homology, so it suffices to prove that p_∞ is surjective in homology.

Given $x \in L_*$ such that $b_\infty(x) = 0$, Lemma 1.9 (13) gives

$$x = p_\infty i_\infty(x) + b_\infty p_\infty h i_\infty(x) - (p_\infty i - 1)(p i_\infty - 1)(x).$$

So we need to show that $(p_\infty i - 1)(p i_\infty - 1)(x)$ lies in the image of b_∞.

Denote $w = (p i_\infty - 1)(x)$, Lemma 1.7 (7) gives

$$b(w) = (p i_\infty - 1) b_\infty(x) = 0.$$

Lemma 1.8 (8) gives

$$i(w) = h i_\infty b_\infty(x) + b h i_\infty(x) = b h i_\infty(x).$$

As i is quasi-isomorphism, we get $w = b(z)$ for some $z \in L_*$. Hence, we have

$$(p_\infty i - 1)(p i_\infty - 1)(x) = (p_\infty i - 1) b(z) \overset{(6)}{=} b_\infty((p_\infty i - 1)(z)).$$

□

Proof of the HE version of Theorem 1.4

Given a small perturbation δ on a HE datum (1) with a homotopy k on (L_*, b) such that

$$pi - 1 = bk + kb.$$

We need to construct a new homotopy k_∞ on (L_*, b_∞) such that

$$p_\infty i_\infty - 1 = b_\infty k_\infty + k_\infty b_\infty.$$

Write $k' = (p_\infty i - 1) k (p i_\infty - 1)$. We obtain

$$\begin{aligned}
& b_\infty k' + k' b_\infty \\
= \;& \underline{b_\infty (p_\infty i - 1)} k (p i_\infty - 1) + (p_\infty i - 1) k \underline{(p i_\infty - 1) b_\infty} \\
\overset{(6)(7)}{=} \;& (p_\infty i - 1) \underline{bk} (p i_\infty - 1) + (p_\infty i - 1) \underline{kb} (p i_\infty - 1) \\
= \;& (p_\infty i - 1)(p i - 1)(p i_\infty - 1) \\
= \;& \underline{(p_\infty i - 1) p}\ \underline{i (p i_\infty - 1)} - (p_\infty i - 1)(p i_\infty - 1) \\
\overset{(8)(9)}{=} \;& (b_\infty p_\infty h + p_\infty h b)(h i_\infty b_\infty + b h i_\infty) - (p_\infty i - 1)(p i_\infty - 1) \\
= \;& b_\infty p_\infty h h i_\infty b_\infty + p_\infty h h i_\infty b_\infty b_\infty + b_\infty p_\infty h b h i_\infty + p_\infty h b h i_\infty b_\infty - (p_\infty i - 1)(p i_\infty - 1) \\
\overset{(13)}{=} \;& b_\infty p_\infty h h i_\infty b_\infty + p_\infty h h i_\infty b_\infty b_\infty + b_\infty p_\infty h b h i_\infty + p_\infty h b h i_\infty b_\infty - b_\infty p_\infty h i_\infty - p_\infty h i_\infty b_\infty \\
& + p_\infty i_\infty - 1.
\end{aligned}$$

Denote

$$k_\infty = (p_\infty i - 1) k (p i_\infty - 1) + p_\infty h i_\infty - p_\infty h b h i_\infty - p_\infty h h i_\infty b_\infty,$$

then we get the desired equality:

$$p_\infty i_\infty - 1 = b_\infty k_\infty + k_\infty b_\infty.$$

□

2. A generalization of killing-contractible-complexe lemma

In this section, given a DR datum (1), we describe the cokernel of i_∞ in the perturbed datum (2). We will see that this is a generalization of the classical killing-contractible-complex lemma [33, Lemma 2.1.6].

Theorem 2.1. *Let R be an associative ring, and let*

$$\cdots \longrightarrow C_n \oplus C'_n \xrightarrow{d=\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}} C_{n-1} \oplus C'_{n-1} \longrightarrow \cdots$$

be a chain complex of R-modules. Assume that $\eta = \eta' + \eta''$ such that (C'_*, η') is a complex and is contractible with contracting homotopy $\sigma : C'_n \longrightarrow C'_{n+1}$. Suppose moreover, that $1+\eta''\sigma : C'_n \longrightarrow C'_n$ is invertible, (e.g. $\eta''\sigma$ is locally nilpotent, that is, for any $x \in C'_n$, there exists a positive integer p such that $(\eta''\sigma)^p(x) = 0$). Denote the map $\lambda = (1 + \eta''\sigma)^{-1}$. Then the following statements hold:

(a) $(C_*, \overline{d} := \alpha - \beta\sigma\lambda\gamma)$ is a complex;
(b) there exist two short exact sequences of complexes

$$0 \to (C'_*, \eta') \xrightarrow{\begin{pmatrix} \beta\sigma \\ 1 + \eta''\sigma \end{pmatrix}} (C_* \oplus C'_*, d) \xrightarrow{g=(1 \ \ -\beta\sigma\lambda)} (C_*, \alpha - \beta\sigma\lambda\gamma) \to 0$$

and

$$0 \to (C_*, \alpha - \beta\sigma\lambda\gamma) \xrightarrow{f=\begin{pmatrix} 1 \\ -\sigma\lambda\gamma \end{pmatrix}} (C_* \oplus C'_*, d) \xrightarrow{(\sigma\gamma \ \ 1+\sigma\eta'')} (C'_*, \eta') \to 0;$$

(c) f and g establish a homotopy equivalence between $(C_*, \overline{d})$ and $(C_* \oplus C'_*, d)$.

Proof Let

$$(L_*, b) = (C_*, 0), (M_*, b) = \Big(C_* \oplus C'_*, \begin{pmatrix} 0 & 0 \\ 0 & \eta' \end{pmatrix}\Big),$$

and

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, p = (1, 0), h = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}, \delta = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta'' \end{pmatrix}.$$

In this case, we have $pi = 1$, $ph = 0$ and $hi = 0$, but not necessarily $h^2 = 0$. So the original datum is a DR datum, but not a SDR datum. Since

$$1 - \delta h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \alpha & \beta \\ \gamma & \eta'' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} = \begin{pmatrix} 1 & \beta\sigma \\ 0 & 1 + \eta''\sigma \end{pmatrix},$$

the invertibility of $1 + \eta''\sigma$ implies that δ is small, and as a DR datum is always a HE datum, we can apply the HE version of Theorem 1.4.

Let us compute the new perturbed datum. We have

$$A = (1 - \delta h)^{-1}\delta = \begin{pmatrix} 1 & \beta\sigma \\ 0 & 1 + \eta''\sigma \end{pmatrix}^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \eta'' \end{pmatrix} = \begin{pmatrix} \alpha - \beta\sigma\lambda\gamma & \beta - \beta\sigma\lambda\eta'' \\ \lambda\gamma & \lambda\eta'' \end{pmatrix};$$

the new differential on C_* is

$$\overline{d} := b_\infty = b + pAi = 0 + (1\ 0)\begin{pmatrix} \alpha - \beta\sigma\lambda\gamma & \beta - \beta\sigma\lambda\eta'' \\ \lambda\gamma & \lambda\eta'' \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha - \beta\sigma\lambda\gamma;$$

we also get two chain maps

$$f := i_\infty = i + hAi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}\begin{pmatrix} \alpha - \beta\sigma\lambda\gamma & \beta - \beta\sigma\lambda\eta'' \\ \lambda\gamma & \lambda\eta'' \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sigma\lambda\gamma \end{pmatrix}$$

and

$$g := p_\infty = p + pAh = (1\ 0) + (1\ 0)\begin{pmatrix} \alpha - \beta\sigma\lambda\gamma & \beta - \beta\sigma\lambda\eta'' \\ \lambda\gamma & \lambda\eta'' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} = \begin{pmatrix} 1 & -\beta\sigma\lambda \end{pmatrix};$$

the map

$$h_\infty = h + hAh = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}\begin{pmatrix} \alpha - \beta\sigma\lambda\gamma & \beta - \beta\sigma\lambda\eta'' \\ \lambda\gamma & \lambda\eta'' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma\lambda \end{pmatrix}$$

realised a homotopy between fg and 1, and the map

$$\begin{aligned}
k_\infty &= (p_\infty i - 1)k(pi_\infty - 1) + p_\infty h i_\infty - p_\infty hbhi_\infty - p_\infty h^2 i_\infty b_\infty \\
&= (p_\infty i - 1)k(pi_\infty - 1) + p_\infty(h - hbh - h^2(b+\delta))i_\infty \\
&= \begin{pmatrix} 1 & -\beta\sigma\lambda \end{pmatrix}\left[\begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \eta' \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}^2\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}\right]\begin{pmatrix} 1 \\ -\sigma\lambda\gamma \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\beta\sigma\lambda \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -\sigma^2\gamma & -\sigma - \sigma\eta'\sigma - \sigma^2\eta \end{pmatrix}\begin{pmatrix} 1 \\ -\sigma\lambda\gamma \end{pmatrix} \\
&= \begin{pmatrix} \beta\sigma\lambda\sigma^2\gamma & \beta\sigma\lambda(\sigma + \sigma\eta'\sigma + \sigma^2\eta) \end{pmatrix}\begin{pmatrix} 1 \\ -\sigma\lambda\gamma \end{pmatrix} \\
&= \beta\sigma\lambda\underline{\sigma^2}\gamma - \beta\sigma\lambda\underline{(\sigma + \sigma\eta'\sigma + \sigma^2\eta)\sigma\lambda}\gamma \\
&= \beta\sigma\lambda(\sigma^2 - \sigma^2\lambda - \sigma\eta'\sigma^2\lambda - \sigma^2\underline{\eta}\sigma\lambda)\gamma \\
&= \beta\sigma\lambda(\underline{\sigma^2 - \sigma^2\lambda} - \sigma\eta'\sigma^2\lambda - \sigma^2\eta'\sigma\lambda\underline{-\sigma^2\eta''\sigma\lambda})\gamma \\
&= \beta\sigma\lambda(\sigma^2\underline{(1 - \lambda - \eta''\sigma\lambda)} - \sigma\eta'\sigma^2\lambda - \sigma^2\eta'\sigma\lambda)\gamma \\
&\overset{(*)}{=} \beta\sigma\lambda(-\sigma\eta'\sigma\sigma\lambda - \sigma\underline{\sigma\eta'}\sigma\lambda)\gamma \\
&= -\beta\sigma\lambda\sigma^2\overline{\lambda}\gamma.
\end{aligned}$$

established a homotopy between gf and 1, where equation $(*)$ use the fact that

$$1 - \lambda - \eta''\sigma\lambda = 1 - \lambda(1 + \eta''\sigma) = 0.$$

This shows (a), (c) and part of (b).

Let us prove the remaining part of (b). Obviously f is injective and g is surjective.
Now we show that the maps

$$\begin{pmatrix} \beta\sigma \\ 1 + \eta''\sigma \end{pmatrix} : (C'_*, \eta') \to \mathrm{Ker}(g) \ \text{ and } \ (\sigma\gamma\ \ 1 + \sigma\eta'') : \mathrm{Cok}(f) \to (C'_*, \eta')$$

are isomorphisms of complexes with inverse maps

$$\begin{aligned}
\begin{pmatrix} \beta\sigma \\ 1 + \eta''\sigma \end{pmatrix}^{-1} &= (0\ \ \lambda) &:& \quad \mathrm{Ker}(g) \to (C'_*, \eta'), \\
(\sigma\gamma\ \ 1 + \sigma\eta'')^{-1} &= \begin{pmatrix} 0 \\ \overline{\lambda} \end{pmatrix} &:& \quad (C'_*, \eta') \to \mathrm{Cok}(f)
\end{aligned}$$

respectively, where $\overline{\lambda} = 1 - \sigma\lambda\eta''$ is the inverse of $1 + \sigma\eta''$

In fact,

$$g \circ \begin{pmatrix} \beta\sigma \\ 1 + \eta''\sigma \end{pmatrix} = \beta\sigma - \beta\sigma\lambda(1 + \eta''\sigma) = 0,$$

so

$$\begin{pmatrix} \beta\sigma \\ 1 + \eta''\sigma \end{pmatrix} : (C'_*, \eta') \to Ker(g)$$

is well-defined. We have

$$\begin{aligned} d\circ\begin{pmatrix} \beta\sigma \\ 1+\eta''\sigma \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}\begin{pmatrix} \beta\sigma \\ 1+\eta''\sigma \end{pmatrix} \\ &= \begin{pmatrix} \alpha\beta\sigma+\beta(1+\eta''\sigma) \\ \gamma\beta\sigma+\eta(1+\eta''\sigma) \end{pmatrix} = \begin{pmatrix} -\beta\eta\sigma+\beta(1+\eta''\sigma) \\ -\eta^2\sigma+\eta(1+\eta''\sigma) \end{pmatrix} \\ &= \begin{pmatrix} \beta(1+\eta''\sigma-\eta\sigma) \\ \eta(1+\eta''\sigma-\eta\sigma) \end{pmatrix} = \begin{pmatrix} \beta\sigma\eta' \\ \eta\sigma\eta' \end{pmatrix} = \begin{pmatrix} \beta\sigma\eta' \\ \eta'+\eta''\sigma\eta' \end{pmatrix} \\ &= \begin{pmatrix} \beta\sigma \\ 1+\eta''\sigma \end{pmatrix}\eta'. \end{aligned}$$

Hence it is a chain map. Now

$$(0\ \ \lambda)\begin{pmatrix} \beta\sigma \\ 1+\eta''\sigma \end{pmatrix} = \lambda(1+\eta''\sigma) = 1;$$

for the other equality, observe that an arbitrary element of Ker(g) has the form $\begin{pmatrix} \beta\sigma\lambda(y) \\ y \end{pmatrix}$ with $y\in C'_*$ and so

$$\begin{pmatrix} \beta\sigma \\ 1+\eta''\sigma \end{pmatrix}(0\ \ \lambda)\begin{pmatrix} \beta\sigma\lambda(y) \\ y \end{pmatrix} = \begin{pmatrix} \beta\sigma\lambda(y) \\ (1+\eta''\sigma)\lambda(y) \end{pmatrix} = \begin{pmatrix} \beta\sigma\lambda(y) \\ y \end{pmatrix}.$$

We have shown that $\begin{pmatrix} \beta\sigma \\ 1+\eta''\sigma \end{pmatrix}$ is an isomorphism of chain complexes.

