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Homological Methods, Representation Theory, and Cluster Algebras

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Ibrahim Assem · Sonia Trepode
Editors

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DE RECHERCHES
MATHÉMATIQUES

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Preface

This volume includes six mini-courses delivered at the 2016 CIMPA (Centre International de Mathématiques Pures et Appliquées) research school held in the Universidad Nacional de Mar del Plata, Mar del Plata, Argentina, from the 7th to the 18th of March 2016. More than 80 mathematicians and students from a dozen countries participated in the event.

This research school was dedicated to the founder of the Argentinian research group in representation theory of algebras, Dr. M.I. Platzeck, on the occasion of her 70th birthday. It was devoted to interactions between representation theory, homological algebra and the new ever-expanding theory of cluster algebras. While homological algebra has always been present as one of the main tools in the study of finite dimensional algebras, the more recent strong connection with cluster algebras quickly established itself as one of the important features of the mathematical landscape. This connection has been fruitful to both areas, representation theory provided a categorification of cluster algebras, while the study of cluster algebras provided representation theory with new objects of study like tilting in the cluster category. This volume stands as a partial testimony to this new and welcome development.

The six courses presented at the the research school were organised as follows. During the first week the more elementary courses were delivered (in this volume, the courses “Introduction to the Representation Theory of Finite-Dimensional Algebras: The Functorial Approach,” “Auslander–Reiten Theory for Finite-Dimensional Algebras” and “Cluster Algebras from Surfaces”), the first two of which form the basis of modern-day representation theory and the third one is an introductory course on an important class of cluster algebras. The more advanced courses, which concentrate on connections between representation theory and cluster algebras, took place during the second week (in this volume, the courses “Cluster Characters,” “A Course on Cluster-Tilted Algebras” and “Brauer Graph Algebras”). We would like to express our gratitude to the authors who submitted contributions and to the referees for their assistance.

The courses in this volume are addressed to graduate students or young researchers with some previous knowledge of noncommutative algebra or homological algebra. This volume will also be of interest to any mathematician who is not a specialist of the topics presented here and would like to access this fast-developing field. Because interactions between topics of the research school can only increase, and the courses presented reflect the diversity as well as the rich activity of the groups working in the area, we hope that this volume will be useful to its readers.

We wish to express our thanks for financial support to the CIMPA, the CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas), the Comisión de Investigaciones Científicas de la Provincia de Buenos Aires, the IMU (International Mathematical Union), the IMJ-PRG (Institut de Mathématiques de Jussieu, Paris Rive Gauche) and the Universidad Nacional de Mar del Plata. We also wish to extend our gratitude to Drs. Galia Dafni and Véronique Hussin of the CRM as well as Elizabeth Loew from Springer for permission to publish this volume in their joint series. We also thank André Montpetit from the CRM for his help in putting the volume in its final form.

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Introduction to the Representation Theory of Finite-Dimensional Algebras: The Functorial Approach

María Inés Platzeck

These are the notes of a course given at the CIMPA School "Homological Methods, Representation Theory and Cluster Algebras," Mar del Plata, Argentina, 2016. The aim of this brief course is to give an introduction to the functorial approach to the representation theory of finite-dimensional algebras, developed by Maurice Auslander and Idun Reiten, and is strongly based on the work "A functorial approach to representation theory," by M. Auslander [4]. No new results are proven here. Further related results can be found in [2, 3, 5–8], works that have been collected in [12].

We will assume throughout the paper that Λ is a finite-dimensional algebra over a field k. The results proven also hold in the more general context of Artin algebras, that is, algebras over a commutative Artinian ring which are finitely generated as a module over the ring. For the sake of simplicity, we will only consider algebras over a field.

All modules considered will be left modules, unless otherwise specified, and $\operatorname{mod}\Lambda$ will denote the category of finitely generated left Λ-modules. Let $(\operatorname{mod}\Lambda)^{\mathrm{op}}$ denote the opposite category of $\operatorname{mod}\Lambda$, and Λ^{op} the opposite algebra of Λ. Inside the category $\big((\operatorname{mod}\Lambda)^{\mathrm{op}}, \mathrm{Ab}\big)$ of contravariant additive functors from $\operatorname{mod}\Lambda$ to the category Ab of abelian groups, we will consider the full subcategory consisting of the finitely presented functors. M. Auslander observed that this subcategory has very good homological properties and has shown with Idun Reiten how the knowledge of this category can shed light in the study of the algebra Λ. I intend here to give an introduction to the study of these ideas.

We will describe next the contents section by section. In the first section we introduce the terminology and prove some general properties of functors in $\big((\operatorname{mod}\Lambda)^{\mathrm{op}}, \mathrm{Ab}\big)$. We start by stating Yoneda's Lemma and prove some consequences of it. Although this lemma can be easily proven, it is a fundamental tool in our study and will be used throughout the paper. In the second section we define and describe simple functors, their projective covers, and show the relation between minimal projective presentations of simple functors in $\big((\operatorname{mod}\Lambda)^{\mathrm{op}}, \mathrm{Ab}\big)$ and right almost split morphisms in $\operatorname{mod}\Lambda$. Section 3 is devoted to studying properties of finitely presented functors. In the last section we prove that simple functors are finitely presented and explain how this fact yields the existence of almost split sequences in $\operatorname{mod}\Lambda$. Finally we turn our attention to the study of the radical of a functor

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and give some further relations between notions and results in the category of functors and in mod Λ.

1 Preliminaries and Notation

Since the k-algebra Λ is finitely generated over k, then Λ is an Artinian ring. Thus the endomorphism ring $\mathrm{End}_\Lambda(M)$ of an indecomposable module M is a local ring, and Krull–Schmidt Theorem holds for Λ. Thus any module can be written as a finite sum of indecomposable Λ-modules and this decomposition is unique, up to order of the summands. We denote by $\mathrm{rad}\,\Lambda$ the Jacobson radical of Λ, that is, the intersection of the maximal left (right) ideals of Λ, which coincides with the nilradical of Λ, and $\mathrm{rad}\,M$ will denote the radical of the Λ-module M. Then $\mathrm{rad}\,M = \mathrm{rad}\,\Lambda.M$. We also recall that projective indecomposable modules have a unique maximal submodule.

Throughout the paper, $\mathrm{ind}\,\Lambda$ denotes a full subcategory of $\mathrm{mod}\,\Lambda$ consisting of a chosen set of representatives of nonisomorphic indecomposable finitely generated Λ-modules, and $\mathrm{add}\,M$ is the full subcategory of $\mathrm{mod}\,\Lambda$ consisting in the finite direct sums of direct summands of M. For unexplained notions and results on ring theory, modules or categories we refer the reader to [1], for homological algebra to [10] or [11]. For a general reference on representation theory of algebras we refer the reader to [9].

