ON SIMPLE-MINDED SYSTEM AND τ-INVARIANT MODULES

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ABSTRACT. For a finite-dimensional self-injective algebra A over an algebraically closed field, we show that modules in the homogeneous tubes of its Auslander-Reiten quiver do not belong to any simple-minded system - a notion of homological generator - of its stable module category. In particular, for most self-injective algebras, their simple-minded systems are also maximal systems of orthogonal stable bricks.

1. RESULTS AND CONSEQUENCES

Let k be a commutative artin ring, and A an artin k-algebra. We denote by mod$A$ the category of all finitely generated left A-modules, and by $\underline{\text{mod}}A$ the stable category of mod$A$, that is, the category with the same class of objects but with morphism spaces being quotiented out by maps factoring through projective modules. *Although most definitions and the problem we consider in this paper can be discussed in this more general setting, we only concentrate, for technical reasons, on the case when k is an algebraically closed field and A is a finite-dimensional self-injective k-algebra.*

Let $\mathcal{S}$ be a class of A-modules. The full subcategory $\langle\mathcal{S}\rangle$ of mod$A$ is the additive closure of $\mathcal{S}$. Denote by $\langle\mathcal{S}\rangle * \langle\mathcal{S}'\rangle$ the class of indecomposable A-modules Y such that there is a short exact sequence $0 \to X \to Y \oplus P \to Z \to 0$ with $X \in \langle\mathcal{S}\rangle, Z \in \langle\mathcal{S}'\rangle$, and P projective. Define $\langle\mathcal{S}\rangle_1 := \langle\mathcal{S}\rangle$ and $\langle\mathcal{S}\rangle_n := \langle\langle\mathcal{S}\rangle_{n-1} * \langle\mathcal{S}\rangle\rangle$ for $n > 1$.

Without loss of generality, we further assume the following throughout the article: A is indecomposable non-simple and contains no nodes (see [10]). Under such setting, the definition of simple-minded systems introduced in [10] can be simplified as follows.

Definition 1.1. ([10]) *Let A be a self-injective algebra over an algebraically closed field. A class of objects $\mathcal{S}$ in $\underline{\text{mod}}A$ is called a simple-minded system (sms) over A if the following conditions are satisfied:*

(1) (orthogonality condition) For any $S, T \in \mathcal{S}$, $\underline{\text{Hom}}_A(S,T) = \begin{cases} 0 & (S \neq T), \\ k & (S = T). \end{cases}$

(2) (generating condition) For each indecomposable non-projective A-module X, there exists some natural number n (depending on X) such that $X \in \langle\mathcal{S}\rangle_n$.

It has been shown in [10] that each sms has finite cardinality and the sms's are invariant under stable equivalences. One of the basic problems about sms is the simple-image problem: Given an sms $\mathcal{S}$ of A, is this the image of the simple modules under some stable

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equivalence $\underline{\mathrm{mod}}B \to \underline{\mathrm{mod}}A$? This has been positively answered in [3] for representation-finite self-injective algebras.

Since simple-minded systems are modelled after the homological behaviour of the set of isomorphism classes of simple modules, it is desirable to show that the special properties of such a set also hold for simple-minded systems as well. In this note, we prove that this is true for one of such properties. Let us be more precise here.

Recall that the *Auslander-Reiten quiver* Γ_A of A is a valued translation quiver where vertices are the isomorphism classes of indecomposable (finitely generated) A-modules, arrows are the irreducible maps valued by their multiplicities, and translation is given by the Auslander-Reiten translate $\tau = DTr$ (see [1]). A (connected) component $\mathcal{C}$ of Γ_A is a *homogeneous tube* if it is of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$ (see [12]). In particular, all modules in a homogeneous tube of ${}_s\Gamma_A$ are of τ-period 1 (or *τ-invariant* for short). Note that none of the simple modules of a self-injective algebra lie in a homogeneous tube.

More generally, recall from Erdmann and Kerner [7] that a component $\mathcal{C}$ of Γ_A *stably quasi-serial of rank n* if its stable part (i.e. the full subquiver obtained by removing all the projective-injective modules) is of the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. In particular, using a result of Hoshino (see the Proof of Corollary 1.5), a stably quasi-serial component of rank 1 is the same as a homogeneous tube of Γ_A for a self-injective algebra A. The following is our main result.