For the statements about $(\sigma\gamma\ \ 1+\sigma\eta'') : \mathrm{Cok}(f)\to(C'_*,\eta')$, we have

$$(\sigma\gamma\ \ 1+\sigma\eta'')\begin{pmatrix} 1 \\ -\sigma\lambda\gamma \end{pmatrix} = \sigma\gamma-(1+\sigma\eta'')\sigma\lambda\gamma = \sigma\gamma-\sigma\lambda\gamma-\sigma(1-\lambda)\gamma = 0.$$

Hence $(\sigma\gamma\ \ 1+\sigma\eta'') : \mathrm{Cok}(f)\to(C'_*,\eta')$ is well-defined.

As

$$(\sigma\gamma\ \ 1+\sigma\eta'')\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix} = (\sigma\gamma\alpha+(1+\sigma\eta'')\gamma\quad \sigma\gamma\beta+(1+\sigma\eta'')\eta) = (\eta'\sigma\gamma\ \ \eta'\sigma\eta)$$

and $\eta'(\sigma\gamma\ \ 1+\sigma\eta'') = (\eta'\sigma\gamma\ \ \eta'+\eta'\sigma\eta'')$, together with the fact that

$$\eta'\sigma\eta = \eta-\sigma\eta'\eta = \eta-\sigma\eta'\eta'' = \eta'+(1-\sigma\eta')\eta'' = \eta'+\eta'\sigma\eta'',$$

we see that $(\sigma\gamma\ \ 1+\sigma\eta'')$ is a chain map.

We have

$$(\sigma\gamma\ \ 1+\sigma\eta'')\begin{pmatrix} 0 \\ \lambda \end{pmatrix} = (1+\sigma\eta'')\overline{\lambda} = 1$$

and for $x\in C_*, y\in C'_*$,

$$\begin{pmatrix} 0 \\ \overline{\lambda} \end{pmatrix}(\sigma\gamma\ \ 1+\sigma\eta'')\overline{\begin{pmatrix} x \\ y \end{pmatrix}} = \overline{\begin{pmatrix} 0 \\ \overline{\lambda}\sigma\gamma(x)+y \end{pmatrix}} = \overline{\begin{pmatrix} x \\ y \end{pmatrix}}$$

where $\overline{\begin{pmatrix} x \\ y \end{pmatrix}}\in Cok(f)$. Then $(\sigma\gamma\ \ 1+\sigma\eta'') : \mathrm{Cok}(f)\to(C'_*,\eta')$ is an isomorphism of complexes.

□

Remark 2.2. In Theorem 2.1, if $\sigma^2=0$, then $h^2=0$ and the original datum is a SDR datum, so the pertubed datum is also a SDR datum by Theorem 1.4. In this case, we have $gf=1$ and the two short exact sequences of Theorem 2.1 (b) split each other.

The following corollary (the so-called *killing-contractible-complexes lemma*) is well-known in homological algebra and we cite it from the textbook [33, Lemma 2.1.6]. Note that this result is stated over a commutative ring k in [33] but it is obviously true over any associative ring. The form we present here is more precise than [33, Lemma 2.1.6], where it is only stated as a quasi-isomorphism.

Corollary 2.3. *Let R be an associative ring, and let*

$$\cdots \longrightarrow C_n \oplus C'_n \xrightarrow{d=\begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}} C_{n-1} \oplus C'_{n-1} \longrightarrow \cdots$$

be a chain complex of R-modules such that (C'_*, η) is a complex and is contractible with contracting homotopy $\sigma : C'_n \longrightarrow C'_{n+1}$. Then the following inclusion of complexes is a homotopy equivalence:

$$\begin{pmatrix} 1 \\ -\sigma\gamma \end{pmatrix} : (C_*, \alpha - \beta\sigma\gamma) \hookrightarrow (C_* \oplus C'_*, d).$$

3. Algebraic Morse theory

In this section, we present a version of algebraic Morse theory which is more general than existing ones in the literature.

Let R be a ring, all modules will be (left) R-modules.

Let X_* be the following complex of R-modules:

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

Suppose that for each $n \in \mathbb{Z}$, there exists a decomposition into direct sums of submodules

$$X_n = \oplus_{i \in I_n} X_{n,i}.$$

So $d_n : X_n \to X_{n-1}$ has a matrix presentation $d_n = (d_{n,ji})$ with $i \in I_n$, $j \in I_{n-1}$ and where $d_{n,ji} : X_{n,i} \to X_{n-1,j}$ is a homomorphism of modules.

We shall construct a weighted quiver $Q = Q_{X_*}$ as follows:

(Q1) The vertices are the pairs (n, i) with $n \in \mathbb{Z}, i \in I_n$;
(Q2) if a map $d_{n,ji}$ with $i \in I_n$, $j \in I_{n-1}$ does not vanish, then draw an arrow from (n, i) to $(n-1, j)$;
(Q3) for an arrow in (Q2), its weight is just the map $d_{n,ji}$.

A *partial matching* is a full subquiver $\mathcal{M}$ of Q such that

(M1) each vertex in Q belongs to at most one arrow of $\mathcal{M}$;
(M2) each arrow in $\mathcal{M}$ has its weight invertible as a homomorphism of modules.

Given a partial matching $\mathcal{M}$, we can construct a new weighted quiver $Q^{\mathcal{M}}$ with additional dotted arrows as follows:

(QM1) Keep everything for all arrows which are not in $\mathcal{M}$ (they will be called *thick arrows*);
(QM2) For an arrow in $\mathcal{M}$, replace it by a new *dotted arrow* in the reverse direction and the weight of this new arrow is the negative inverse of the weight of the original arrow.

Note that both Q and $Q^{\mathcal{M}}$ have no multiple arrows.

We need to fix some notations: For $n \in \mathbb{Z}$, denote $\mathcal{V}_n = \{(n, i) : i \in I_n\}$. Call a vertex of Q a *critical vertex* (with respect to $\mathcal{M}$), if it is not incident to any arrow in $\mathcal{M}$. Denote by $\mathcal{V}_n^{\mathcal{M}}$ the set of critical vertices in $\mathcal{V}_n$. Denote by $\mathcal{U}_n \subseteq \mathcal{V}_n$ the set of vertices which appear as the starting vertex of an arrow in $\mathcal{M}$, by $\mathcal{D}_n \subseteq \mathcal{V}_n$ the set of vertices which appear as the ending vertex of an arrow in $\mathcal{M}$. Clearly, we have $\mathcal{V}_n = \mathcal{V}_n^{\mathcal{M}} \bigcup \mathcal{U}_n \bigcup \mathcal{D}_n$ (disjoint union). For a path p in $Q^{\mathcal{M}}$, we can define a composition which is the composition of all maps appearing as the weights of all arrows in p, denote this composition as $\varphi_p^{\mathcal{M}}$.

A path in $Q^{\mathcal{M}}$ is called zigzag if dotted arrows and thick arrows appear alternately. For any two vertices $(n, i) \in \mathcal{V}_n$ and $(m, j) \in \mathcal{V}_m$, denote by $\mathcal{P}^{\mathcal{M}}((n, i), (m, j))$ the set of all zigzag paths from (n, i) to (m, j) in $Q^{\mathcal{M}}$; notice that in this case, we have necessarily $m = n, n-1$ or $n+1$. For any two vertices $(n, i) \in \mathcal{D}_n$ and $(n, j) \in \mathcal{V}_n$, denote by $\mathcal{P}_1^{\mathcal{M}}((n, i), (n, j))$ the set of all zigzag paths from (n, i) to (n, j) in $Q^{\mathcal{M}}$ which begin with a dotted arrow (and which necessarily end with a thick arrow). Similarly, we denote by

$\mathcal{P}_2^{\mathcal{M}}((n,i),(n,j))$ the set of all zigzag paths from (n,i) to (n,j) in $Q^{\mathcal{M}}$ which begin with a thick arrow (and which necessarily end with a dotted arrow).

We impose a *local finiteness hypothesis* (LFH) which is more general than conditions imposed previously; for the comparison between this condition with those of [25, 38, 24], see Remarks 3.5, 3.6 and 3.7. A *Morse matching* is a partial matching which satisfies the following local finiteness hypothesis:

(LFH) Given an arbitrary vertex $(n,i) \in \mathcal{D}_n$, for each vertex $(n,j) \in \mathcal{V}_n$ and for each element $x \in X_{n,i}$, the sum

$$\sum_{p\in\mathcal{P}_1^{\mathcal{M}}((n,i),(n,j))} \varphi_p^{\mathcal{M}}(x)$$

exists (for instance, it may be a finite sum or it is convergent in a certain norm); moreover, the number of vertices $(n,j) \in \mathcal{V}_n$ such that

$$\sum_{p\in\mathcal{P}_1^{\mathcal{M}}((n,i),(n,j))} \varphi_p^{\mathcal{M}}(x) \neq 0$$

is finite.

Remark 3.1. Since for each element $x \in X_{n,i}$ the value $d_n(x)$ is a finite sum in $X_{n-1} = \oplus_{j\in I_{n-1}} X_{n-1,j}$, it is easy to see that the condition (LFH) will imply the following three conditions:

(LFH1) Given an arbitrary vertex $(n,i) \in \mathcal{V}_n^{\mathcal{M}}$, for each vertex $(n-1,j) \in \mathcal{V}_{n-1}^{\mathcal{M}}$ and for each element $x \in X_{n,i}$, the sum

$$\sum_{p\in\mathcal{P}^{\mathcal{M}}((n,i),(n-1,j))} \varphi_p^{\mathcal{M}}(x)$$

exists; moreover, the number of vertices $(n-1,j) \in \mathcal{V}_{n-1}^{\mathcal{M}}$ such that

$$\sum_{p\in\mathcal{P}^{\mathcal{M}}((n,i),(n-1,j))} \varphi_p^{\mathcal{M}}(x) \neq 0$$

is finite.

(LFH2) Given an arbitrary vertex $(n,i) \in \mathcal{D}_n$, for each vertex $(n+1,j) \in \mathcal{U}_{n+1}$ and for each element $x \in X_{n,i}$, the sum

$$\sum_{p\in\mathcal{P}^{\mathcal{M}}((n,i),(n+1,j))} \varphi_p^{\mathcal{M}}(x)$$

exists; moreover, the number of vertices $(n+1,j) \in \mathcal{U}_{n+1}$ such that

$$\sum_{p\in\mathcal{P}^{\mathcal{M}}((n,i),(n+1,j))} \varphi_p^{\mathcal{M}}(x) \neq 0$$

is finite.

(LFH3) Given an arbitrary vertex $(n,i) \in \mathcal{V}_n^{\mathcal{M}}$, for each vertex $(n,j) \in \mathcal{V}_n$, and for each element $x \in X_{n,i}$, the sum

$$\sum_{p\in\mathcal{P}_2^{\mathcal{M}}((n,i),(n,j))} \varphi_p^{\mathcal{M}}(x)$$

exists; moreover, the number of vertices $(n,j) \in \mathcal{V}_n$ such that

$$\sum_{p\in\mathcal{P}_2^{\mathcal{M}}((n,i),(n,j))} \varphi_p^{\mathcal{M}}(x) \neq 0$$

is finite.

The following sufficient condition for the Morse matching is frequently useful.

Proposition 3.2. *Let $\mathcal{M}$ be a partial matching of Q. If any zigzag path from (n,i) is of finite length for each vertex (n,i) in $Q^{\mathcal{M}}$, then $\mathcal{M}$ is a Morse matching.*

Proof Let $(n,i) \in \mathcal{D}_n$. We first prove that the set of all zigzag paths from (n,i) which begin with a dotted arrow is a finite set. Assume that there are infinite such zigzag paths in $Q^{\mathcal{M}}$. The condition (M1) of the partial matching $\mathcal{M}$ guarantees that the first dotted arrow of these zigzag paths coincide which is of the form $(n,i) \dashrightarrow (n+1,j)$. Since each term X_n in the complex X_* is a direct sum, there are only finitely many thick arrows leaving from $(n+1,j)$. So there exists at least one thick arrow $(n+1,j) \to (n,k)$ such that there are infinite zigzag paths which begin with

$$(n,i) \dashrightarrow (n+1,j) \to (n,k).$$

Repeating the above process for the zigzag paths from (n,k) and by induction, we obtain a zigzag path from (n,i) of infinite length, a contradiction. Hence the set of all zigzag paths from (n,i) which begin with a dotted arrow is a finite set. In particular, for each element $x \in X_{n,i}$, the set

$$\{p \mid p \text{ is a zigzag path which begins with a dotted arrow from (n,i)}, \varphi_p^{\mathcal{M}}(x) \neq 0\}$$

is a finite set. Therefore the condition (LFH) follows.