A finitely generated module X is also finitely generated when considered as k-vector space, because the algebra Λ is finite-dimensional over k. Thus the Λ^{op}-module $D(X) = \mathrm{Hom}_k(X, k)$ is also finitely generated and $D = \mathrm{Hom}_k(-, k)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{\mathrm{op}}$ is a duality, which induces an equivalence of categories $\big((\mathrm{mod}\,\Lambda)^{\mathrm{op}}, \mathrm{Ab}\big) \to (\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$. To simplify notation we will write $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$ instead of $\big((\mathrm{mod}\,\Lambda)^{\mathrm{op}}, \mathrm{Ab}\big)$.

Let C in $\mathrm{mod}\,\Lambda$. We denote by $(-, C)$ the representable functor $\mathrm{Hom}_\Lambda(-, C)\colon \mathrm{mod}\,\Lambda^{\mathrm{op}} \to \mathrm{Ab}$ defined by $\mathrm{Hom}_\Lambda(-, C)(X) = \mathrm{Hom}_\Lambda(X, C)$. For functors F, G in $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$ we denote by (F, G) the group of morphisms (natural transformations) from F to G. Notice that this is a set because the isomorphism classes of objects in $\mathrm{mod}\,\Lambda$ constitute a set. If $\epsilon\colon F \to G$ and $h\colon C_1 \to C_2$ we denote by $\epsilon_C\colon F(C) \to G(C)$ and $F(h)\colon F(C_2) \to F(C_1)$ the corresponding morphisms in Ab.

Of fundamental importance is the following result due to Yoneda, whose proof we leave to the reader:

Lemma 1.1 (Yoneda). *Let C in $\mathrm{mod}\,\Lambda$, F in $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$. The correspondence $\theta = \theta_{C,F}\colon ((-, C), F) \to F(C)$ defined by $\theta(\alpha) = \alpha_C(\mathrm{id}_C)$ is a group isomorphism, functorial in C and in F. That is, for $f \in (C_1, C_2)$ and $\epsilon \in (F_1, F_2)$, the diagrams*

$$\begin{array}{ccc} \big((-, C_2), F\big) & \xrightarrow{\theta_{C_2,F}} & F(C_2) \\ {\scriptstyle ((-,f),F)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \big((-, C_1), F\big) & \xrightarrow{\theta_{C_1,F}} & F(C_1) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \big((-, C), F_1\big) & \xrightarrow{\theta_{C,F_1}} & F_1(C) \\ {\scriptstyle ((-,C),\epsilon)}\downarrow & & \downarrow{\scriptstyle \epsilon_C} \\ \big((-, C), F_2\big) & \xrightarrow{\theta_{C,F_2}} & F_2(C) \end{array}$$

commute.

The following consequences of Yoneda's Lemma will be very useful.

Corollary 1.2. *The correspondence* $(X,Y) \to \big((-,X),(-,Y)\big)$ *mapping* f *to* $(-,f)$, *for* $f \in (X,Y)$, *is an isomorphism, for all* X, Y *in* $\operatorname{mod}\Lambda$.

Proof. $\theta_{X,(-,Y)}\big((-,f)\big) = (-,f)_X(\mathrm{id}_X) = (X,f)(\mathrm{id}_X) = f.\mathrm{id}_X = f$, so the morphism considered is the inverse of the isomorphism $\theta_{X,(-,Y)}$.

Corollary 1.3. *Let* X, Y *in* $\operatorname{mod}\Lambda$, *let* F *in* $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$ *be a subfunctor of* $(-,Y)$ *and* $f \in (X,Y)$. *Then the following conditions are equivalent*:

(a) $f \in F(X)$
(b) $\operatorname{Im}\big((-,f)\colon (-,X) \to (-,Y)\big) \subseteq F$

Proof. The inclusion $j\colon F \to (-,Y)$ induces a commutative diagram

$$\begin{array}{ccc} \big((-,X),F\big) & \xrightarrow{\theta_{X,F}} & F(X) \\ \Big\downarrow{\scriptstyle ((-,X),j)} & & \Big\downarrow{\scriptstyle j_X} \\ \big((-,X),(-,Y)\big) & \xrightarrow{\theta_{X,(-,Y)}} & (X,Y)\,. \end{array}$$

We have that (b) holds, that is, $\operatorname{Im}(-,f) \subseteq F$, if and only if $(-,f) \in \operatorname{Im}\big((-,X),j\big)$. Since the horizontal arrows in the diagram are isomorphisms, it follows that $(-,f) \in \operatorname{Im}\big((-,X),j\big)$ if and only if $\theta_{X,(-,Y)}\big((-,f)\big) \in \operatorname{Im} j_X$, that is, whenever $f \in F(X)$. Thus (a) and (b) are equivalent. □

The category $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$ is an abelian category. We will give definitions, which in fact are characterizations of the corresponding notions defined categorically.

A sequence $F_1 \xrightarrow{f} F_2 \xrightarrow{g} F_3$ in $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$ is *exact* if the sequence $F_1(C) \xrightarrow{f_C} F_2(C) \xrightarrow{g_C} F_3(C)$ is exact, for every $C \in \operatorname{mod}\Lambda$. Moreover, $\operatorname{Ker} f$, $\operatorname{Coker} f$ are the functors such that the sequence $0 \to \operatorname{Ker} f \to F_1 \xrightarrow{f} F_2 \to \operatorname{Coker} f \to 0$ is exact in $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$, and f is a monomorphism whenever $\operatorname{Ker} f = 0$, an epimorphism if $\operatorname{Coker} f = 0$.

The notion of projective functor is defined as usual: P in $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$ is *projective* whenever the functor $(P,-)$ preserves epimorphisms.

It follows from Yoneda's Lemma that $F_1 \xrightarrow{f} F_2 \xrightarrow{g} F_3$ is exact in $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$ if and only if $\big((-,C),F_1\big) \xrightarrow{f} ((-C),F_2) \xrightarrow{g} ((-,C),F_3)$ is exact for every C in $\operatorname{mod}\Lambda$. This shows that the representable functor $(-,C)$ is a projective object in $(\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$.

Moreover for every $F \in (\operatorname{mod}\Lambda^{\mathrm{op}},\mathrm{Ab})$ there exist C_i in $\operatorname{mod}\Lambda$ and an epimorphism $\coprod_{i\in I}(-,C_i) \to F$. This result follows using again Yoneda's Lemma. In fact, for C in $\operatorname{mod}\Lambda$ and $z \in F(C)$, let $h_z\colon (-,C) \to F$ be such that $h_z(\mathrm{id}_C) = z$. Then $\coprod_{z\in C} h_z\colon \coprod_{z\in C}(-,C) \to F$ is such that $(\coprod_{z\in C} h_z)_C\colon \coprod_{z\in C}(C,C) \to F(C)$ is surjective. Moreover, if $C \simeq C'$, then $(\coprod_{z\in C} h_z)_{C'}$ is also surjective. Thus

$$\coprod_C \coprod_{z \in C} h_z : \coprod_C \coprod_{z \in C} (-, C) \to F \ ,$$

where C runs over a complete set of representatives of the isomorphism classes of Λ-modules, is an epimorphism.