Theorem 1.2. *Let A be a self-injective algebra over an algebraically closed field. Then none of the objects in an sms of A lie in the homogeneous tubes of the Auslander-Reiten quiver of A.*

According to a result of Crawley-Boevey [4], if A is tame, then almost all modules (that is, for each $d > 0$, all but a finite number of isomorphism classes of indecomposable A-modules of dimension d) lie in homogeneous tubes. Therefore, our result excludes most of the modules of a tame self-injective algebra from forming an sms.

We remark that it is possible for modules lying in a stably quasi-serial component of higher rank to form an sms in general. For example, let A be the group algebra of the alternating group A_4 over a field of characteristic 2. It is known that A is a symmetric special biserial algebra, and its Auslander-Reiten quiver Γ_A of A is fully described in [6, p.62-63]. There are three simple A-modules (up to isomorphism) which we denote by $k, 1, 2$ respectively. It is well-known that there is a stable equivalence between A and the principal block B of the group algebra of A_5, given by the induction and restriction functor. The set of isomorphism-class-representatives of simple B-modules is sent to $\mathcal{S} = \{k, \frac{1}{2}, \frac{2}{1}\}$ under this stable equivalence. So, $\mathcal{S}$ is an sms of A with the modules $\frac{1}{2}$ and $\frac{2}{1}$ lying in the stably quasi-serial components of rank 3.

In fact, there are many self-injective algebras whose Auslander-Reiten quiver consists only of stably quasi-serial components (see, for example, [2]). In such situations, it follows from our result that any member of an sms lies in some stably quasi-serial component of rank bigger than 1.

While it is possible to have indecomposable modules in a stably quasi-serial component of higher rank forming an sms, we know from [7] that the quasi-lengths of such modules are not more than the rank (see Lemma 2.1). In fact, this is the first step of our proof of the main theorem.

1.1. **Relations between generators of stable module categories.** In [5], Dugas defines simple-minded systems in a more general setting: Hom-finite Krull-Schmidt triangulated k-category $\mathcal{T}$. We now recall his definition in the case when $\mathcal{T}$ is the stable category $\underline{\mathrm{mod}}A$ of a self-injective algebra A. For two classes of objects $\mathcal{S}, \mathcal{S}'$ in $\underline{\mathrm{mod}}A$, we set

$$\mathcal{S} *_{\triangle} \mathcal{S}' = \{Y \in \underline{\mathrm{mod}}A | \text{ there is a triangle } X \to Y \to Z \text{ with } X \in \mathcal{S}, Z \in \mathcal{S}'\}.$$

It is shown in [5, Lemma 2.1] that $*_{\triangle}$ satisfies the associative law. Define $(\mathcal{S})_0 := \{0\}$ and $(\mathcal{S})_n := (\mathcal{S})_{n-1} *_{\triangle} (\mathcal{S} \cup \{0\})$ for $n \geq 1$. It can be shown that $(\mathcal{S})_n *_{\triangle} (\mathcal{S})_m = (\mathcal{S})_{n+m}$ for all $n, m \geq 0$ (see [5, Lemma 2.3]).

Definition 1.3. ([5]) *Let A be a self-injective algebra over an algebraically closed field. A class of objects $\mathcal{S}$ in $\underline{\mathrm{mod}}A$ is called a simple-minded system (sms) over A if the following conditions are satisfied:*

(1) (orthogonality condition) For any $S, T \in \mathcal{S}$, $\underline{\mathrm{Hom}}_A(S,T) = \begin{cases} 0 & (S \neq T), \\ k & (S = T). \end{cases}$

(2) (generating condition) For each object X in $\underline{\mathrm{mod}}A$, there exists some natural number n (depending on X) such that $X \in (\mathcal{S})_n$.

We first remark on the relation between the two definitions of sms's. The main difference is the "speed" they generate $\underline{\mathrm{mod}}A$: If $X \in \mathcal{S}$ and $n \geq 1$, then $X^n := X^{\oplus n} \in (\mathcal{S})_n \setminus (\mathcal{S})_{n-1}$, while $X^n \in \langle \mathcal{S} \rangle_m$ for any $m \geq 1$. In spite of this, Definition 1.1 and Definition 1.3 are equivalent; the reason is as follows. It is clear from the definitions that $(\mathcal{S})_n \subseteq \langle \mathcal{S} \rangle_n$ for any $n \geq 1$. Therefore, a class $\mathcal{S}$ which satisfies Definition 1.3 also satisfies Definition 1.1. Conversely, one can see that by induction on n that for each $X \in \langle \mathcal{S} \rangle_n$, there is some $n' \gg n$ with $X \in (\mathcal{S})_{n'}$, and the claim follows.