□

Given a Morse matching $\mathcal{M}$, we can construct a new complex $(X_*^{\mathcal{M}}, d^{\mathcal{M}})$ as follows:

The complex $X_*^{\mathcal{M}}$ has its n-th component $X_n^{\mathcal{M}} = \oplus_{(n,i)\in\mathcal{V}_n^{\mathcal{M}}} X_{n,i}$ and the differential $d_n^{\mathcal{M}} : X_n^{\mathcal{M}} \to X_{n-1}^{\mathcal{M}}$ has the matrix presentation $d_n^{\mathcal{M}} = (d_{n,ji}^{\mathcal{M}})$ with $(n,i) \in \mathcal{V}_n^{\mathcal{M}}, (n-1,j) \in \mathcal{V}_{n-1}^{\mathcal{M}}$ and where $d_{n,ji}^{\mathcal{M}} : X_{n,i} \to X_{n-1,j}$ is defined to be

$$d_{n,ji}^{\mathcal{M}} = \sum_{p\in\mathcal{P}^{\mathcal{M}}((n,i),(n-1,j))} \varphi_p^{\mathcal{M}}.$$

Note that $d^{\mathcal{M}}$ exists by (LFH1).

Now we can state the main result of algebraic Morse theory, which contains as special cases and refines all versions appeared in the literature [25, 39, 24].

Theorem 3.3. *(a) Within the above setup, $(X_*^{\mathcal{M}}, d^{\mathcal{M}})$ is a complex.*

(b) Define maps

$$f_n : X_n^{\mathcal{M}} \to X_n$$
$$x \in X_{n,i} \mapsto f_n(x) := x + \sum_{(n,j)\in\mathcal{U}_n} \sum_{p\in\mathcal{P}_2^{\mathcal{M}}((n,i),(n,j))} \varphi_p^{\mathcal{M}}(x),$$

and

$$g_n : X_n \to X_n^{\mathcal{M}}$$
$$x \in X_{n,i} \mapsto g_n(x) := \begin{cases} \sum_{(n,j)\in\mathcal{V}_n^{\mathcal{M}}} \sum_{p\in\mathcal{P}_1^{\mathcal{M}}((n,i),(n,j))} \varphi_p^{\mathcal{M}}(x), & (n,i) \in \mathcal{D}_n \\ x, & (n,i) \in \mathcal{V}_n^{\mathcal{M}} \\ 0 & (n,i) \in \mathcal{U}_n \end{cases}$$

Then $f_* : X_*^{\mathcal{M}} \to X_*$ and $g_* : X_* \to X_*^{\mathcal{M}}$ are chain maps which are homotopy equivalent: $gf = \mathrm{Id}_{X_*^{\mathcal{M}}}$ and $fg \sim \mathrm{Id}_{X_*}$ via the homotopy

$$\theta_n : X_n \to X_{n+1}$$
$$x \in X_{n,i} \mapsto \theta_n(x) := \begin{cases} \sum_{(n+1,j)\in\mathcal{U}_{n+1}} \sum_{p\in\mathcal{P}^{\mathcal{M}}((n,i),(n+1,j))} \varphi_p^{\mathcal{M}}(x), & (n,i) \in \mathcal{D}_n \\ 0 & \text{otherwise} \end{cases}$$

(c) We have a decomposition

$$(X_*, d) \cong (X_*^{\mathcal{M}}, d^{\mathcal{M}}) \oplus (Y_*, d^Y)$$

where (Y_*, d^Y) is a null homotopic complex.

Proof We will use Theorem 2.1 to prove the result.

Construct the chain complex of R-modules $(C_n \oplus C'_n, d)$, $d = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}$, as follows:

let

$$C_n = \oplus_{(n,i)\in \mathcal{V}_n^{\mathcal{M}}} X_{n,i}, C'_n = \oplus_{(n,i)\in \mathcal{V}_n \setminus \mathcal{V}_n^{\mathcal{M}}} X_{n,i},$$

$\alpha, \beta, \gamma, \eta$ be the map defined by corresponding arrows in Q respectively, η' be the map defined by arrows in $\mathcal{M}$, $\eta'' = \eta - \eta'$ be the weight of the arrows which are not in $\mathcal{M}$; for an arrow $d_{n,ji} : X_{n,i} \to X_{n-1,j}$ lying in $\mathcal{M}$, then $\sigma = d_{n,ji}^{-1} : X_{n-1,j} \to X_{n,i}$ is the homotopy for the "piece" $d_{n,ji} : X_{n,i} \to X_{n-1,j}$, so the weight of the dotted arrow is $-\sigma$. Note that we have always $\sigma^2 = 0$.

It's obvious that $(C_n \oplus C'_n, d)$ is just the original chain complex X_*; the fact that $\eta'^2 = 0$ and that σ is contracting homotopy of (C'_n, η') follow from the definition of Morse matching; it is interesting to see that powers of $\eta''\sigma$ are zigzag paths in $Q^{\mathcal{M}}$, so for each $x \in X_{n,i}$ with $(n, i) \in \mathcal{D}_n$,

$$\lambda(x) := (1 + \eta''\sigma)^{-1}(x) = \sum_{i=0}^{\infty} (\eta''(-\sigma))^i(x)$$

exists by the axiom (LFH).

Note that the existence of f, g and θ follows from (LFH3), (LFH) and (LFH2) respectively. It is not difficult to check that f and g are exactly the chain maps f and g in Theorem 2.1, and that the homotopy θ is just

$$h_\infty = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma\lambda \end{pmatrix}$$

in Theorem 2.1. Now by applying Theorem 2.1 and Remark 2.2, and by noting that the differential in the new complex $X_*^{\mathcal{M}}$ is just $\alpha - \beta\sigma\lambda\gamma$, we get the desired result.

□

Remark 3.4. By Remark 2.2, the above theorem in fact gives a SDR datum

$$X_*^{\mathcal{M}} \underset{g}{\overset{f}{\rightleftarrows}} X_* \circlearrowright \theta .$$

Remark 3.5. In the paper [25], D. N. Kozlov defined a Morse matching to be a partial matching satisfying the following conditions:

(K1) Each I_n is finite;

(K2) for each vertex $(n, i) \in \mathcal{D}_n$, $\mathcal{P}_1^{\mathcal{M}}((n, i), (n, i)) = \emptyset$, i.e. there is no zigzag path, which begins with a dotted arrow and ends with a thick arrow, from (n, i) to itself.

In fact, D. N. Kozlov also asked that R is commutative ring and that $X_n = 0$ for $n << 0$. It is obvious that the first condition is unnecessary and the latter one is superfluous by [25, Remark 2].

It is easy to see that whenever the conditions (K1)(K2) hold, so does the axiom (LFH).

Remark 3.6. In the article [38], E. Sköldberg defined a Morse matching to be a partial matching satisfying the following condition:

(S) Each I_n has a well-founded partial order $\prec$ such that for $a, c \in I_n$, $c \prec a$ whenever there is a zigzag path of length two $a \to b \to c$ in $Q^{\mathcal{M}}$ (note that one of the arrows should be dotted).

It is easy to see that the condition (S) is more general than (K1)(K2) in Remark 3.5. The condition (S) also implies the axiom (LFH) by Proposition 3.2. In fact, for every (n, i) in Q, by the well-founded partial order condition, each zigzag path from (n, i) has finite length.

Remark 3.7. In the article [24], M. Joellenbeck and V. Welker defined a Morse matching to be a partial matching satisfying the following conditions:

(JW1) There is no directed cycles in $Q^{\mathcal{M}}$;

(JW2) the finiteness condition in [24, Definition 2.3].

It is not hard to see that whenever the conditions (JW1)(JW2) hold, the condition (LFH) also holds.

4. Two-sided Anick resolutions

In this section, we generalise the construction of two-sided Anick resolutions of E. Sköldberg in [38] from one vertex algebras to algebras given by quotients of path algebras of quivers.

Let k be a fixed field. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver with vertex set Q_0 and arrow set Q_1 and where $s : Q_1 \to Q_0$ (resp. $t : Q_1 \to Q_0$) gives the starting vertex (resp. the target vertex) of an arrow. We will write paths from left to right, that is, the notation $p = \alpha_1\alpha_2 \cdots \alpha_r$ means that for $1 \leq i \leq r-1$, $s(\alpha_{i+1}) = t(\alpha_i)$. The length $l(p)$ of the above path p is defined to be r and the vertices are viewed as paths of length 0. For $n \geq 0$, Q_n denotes the set of all paths of length n and $Q_{\geq n}$ is the set of all paths with length at least n. Denote kQ its path algebra, say the space generated by all paths of finite length and endowed with the multiplication given by concatenation of paths.

Let us first briefly recall the Gröbner-Shirshov basis theory for path algebra kQ following E. Green's paper [19]. Let $\mathcal{B} := Q_{\geq 0}$ denote the set of all finite (directed) paths in Q. Then $\mathcal{B}$ is a multiplicative k-basis of kQ. Write $\mathcal{B}_+ = \mathcal{B} \setminus Q_0$. Fix an admissible well-order $\prec$ on $\mathcal{B}$, that is, a well-order on $\mathcal{B}$ which is compatible with multiplication. For instance, we can take a left length-lexicographic order. For a linear combination r of paths, its tip $\mathrm{Tip}(r)$ is by definition the maximal monomial appearing with nonzero coefficients in r. For a nonempty subset X of kQ, put $\mathrm{Tip}(X) = \{\mathrm{Tip}(r) \mid r \in X, r \neq 0\}$.

Let I be a two-sided ideal in kQ contained in $kQ_{\geq 2}$. Write $\mathrm{NonTip}(I)$ the complement set of $\mathrm{Tip}(I)$ in $\mathcal{B}$. Then there exists a decomposition of vector spaces

$$kQ = I \oplus \mathrm{Span}_k(\mathrm{NonTip}(I)).$$

So $\mathrm{NonTip}(I)$ is a basis of the quotient algebra $A = kQ/I$. Recall that a Gröbner-Shirshov basis of I with respect to the admissible order $\prec$ is a subset $\mathcal{G} \subseteq I$ such that $W := \mathrm{Tip}(\mathcal{G})$ generates the initial ideal $\langle \mathrm{Tip}(I) \rangle$. Note that in this case $I = \langle \mathcal{G} \rangle$. A Gröbner-Shirshov basis $\mathcal{G}$ for the ideal I is reduced if the following three conditions are satisfied:

- (R1) For any $g \in \mathcal{G}$, the coefficient of $\mathrm{Tip}(g)$ is 1;
- (R2) For any $g \in \mathcal{G}$, $g - \mathrm{Tip}(g) \in \mathrm{Span}_k(\mathrm{NonTip}(I))$;
- (R3) No element in $W = \mathrm{Tip}(\mathcal{G})$ is a factor of another element in W.

It is easy to see that under the given admissible order, I has a unique reduced Gröbner-Shirshov basis, and in this case W is a minimal generator set of $\langle \mathrm{Tip}(I) \rangle$; moreover, $b \in \mathcal{B}$ lies in $\mathrm{NonTip}(I)$ if and only if b does not divide by any element of W. In the following, we always assume that $W = \mathrm{Tip}(\mathcal{G})$ for a reduced Gröbner-Shirshov basis $\mathcal{G}$ of I.

Similar as in [38], we define a new quiver $Q_W = (V, E)$ (with respect to W), called the Ufnarovskiĭ graph [41], with vertex set V and arrow set E as follows:

$$V = Q_0 \cup Q_1 \cup \{u \in \mathcal{B} \mid u \text{ is a proper right factor of some } v \in W\},$$

and E is the union of $\{e \to x | e \in Q_0, x = ex \in Q_1\}$ with

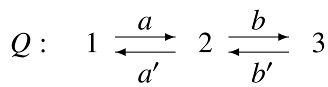
$$\{u \to v \mid uv \in \mathcal{B}, uv \in \langle \mathrm{Tip}(I) \rangle, w \notin \langle \mathrm{Tip}(I) \rangle \text{ for all proper left factors } w \text{ of } uv\}.$$

Note that the above condition for the arrow $u \to v$ is equivalent to the following: $uv \in \mathcal{B}$ and uv has a unique factor $w \in W$ which is a right factor of uv. Clearly the above condition implies the following: if $u \to v$ is an arrow in Q_W, then $u \to v_1$ can not be an arrow in Q_W for any proper left factor v_1 of v. The set of i-chains $W^{(i)}, i \geq 0$ (also called Anick chains) consists of all sequences $(v_1, \cdots, v_i, v_{i+1})$ in $\mathcal{B}_+^{i+1}$ such that

$$e \to v_1 \to \cdots \to v_i \to v_{i+1}$$

is a path in Q_W, where $e \in Q_0$ and $\mathcal{B}_+ = \mathcal{B} \setminus Q_0$; by convention $W^{(-1)} = Q_0$. For $w = (w_1, \cdots, w_m) \in W^{(m-1)}$, the length of the path $w_1 \cdots w_m$ is called the degree of w and m is called the weight of w. In order to have an intuition, let us consider a concrete example of the Ufnarovskiĭ graph.