When the set I can be chosen to be finite the functor F is said to be *finitely generated*. In this case, if I is finite and $C = \coprod_{i \in I} C_i$ then $(-, C)$ maps onto F. Thus the functor F in $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$ is finitely generated if there exists an epimorphism $h: (-, C) \to F$, with C in $\mathrm{mod}\,\Lambda$. In case F is also projective the epimorphism $(-, C) \to F$ splits, so there is $t: F \to (-, C)$ such that $ht = id_F$. Then $p = th$ is an idempotent endomorphism of $(-, C)$ such that $\mathrm{Im}\, p \simeq F$. By Yoneda's Lemma we obtain that $p = (-, g)$, for some idempotent endomorphism g of C, and $F \simeq (-, \mathrm{Im}\, g)$ is a representable functor. So the finitely generated projective functors coincide with the representable functors.

As it occurs for modules, the following result holds for an exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ in $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$:

(a) If F_1, F_3 are finitely generated, then F_2 is finitely generated.
(b) If F_2 is finitely generated, then F_3 is finitely generated.

2 Projective Covers and Simple Functors

Given an additive category $\mathcal{C}$, we recall that a morphism $\alpha: C_1 \to C_2$ in $\mathcal{C}$ is *right minimal* if for any morphism $\beta: C_1 \to C_1$ in $\mathcal{C}$ such that $\alpha\beta = \alpha$ we have that β is an isomorphism. The notion of *left minimal* morphism is defined dually: $\alpha: C_1 \to C_2$ in $\mathcal{C}$ is *left minimal* if any morphism $\beta: C_2 \to C_2$ in $\mathcal{C}$ such that $\beta\alpha = \alpha$ is an isomorphism.

It is not difficult to prove that when $\mathcal{C}$ is the category of Λ-modules, an epimorphism $f: P \to M$ in $\mathrm{mod}\,\Lambda$ with P projective is a projective cover of M if and only if f is right minimal.

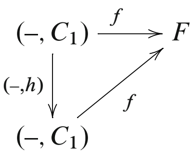
We say that an epimorphism $P \xrightarrow{f} F$ in $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$ is a *projective cover* of F if P is projective and f is a right minimal morphism.

We will prove next that projective covers of finitely generated functors exist and are unique up to isomorphism in $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$.

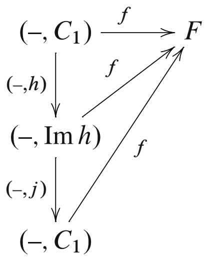
Proposition 2.1. *Let F be a finitely generated functor in* $(\mathrm{mod}\,\Lambda^{\mathrm{op}}, \mathrm{Ab})$. *Then*

(a) *F has a projective cover.*
(b) *If $f: (-, C) \to F$ and $f_1: (-, C_1) \to F$ are projective covers of F then there is an isomorphism $t: (-, C_1) \to (-, C)$ such that $f_1 = ft$.*

Proof. (a) Since F is finitely generated there exists an epimorphism $(-, C) \to F$, with C in $\mathrm{mod}\,\Lambda$. We choose amongst all such epimorphisms one, say $f: (-, C_1) \to F$, such that C_1 has smallest dimension. We prove next that f is a right minimal morphism. So we assume that $h: C_1 \to C_1$ is a morphism such that the diagram



commutes. Then we obtain a commutative diagram



where $j: \operatorname{Im} h \to C_1$ is the inclusion morphism. Then $f(-,j): (-,\operatorname{Im} h) \to F$ is an epimorphism, and the minimality of the dimension of C_1 implies that $\dim \operatorname{Im} h = \dim C_1$, so $\operatorname{Im} h = C_1$, and therefore h is an isomorphism. This proves that $f: (-,C_1) \to F$ is right minimal and it is thus a projective cover of F.

(b) Assume that $f: (-,C) \to F$, $f_1: (-,C_1) \to F$ are projective covers of F. Since f is an epimorphism and $(-,C_1)$ is projective, there is a morphism $t: (-,C_1) \to (-,C)$ such that $f_1 = ft$. Using that f_1 is an epimorphism and $(-,C)$ is projective we find $h: (-,C) \to (-,C_1)$ such that $f_1 h = f$. Then $f_1 = f_1 ht$, $f = fth$ and the minimality of f_1 and f imply that ht and th are isomorphisms. This proves that t is an isomorphism and ends the proof of (b). □

The notions of simple functor and indecomposable functor, as well as the notion of functor of finite length, are defined as in the category of Λ-modules. We will next study simple functors. The functor S in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ is *simple* if $S \neq 0$ and S contains no proper subfunctors. It follows directly from the definition that a nonzero functor S is simple if and only if any nonzero morphism $F \to S$ in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ is an epimorphism.

We start by proving that simple functors are finitely generated. First we state an important remark, whose proof is straightforward.

Remarks 2.2. Let F in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ and X in $\operatorname{mod} \Lambda$. Then $F(X)$ has a natural structure of $\operatorname{End}_\Lambda(X)^{\mathrm{op}}$-module, defined by $f.z = F(f)(z)$, for $f \in \operatorname{End}_\Lambda(X)$ and $z \in F(X)$. Moreover, if $f: F \to G$ is a morphism in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$, then $f_X: F(X) \to G(X)$ is a morphism of $\operatorname{End}_\Lambda(X)^{\mathrm{op}}$-modules, for every X in $\operatorname{mod} \Lambda$.

Let S in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ be simple. Then $S \neq 0$, so there is C in $\operatorname{mod} \Lambda$ such that $S(C) \neq 0$, and since S is an additive functor we may assume that C is an indecomposable module. Let $0 \neq x \in S(C)$. By Yoneda's Lemma we know that there is $f: (-,C) \to S$ such that $f_C(\mathrm{id}_C) = x$. Then f is an epimorphism because S is a simple functor, and thus $f_C: (C,C) \to S(C)$ is also an epimorphism in $\operatorname{mod} \operatorname{End}_\Lambda(C)^{\mathrm{op}}$.