The problem we consider in this paper can be generalised as follows. If a Hom-finite Krull-Schmidt k-linear triangulated category exhibits Auslander-Reiten *triangles*, is there a simple-minded system which contains object(s) lying in the homogeneous tube(s)? We will remark on the difficulty of this problem at the end of this introduction.

In [10], a weaker version of sms has also been introduced, and it has been shown that when A is representation-finite self-injective, the following system is sufficient for defining (hence equivalent to) an sms.

Definition 1.4. ([10]) *Let A be a self-injective algebra over an algebraically closed field. A class of objects $\mathcal{S}$ in $\underline{\mathrm{mod}}A$ is called a weakly simple-minded system (wsms) if the following two conditions are satisfied:*

(a) (orthogonality condition) For any $S, T \in \mathcal{S}$, $\underline{\mathrm{Hom}}_A(S,T) = \begin{cases} 0 & (S \neq T), \\ k & (S = T). \end{cases}$

(b) (weak generating condition) For any indecomposable non-projective A-module X, there exists some $S \in \mathcal{S}$ (depends on X) such that $\underline{\mathrm{Hom}}_A(X,S) \neq 0$.

In general, we do not know if there exists a wsms which is not an sms. The generating condition in sms is much stronger as it encodes the homological structure of $\underline{\mathrm{mod}}A$. Hence, it is more effective in proving various results. For example, the fact that each sms is of finite cardinality follows easily from this generating condition ([10, Proposition 2.7]). Another note-worthy example is that we can easily determine the sms's of an infinite series of 4-dimensional weakly symmetric local algebras (see the discussion before [10, Corollary 3.3]), but to determine the wsms's over them is a much more complicating task.

Closely related to wsms is Pogorzaly's maximal system of orthogonal stable bricks [11]. Simply put, it is a wsms with an extra condition: no objects in the system is τ-invariant.

While it is easy to see that any sms is a wsms, there is no apparent relations between sms's and maximal systems of orthogonal stable bricks, that is, it is not clear if an object in an sms can be τ-invariant. It follows easily from Theorem 1.2 that the implication from being an sms to being a wsms actually "factors through" being a maximal system of orthogonal stable bricks, for almost all self-injective algebras.

Corollary 1.5. *Let A be a self-injective algebra over an algebraically closed field which is not a local Nakayama algebra. Any simple-minded system of A is a maximal system of orthogonal stable bricks.*

Proof. We have already mentioned that modules in a homogeneous tube are τ-invariant. Here we only need an almost converse of this proved by Hoshino in [9, Theorem 1]. His result asserts that for a basic (indecomposable) artin algebra Λ over an algebraically closed field, if there is an indecomposable τ-invariant module M, then either Λ is a local Nakayama algebra, or M lies in a component of Γ_A which consists only of τ-invariant modules, i.e. a homogeneous tube. The claim now follows from applying this result to Theorem 1.2. □

Our proof of Theorem 1.2 relies heavily on known results of the Auslander-Reiten theory of finite dimensional self-injective algebras from [7, 9], and a few from [10]. The lemmas needed and derived from these articles will be presented in the next section. We will present the core part of the proof of Theorem 1.2 in the final section. We remark finally that the need of these strong results in our proof shows some major obstacles in generalising Theorem 1.2 and Corollary 1.5 to simple-minded systems of Hom-finite Krull-Schmidt k-linear triangulated categories.

2. Technical Lemmas

For general properties of stable categories for self-injective algebras and of Auslander-Reiten theory we refer to [1, 6, 12]. It is well-known that the stable category of a self-injective algebra is triangulated, with the suspension functor given by the inverse syzygy functor Ω^{-1} (see [8]). We will freely use the properties of this triangulated structure.

As in Section 1, A will always be an indecomposable non-simple self-injective algebra.