Example 4.1. Let $A = kQ/I$ be a finite dimensional algebra defined by the following quiver



with relations

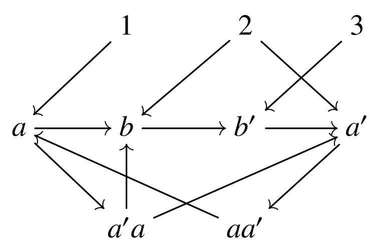
$$ab = b'a' = a'a - bb' = 0.$$

Consider the left length-lexicographic order with $a > b > b' > a'$. Then $\mathcal{G} = \{ab, b'a', bb'-a'a, aa'a, a'aa'\}$ is a reduced Gröbner-Shirshov basis of I with respect to this order. Therefore we have the following:

$$W = \mathrm{Tip}(\mathcal{G}) = \{ab, b'a', bb', aa'a, a'aa'\}, Q_0 = \{1, 2, 3\} := \{e_1, e_2, e_3\}, Q_1 = \{a, b, a', b'\},$$

$$\{u \text{ is a proper right factor for some } v \in W\} \setminus Q_1 = \{a'a, aa'\}, \mathrm{NonTip}(I) \setminus Q_0 = \{a, b, b', a', a'a, aa', b'b\}.$$

The associated Ufnarovskiĭ graph Q_W is



Now let $A = kQ/I$ be an arbitrary algebra as before. Then $E = \oplus_{e \in Q_0} ke$ is a semisimple subalgebra of A such that $A = E \oplus A_+$ as spaces, where $A_+ = \mathrm{Span}_k\{\mathrm{NonTip}(I) \setminus Q_0\}$. Recall the reduced two-sided bar resolution $B(A, A)$ of the algebra A in the sense of C. Cibils [10]:

$$B(A, A)_0 = A \otimes_E A, \text{ and for } n \geq 1, B(A, A)_n = A \otimes_E (A_+)^{\otimes_E n} \otimes_E A \cong A^e \otimes_{E^e} (A_+)^{\otimes_E n},$$

where $A^e = A \otimes_k A^{op}$ and $E^e = E \otimes_k E^{op}$, and the differential is defined by (for $n \geq 1$)

$$d([a_1|\cdots|a_n]) = a_1[a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\cdots|a_ia_{i+1}|\cdots|a_n] + (-1)^n[a_1|\cdots|a_{n-1}]a_n, \tag{14}$$

where $[a_1|\cdots|a_n] = 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$. We decompose the above resolution as follows:

$$B(A, A)_n = \bigoplus A^e \otimes_{E^e} k[w_1|\cdots|w_n] = \bigoplus A^e \cdot [w_1|\cdots|w_n],$$

where the sum takes over all the sequences $(w_1, \cdots, w_n)$ such that all $w_i \in \mathrm{NonTip}(I) \setminus Q_0$ and that $w_1 \cdots w_n$ is a path in Q. Note that if $w_1 \cdots w_n = e_i w_1 \cdots w_n e_j$, then $A^e \cdot [w_1|\cdots|w_n]$ is isomorphic to the indecomposable projective A^e-module $Ae_i \otimes_k e_jA$. We write $(w_1, \cdots, w_n)$ instead of $[w_1|\cdots|w_n]$ for the vertices in the decorated quiver $Q_B = Q_{B(A,A)}$ (cf. Section 3).

In general, an arrow in Q_B may contains the information of several terms in the expression (14) of the differential (for example $d([a]) = a[]1 - 1[]a$ and its corresponding arrow is $(a) \overset{a\otimes 1 - 1\otimes a}{\to} e_1 \in Q_0$). For the sake of clarity, it is necessary to view each term of the expression of the differential d as an arrow. That is to say we will use a new weighted quiver $\overline{Q_B}$ to construct the two-sided Anick resolution of A via Theorem 3.3 instead of Q_B. The weighted quiver $\overline{Q_B}$ has the same vertices set as that of Q_B. And the arrows of $\overline{Q_B}$ are listed as follows. For a vertex $(w_1, \cdots, w_n)$ in $\overline{Q_B}$,

(a) we denote d_n^0 the arrow

$$(w_1, w_2, \cdots, w_n) \xrightarrow[w_1 \otimes 1]{d_n^0} (w_2, \cdots, , w_n)$$

with weight $w_1 \otimes 1$, which is put under the arrow (we will do the same in the following);

(b) for $1 \leq i \leq n-1$, assume that $w_i w_{i+1} \equiv \sum_j \lambda_j u_j \mod I$ with all $u_j \in \mathrm{NonTip}(I) \backslash Q_0$ and $\lambda_j \in k^* = k \setminus \{0\}$; by abuse of notations, we still denote by d_n^i all arrows of the form:

$$(w_1, \cdots, w_{i-1}, w_i, w_{i+1}, w_{i+2}, \cdots, w_n) \xrightarrow[(-1)^i \cdot \lambda_j \otimes 1]{d_n^i} (w_1, \cdots, w_{i-1}, u_j, w_{i+2}, \cdots, w_n);$$

(c) the arrow

$$(w_1, \cdots, w_{n-1}, w_n) \xrightarrow[(-1)^n 1 \otimes w_n]{d_n^n} (w_1, \cdots, , w_{n-1})$$

with weight $(-1)^n 1 \otimes w_n$ is denoted by d_n^n. Note that the weight here should be written as $(-1)^n 1 \otimes w_n^{op}$, but for simplicity, we just write it as $(-1)^n 1 \otimes w_n$.

Remark 4.2. We can define all the conceptions in Section 3 for $\overline{Q_B}$ similarly. Notice that the only difference between Q_B and $\overline{Q_B}$ is that there exist parallel (or multiple) arrows in $\overline{Q_B}$ whose weights sum to that of the arrow with the same starting vertex and target in Q_B. It follows that Theorem 3.3 yields the same result if we use $\overline{Q_B}$ instead of Q_B. Note that the notion $\overline{Q_B}$ and the notion $\overline{Q_B}^{\mathcal{M}}$ below in the present paper is only valid for the reduced two-sided bar resolution $B(A, A)$.

For $w \in \mathcal{B}$, let $V_{w,i}$ be the vertices $(w_1, \cdots, w_n)$ in $\overline{Q_B}$ such that $w = w_1 \cdots w_n$ and i is the largest integer $i \geq -1$ such that $(w_1, \cdots, w_{i+1})$ is an i-chain. Let $V_w = \bigcup_i V_{w,i}$. Thus $(w_1, \cdots, w_n) \in V_{w,-1}$ if and only if $w_1 \notin Q_1$, $(w_1, \cdots, w_n) \in V_{w,0}$ if and only if $w_1 \in Q_1$ and (w_1, w_2) is not a 1-chain, etc.

We define a partial matching $\mathcal{M}$ to be the set of arrows of the following form in $\overline{Q_B}$:

$$(w_1, \cdots, w_{i+1}, w'_{i+2}, w''_{i+2}, w_{i+3}, \cdots, w_n) \to (w_1, \cdots, w_{i+1}, w_{i+2}, w_{i+3}, \cdots, w_n), \tag{15}$$

where $(w_1, \cdots, w_n) \in V_{w,i}$, $w'_{i+2} w''_{i+2} = w_{i+2}$ and $(w_1, \cdots, w_{i+1}, w'_{i+2}, w''_{i+2}, w_{i+3}, \cdots, w_n) \in V_{w,i+1}$. Note that in this case i is necessarily less than $n-1$. Indeed, $\mathcal{M}$ is a partial matching: clearly no vertex in $\mathcal{M}$ is the origin of more than one edges in $\mathcal{M}$; if a vertex $(w_1, \cdots, w_{i+1}, w_{i+2}, w_{i+3}, \cdots, w_n)$ in $\mathcal{M}$ is the terminus of more than one edges in $\mathcal{M}$, then there would have two different decompositions $w_{i+2} = w'_{i+2} w''_{i+2} = v'_{i+2} v''_{i+2}$ such that $w_{i+1} w'_{i+2}$ has a unique factor in W which is a right factor of $w_{i+1} w'_{i+2}$ and also that $w_{i+1} v'_{i+2}$ has a unique factor in W which is a right factor of $w_{i+1} v'_{i+2}$, and this is a contradiction; the situation

$$(w_1, \cdots, w_{i+1}, w'_{i+2}, w''_{i+2}, w_{i+3}, \cdots, w_n) \to (w_1, \cdots, w_{i+1}, w_{i+2}, w_{i+3}, \cdots, w_n) \in V_{w,i},$$

$$(w_1, \cdots, w_{i+1}, w'_{i+2}, w^{(3)}_{i+2}, w^{(4)}_{i+2}, w_{i+3}, \cdots, w_n) \to (w_1, \cdots, w_{i+1}, w'_{i+2}, w''_{i+2}, w_{i+3}, \cdots, w_n) \in V_{w,i+1}$$

cannot occur since this would imply w_{i+2} lies in $\langle \mathrm{Tip}(I) \rangle$; moreover, the arrow (15) in $\mathcal{M}$ represents an invertible homomorphism since its weight is $(-1)^{i+2}$.

Recall from last section that the above partial matching gives a new decorated quiver $\overline{Q_B}^{\mathcal{M}}$ from the original decorated quiver $\overline{Q_B}$ by reversing the arrows in $\mathcal{M}$. The arrows d_n^i in $\overline{Q_B}$ which remain unchanged in $\overline{Q_B}^{\mathcal{M}}$ will be drawn by thick arrows, still denoted by d_n^i; an arrow d_n^i lying in $\mathcal{M}$ will be drawn in dotted arrows with reverse direction, denoted by d_n^{-i}. By definition arrow d_n^i in $\mathcal{M}$ necessarily has $1 \leq i \leq n-1$ and $w_i w_{i+1} \in \mathrm{NonTip}(I)\backslash Q_0$, so the arrow d_n^{-i} has the form

$$\begin{array}{c} (w_1, \cdots, w_{i-1}, w_i w_{i+1}, w_{i+2}, \cdots, w_n) \\ \big\downarrow {\scriptstyle d_n^{-i}},\ {\scriptstyle (-1)^{i+1}} \\ (w_1, \cdots, w_{i-1}, w_i, w_{i+1}, w_{i+2}, \cdots, w_n) \end{array}.$$

The following result should be compared with [38, Lemma 9].

Theorem 4.3. *The partial matching $\mathcal{M}$ defined above is a Morse matching of $\overline{Q_B}$ such that the set of critical vertices in n-th component is identified with the set $W^{(n-1)}$ of $n-1$-chains.*

Proof First we identify the critical vertices in n-th component. Suppose $(w_1, \cdots, w_n) \in V_{w,i}$ is a critical vertex. Then $(w_1, \cdots, w_{i+1})$ is an i-chain. Suppose $i < n-1$. There are two cases: $w_{i+1}w_{i+2} \in \langle \mathrm{Tip}(I)\rangle$ or $w_{i+1}w_{i+2} \notin \langle \mathrm{Tip}(I)\rangle$. When $w_{i+1}w_{i+2} \in \langle \mathrm{Tip}(I)\rangle$, there is a decomposition $w_{i+2} = w'_{i+2}w''_{i+2}$ with w'_{i+2} minimal such that $w_{i+1}w'_{i+2} \in \langle \mathrm{Tip}(I)\rangle$. Since $(w_1, \cdots, w_{i+2})$ is not an $i+1$-chain, w''_{i+2} has nonzero length, which means that there is an edge $(w_1, \cdots, w_{i+1}, w'_{i+2}, w''_{i+2}, \cdots, w_n) \to (w_1, \cdots, w_n)$ in $\mathcal{M}$. This contradicts that $(w_1, \cdots, w_n)$ is a critical vertex. When $w_{i+1}w_{i+2} \notin \langle \mathrm{Tip}(I)\rangle$, $(w_1, \cdots, w_{i+1}w_{i+2}, \cdots, w_n) \in V_{w,i-1}$ and there is an edge $(w_1, \cdots, w_n) \to (w_1, \cdots, w_{i+1}w_{i+2}, \cdots, w_n)$ in $\mathcal{M}$. This is also a contradiction. Hence $i = n-1$ and $(w_1, \cdots, w_n) \in W^{(n-1)}$. On the other hand, it is obvious that all vertices $(w_1, \cdots, w_n)$ in $W^{(n-1)}$ are critical vertices in n-th component. Thus the set of critical vertices in n-th component is identified with the set $W^{(n-1)}$ of $n-1$-chains.