We will prove next that the $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$-module $S(C)$ is simple. Let $y \in S(C)$, and let $h: C \to C$ be such that $f_C(h) = y$. From the commutative diagram

$$\begin{array}{ccc} (C,C) & \xrightarrow{f_C} & S(C) \\ \big\downarrow{\scriptstyle (h,C)} & & \big\downarrow{\scriptstyle S(h)} \\ (C,C) & \xrightarrow{f_C} & S(C) \end{array}$$

we obtain that

$$y = f_C(h) = f_C(h,C)(\mathrm{id}_C) = S(h) f_C(\mathrm{id}_C) = S(h)(x) = h.x\ .$$

Thus $y = h.x$. So we prove that x generates the $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$-module $S(C)$. Since x is an arbitrary nonzero element of $S(C)$, it follows that $S(C)$ is a simple module over $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$.

We give now a characterization of simple functors.

Proposition 2.3. *A functor S in* $(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$ *is simple if and only if there exists a unique C in* $\operatorname{ind}\Lambda$ *such that $S(C) \neq 0$ and $S(C)$ is a simple module over* $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$.

Moreover, if S in $(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$ *is simple and the indecomposable module C is such that $S(C) \neq 0$, then $(-,C) \to S$ is a projective cover.*

Proof. We proved above that when S is simple then there exists C as required. Since C is indecomposable and $(-,C)$ is projective, it follows that $(-,C) \to S$ is a projective cover. Then the uniqueness of such C follows from the uniqueness of projective covers.

To prove the converse we assume that there exists a unique C in $\operatorname{ind}\Lambda$ such that $S(C) \neq 0$ and $S(C)$ is a simple module over $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$. Let $0 \neq S_1 \subseteq S$ in $(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$. Then $S_1(C') \subseteq S(C') = 0$ for any indecomposable module C' in $\operatorname{mod}\Lambda$ not isomorphic to C. On the other hand, since $S_1 \neq 0$ and $S_1(C) \subseteq S(C)$ with $S(C)$ simple over $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$, it follows that $S_1(C) = S(C)$, so that $S_1 = S$. Thus S is a simple functor. □

So a simple functor S determines a unique module C in $\operatorname{ind}\Lambda$ such that $S(C) \neq 0$. We will show that the converse holds, that is, given C in $\operatorname{ind}\Lambda$ there is a simple functor S such that $S(C) \neq 0$, and such S is unique up to isomorphism.

We know that indecomposable projective Λ-modules have a unique maximal submodule. A similar result holds in $(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$. In fact, we will show that a finitely generated indecomposable projective functor $(-,C)$ has a unique maximal subfunctor.

We recall that a morphism $f: X \to Y$ in an additive category $\mathcal{C}$ is said to be a *split epimorphism* if it is an epimorphism and there is a morphism $g: Y \to X$ such that the composition fg is the identity morphism $\mathrm{id}_Y: Y \to Y$. And f is a *split monomorphism* if it is a monomorphism and there is a morphism $g: Y \to X$ such that the composition gf is the identity morphism $\mathrm{id}_X: X \to X$.

Exercise 2.4. Let $g: B \to C$ in $\operatorname{mod}\Lambda$, with C indecomposable. Then the following conditions are equivalent.

(a) $(-,g): (-,B) \to (-,C)$ is an epimorphism.

(b) $(-, g): (-, B) \to (-, C)$ is a split epimorphism.
(c) $g: B \to C$ is a split epimorphism.

If we assume, moreover, that B is indecomposable then the above conditions are equivalent to

(d) $g: B \to C$ is an isomorphism.

Thus, if we want to prove that $(-, C)$ has a unique maximal subfunctor $\mathcal{R}$, then $\mathcal{R}$ must contain $\operatorname{Im}(-, g)$, for any $g: B \to C$ which is not a split epimorphism. In the next proposition we show that these images generate a unique maximal submodule of $(-, C)$.

Proposition 2.5. *Let C in* ind Λ, *and let $\{f_i: B_i \to C\}_{i \in I}$ be the family of all nonisomorphisms $f_i: B_i \to C$ with B_i in* ind Λ. *Then*

(a) $\operatorname{Im}\big(\coprod_{i \in I}(-, B_i) \xrightarrow{\coprod(-, f_i)_{i \in I}} (-, C)\big)$ *is the unique maximal subfunctor* $\operatorname{rad}(-, C)$ *of* $(-, C)$.
(b) $S_C = (-, C)/\operatorname{rad}(-, C)$ *is the unique simple functor not vanishing on C.*

Proof. Let $f = \coprod(-, f_i)_{i \in I}: \coprod_{i \in I}(-, B_i) \to (-, C)$. First we will prove that f is not an epimorphism. Otherwise $f_C = \coprod(C, f_i)_{i \in I}: \coprod_{i \in I}(C, B_i) \to (C, C)$ would be an epimorphism. Since $\operatorname{End}_\Lambda(C)^{\text{op}}$ is a local ring, then it contains a unique maximal ideal $\mathcal{M}$, and then there is $i \in I$ such that $\operatorname{Im}(C, f_i)$ is not contained in $\mathcal{M}$. Thus $\operatorname{Im}(C, f_i) = \operatorname{End}_\Lambda(C)^{\text{op}}$ and then $(C, f_i): (C, B_i) \to (C, C)$ is an epimorphism. Since B_i is an indecomposable module, we obtain from Exercise 2.4(d) that $f_i: B_i \to C$ is an isomorphism. This contradicts the hypothesis and ends the proof that f is not an epimorphism. Thus $\operatorname{Im}(f)$ is a proper subfunctor of $(-, C)$, and we will prove that it contains all proper subfunctors of $(-, C)$.

Let F be a proper subfunctor of $(-, C)$. We will prove that $F \subseteq \operatorname{Im}(f) = \mathcal{R}$. Let B in ind Λ and $h \in F(B) \subseteq (B, C)$. By Corollary 1.3 we know that $\operatorname{Im}(-, h) \subseteq F$, and since we assumed that F is a proper subfunctor of $(-, C)$ it follows that $(-, h)$ is not an epimorphism. Using again Exercise 2.4(d) we obtain that h is not an isomorphism, so h is one of the $f_i: B_i \to C$ and $B = B_i$, with $i \in I$. Therefore $\operatorname{Im}((-, h): (-, B) \to (-, C)) \subseteq \mathcal{R}$ and from Corollary 1.3 we obtain that $h \in \mathcal{R}(B)$. Thus $F(B) \subseteq \mathcal{R}(B)$ for every B in ind Λ. Since F and $\mathcal{R}$ are additive functors it follows that $F \subseteq \mathcal{R}$.

This proves that $\mathcal{R}$ is the only maximal subfunctor $\operatorname{rad}(-, C)$ of $(-, C)$, that is, (a) holds. This ends the proof of the proposition, because (b) is a direct consequence of (a). □

Corollary 2.6. *Let C in* ind Λ

(a) $\operatorname{rad}(-, C)(X) = \{f: X \to C \mid f$ *is not an isomorphism*$\}$, *for X in* ind Λ.
(b) $\operatorname{rad}(-, C)(X) = \{f: X \to C \mid f$ *is not a split epimorphism*$\}$, *for X in* mod Λ.