For completeness, we now recall various notations and discussions in [7, Section 2]. Let $\mathcal{C}$ be a stably quasi-serial component of rank $n \geq 1$ of the stable Auslander-Reiten quiver of A. We use the same notations for specifying modules in $\mathcal{C}$. Namely, if X is an indecomposable non-projective module lying at the end (i.e. the *mouth* of $\mathcal{C}$, then for any natural number $r \geq 1$, there is a unique infinite sectional path

$$X = X(1) \to X(2) \to \cdots \to X(r) \to X(r+1) \to \cdots,$$

and dually, there is a unique infinite sectional path in $\mathcal{C}$ with

$$\cdots \to [r+1]X \to [r]X \to \cdots \to [2]X \to [1]X = X.$$

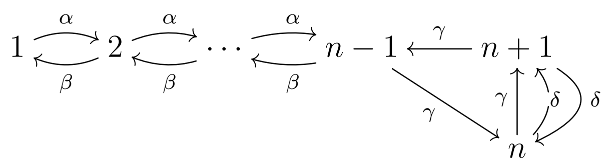
We say that $X(r)$ (resp. $[r]X$) is of quasi-length r. For notational convenience, we treat $X(0) = [0]X = 0$. Note that if $\mathcal{C}$ is homogeneous, then $X(r) = [r]X$ for all $r \geq 1$.

Lemma 2.1. ([7, Lemma 3.5.1]) *Suppose $X = X(1)$ is an indecomposable non-projective module lying on the mouth of a stably quasi-serial component of Γ_A of rank n. Then we* *have* $\dim \underline{\mathrm{End}}_A(X(mn+s)) > m$ *for any integer* $m \geq 0$ *and* $s \in \{1, \ldots, n\}$. *In particular, by the orthogonality condition, if* $X(r)$ *belongs to an sms of* $\underline{\mathrm{mod}}A$, *then* $r \leq n$.

This result follows from some special properties of the function $\dim \underline{\mathrm{Hom}}_A(W, -)$ on the modules in certain Auslander-Reiten sequences, and we recommend the interested reader to consult [7, Section 2] for the details.

We do not know if a module of quasi-length n can belong to an sms in general. It follows from Theorem 1.2 that this is not the case when $n = 1$. On the other hand, it is possible for a module in an sms of A to have quasi-length $n-1$. For example, we can define the following quiver Q for any integer $n \geq 2$.

$Q:$



Let A be the bounded path algebra of Q with relations $\alpha^2 = \beta^2 = 0$, $\alpha\gamma = \gamma\beta = 0$, $\delta\gamma = \gamma\delta = 0$, $\alpha\beta = \beta\alpha$, $\beta\alpha = \gamma^3$, and $\delta^2 = \gamma^3$, whenever they make sense. Then A is a (tame) symmetric special biserial algebra, where the simple module corresponding to the vertex $n-1$ is a module of quasi-length $n-1$ lying on a stably quasi-serial tube of rank n.

From now on, we concentrate only on the case when $\mathcal{C}$ is a stably quasi-serial component of rank 1. We fix the notation X as the unique (non-projective) indecomposable module lying at the end of $\mathcal{C}$.

As explained in the proof of Corollary 1.5, $\mathcal{C}$ contains only τ-invariant modules, i.e. it contains no projective module and is a homogeneous tube. Consequently, any τ-invariant module in an sms of A will lies at the end of a homogeneous tube by Lemma 2.1. Using [7, Proposition 2.3, Lemma 2.3.1], we obtain the following short exact sequence

$$0 \to X(i) \xrightarrow{\epsilon} X(i+j) \xrightarrow{\pi} X(j) \to 0 \tag{1}$$

for any $i, j > 0$, where ϵ (resp. π) is given by the composition of the irreducible maps on the sectional path starting from $X(i)$ (resp. $X(i+j)$) and ending at $X(i+j)$ (resp. $X(j)$). Since a monomorphism (resp. an epimorphism) between non-injective (resp. non-projective) indecomposable modules does not factor through an (resp. a projective) injective module, the corresponding morphisms $\underline{\epsilon}$ and $\underline{\pi}$ in $\underline{\mathrm{mod}}A$ are non-zero.

Lemma 2.2. *If* S *is a non-projective indecomposable module with* $\underline{\mathrm{Hom}}_A(X, S) = 0$, *then* $\underline{\mathrm{Hom}}_A(X(r), S) = 0$ *for all* $r \geq 1$. *Dually, if* $\underline{\mathrm{Hom}}_A(S, X) = 0$, *then so is* $\underline{\mathrm{Hom}}_A(S, X(r))$ *for all* $r \geq 1$.