Next, we show that any zigzag path in $\overline{Q_B}^{\mathcal{M}}$ is of finite length. Consider a vertex $(w_1, \cdots, w_n) \in V_w$, and look at the corresponding differential

$$d([w_1|\cdots|w_n]) = w_1[w_2|\cdots|w_n] + \sum_{i=1}^{n-1}(-1)^i[w_1|\cdots|w_iw_{i+1}|\cdots|w_n] + (-1)^n[w_1|\cdots|w_{n-1}]w_n.$$

The element $w_1[w_2|\cdots|w_n]$ (resp. $[w_1|\cdots|w_{n-1}]w_n$) is in the component corresponding to the vertex $(w_2, \cdots, w_n)$ (resp. $(w_1, \cdots, w_{n-1})$), and $w_2 \cdots w_n \prec w_1 \cdots w_n$ (resp. $w_1 \cdots w_{n-1} \prec w_1 \cdots w_n$). The elements $[w_1|\cdots|w_iw_{i+1}|\cdots|w_n]$ can all be written as linear combinations of elements in components corresponding to $(w_1, \cdots, w_{i-1}, u, w_{i+2}, \cdots, w_n)$, where $w_1 \cdots w_{i-1}uw_{i+2} \cdots w_n \preceq w_1 \cdots w_n$, with equality or inequality depending on whether $w_iw_{i+1} \notin \langle \mathrm{Tip}(I)\rangle$ (this is the only case such that the vertex $(w_1, \cdots, w_{i-1}, u, w_{i+2}, \cdots, w_n)$ remains in V_w with $u = w_iw_{i+1}$) or not (in this case $w_iw_{i+1} \in \mathrm{Tip}(I)$ and $w_iw_{i+1} = \sum_j \lambda_j u_j \pmod I$ with all $u_j \prec w_iw_{i+1}$ or $w_iw_{i+1} = 0$). So for a thick arrow $v \to v'$ in $\overline{Q_B}^{\mathcal{M}}$ with $v \in V_w$ and $v' \in V_{w'}$, we have $w \succeq w'$. On the other hand, for a dotted arrow $v \dashrightarrow v'$ in $\overline{Q_B}^{\mathcal{M}}$, we have $v, v' \in V_w$ for some $w \in \mathcal{B}$ by the definition of $\mathcal{M}$.

Let $p = \alpha_1\alpha_2\cdots$ be a zigzag path in $\overline{Q_B}^{\mathcal{M}}$. We are going to prove that p has finite length. By the well-ordering of $\preceq$ and the observations above, without loss of generality we may assume that the starting vertex of each arrow in p belongs to V_w with the same $w \in \mathcal{B}$. Let

$$v_k \overset{\alpha_k}{\dashrightarrow} v_{k+1} \overset{\alpha_{k+1}}{\longrightarrow} v_{k+2}$$

be the segment of p of length 2. By the construction of $\mathcal{M}$, the dotted arrow α_k has the form

$$d_{n+1}^{-(i+2)} : v_k = (w_1, \cdots, w_{i+1}, w_{i+2}, w_{i+3}, \cdots, w_n) \dashrightarrow v_{k+1} = (w_1, \cdots, w_{i+1}, w'_{i+2}, w''_{i+2}, w_{i+3}, \cdots, w_n)$$

with $v_k \in V_{w,i}$ and $v_{k+1} \in V_{w,i+1}$, $-1 \leq i \leq n-2$. As $v_{k+2} \in V_w$, the thick arrow $\alpha_{k+1} \neq d_{n+1}^l, 0 \leq l \leq i+1$. Notice that $\alpha_k = d_{n+1}^{-(i+2)}$, thus the arrow d_{n+1}^{i+2} with starting vertex v_{k+1} does not exist in $\overline{Q_B}^{\mathcal{M}}$. Hence $\alpha_{k+1} = d_{n+1}^l$ with $i+3 \leq l \leq n+1$ which shows that the first $i+2$ components of v_{k+1} and v_{k+2} coincide. Then we have $v_{k+2} \in V_{w,i+j}$ with $j > 0$. As the length of $w \in \mathcal{B}$ is finite, the subscript i of $V_{w,i}$ has a finite upper bound. So the zigzag path p is of finite length. Hence $\mathcal{M}$ is a Morse matching in terms of Proposition 3.2.

□

Hence by Theorem 3.3, the reduced two-sided bar resolution $B(A, A)$ of the algebra A is homotopy equivalent to a complex $(B(A, A)^{\mathcal{M}}, d^{\mathcal{M}})$ associated to the quiver $\overline{Q_B}^{\mathcal{M}}$. We will call $(B(A, A)^{\mathcal{M}}, d^{\mathcal{M}})$ a two-sided Anick resolution of A according to [38, Theorem 4], and this resolution can be described as follows: for $n \geq 0$, the n-th component is $A \otimes_E kW^{(n-1)} \otimes_E A$, and the differential from n-th component to $(n-1)$-th component corresponds to the sum

$$\sum_{\substack{p \in \mathcal{P}^{\mathcal{M}}(w,w') \\ w \in W^{(n-1)}, w' \in W^{(n-2)}}} \varphi_p^{\mathcal{M}}$$

of all zigzag paths in (LFH1). Theorem 3.3 has the following obvious corollary.

Corollary 4.4. *If W is a finite set and the Ufnarovskiĭ-graph Q_W has no oriented cycles, that is, Q_W is a finite acyclic quiver, then the algebra A has finite global dimension.*

Example 4.5. A concrete calculation shows that the rightmost part of the two-sided Anick resolution of the algebra A in Example 4.1 is the following:

$$\cdots \longrightarrow P_2 \xrightarrow{d_2^{\mathcal{M}}} P_1 \xrightarrow{d_1^{\mathcal{M}}} P_0 \xrightarrow{\epsilon} A \longrightarrow 0,$$

where

$$P_n := A \otimes_E kW^{(n-1)} \otimes_E A \cong \bigoplus_{(w_1, \cdots, w_n) \in W^{(n-1)}} A^e \cdot [w_1 | \cdots | w_n],$$

$$W^{(-1)} = \{e_1, e_2, e_3\}, W^{(0)} = \{(a), (b), (a'), (b')\}, W^{(1)} = \{(a, b), (b, b'), (b', a'), (a, a'a), (a', aa')\},$$

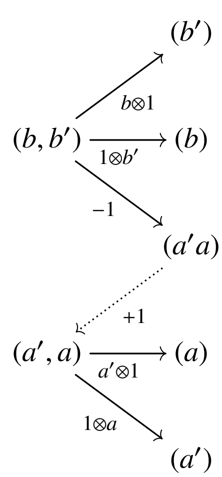
ϵ is the multiplication, $d_1^{\mathcal{M}}$ is given by the matrix (here the element $b \otimes 1$ means that the generator (b) in P_1 maps to $b \otimes 1$ times the generator e_3 in P_0, etc.)

$$\begin{pmatrix} -1 \otimes a & a \otimes 1 & 0 \\ 0 & -1 \otimes b & b \otimes 1 \\ a' \otimes 1 & -1 \otimes a' & 0 \\ 0 & b' \otimes 1 & -1 \otimes b' \end{pmatrix},$$

$d_2^{\mathcal{M}}$ is given by the matrix

$$\begin{pmatrix} 1 \otimes b & a \otimes 1 & 0 & 0 \\ -a' \otimes 1 & 1 \otimes b' & -1 \otimes a & b \otimes 1 \\ 0 & 0 & b' \otimes 1 & 1 \otimes a' \\ 1 \otimes a'a + aa' \otimes 1 & 0 & a \otimes a & 0 \\ a' \otimes a' & 0 & 1 \otimes aa' + a'a \otimes 1 & 0 \end{pmatrix}.$$

For example, all the zigzag paths from the vertex (b, b') can be calculated using the following diagram:



5. Minimal criterion for two-sided Anick resolutions

In this section we will consider in which case a two-sided Anick resolution is minimal.

Definition 5.1. *Let $A = kQ/I$ be an algebra as in Section 4, $E = \oplus_{e\in Q_0} ke$ be its semisimple subalgebra. Then A is an augmented E-algebra with augmented ideal A_+.*

A projective resolution (P_*, d_*) of a left A-module M is minimal if the induced map

$$1 \otimes d_* : E \otimes_A P_* \to E \otimes_A P_{*-1}$$

is zero. A projective resolution (P'_*, d'_*) of an A-bimodule M' is minimal if the induced map

$$1 \otimes d'_* \otimes 1 : E \otimes_A P'_* \otimes_A E \to E \otimes_A P'_{*-1} \otimes_A E$$

is zero.

Our definition of minimality is consistent with the usual one in literature, see, for example [2, Page 325].

We keep the notations in Section 4.

Definition 5.2. *Let $w, w' \in \mathcal{B}$. Define a reduction step from w to w' with coefficient $\lambda \in k^*$, denoted by $w \Longrightarrow_\lambda w'$, if there exist $u, v \in \mathcal{B}$ and $f \in I$ such that*

(a) $\mathrm{Tip}(f) \in W$;
(b) $w = \mathrm{Tip}(ufv) = u\mathrm{Tip}(f)v$;
(c) $-\lambda w' = upv$ *where* $p \neq \mathrm{Tip}(f)$ *is a monomial appearing in* f.

We say w converges to w' if there is a sequence of reduction steps $w \Longrightarrow_{\lambda_1} u_1 \Longrightarrow_{\lambda_2} \cdots \Longrightarrow_{\lambda_m} u_m \Longrightarrow_{\lambda_{m+1}} w'$ with $u_1, \cdots, u_m \in \mathcal{B}$.

Remark 5.3. For a thick arrow

$$(u_1, \cdots, u_n) \xrightarrow{d_n^i} (v_1, \cdots, v_{n-1})$$

with $1 \le i \le n-1$, we have either $u = u_1 \cdots u_n \Longrightarrow_\lambda v = v_1 \cdots v_{n-1}$ or $u = v$. Similarly, for a dotted arrow

$$(u_1, \cdots, u_{n-1}) \overset{d_n^{-i}}{\dashrightarrow} (u_1, \cdots, u_{i-1}, u'_i, u''_i, u_{i+1}, \cdots, u_{n-1})$$

with $1 \le i \le n-1$, we have $u_1 \cdots u_{n-1} = u_1 \cdots u_{i-1} u'_i u''_i u_{i+1} \cdots u_n$.

The following result says that if an arbitrary $(n-1)$-chain can not converge to a $(n-2)$-chain, then the two-sided Anick resolution is minimal, thus providing a handy minimal criterion.

Theorem 5.4 (Minimal criterion)**.** *With the notations in Section 4. For an arbitrary $(n-1)$-chain $(w_1, \cdots, w_n)$ with $n \geq 1$, if $w = w_1 \cdots w_n$ can not converge to $u = u_1 \cdots u_{n-1}$ for any $(u_1, \cdots, u_{n-1}) \in W^{(n-2)}$, then the two-sided Anick resolution $(B(A,A)^{\mathcal{M}}, d^{\mathcal{M}})$ is minimal.*

Proof It is clear that the two-sided Anick resolution $(B(A,A)^{\mathcal{M}}, d^{\mathcal{M}})$ is minimal if and only if

$$\mathrm{Im}(d_n^{\mathcal{M}}) \subseteq (A^e)_+ \otimes_{E^e} B(A,A)_n^{\mathcal{M}}$$

for $n \geq 0$. So it suffices to prove that each zigzag path from $(w_1, \cdots, w_n) \in W^{(n-1)}$ to $(w'_1, \cdots, w'_{n-1}) \in W^{(n-2)}$ in $\overline{Q_B}^{\mathcal{M}}$ has weight $\lambda \in (A^e)_+$ for $n \geq 1$. Let $p = \alpha_1\alpha_2 \cdots \alpha_m$ be such a zigzag path. By Remark 5.3, if the thick arrows appearing in p have the form d_n^i, $1 \le i \le n-1$, necessarily $w = w_1 \cdots w_n$ converges to $w' = w'_1 \cdots w'_{n-1}$. Thus, by our assumption, there exists a thick arrow $\alpha_j = d_n^0$ or $\alpha_j = d_n^n$ whose weight lies in $(A^e)_+$. As the weight of p is the product of the weight of all arrows appearing in p. Hence the weight of p lies in $(A^e)_+$.

□

In practise, when the Gröbner-Shirshov basis is homogeneous, it is sometimes helpful to use Theorem 5.4 to determine the two-sided Anick resolution is minimal by calculating the degrees of the Anick chains; for example, if each n-chain has different degree with any $(n-1)$-chain, it immediately follows that the two-sided Anick resolution is minimal.

Remark 5.5. Notice that the hypothesis of the above result holds unconditionally for $n = 0, 1, 2$, so the two-sided Anick resolution is minimal in degree $0, 1, 2$.

Remark 5.6. The original papers of Anick [1] and Anick-Green [2] only consider one-sided Anick resolutions. It is easy to see that the minimal criterion Theorem 5.4 still works for one-sided Anick resolutions.

Remark 5.7. In the article [24], the authors gave a sufficient condition for the minimality of one-sided Anick resolutions:

- There does not exist an arrow of type II in $\overline{Q_B}^{\mathcal{M}}$, that is, an arrow of the form
$$d_n^i : (w_1, \cdots, w_n) \to (w_1, \cdots, w_{i-1}, v, w_{i+2}, \cdots, w_n)$$
with $w_i w_{i+1} \Longrightarrow_\lambda v$ and $(w_1, \cdots, w_{i-1}, v, w_{i+2}, \cdots, w_n) \in W^{(n-2)}$ (Note that they do not ask $(w_1, \cdots, w_n)$ to be an $(n-1)$-chain).