Proof. (a) Follows from (a) in the proposition and Corollary 1.3.

(b) Follows from (a) in the proposition and the following lemma. □

Lemma 2.7. *Let C in* ind Λ, *X in* mod Λ, $X = \coprod_{i=1}^n X_i$, $f = \coprod_{i=1}^n f_i: \coprod_{i=1}^n X_i \to C$. *Then the following conditions are equivalent:*

(a) *f is a split epimorphism.*
(b) *There exists $i \in \{1, \dots, n\}$ such that f_i is an isomorphism.*

Proof. Assume that f is a split epimorphism and let $g: C \to \coprod_{i=1}^{n} X_i$ be such that $fg = \mathrm{id}_C$. We use matrix notation to denote f and g, that is, we write $f = \begin{bmatrix} f_1 \cdots f_n \end{bmatrix}$, $g = \begin{bmatrix} g_1 \cdots g_n \end{bmatrix}^{\mathrm{t}}$. Then $fg = \sum_{i=1}^{n} f_i g_i$ is an isomorphism. Since $\mathrm{End}_{\Lambda}(C)$ is a local ring, it follows that there is i such that $f_i g_i$ is an isomorphism. Thus f_i is a split epimorphism, and therefore an isomorphism because X_i is indecomposable. So (a) implies (b).

To prove the converse we assume that one of the f_i, say f_1, is a split epimorphism. Let $g_1: C \to X_1$ be such that $f_1 g_1 = \mathrm{id}_C$ and let $g = \begin{bmatrix} g_1 \, 0 \cdots 0 \end{bmatrix}$. Then $\begin{bmatrix} g_1 \, 0 \cdots 0 \end{bmatrix} \begin{bmatrix} f_1 \; f_2 \cdots f_n \end{bmatrix}^{\mathrm{t}} = \mathrm{id}_C$. That is, $fg = \mathrm{id}_C$, so (a) holds. □

We would like to know when $\mathrm{rad}(-, C)$ is a finitely generated functor. That is, when is there a morphism $f: B \to C$ inducing an epimorphism $(-, B) \to \mathrm{rad}(-, C)$? The answer to this question is given in the next corollary.

Corollary 2.8. *Let $f: B \to C$ in $\mathrm{mod}\,\Lambda$ with C indecomposable. Then $\mathrm{Im}(-, f) = \mathrm{rad}(-, C)$ if and only if f satisfies the following conditions*:

(a) *f is not a split epimorphism.*
(b) *If $h: X \to C$ is not a split epimorphism then there exists $s: X \to B$ such that $h = fs$.*

Proof. Assume that $\mathrm{Im}(-, f) = \mathrm{rad}(-, C)$. From Corollary 1.3 we obtain that $f \in \mathrm{rad}(-, C)(B)$, so f is not a split epimorphism, by Corollary 2.6(b), thus condition (a) holds. To prove (b) we consider a morphism $h: X \to C$ in $\mathrm{mod}\,\Lambda$ which is not a split epimorphism. Using again Corollary 2.6(b) and our assumption we have that $h \in \mathrm{rad}(-, C)(X) = \mathrm{Im}\big((X, f): (X, B) \to (X, C)\big)$, so there is $s \in (X, B)$ such that $(X, f)(s) = h$, that is, there is s such that $fs = h$. This ends the proof of (b).

The proof of the converse is similar and is left to the reader. □

Definition 2.9. A morphism $f: B \to C$ in $\mathrm{mod}\,\Lambda$ is called a right almost split morphism if it satisfies the following conditions:

(a) f is not a split epimorphism.
(b) If $h: X \to C$ is not a split epimorphism then there exists $s: X \to B$ such that $h = fs$.

If, moreover, the morphism f is right minimal, then f is called a minimal right almost split morphism.

Left almost split morphisms are defined dually:

Definition 2.10. A morphism $f: A \to B$ in $\mathrm{mod}\,\Lambda$ is called a left almost split morphism if it satisfies the following conditions:

(a) f is not a split monomorphism.
(b) If $h: A \to X$ is not a split monomorphism then there exists $s: B \to X$ such that $h = sf$.

If, moreover, the morphism f is left minimal, then f is called a minimal left almost split morphism.

Given a module C, the existence of a right almost split morphism $f: B \to C$ implies that C is indecomposable and f is unique in the sense stated in the next exercise.

Exercise 2.11. Let $f: B \to C$ be a right almost split morphism. Prove

(a) The module C is indecomposable.
(b) If f is minimal and $f': B' \to C$ is another minimal right almost split morphism, then there exists an isomorphism $\sigma: B' \to B$ such that $f' = f\sigma$
(c) If C is not projective then f is an epimorphism.

The previous corollary can be stated in the following way:

Corollary 2.12. *Let C be an indecomposable module. A morphism $f: B \to C$ in* $\operatorname{mod} \Lambda$ *satisfies that* $\operatorname{Im}(-, f) = \operatorname{rad}(-, C)$ *if and only if f is a right almost split morphism.*

In analogy with the definition of finitely presented modules, finitely presented functors are defined as follows.

Definition 2.13. A functor F in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ is finitely presented if there exists an exact sequence $(-, B) \xrightarrow{f} (-, C) \xrightarrow{g} F \to 0$ with B, C in $\operatorname{mod} \Lambda$. If this is the case, the sequence $(-, B) \xrightarrow{f} (-, C) \xrightarrow{g} F \to 0$ is called a projective presentation of F. If, moreover, f and g are right minimal, then the projective presentation is called minimal.

Let S in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ be a simple functor and let C be the unique module in $\operatorname{ind} \Lambda$ such that $S(C) \neq 0$. We know by Proposition 2.3 that $(-, C) \to S$ is a projective cover, and we proved in Proposition 2.5 that $(-, C)$ has a unique maximal subfunctor $\operatorname{rad}(-, C)$, so that $S = (-, C)/\operatorname{rad}(-, C)$. It follows that S is a finitely presented functor if and only if $\operatorname{rad}(-, C)$ is finitely generated. This is, if and only if there exists a morphism $f: B \to C$ such that $\operatorname{Im}(-, f) = \operatorname{rad}(-, C)$. From the previous corollary we know that this amounts to saying that there exists an almost split morphism $f: B \to C$ in $\operatorname{mod} \Lambda$. We state this result in the following proposition.

Proposition 2.14. *Let S in* $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ *be a simple functor and let C be the unique module in* $\operatorname{ind} \Lambda$ *such that* $S(C) \neq 0$. *Then the following conditions are equivalent*:

(a) *S is finitely presented.*
(b) *There exists a right almost split morphism $f: B \to C$ in* $\operatorname{mod} \Lambda$.