Proof We prove by induction on r. For $r = 1$, $X(r) = X$ and the claim follows by the assumption that, $\underline{\mathrm{Hom}}_A(X, S) = 0$. Now $\underline{\mathrm{Hom}}_A(X(r), S) = 0$ follows from applying [10, Lemma 2.5] using the induction hypothesis on the sequence:

$$0 \to X(r-1) \xrightarrow{\epsilon_r} X(r) \xrightarrow{\pi_r} X \to 0, \tag{2}$$

obtained from (1) with $i = r-1$ and $j = 1$. The claim on $\underline{\mathrm{Hom}}_A(S, X(r))$ can be proved dually.

$\square$

Let $\nu := DHom_A(-, A)$ denote the Nakayama functor. Then using Auslander-Reiten duality and the well-known fact that $\tau \cong \nu\Omega^2$, we have the following k-space isomorphisms for all $M, N \in \underline{\mathrm{mod}}A$:

$$\underline{\mathrm{Hom}}_A(M, N) \cong DExt^1_A(N, \tau M) \cong D\underline{\mathrm{Hom}}_A(N, \nu\Omega M). \tag{3}$$

In other words, there is a Serre duality in the triangulated category $\underline{\mathrm{mod}}A$ with Serre functor $\nu\Omega$.

Lemma 2.3. *If S is a non-projective indecomposable module with $\underline{\mathrm{Hom}}_A(S, X) = 0$, then $\underline{\mathrm{Hom}}_A(\Omega(X(r)), S) = 0$ for all $r \geq 1$.*

Proof Take M, N as $\Omega(X(r)), S$ respectively in (3), then we get

$$\dim \underline{\mathrm{Hom}}_A(\Omega(X(r)), S) = \dim \underline{\mathrm{Hom}}_A(S, \nu\Omega(\Omega X(r))).$$

Since $\nu\Omega^2 X(r) \cong \tau X(r) \cong X(r)$, the later space is just $\underline{\mathrm{Hom}}_A(S, X(r))$, which is zero by Lemma 2.2.

□

Lemma 2.4. *If $\dim \underline{\mathrm{End}}_A(X) = 1$, then for all $r \geq 1$, the following vector spaces have dimension 1:*

(i) $\underline{\mathrm{Hom}}_A(X(r), X)$, *(i')* $\underline{\mathrm{Hom}}_A(X, X(r))$, *(ii)* $\underline{\mathrm{Hom}}_A(\Omega(X(r)), X)$.

In particular, if X belongs to an sms $\mathcal{S}$ of $\underline{\mathrm{mod}}A$, then $\Omega(X(r)) \notin \mathcal{S} \setminus \{X\}$ for all $r \geq 1$.

Proof (i): We proceed by induction on r. For $r = 1$ this is trivial. Suppose that $r > 1$. Applying $\underline{\mathrm{Hom}}(-, X)$ to (2), we obtain an exact sequence

$$\underline{\mathrm{Hom}}(X, X) \xrightarrow{\underline{\pi_r}^*} \underline{\mathrm{Hom}}(X(r), X) \xrightarrow{\underline{\epsilon_r}^*} \underline{\mathrm{Hom}}(X(r-1), X).$$

Since $\dim \underline{\mathrm{Hom}}(X, X) = 1$, and $\underline{\pi_r}^*(\underline{id_X}) = \underline{\pi_r} \neq 0$, $\underline{\pi_r}^*$ is injective. By the induction hypothesis, $\underline{\mathrm{Hom}}(X(r-1), X) \cong k$. By the exactness of the above sequence, this means that $\underline{\mathrm{Hom}}(X(r), X)$ is a k-space of dimension one or two. Since $\underline{\mathrm{Hom}}(X(r), X)$ has dimension 2 is equivalent to $\underline{\epsilon_r}^*$ being surjective, which in turn is equivalent to $\mathrm{Hom}(\epsilon_r, X)$ being surjective by standard argument[a]. It suffices to show that π_r does not factor through ϵ_r to finish the proof. Suppose the contrary. Then there is a non-zero map $t : X(r) \to X$ with the commutative diagram:

$$\begin{array}{ccc} X(r-1) & \xrightarrow{\epsilon_r} & X(r) \\ & {\scriptstyle \pi_r}\searrow \quad \swarrow{\scriptstyle t} & \\ & X & \end{array}$$

Using the short exact sequence (1) again, we can quotient out $\ker(\pi_r) \cong X(r-2)$ in the top row and obtain new commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_2} & X(2) \\ & {\scriptstyle id_X}\searrow \quad \swarrow{\scriptstyle \bar{t}} & \\ & X & \end{array}$$

This means that the identity map factors through an irreducible map ϵ_2, a contradiction.