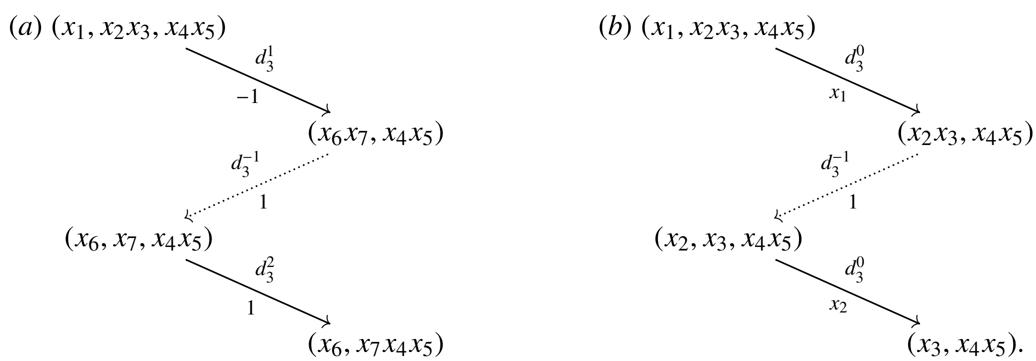
However, there is a counter-example.

Let $A = k\langle x_1, x_2, \cdots, x_7\rangle / I$ with $I = \langle x_1x_2x_3 - x_6x_7, x_3x_4x_5, x_6x_7x_4x_5\rangle$. Fix the order $x_1 > x_2 > \cdots > x_7$. The set of the reduced monomial generators of $\langle \mathrm{Tip}(I)\rangle$ with respect to the left length lexicographic order is $W = \{x_1x_2x_3, x_3x_4x_5, x_6x_7x_4x_5\}$. We list all the i-chains ($i \geq -1$) as follows:

(a) $W^{(-1)}$ is a single point set;
(b) $W^{(0)} = \{x_1, \cdots, x_7\}$;
(c) $W^{(1)} = \{(x_1, x_2x_3), (x_3, x_4x_5), (x_6, x_7x_4x_5)\}$;
(d) $W^{(2)} = \{(x_1, x_2x_3, x_4x_5)\}$;
(e) $W^{(i)} = \emptyset, i \geq 3$.

The (one-sided) Anick resolution of k has the form
$$0 \to A \otimes W^{(2)} \xrightarrow{d_3^{\mathcal{M}}} A \otimes W^{(1)} \xrightarrow{d_2^{\mathcal{M}}} A \otimes W^{(0)} \xrightarrow{d_1^{\mathcal{M}}} A \to k \to 0.$$

We now compute the differential $d_3^{\mathcal{M}}$. There are exactly two zigzag paths starting from a 2-chain to an 1-chain as follows:



We have $d_3^{\mathcal{M}}(1 \otimes (x_1, x_2x_3, x_4x_5)) = -1 \otimes (x_6, x_7x_4x_5) + x_1x_2 \otimes (x_3, x_4x_5)$. It follows that the induced map $1 \otimes d_3^{\mathcal{M}} \neq 0$. So the resolution is not minimal.

However, since all reduction steps are induced by the reduction $x_1x_2x_3 \Longrightarrow_1 x_6x_7$, no reduction of type *II* appears because there is no Anick chain of the form $(w_1, \cdots, ux_6x_7v, \cdots, w_n)$.

Remark 5.8. In general, the criterion in Theorem 5.4 is NOT a necessary condition.

Let $B = k\langle x_1, x_2, x_3, x_4, x_5\rangle/I$ with $I = \langle x_1x_2x_3 - x_1x_5, x_2x_3x_4 - x_5x_4, x_1x_5x_4\rangle$. Fix the order $x_1 > x_2 > x_3 > x_4 > x_5$. The set of the reduced monomial generators of $\langle \mathrm{Tip}(I)\rangle$ with respect to the left length lexicographic order is $W = \{x_1x_2x_3, x_2x_3x_4, x_1x_5x_4\}$. We list all the i-chains ($i \geq -1$) as follows:

(a) $W^{(-1)}$ is a single point set;
(b) $W^{(0)} = \{x_1, \cdots, x_5\}$;
(c) $W^{(1)} = \{(x_1, x_2x_3), (x_2, x_3x_4), (x_1, x_5x_4)\}$;
(d) $W^{(2)} = \{(x_1, x_2x_3, x_4)\}$;
(e) $W^{(i)} = \emptyset, i \geq 3$.

The two-sided Anick resolution of A has the form

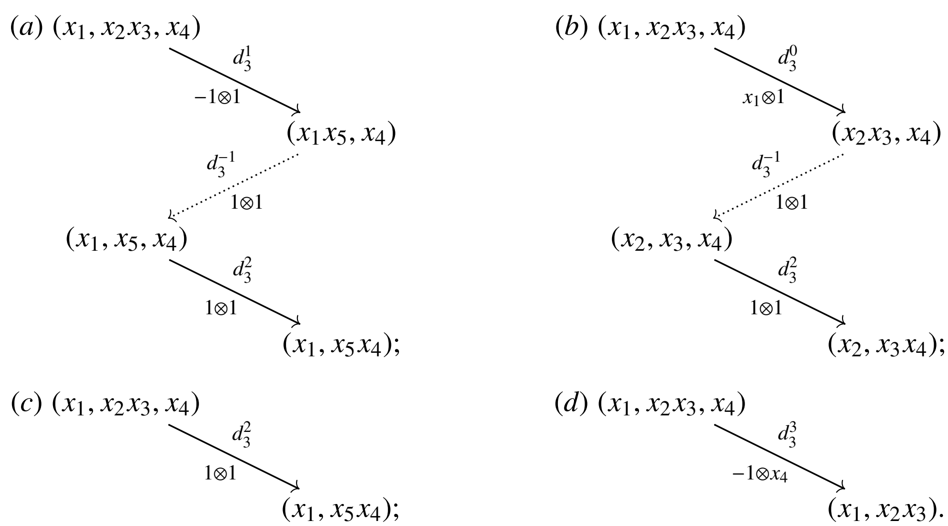
$$0 \to A \otimes W^{(2)} \otimes A \xrightarrow{d_3^{\mathcal{M}}} A \otimes W^{(1)} \otimes A \xrightarrow{d_2^{\mathcal{M}}} A \otimes W^{(0)} \otimes A \xrightarrow{d_1^{\mathcal{M}}} A \otimes A \to A \to 0.$$

For the 2-chain (x_1, x_2x_3, x_4), there are two reduction steps converging to the 1-chain (x_1, x_5x_4):

$$w = \underline{x_1x_2x_3}x_4 \Longrightarrow_1 x_1x_5x_4 \text{ and } w = x_1\underline{x_2x_3x_4} \Longrightarrow_1 x_1x_5x_4.$$

So the assumption of Theorem 5.4 is not fulfilled.

However, this two-sided Anick resolution of A is minimal. For the differential $d_3^{\mathcal{M}}$, there are four zigzag paths starting from a 2-chain to an 1-chain as follows:



We have thus

$$\begin{aligned} &d_3^{\mathcal{M}}(1 \otimes (x_1, x_2x_3, x_4) \otimes 1) \\ &= -1 \otimes (x_1, x_5x_4) \otimes 1 + 1 \otimes (x_1, x_5x_4) \otimes 1 + x_1 \otimes (x_2, x_3x_4) \otimes 1 - 1 \otimes (x_1, x_2x_3) \otimes x_4 \\ &= x_1 \otimes (x_2, x_3x_4) - 1 \otimes (x_1, x_2x_3) \otimes x_4. \end{aligned}$$

So the induced map $1_k \otimes d_3^{\mathcal{M}} \otimes 1_k = 0$. By Remark 5.5, $1_k \otimes d_2^{\mathcal{M}} \otimes 1_k$ and $1_k \otimes d_1^{\mathcal{M}} \otimes 1_k$ vanish as well. Hence the two-sided Anick resolution is minimal.

6. Homological properties of Chinese algebras

In this section, we will use the two-sided Anick resolution to study the homological properties of Chinese algebras of finite rank [7, 8].

Let $X = \{x_1, \cdots, x_n\}$ with $n \geq 1$, X^* be the free monoid generated by X. The Chinese congruence is the congruence on X^* generated by the following relations:

(a) $x_ix_jx_k = x_ix_kx_j = x_jx_ix_k, \forall i > j > k$,

(b) $x_ix_jx_j = x_jx_ix_j$, $x_ix_ix_j = x_ix_jx_i$, $\forall i > j$.

The Chinese monoid $\mathrm{CH}(X)$ (of rank n) is the quotient monoid of the free monoid X^* by the Chinese congruence.

Definition 6.1 (Chinese algebra)**.** *Let k be a field. The Chinese algebra A (of rank n) is the semigroup algebra of the Chinese monoid* $\mathrm{CH}(X)$.

Equivalently, A is the algebra with relation by $k\langle X|T\rangle = k\langle x_1, \cdots, x_n\rangle/I$ with

$$I = \langle x_ix_jx_k - x_jx_ix_k, x_ix_kx_j - x_jx_ix_k, x_ix_jx_j - x_jx_ix_j, x_ix_ix_j - x_ix_jx_i\rangle_{i>j>k}.$$

Y. Chen and J. Qiu obtained the Gröbner-Shirshov basis $\mathcal{G}$ with respect to the left length lexicographic order on X^* generated by

$$x_n > x_{n-1} > \cdots > x_1$$

as follows:

Theorem 6.2 ([8])**.** *The Gröbner-Shirshov basis $\mathcal{G}$ for the ideal I with respect to the left length lexicographic order on X^* consists of*

(a) $x_ix_jx_k - x_jx_ix_k, x_ix_kx_j - x_jx_ix_k$, $\forall\ i > j > k$;
(b) $x_ix_jx_j - x_jx_ix_j, x_ix_ix_j - x_ix_jx_i$, $\forall\ i > j$;
(c) $x_ix_jx_ix_k - x_ix_kx_ix_j$, $\forall\ i > j > k$.

Note that the above Gröbner-Shirshov basis of the Chinese algebra A is homogeneous and the reduced monomial generators of $\mathrm{Tip}(I)$ is

$$W = \mathrm{Tip}(\mathcal{G}) = \{x_ix_jx_k, x_ix_kx_j, x_ix_jx_ix_k\}_{i>j>k} \cup \{x_ix_jx_j, x_ix_ix_j\}_{i>j}.$$

By Theorem 4.3 and 3.3, we obtain a free bimodule resolution (P_*, d_*) of A (For simplicity here and in the following we often write $d_*^{\mathcal{M}}$ as d_*) with

$$P_i = A \otimes_k kW^{(i-1)} \otimes_k A, i \geq 0.$$

The following lemma implies that the length of the resolution (P_*, d_*) is equal to $\frac{n(n+1)}{2}$.

Lemma 6.3. *With the notation above, the sets of Anick chains $W^{(i)}$ have the following properties:*

(a) $|W^{(\frac{n(n+1)}{2}-1)}| = 1$ *and the unique element in $W^{(\frac{n(n+1)}{2}-1)}$ is of maximal degree among all the Anick chains;*
(b) $W^{(k)} = \emptyset$, $\forall k \geq \frac{n(n+1)}{2}$;
(c) the Ufnarovskiĭ graph Q_W (cf. Section 4) is a finite quiver and it has no oriented cycle;
(d) each $W^{(i)}$ is a finite set.

Proof We prove (a)(b) by induction on n, the rank of the Chinese algebra A.

The case $n = 1$ is trivial as $W = \emptyset$ and there are only two Anick chains: one (-1)-chain and one 0-chain x_1.

For the case $n = 2$, we have $W = \{x_2x_2x_1, x_2x_1x_1\}$. It follows that $W^{(2)} = \{(x_2, x_2x_1, x_1)\}$, $W^{(1)} = \{(x_2, x_2x_1), (x_2, x_1x_1)\}$, $W^{(0)} = \{x_1, x_2\}$, $W^{(-1)} = \{*\}$ and $W^{(k)} = \emptyset$ for $k \geq 3$. Thus (a)(b) hold for $n = 2$.

Inductively, let $w = (w_1, \cdots, w_m)$ be an Anick chain of the maximal weight. It is clear that $w_1 \in \{x_1, \cdots, x_n\}$. If $w_1 \neq x_n$, we can construct a new Anick chain $(x_n, x_nx_{n-1}, w_1, \cdots, w_m)$ whose weight and degree are greater than those of w. So we have $w_1 = x_n$.

It follows from the definition of Anick chains that $w_1w_2 = x_nw_2 \in W$, then we have

$$w_2 \in \{x_jx_k, x_kx_j, x_jx_nx_k\}_{k<j<n} \cup \{x_jx_j, x_nx_j\}_{j<n}.$$

Let's discuss these cases except for the case $w_2 = x_nx_{n-1}$.

(i) $w_2 = x_jx_k$ with $k < j < n$. One can see that the sequence $(x_n, x_nx_{n-1}, x_j, w_2, w_3, \cdots, w_m)$ is an Anick chain whose weight and degree are greater than those of w.

(ii) $w_2 = x_k x_j$ with $k < j < n$. As there is no monomial beginning with $x_k x_j$ in W, we have $x_j w_3 \in W$. Thus, the weight and degree of the Anick chain $(x_n, x_n x_{n-1}, x_j, w_3, \cdots, w_m)$ is greater than those of w.