Moreover, if S is finitely presented, then $(-, B) \xrightarrow{(-,f)} (-, C) \to S \to 0$ *is a projective presentation of S if and only if $f: B \to C$ is a right almost split morphism, and it is a minimal projective presentation if and only if $f: B \to C$ is a minimal right almost split morphism.*

The previous proposition shows that proving that almost split morphisms $B \to C$ exist in $\operatorname{mod} \Lambda$ for every C in $\operatorname{ind} \Lambda$ is equivalent to prove that simple functors in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ are finitely presented. In the next section we study properties of finitely presented functors, and in Sect. 4 we will prove that simple functors are amongst them.

3 Finitely Presented Functors

Finitely presented functors play an important role in the study of the category of finitely generated modules, and we devote this section to study some of their properties, which will be needed later.

Finitely presented functors are the cokernel of morphisms between projective functors, and projective functors are representable. Thus, a finitely presented functor is given by a morphism between representable functors. It follows from Yoneda's Lemma that such a morphism is given by a morphism between the corresponding modules. This provides a way to study properties of finitely presented functors using properties of morphisms in mod Λ and conversely. A nice and simple illustration of this idea is given in the following remark.

Finitely presented functors have projective dimension at most 2. In fact, if F in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ is finitely presented, $(-, B) \xrightarrow{f} (-, C) \to F \to 0$ is exact and $K = \text{Ker}\, f$, then $0 \to (-, K) \to (-, B) \xrightarrow{f} (-, C) \to F \to 0$ is a projective resolution of F.

In this case, the existence of kernels in mod Λ translates in a property about the projective dimension of finitely presented functors.

We start by giving some characterizations of finitely presented functors. We leave the proof to the reader, since it is analogous to the proof of the corresponding result for finitely presented modules.

Proposition 3.1. *The following conditions are equivalent for F in* $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$:

(a) *F is finitely presented.*
(b) *There exists an exact sequence $0 \to F_1 \to F_0 \to F \to 0$ in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ with F_0 finitely generated projective, F_1 finitely generated.*
(c) *F is finitely generated and given an exact sequence $0 \to F_1 \to F_0 \to F \to 0$ in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$, then F_0 finitely generated implies that F_1 is finitely generated.*

To prove the next proposition we will use again Yoneda's Lemma.

Proposition 3.2. *Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ with F_1 finitely generated, F_2 finitely presented. Then F_1 and F_3 are finitely presented.*

Proof. Assume that $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ with F_1 finitely generated, F_2 finitely presented. To prove that F_3 is finitely presented we consider A in mod Λ such that $(-, A) \to F_2 \to 0$, which exists because F_2 is finitely generated. We obtain a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & (-,A) & \xrightarrow{=} & (-,A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_1 & \to & F_2 & \to & F_3 & \to & 0 \\
 & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\
 & & F_1 & & 0 & & 0\,. & &
\end{array}
$$

Using the Snake Lemma (see [11, III, 5.1] or [10, III, 3]) we obtain an exact sequence $0 \to K \to L \to F_1 \to 0$. Since F_2 is finitely presented, then by the previous proposition we know that K is finitely generated. On the other hand, we know by hypothesis that F_1 is finitely generated. Thus the middle term L of the sequence $0 \to K \to L \to F_1 \to 0$ is also finitely generated. It follows from the exactness of the last column of the above diagram and the previous proposition that F_3 is finitely presented.

We prove next that F_1 is finitely presented.

Assume first that $F_2 = (-, C)$, with $C \in \operatorname{mod} \Lambda$. Since F_1 is finitely generated, there exists an epimorphism $(-, B) \to F_1 \to 0$, with B in $\operatorname{mod} \Lambda$. We know from Yoneda's Lemma that the composition $(-, B) \to F_1 \to F_2 = (-, C)$ induces a morphism $f : B \to C$ such that $\operatorname{Im}(-, f) = F_1$. Then we obtain an exact sequence

$$0 \to (-, \operatorname{Ker} f) \to (-, B) \to \operatorname{Im}(-, f) = F_1 \to 0\,,$$

and this shows that F_1 is finitely presented.

Now we prove the general case, so we assume that F_1 is finitely generated and F_2 is finitely presented. Then there exists an epimorphism $(-, C) \to F_2 \to 0$ with C in $\operatorname{mod} \Lambda$. We obtain a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & \xrightarrow{=} & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & L & \to & (-,C) & \to & F_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & \\
0 & \to & F_1 & \to & F_2 & \to & F_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & 0\,. & & & &
\end{array}
$$

We know that K is finitely generated because F_2 is finitely presented. We proved that F_3 is finitely presented, so from the exact sequence $0 \to L \to (-, C) \to F_3 \to 0$ and the case just proved it follows that L is finitely presented. Since K is finitely generated, the first nonzero column in the diagram shows that F_1 is finitely presented. □

4 Simple Functors Are Finitely Presented

To prove the results in this section we will need the notion of dual of a finitely generated functor. For this reason we will restrict to an appropriate subcategory of $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ containing the finitely generated functors, and with the property that there is a duality between this subcategory and its opposite category.

Let F in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ and X in $\text{mod}\,\Lambda^{\text{op}}$. Then X is a finite-dimensional vector space over k, and $F(X)$ is a k-vector space. We denote by $(\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k)$ the full subcategory of $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ consisting of the functors F such that the k-vector space $F(X)$ is finitely generated, for every X in $\text{mod}\,\Lambda$.

Given an exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$, then F_2 is in $(\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k)$ if and only if F_1, F_3 are in $(\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k)$. Also, representable functors are in $(\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k)$, and thus all finitely generated functors are also in $(\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k)$.

Let $D = \text{Hom}(-, k)\colon \text{mod}\,\Lambda \to \text{mod}\,\Lambda^{\text{op}}$ be the usual duality, where $D(M)$ is considered as a Λ-module by $(\lambda.f)(m) = f(\lambda m)$, for any M in $\text{mod}\,\Lambda$, $m \in M$, $f \in \text{Hom}_k(M, k)$ and $\lambda \in \Lambda$. We define a duality $D\colon (\text{mod}\,\Lambda, \text{mod}\,k) \to (\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k)$ by $D(F)(X) = D\big(F(X)\big)$, for F in $(\text{mod}\,\Lambda, \text{mod}\,k)$ and $X \in \text{mod}\,\Lambda$, and $(Dh)_X = D(h_X)$, for a morphism $h\colon F_1 \to F_2$ in $(\text{mod}\,\Lambda, \text{mod}\,k)$, X in $\text{mod}\,\Lambda$. The inverse duality $D\colon (\text{mod}\,\Lambda^{\text{op}}, \text{mod}\,k) \to (\text{mod}\,\Lambda, \text{mod}\,k)$ is defined in an analogous way.

We will use this duality to prove that simple functors in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ are finitely presented. We start by proving that the duality D preserves finitely presented functors.