(i'): This is dual to (i).

[a] $\underline{\mathrm{Hom}}(\epsilon_r, X)$ being surjective is equivalent to $\underline{\mathrm{Hom}}(\Omega(\pi_r), X) \cong \mathrm{Ext}^1(\pi_r, X)$ being injective, which is again equivalent to $\mathrm{Hom}(\epsilon_r, X)$ being surjective.

(ii): Take M, N as $\Omega(X(r)), X$ respectively in (3). The claim now follows from (i'), using similar argument as in Lemma 2.3.

The final statement follows immediately from the orthogonality condition of $\mathcal{S}$.

□

Lemma 2.5. *For every positive integer $r \geq 1$, there is a non-split triangle in $\underline{\text{mod}}A$:*

$$\Omega(X(r+1)) \to \Omega(X(r)) \to X \to . \tag{4}$$

Proof Take (1) with $i = 1$ and $j = r$, we obtain the non-split triangle

$$X \xrightarrow{\epsilon} X(r+1) \xrightarrow{\pi} X(r) \to,$$

which can be rotated to form a triangle with terms agreeing to those in (4). So it remains to show that the connecting morphism $\Omega(X(r)) \to X$ of the above triangle is non-zero. Suppose the contrary, then the cone of the zero connecting morphism will be $X(r) \oplus X \ncong X(r+1)$, hence a contradiction.

□

Lemma 2.6. *If $\Omega(X) \cong X$, then* $\dim \underline{\text{End}}_A(X) \neq 1$*. In particular, an sms of $\underline{\text{mod}}A$ does not contain X with $\tau X \cong X$ and $\Omega(X) \cong X$.*

Proof. By the previous Lemma 2.5, for any $r \geq 1$ we have triangle $\Omega(X(r)) \xrightarrow{\underline{f}} X \to X(r+1) \to$ for some non-zero morphism $\underline{f} \in \underline{\text{Hom}}_A(\Omega(X(r)), X)$. On the other hand, by taking $(i, j) = (r-1, 1)$ in (1), we obtain another triangle $X(r) \xrightarrow{\underline{\pi}} X \to \Omega^{-1}(X(r-1))$ where $\underline{\pi}$ is non-zero. Note that in the case $r = 1$, we regard $X(r-1) = 0$.

Since Ω is a triangulated auto-equivalence commuting with τ, $\Omega(\mathcal{C})$ is also a homogeneous tube and the quasi-length is invariant under Ω. In particular, $\Omega(X) \cong X$ implies that $\Omega^{\pm}(X(r)) \cong X(r)$ in $\underline{\text{mod}}A$, and the two triangles above can be rewritten as:

$$X(r) \xrightarrow{\underline{f}} X \longrightarrow X(r+1) \longrightarrow$$

$$X(r) \xrightarrow{\underline{\pi}} X \longrightarrow X(r-1) \longrightarrow$$

If $\dim \underline{\text{End}}_A(X) = 1$, then by Lemma 2.4, $\dim \underline{\text{Hom}}_A(\Omega(X(r)), X) = 1$. Therefore, $\underline{\pi}$ is a scalar multiple of $\underline{f}$, and we have $X(r+1) \cong X(r-1)$, which is absurd. □

We are now going to prove our main result Theorem 1.2.

3. Proof of Theorem 1.2

Let $\mathcal{S}$ be an sms of $\underline{\text{mod}}A$ for a non-simple indecomposable basic self-injective algebra A. Suppose on the contrary that $\mathcal{S}$ contains an A-module X which lies in a homogeneous tube $\mathcal{C}$ of Γ_A. By Lemma 2.1, X must lie at the end of $\mathcal{C}$, and the component $\mathcal{C}$ consists of modules $X = X(1), X(2), \cdots$. Moreover, by Lemma 2.6, we have $\Omega(X) \ncong X$.