(iii) $w_2 = x_j x_n x_k$ with $k < j < n$. Notice that $x_j x_n x_k w_3 \notin W$ for any $w_3 \in X^*$, so we have either $x_n x_k w_3 \in W$ or $x_k w_3 \in W$. If $x_n x_k w_3 \in W$, the weight and degree of the Anick chain $(x_n, x_n x_{n-1}, x_n x_{n-2}, w_3, \cdots, w_m)$ are greater than those of w. Now we assume that $x_k w_3 \in W$. It follows that the degree of w_3 is 2 or 3 and $w_3 = x_l w_3'$ with $l \leq k$. Thus we have $x_n x_k x_l \in W$ which contradicts to the fact that w is an Anick chain.

(iv) $w_2 = x_j x_j$ with $j < n$. We have either $x_j x_j w_3 \in W$ or $x_j w_3 \in W$. If $x_j x_j w_3 \in W$, then $w_3 = x_\ell$ with $\ell < j$. Thus the weight and degree of the Anick chain $(x_n, x_n x_{n-1}, x_n x_{n-2}, w_3, \cdots, w_m)$ are greater than those of w. If $x_j w_3 \in W$, it is easy to see that the Anick chain $(x_n, x_n x_{n-1}, x_j, w_3, \cdots, w_m)$ has the greater weight and degree.

(v) $w_2 = x_n x_j$ with $j < n-1$. We can construct a new Anick chain $(x_n, x_n x_{n-1}, w_2, \cdots, w_m)$ whose weight and degree are greater than those of w.

Above discussion tells us that $w_2 = x_n x_{n-1}$.

Now lets consider w_3. If $x_{n-1} w_3 \in W$, the Anick chain $(x_n, x_n x_{n-1}, x_{n-1}, w_3, \cdots, w_m)$ whose weight and degree are greater than those of w. So it must be $x_n x_{n-1} w_3 \in W$. It follows that either $w_3 = x_j$, $j < n$ or $w_3 = x_n x_k$, $k < n-1$. If $w_3 = x_j$ with $j < n$, we can construct a new Anick chain $(x_n, x_n x_{n-1}, x_n x_{n-2}, w_3, \cdots, w_m)$ with the greater weight and degree. If $w_3 = x_n x_k$ with $k < n-2$, the Anick chain $(x_n, x_n x_{n-1}, x_n x_{n-2}, w_3, \cdots, w_m)$ whose weight and degree are greater than those of w. Hence we have $w_3 = x_n x_{n-2}$.

Similarly, one can show that

$$w_4 = x_n x_{n-3}, \cdots, w_n = x_n x_1.$$

Notice that $x_1 w_{n+1} \notin W$ for any $w_{n+1} \in X^*$, so it must be $x_n x_1 w_{n+1} \in W$ which shows that $w_{n+1} \in \{x_1, \cdots, x_{n-1}\}$. If $w_{n+1} \neq x_{n-1}$, the Anick chain

$$(x_n, x_n x_{n-1}, \cdots, x_n x_1, x_{n-1}, x_{n-1} x_{n-2}, w_{n+1}, w_{n+2}, \cdots, w_m)$$

has the greater weight and degree. Hence $w_{n+1} = x_{n-1}$.

As the first letter is maximal among all letters in any monomial of W, each w_i does not contain x_n for $i \geq n+1$. By the induction hypothesis, the segment $(x_{n-1}, w_{n+2}, \cdots, w_m)$ of w is of weight $\frac{n(n-1)}{2}$ and its degree is greater than that of any other Anick chain of the form $(x_{n-1}, w_2', \cdots, w_p')$. It follows that w is of weight $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ and has the maximal degree. Hence (a)(b) holds.

Since W is finite, the Ufnarovskiĭ graph Q_W is a finite quiver and the nonexistence of oriented cycles can be deduced from (b). Thus (c) holds.

The statement (d) follows from (c).

□

Remark 6.4. It should be noted that in the proof of Lemma 6.3, we constructed the unique Anick chain of maximal weight and degree

$$w_{\max} = (x_n, x_n x_{n-1}, \cdots, x_n x_1, x_{n-1}, x_{n-1} x_{n-2}, \cdots, x_2, x_2 x_1, x_1)$$

whose degree is equal to n^2.

Recall that an algebra is homologically smooth, if it admits a finite length resolution by finitely generated projective bimodules [18, Definition 3.1.3].

Theorem 6.5. *The Chinese algebra A of rank $n \geq 1$ is homologically smooth and of global dimension $\frac{n(n+1)}{2}$.*

Proof By Lemma 6.3, each $P_i = A \otimes_k kW^{(i-1)} \otimes_k A$ is a finitely generated free bimodule and the two-sided Anick resolution (P_*, d_*) is of length $\frac{n(n+1)}{2}$. This implies that A is homologically smooth.

For any left A module M, applying the functor $-\otimes_A M$ to the two-sided Anick resolution yields a free resolution $(P_*\otimes_A M, d_*\otimes 1_M)$ of $A\otimes_A M\cong M$ with length $\frac{n(n+1)}{2}$. So we have $\mathrm{gldim}(A)\le \frac{n(n+1)}{2}$.

On the other hand, in Lemma 6.3, we have shown that the degree of generator w_{max} of $P_{\frac{n(n+1)}{2}}$ is greater than any other generator of the resolution (P_*, d_*). The homogeneity for the relation of A implies that w_{max} cannot be reduced to any other Anick chain. In terms of the proof of Theorem 5.4, the map $1\otimes_A d_{\frac{n(n+1)}{2}-1}\otimes_A 1$ induced by applying functor $k\otimes_A - \otimes_A k$ to the resolution (P_*, d_*) is a zero map. As $\mathrm{Tor}^A_m(k,k)\cong \mathrm{H}_m(k\otimes_A P'_*\otimes_A k, 1\otimes_A d'_*\otimes_A 1)$ for any two-sided projective resolution (P'_*, d'_*) of A. So, by the fact that $d_{\frac{n(n+1)}{2}}=0$ and $1\otimes_A d_{\frac{n(n+1)}{2}-1}\otimes_A 1=0$, we have the isomorphisms

$$\mathrm{Tor}^A_{\frac{n(n+1)}{2}}(k,k)\cong k\otimes_A P_{\frac{n(n+1)}{2}}\otimes_A k\cong k\otimes_A A\otimes kW^{(\frac{n(n+1)}{2}-1)}\otimes A\otimes_A k\cong kW^{(\frac{n(n+1)}{2}-1)}\neq 0.$$

It implies that the projective dimension of k is as least $\frac{n(n+1)}{2}$. Hence we have $\mathrm{gldim}(A)\ge \frac{n(n+1)}{2}$. This finishes the proof.

□

Remark 6.6. The above proof indicates that the two-sided Anick resolution of the Chinese algebra of rank $n\ge 3$ is of minimal length, however, it is NOT minimal. In fact, $1_k\otimes_A d_3\otimes_A 1_k\neq 0$.

Remark 6.7. Although the Chinese algebra A of rank $n\ge 3$ is cubic, it is NOT a 3-Koszul algebra [3, 42, 20]. In fact, assume that A is a 3-Koszul algebra generated in degree 1, then, by definition, $\mathrm{Tor}^A_i(k,k)$ has a basis in degree $\delta(i)$, where

$$\delta(i)=\begin{cases}\frac{3i}{2} & \text{if } i \text{ is even}\\ \frac{3(i-1)}{2}+1 & \text{if } i \text{ is odd}\end{cases}.$$

We have proved that $\mathrm{Tor}^A_{\frac{n(n+1)}{2}}(k,k)=k\{(x_n, x_nx_{n-1},\cdots,x_nx_1,x_{n-1},\cdots,x_1)\}$ in Theorem 6.5 and the degree of the unique generator of $\mathrm{Tor}^A_{\frac{n(n+1)}{2}}(k,k)$ is n^2 by Remark 6.4 which is not equal to $\delta(\frac{n(n+1)}{2})$, thus giving a contradiction.

In a forthcoming paper, we will try to construct a minimal two-sided projective resolution of the Chinese algebra and compute its Hochschild cohomology.

7. A Koszul algebra whose Anick resolution is not minimal

In order to answer two question from the book [35], N. Iyudu and S. Shkarin [23] classied Hilbert series of Koszul algebras with three generators and three relations. They introduces a new Koszul algebra

$$A=k\langle x,y,z\mid x^2+yx, xz, zy\rangle.$$

V. Dotsenko and S. R. Chowdhury [12] calculated the bar homology $\mathrm{Tor}^A_*(k,k)$ of A through the (one-sided) Anick resolution. In this section, we will use the algebraic Morse theory to construct the two-sided minimal resolution of A from the two-sided Anick resolution which itself is not minimal.

Lemma 7.1. [12] *Let A as above. The Gröbner-Shirshov basis for the ideal I with respect to the left length lexicographic order generated by $x>y>z$ is*

$$\mathcal{G}=\{xy^kx+y^{k+1}x, xz, zy\}_{k\ge 0}.$$

It follows that the reduced monomial generators of $\mathrm{Tip}(I)$ is

$$W=\mathrm{Tip}(\mathcal{G})=\{xy^kx, xz, zy\}_{k\ge 0}.$$

It produces the list of all Anick chains as follows.

(a) the set of (-1)-chain $W^{(-1)}$ is a singleton;
(b) the set of 0-chains $W^{(0)}$ consists of x, y, z;

(c) the set of 1-chains $W^{(1)}$ consists of $(x, y^k x), (x, z), (z, y)$ with $k \geq 0$;
(d) for $n \geq 2$, the set of n-chains $W^{(n)}$ consists of

$$(x, y^{k_1}x, \cdots, y^{k_n}x), (x, y^{k_1}x, \cdots, y^{k_{n-1}}x, z), (x, y^{k_1}x, \cdots, y^{k_{n-2}}x, z, y)$$

with $k_1, \cdots, k_n \geq 0$.

By Theorem 3.3 and 4.3, we obtain a free bimodule resolution (P_*, d_*) of A with

$$P_i = A \otimes kW^{(i-1)} \otimes A, i \geq 0.$$

Proposition 7.2. *The differential of the two-sided resolution is given by the following.*

(a) $d_0(1 \otimes 1) = 1$;
(b) $d_1(1 \otimes a \otimes 1) = a \otimes 1 - 1 \otimes a$, $a = x, y, z$;
(c) $d_2(1 \otimes (x, y^k x) \otimes 1) = (xy^k + y^{k+1}) \otimes x \otimes 1 + 1 \otimes x \otimes y^k x + 1 \otimes y \otimes y^k x + \sum_{i=1}^{k}(xy^{i-1} + y^i) \otimes y \otimes y^{k-i}x$,
$d_2(1 \otimes (x, z) \otimes 1) = x \otimes z \otimes 1 + 1 \otimes x \otimes z$,
$d_2(1 \otimes (z, y) \otimes 1) = z \otimes y \otimes 1 + 1 \otimes z \otimes y$;
(d) *for* $n \geq 3$,

$$\begin{aligned} & d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_n}x) \otimes 1) \\ =& (xy^{k_1} + y^{k_1+1}) \otimes (x, y^{k_2}x, \cdots, y^{k_n}x) \otimes 1 + (-1)^{n+1} \otimes (x, y^{k_1}x, \cdots, y^{k_{n-1}}x) \otimes y^{k_n}x \\ & + \sum_{i=1}^{n-1}(-1)^i \otimes (x, y^{k_1}x, \cdots, y^{k_{i-1}}x, y^{k_i+k_{i+1}+1}x, y^{k_{i+2}}x, \cdots, y^{k_n}x) \otimes 1, \end{aligned} \tag{16}$$

$$\begin{aligned} & d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_{n-1}}x, z) \otimes 1) \\ =& (xy^{k_1} + y^{k_1+1}) \otimes (x, y^{k_2}x, \cdots, y^{k_{n-1}}x, z) \otimes 1 + (-1)^{n+1} \otimes (x, y^{k_1}x, \cdots, y^{k_{n-1}}x) \otimes z \\ & + \sum_{i=1}^{n-2}(-1)^i \otimes (x, y^{k_1}x, \cdots, y^{k_{i-1}}x, y^{k_i+k_{i+1}+1}x, y^{k_{i+2}}x, \cdots, y^{k_{n-1}}x, z) \otimes 1, \end{aligned} \tag{17}$$

$$\begin{aligned} & d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_{n-2}}x, z, y) \otimes 1) \\ =& (xy^{k_1} + y^{k_1+1}) \otimes (x, y^{k_2}x, \cdots, y^{k_{n-2}}x, z, y) \otimes 1 + (-1)^{n+1} \otimes (x, y^{k_1}x, \cdots, y^{k_{n-2}}x, z) \otimes y \\ & + \sum_{i=1}^{n-3}(-1)^i \otimes (x, y^{k_1}x, \cdots, y^{k_{i-1}}x, y^{k_i+k_{i+1}+1}x, y^{k_{i+2}}x, \cdots, y^{k_{n-2}}x, z, y) \otimes 1. \end{aligned} \tag{18}$$

Proof At the risk of being repetitive, again, as we are working with Theorem 3.3, the differential of (P_*, d_*) is determined by all the zigzag paths between two critical vertices in the quiver $\overline{Q_B}^{\mathcal{M}}$ (cf. Section 3 and Remark 4.2).