Proposition 4.1. *Let F in $(\text{mod}\,\Lambda, \text{mod}\,k)$ be a finitely presented functor. Then DF is also a finitely presented functor.*

Proof. Let $(B, -) \to (A, -) \to F \to 0$ be an exact sequence, with $A, B \in \text{mod}\,\Lambda$.

Then the sequence $0 \to DF \to D(A, -) \to D(B, -)$ is exact. To prove that DF is finitely presented it suffices to prove that $D(A, -)$ and $D(B, -)$ are finitely presented. So we will prove that the dual of any representable functor is finitely presented. Let $X \in \text{mod}\,\Lambda$, and consider an exact sequence $\Lambda^m \to \Lambda^n \to X \to 0$. Then we obtain an exact sequence $0 \to (X, -) \to (\Lambda^n, -) \to (\Lambda^m, -)$, which leads to the exact sequence $D(\Lambda^m, -) \to D(\Lambda^n, -) \to D(X, -) \to 0$.

Thus to prove that $D(X, -)$ is finitely presented we only need to prove that $D(\Lambda, -)$ is finitely presented. Let Y in $\text{mod}\,\Lambda$. Using that there is an isomorphism of functors $(\Lambda, -) = \text{Hom}_\Lambda(\Lambda, -) \simeq \text{Id}$ we obtain $D(\Lambda, -)(Y) = D\big(\text{Hom}_\Lambda(\Lambda, Y)\big) \simeq DY \simeq \text{Hom}_\Lambda(\Lambda, DY) \simeq \text{Hom}_{\Lambda^{\text{op}}}(Y, D\Lambda)$. These isomorphisms are functorial and hold for any Y in $\text{mod}\,\Lambda$. Thus $D(\Lambda, -) \simeq (-, D\Lambda)$. So $D(\Lambda, -)$ is representable and therefore it is finitely presented. □

The following corollary will be very useful to prove that simple functors in $(\text{mod}\,\Lambda^{\text{op}}, \text{Ab})$ are finitely presented.

Corollary 4.2. *The functor F in* $(\operatorname{mod} \Lambda^{\mathrm{op}}, \operatorname{mod} k)$ *is finitely presented if and only if F and DF are finitely generated.*

Proof. Assume that F, DF are finitely generated, and let C in $\operatorname{mod} \Lambda$ be such that there is an epimorphism $(–, C) \to F$. Then we obtain an exact sequence $0 \to DF \to D(–, C)$. By the previous proposition we know that $D(–, C)$ is finitely presented, because $(–, C)$ is finitely presented. Since we are assuming that DF is finitely generated, it follows from Proposition 3.2 that DF is finitely presented. Using again the previous proposition we obtain that $F \simeq DDF$ is finitely presented. This proves that if F and DF are finitely generated then F is finitely presented. The converse follows directly from the previous proposition. □

Theorem 4.3. *The simple functors in* $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ *are finitely presented.*

Proof. The simple functors in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ are finitely generated and thus belong to the subcategory $(\operatorname{mod} \Lambda^{\mathrm{op}}, \operatorname{mod} k)$. Let S in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ be a simple functor. This amounts to saying that every nonzero morphism $S \to F$ in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ is a monomorphism. Then every nonzero morphism $G \to DS$ in $(\operatorname{mod} \Lambda, \mathrm{Ab})$ is an epimorphism, and this proves that DS is also a simple functor and consequently it is finitely generated. So we can apply the previous corollary to S and conclude that S is finitely presented. □

We are now in a position to prove the existence of almost split morphisms.

Theorem 4.4. *Let C be an indecomposable finitely generated Λ-module. Then there exists a minimal right almost split morphism $f: B \to C$.*

Proof. We know by Proposition 2.5 that $S_C = (–, C)/\operatorname{rad}(–, C)$ is a simple functor. By the previous theorem we know that it is finitely presented, and let $(–, B) \to (–, C) \xrightarrow{(–,f)} S_C \to 0$ be a minimal projective presentation of S_C in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$. Then Proposition 2.14 states that $f: B \to C$ is a minimal right almost split morphism. □

Lemma 4.5. *Let $f: B \to C$ be a minimal right almost split morphism. Then*

(a) *If C is not projective then $A = \operatorname{Ker} f$ is indecomposable.*
(b) *If $C = P$ is projective then there is an isomorphism $\sigma: B \to \operatorname{rad} P$ and the diagram*

$$\begin{array}{ccc} B & \xrightarrow{f} & P \\ {\scriptstyle\sigma}\downarrow & \nearrow_{j} & \\ \operatorname{rad} P & & \end{array}$$

commutes, where j denotes the inclusion morphism.

Proof. (a) Assume that C is not projective. Let $A = \operatorname{Ker} f$, and assume $A = \coprod_{i=1}^{n} A_i$, with A_i in $\operatorname{mod} \Lambda$ indecomposable for $i = 1, \dots, n$. Consider the exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$. Since f is right almost split, then it is not a split epimorphism, so g is not a split monomorphism. Then there is i such that the projection $\pi_i: A \to A_i$ induced by the given decomposition of A as as direct sum of the A_i does not factor through g. In the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & A_i & \xrightarrow{s} & L & \xrightarrow{t} & C & \longrightarrow & 0 \end{array}$$

the morphism s is not a split monomorphism. Otherwise there exists $\rho\colon L \to A_i$ such that $\rho s = \mathrm{id}_{A_i}$. Then, from $s\pi_i = vg$ we get $\pi_i = \rho s \pi_i = \rho v g$, so $\pi_i = (\rho v)g$. This contradicts the fact that $\pi_i\colon A \to A_i$ does not factor through g. Thus s is not a split monomorphism, so t is not a split epimorphism.

Since f is a right almost split morphism, there is $h\colon L \to B$ such that $t = fh$. Therefore $f = tv = fhv$. Since f is minimal, from $f = f(hv)$, we obtain that hv is an isomorphism. Thus v is a split monomorphism and therefore an isomorphism, because $l(L) \leq l(B)$. Then π_i is also an isomorphism, therefore $A \simeq A_i$, so $n = 1$ and $A = \operatorname{Ker} f$ is indecomposable.