We prove the theorem by showing that there is no positive integer n such that the non-projective indecomposable module $\Omega(X)$ is in $\langle \mathcal{S} \rangle_n$. Hence, contradicting the generating condition of the sms $\mathcal{S}$.

For all positive integer $r \geq 1$, we have by Lemma 2.4 (and $\Omega(X) \ncong X$) that $\Omega(X(r)) \notin \mathcal{S}$, hence not in $\langle \mathcal{S} \rangle_1$. By Lemma 2.3 and the orthogonality condition of $\mathcal{S}$, there is no non-zero morphism from $\Omega(X(r))$ to any S in $\mathcal{S} \setminus \{X\}$. Combining with Lemma 2.4 (ii), the last step in generating $\Omega(X(r))$ from $\mathcal{S}$ is then given by the following triangle in $\underline{\text{mod}}A$:

$$Y \to \Omega(X(r)) \xrightarrow{\underline{f}} X^l \to \tag{5}$$

for some non-zero object Y. We can assume that no direct summand of Y is isomorphic to $\Omega(X(r))$; otherwise, the triangle (5) does not generate $\Omega(X(r))$, contradicting the construction.

Lemma 2.4 (ii) implies that the map $\underline{f}$ is of the form

$$\underline{f} = (a_1\underline{f_0}, a_2\underline{f_0}, \cdots, a_l\underline{f_0})^t,$$

where $\underline{f_0}$ spans $\underline{\text{Hom}}_A(\Omega(X(r)), X)$, and $a_1, a_2, \cdots, a_l$ are scalars in k. Note that not all a_i's are zero, otherwise there is a direct summand of Y isomorphic to $\Omega(X(r))$. Without loss of generality, we may assume that $a_1 \neq 0$. Then we have the following commutative diagram

$$\begin{array}{ccc}
\Omega(X(r)) & \xrightarrow{\quad \underline{f} \quad} & X^l \\
\downarrow{\scriptstyle \underline{id_{\Omega(X(r))}}} & & \downarrow{\scriptstyle \theta} \\
\Omega(X(r)) & \xrightarrow{(\underline{f_0},0,\ldots,0)^t} & X^l,
\end{array}$$

where θ is the matrix of maps

$$\begin{pmatrix}
a_1^{-1}\underline{id_X} & 0 & \cdots & 0 \\
a_2a_1^{-1}\underline{id_X} & \underline{id_X} & & 0 \\
\vdots & & \ddots & \\
a_la_1^{-1}\underline{id_X} & 0 & & \underline{id_X}
\end{pmatrix}.$$

Clearly, $\underline{id_{\Omega(X(r))}}$ and θ are isomorphisms. By the axioms of triangulated categories, we obtain an isomorphism between the triangle (5) and the direct sum of the following two triangles:

$$Y' \to \Omega(X(r)) \xrightarrow{\underline{f_0}} X \to$$

and

$$\Omega(X^{l-1}) \to 0 \to X^{l-1} \to .$$

This implies that $Y \cong Y' \oplus \Omega(X^{l-1}) \cong Y' \oplus \Omega(X)^{l-1}$ as objects in $\underline{\text{mod}}A$. Note that if $r = 1$, then $l = 1$ as there is no direct summand of Y isomorphic to $\Omega(X(r))$. Also, it follows from Lemma 2.5, and $\underline{f_0}$ being the unique (up to scalar) non-zero morphism in $\underline{\text{Hom}}_A(\Omega(X(r)), X)$, that $Y' \cong \Omega(X(r+1))$ in $\underline{\text{mod}}A$.

Since $\mathcal{S}$ is an sms, there is a positive integer n_r such that $X(r) \in \langle \mathcal{S} \rangle_{n_r}$ and $X(r) \notin \langle \mathcal{S} \rangle_{n_r-1}$ for each $r \geq 1$. The conclusion from the above argument is that, using the description of Y, we have $n_1 > n_2$, and $n_r > \max\{n_{r+1}, n_1\} \geq n_{r+1}$ for all $r > 1$. In particular, we have a strictly decreasing chain of infinitely many positive integers $n_1 > n_2 > n_3 > \cdots$, which is absurd. This ends the proof of Theorem 1.2.

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