The maps d_0 and d_1 are easy. Now let us consider $d_2(1 \otimes (x, y^k x) \otimes 1)$, or equivalently, all the zigzag paths in $\overline{Q_B}^{\mathcal{M}}$ with starting vertex being $(x, y^k x)$ and target lying in $W^{(0)}$. Let $p = \alpha_1 \cdots \alpha_m$ be such a zigzag path. Notice that α_1 and α_m must be the thick arrow. We discuss α_1 in three cases (the notations d_n^j below are given in Section 4).

(i) $\alpha_1 = d_2^2 : (x, y^k x) \overset{1 \otimes y^k x}{\rightarrow} x$. As $x \in W^{(0)}$ is a critical vertex, there is no dotted arrow with x as its starting vertex. Thus the zigzag path $p = \alpha_1$ and it gives the term $1 \otimes x \otimes y^k x$ of $d_2(1 \otimes (x, y^k x) \otimes 1)$;

(ii) $\alpha_1 = d_2^0 : (x, y^k x) \overset{x \otimes 1}{\rightarrow} y^k x$. Then $\alpha_2 = d_2^{-1} : y^k x \overset{1 \otimes 1}{\dashrightarrow} (y, y^{k-1}x)$ is the unique dotted arrow with $y^k x$ as its starting vertex. It follows that α_3 is either d_2^0 or d_2^2 as the arrow $d_2^1 : (y, y^{k-1}x) \overset{-1 \otimes 1}{\rightarrow} y^k x$ does not exist in $\overline{Q_B}^{\mathcal{M}}$.

(ii-1) if $\alpha_3 = d_2^2 : (y, y^{k-1}x) \overset{1 \otimes y^{k-1}x}{\rightarrow} y$, then the zigzag path is $p = \alpha_1\alpha_2\alpha_3$ and it gives the term $x \otimes y \otimes y^{k-1}x$ of $d_2(1 \otimes (x, y^k x) \otimes 1)$;

(ii-2) if $\alpha_3 = d_2^0 : (y, y^{k-1}x) \overset{y\otimes 1}{\to} y^{k-1}x$, then the discussion of the zigzag path $p' = \alpha_3 \cdots \alpha_m$ is similar as that in the case (ii).

It follows by induction that all the zigzag paths with $\alpha_1 = d_2^0$ as their first thick arrows give the terms

$$\sum_{i=1}^{k} xy^{i-1} \otimes y \otimes y^{k-i}x + xy^k \otimes x \otimes 1$$

of $d_2(1 \otimes (x, y^k x) \otimes 1)$;

(iii) $\alpha_1 = d_2^1 : (x, y^k x) \overset{1\otimes 1}{\to} y^{k+1}x$. This case is also similar to the case (ii) and one can show that the zigzag paths with first thick arrows being $\alpha_1 = d_2^1$ give the terms

$$\sum_{i=1}^{k+1} y^{i-1} \otimes y \otimes y^{k-i+1}x + y^{k+1} \otimes x \otimes 1$$

of $d_2(1 \otimes (x, y^k x) \otimes 1)$.

Combining the results above, we obtain the expression of d_2 on $1 \otimes (x, y^k x) \otimes 1$. The formulas of $d_2(1 \otimes (x, z) \otimes 1)$ and $d_2(1 \otimes (z, y) \otimes 1)$ can be proved similarly.

For $n \geq 2$, let $p = \alpha_1 \cdots \alpha_m$ be a zigzag path with starting vertex being $(x, y^{k_1}x, \cdots, y^{k_n}x)$ and target lying in $W^{(n-1)}$. We discuss α_1 in four cases.

(i) $\alpha_1 = d_{n+1}^{n+1} : (x, y^{k_1}x, \cdots, y^{k_n}x) \overset{(-1)^{n+1}\otimes y^{k_n}x}{\to} (x, y^{k_1}x, \cdots, y^{k_{n-1}}x) \in W^{(n-1)}$. We have $p = \alpha_1$ and this zigzag path gives the term $(-1)^{n+1} \otimes (x, y^{k_1}x, \cdots, y^{k_{n-1}}x) \otimes y^{k_n}x$ of $d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_n}x) \otimes 1)$;

(ii) $\alpha_1 = d_{n+1}^{i+1} : (x, y^{k_1}x, \cdots, y^{k_n}x) \overset{(-1)^i\otimes 1}{\to} (x, y^{k_1}x, \cdots, y^{k_{i-1}}x, y^{k_i+k_{i+1}+1}x, y^{k_{i+2}}x, \cdots, y^{k_n}x) \in W^{(n-1)}$ with $1 \leq i \leq n-1$. It follows that $p = \alpha_1$ and these zigzag paths give the terms $(-1)^i \otimes (x, y^{k_1}x, \cdots, y^{k_{i-1}}x, y^{k_i+k_{i+1}+1}x, y^{k_{i+2}}x, \cdots, y^{k_n}x) \otimes 1$ of $d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_n}x) \otimes 1)$;

(iii) $\alpha_1 = d_{n+1}^0 : (x, y^{k_1}x, \cdots, y^{k_n}x) \overset{x\otimes 1}{\to} (y^{k_1}x, \cdots, y^{k_n}x)$. Then $\alpha_2 = d_{n+1}^{-1} : (y^{k_1}x, \cdots, y^{k_n}x) \overset{1\otimes 1}{\dashrightarrow} (y, y^{k_1-1}x, \cdots, y^{k_n}x)$. If α_3 is of the form $d_{n+1}^i : (y, y^{k_1-1}x, \cdots, y^{k_n}x) \to (y, y^{k_1-1}x, \cdots)$ with $2 \leq i \leq n+1$, there is no dotted arrow with starting vertex being $(y, y^{k_1-1}x, \cdots)$. Meanwhile, α_3 could not be d_{n+1}^1. Thus α_3 can only be $d_{n+1}^0 : (y, y^{k_1-1}x, \cdots, y^{k_n}x) \overset{y\otimes 1}{\to} (y^{k_1-1}x, \cdots, y^{k_n}x)$. Repeating the discussion in this case for $p = \alpha_3 \cdots \alpha_m$ and using the induction on it, we can see that the zigzag path in this case is $p = d_{n+1}^0 d_{n+1}^{-1} d_{n+1}^0 \cdots d_{n+1}^{-1} d_{n+1}^0$ and it gives the term $xy^{k_1} \otimes (x, y^{k_2}x, \cdots, y^{k_n}x) \otimes 1$ of $d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_n}x) \otimes 1)$;

(iv) $\alpha_1 = d_{n+1}^1 : (x, y^{k_1}x, \cdots, y^{k_n}x) \overset{1\otimes 1}{\to} (y^{k_1+1}x, \cdots, y^{k_n}x)$. Similar to the discussion in case (iii), one can show that $p = d_{n+1}^1 d_{n+1}^{-1} d_{n+1}^0 d_{n+1}^{-1} \cdots d_{n+1}^{-1} d_{n+1}^0$ and it gives the term $y^{k_1+1} \otimes (x, y^{k_2}x, \cdots, y^{k_n}x) \otimes 1$ of $d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_n}x) \otimes 1)$.

Above discussions prove the formula of $d_{n+1}(1 \otimes (x, y^{k_1}x, \cdots, y^{k_n}x) \otimes 1)$. The rest formulas can be proved similarly.

□

Denote the new weighted quiver with respect to the Anick resolution (P_*, d_*) by Q_P (cf. Section 3). Its vertices set is the set of all the Anick chains and arrows set is determined by the differential in Proposition 7.2. We define a full subquiver $\mathcal{M}$ of Q_P as the union of the following sets of arrows.

$$\{(x, x, y^{\ell_1}x, \cdots, y^{\ell_n}x) \overset{-1\otimes 1}{\to} (x, y^{\ell_1+1}x, \cdots, y^{\ell_n}x); n \geq 1, \ell_1, \cdots, \ell_n \geq 0\}$$
$$\{(x, x, y^{\ell_1}x, \cdots, y^{\ell_n}x, z) \overset{-1\otimes 1}{\to} (x, y^{\ell_1+1}x, \cdots, y^{\ell_n}x, z); n \geq 1, \ell_1, \cdots, \ell_n \geq 0\}$$
$$\{(x, x, y^{\ell_1}x, \cdots, y^{\ell_n}x, z, y) \overset{-1\otimes 1}{\to} (x, y^{\ell_1+1}x, \cdots, y^{\ell_n}x, z, y); n \geq 1, \ell_1, \cdots, \ell_n \geq 0\}$$

It is easy to see that each arrow in $\mathcal{M}$ has weight $-1 \otimes 1$ and two different arrows in $\mathcal{M}$ do not share the endpoints. Thus we have the following.

Lemma 7.3. *$\mathcal{M}$ is a partial matching.*

In the quiver $Q_P^{\mathcal{M}}$, two endpoints of a dotted arrow share the degree (number of variables) and for a thick arrow, the degree of its starting vertex is greater or equal to that of its target. Let $p = \alpha_1 \cdots \alpha_m$ be a zigzag path in $Q_P^{\mathcal{M}}$. Without loss of generality, we may assume that all arrows in p keep the degree of vertices. Consider the segment $v_i \overset{\alpha_i}{\dashrightarrow} v_{i+1} \overset{\alpha_{i+1}}{\to} v_{i+2}$ of p. By the construction of $\mathcal{M}$, α_i has the form $\alpha_i : (x, y^{\ell_1+1}x, \cdots) \dashrightarrow (x, x, y^{\ell_1}x, \cdots)$. Note that we are concerned with the quiver $Q_P^{\mathcal{M}}$ rather than $Q_B^{\mathcal{M}}$, the thick arrow α_{i+1} is given by the differential of the two-sided Anick resolution (16) (17) (18). Assume that α_{i+1} corresponds to the term $(x + y) \otimes (x, y^{\ell_1}x, \cdots) \otimes 1$ of $d(1 \otimes (x, x, y^{\ell_1}x, \cdots) \otimes 1)$, that is, the terminal vertex v_{i+2} of α_{i+1} is $(x, y^{\ell_1}x, \cdots)$, it follows that two endpoints of α_{i+1} do not share the degree. According to the construction of $Q_P^{\mathcal{M}}$, we have $v_{i+2} \neq v_i$. Therefore, under our assumption that α_{i+1} keeps the degree of vertices, v_{i+2} has the form $(x, x, y^{\ell_1}x, \cdots)$. By the chosen of the arrows of $\mathcal{M}$ there is no dotted arrow with v_{i+2} as its starting vertex. Thus the zigzag path p is of finite length. According to Proposition 3.2, we have

Proposition 7.4. *$\mathcal{M}$ is a Morse matching.*

Obviously, the set of critical vertices $\mathcal{V}^{\mathcal{M}}$ consists of the Anick chains without being $(x, x, y^{\ell_1}x, \cdots)$ and $(x, y^{\ell_1+1}x, \cdots)$ with $\ell_1 \geq 0$. This observation filters out most of vertices and we list the vertices left as follows.

$$\mathcal{V}_4^{\mathcal{M}} = \{(x, x, z, y)\}, \mathcal{V}_3^{\mathcal{M}} = \{(x, x, z), (x, z, y)\}, \mathcal{V}_2^{\mathcal{M}} = \{(x, x), (x, z), (z, y)\}, \mathcal{V}_1^{\mathcal{M}} = \{x, y, z\}, \mathcal{V}_0^{\mathcal{M}} = \{*\}.$$

By Theorem 3.3, we obtain the two-sided resolution of A

$$0 \to A \otimes k\mathcal{V}_4^{\mathcal{M}} \otimes A \overset{d_4'}{\to} A \otimes k\mathcal{V}_3^{\mathcal{M}} \otimes A \overset{d_3'}{\to} A \otimes k\mathcal{V}_2^{\mathcal{M}} \otimes A \overset{d_2'}{\to} A \otimes k\mathcal{V}_1^{\mathcal{M}} \otimes A \overset{d_1'}{\to} A \otimes A \overset{\epsilon}{\to} A,$$

The differential listed below can be easily calculated by the enumeration of zigzag paths of $Q_P^{\mathcal{M}}$.

$$\begin{aligned}
&d_4'(1 \otimes (x, x, z, y) \otimes 1) = (x + y) \otimes (x, z, y) \otimes 1 + 1 \otimes (x, x, z) \otimes y,\\
&d_3'(1 \otimes (x, x, z) \otimes 1) = (x + y) \otimes (x, z) \otimes 1 - 1 \otimes (x, x) \otimes z,\\
&d_3'(1 \otimes (x, z, y) \otimes 1) = x \otimes (z, y) \otimes 1 - 1 \otimes (x, z) \otimes y,\\
&d_2'(1 \otimes (x, x) \otimes 1) = (x + y) \otimes x \otimes 1 + 1 \otimes x \otimes x + 1 \otimes y \otimes x,\\
&d_2'(1 \otimes (x, z) \otimes 1) = x \otimes z \otimes 1 + 1 \otimes x \otimes z,\\
&d_2'(1 \otimes (z, y) \otimes 1) = z \otimes y \otimes 1 + 1 \otimes z \otimes y,\\
&d_1'(1 \otimes a \otimes 1) = a \otimes 1 - 1 \otimes a, a = x, y, z,
\end{aligned}$$

ϵ is the multiplication. Hence this resolution is minimal. Note that by tensoring k over A from right we get a minimal one-sided resolution which is smaller than the one-sided Anick resolution obtained in [12].

In a forthcoming paper, the Hochschild cohomology groups of A as well as its Gerstenhaber algebra structure will be determined explicitly.

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