(b) Let $C = P$ be projective. We know by Exercise 2.11 that P is indecomposable, because there is a right almost split morphism $B \to P$. We prove next that the inclusion $j\colon \operatorname{rad} P \to P$ is a right almost split morphism. Let $h\colon X \to P$ be a morphism in $\operatorname{mod} \Lambda$ which is not a split epimorphism. Then h is not an epimorphism because P is projective, so $\operatorname{Im} h$ is contained in the unique maximal submodule $\operatorname{rad} P$ of P. If we denote also by h the induced morphism $X \to \operatorname{rad} P$ we have a commutative diagram

$$\begin{array}{ccc} \operatorname{rad} P & \xrightarrow{j} & P \\ \uparrow{\scriptstyle h} & \nearrow{\scriptstyle h} & \\ X & & \end{array} .$$

This proves that $\operatorname{rad} P \to P$ is a right almost split morphism, and a straightforward argument shows that f is also minimal. Now, (b) follows from the uniqueness of minimal right almost split morphisms (See Exercise 2.11). □

Definition 4.6. An exact sequence $0 \to A \to B \xrightarrow{g} C \to 0$ in $\operatorname{mod} \Lambda$ is an almost split sequence if

(a) The sequence does not split.
(b) The modules A and C are indecomposable.
(c) g is a right almost split morphism.

Remarks 4.7. Almost split sequences were defined by Maurice Auslander and Idun Reiten in [5] and are also called AR-sequences. They are a fundamental tool in the study of finite-dimensional algebras.

We give next some characterizations of almost split sequences. We leave the proof as an exercise for the reader.

Proposition 4.8. *The following conditions are equivalent for a non-split exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* mod Λ.

(a) *The sequence is an almost split sequence.*
(b) *The modules A and C are indecomposable and f is a left almost split morphism.*
(c) *The morphism g is minimal right almost split.*
(d) *The morphism f is minimal left almost split.*

We are in a position to prove an important theorem, which proves the existence of almost split sequences.

Theorem 4.9. *For any nonprojective indecomposable module C (noninjective indecomposable module A) there exists an almost split sequence* $0 \to A \to B \to C \to 0$ *in* mod Λ.

Proof. Given an indecomposable nonprojective module C we know by Theorem 4.4 that there exists a minimal almost split morphism $g\colon B \to C$, which is an epimorphism by Exercise 2.11(c). From Lemma 4.5 we know that $A = \operatorname{Ker} g$ is indecomposable, so that the sequence $0 \to A \to B \to C \to 0$ is almost split.

When A is an indecomposable noninjective module the statement follows by duality, using Proposition 4.8. □

Almost split sequences are determined, up to isomorphism, by the indecomposable nonprojective module C (the indecomposable noninjective module A), as we state next.

Proposition 4.10. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, $0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ *in* mod Λ *be almost split sequences. Then the following conditions are equivalent*:

(a) $A \simeq A'$
(b) *There exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_3} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

with $\sigma_1, \sigma_2, \sigma_3$ isomorphisms.
(c) $C \simeq C'$.

Proof. The proof is straightforward using the definitions and is left as an exercise. □

Exercise 4.11. Let C be a nonprojective indecomposable module. Prove that the sequence $0 \to A \to B \to C \to 0$ is an almost split sequence if and only if $0 \to (-, A) \to (-, B) \to (-, C) \to S_C \to 0$ is a minimal projective resolution of the simple functor S_C which is nonzero at C.

This exercise shows us how an appropriately chosen well-known notion in the category of modules, when applied to functors, can be translated into a new notion in the category of modules. In this case the notion of minimal projective resolution, known for modules, is carried over to the category of functors and we look at the minimal projective resolution of a simple functor S. Since projective functors are representable, we obtain an exact sequence $0 \to (-, A) \to (-, B) \to (-, C) \to S \to 0$. Then Yoneda's Lemma allows us to go back to the category of modules, and we obtain the exact sequence $0 \to A \to B \to C$. Then $A \neq 0$ if and only if C is not projective and in this case $0 \to A \to B \to C \to 0$ is an almost split sequence. So the fact that the exact sequence of functors is a projective resolution of a simple functor, when expressed in the category of modules leads to the notion of almost split sequence.

We will give another example of this situation. We start by recalling some definitions. A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph, where Q_0 denotes the set of vertices of Q and Q_1 the set of arrows between vertices. We will assume always that Q is a finite quiver, that is, both Q_0, Q_1 are finite sets. P. Gabriel associated a quiver Q_Λ to a basic algebra Λ, which turned out to be a fundamental tool in the study of the representation theory of finite-dimensional algebras. We recall that the algebra Λ is said to be *basic* if given a decomposition $\Lambda = \coprod_{i=1}^{n} P_i$ of Λ as a direct sum of indecomposable projective modules, then P_i is not isomorphic to P_j for $i \neq j$. It is known that given a finite-dimensional algebra Λ there exists a basic algebra Γ such that the categories $\operatorname{mod} \Lambda$ and $\operatorname{mod} \Gamma$ are equivalent (see, for example, [9, II.2]).

Let Λ be a basic finite-dimensional algebra and write $\Lambda = \coprod_{i=1}^{n} P_i$, with P_i indecomposable and projective for $i = 1, \dots, n$. Then the set of vertices of Q_Λ is $\{1, \dots, n\}$, where i denotes the vertex associated to the projective P_i, and the set of arrows of Q_Λ is defined as follows. Let $P_i = \Lambda e_i$. Then there is an arrow from the vertex i to the vertex j in $(Q_\Lambda)_1$ whenever $e_j(\operatorname{rad} \Lambda / \operatorname{rad}^2 \Lambda)e_i \neq 0$. This can be stated in terms of morphisms between projective modules: there is an arrow from the vertex i to the vertex j in $(Q_\Lambda)_1$ if and only if there is a morphism $f \colon P_j \to P_i$ in $\operatorname{mod} \Lambda$ such that $\operatorname{Im} f \subseteq \operatorname{rad} P_i$ and $\operatorname{Im} f \nsubseteq \operatorname{rad}^2 P_i$.

So morphisms between indecomposable projective modules such that their image is contained in the radical of the codomain and is not contained in the square of such radical are important in $\operatorname{mod} \Lambda$. We will carry over this notion to the category $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$. To do so, we define the radical of a functor in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ and study some of its properties.

Definition 4.12. Let F in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$. Then the radical of F is the intersection of all the maximal subfunctors of F.

We will denote the radical of F by $\operatorname{rad} F$. One can easily prove that, if $f \colon F \to G$ is a morphism in $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$, then $f(\operatorname{rad} F) \subseteq \operatorname{rad} G$ and, moreover, the equality holds when f is an epimorphism. Thus we have a functor rad: $(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab}) \to (\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$, sending F to $\operatorname{rad} F$ and $f \colon F \to G$ to $f|_{\operatorname{rad} F} \colon \operatorname{rad} F \to \operatorname{rad} G$. Then rad^n is defined by induction: $\operatorname{rad}^1 = \operatorname{rad}$, $\operatorname{rad}^{n+1} = \operatorname{rad} \operatorname{rad}^n$. The functor radical is additive, and in particular $\operatorname{rad}(\coprod_{i=1}^{n} F_i) \simeq \coprod_{i=1}^{n} \operatorname{rad}(F_i)$.

Let C in $\operatorname{mod} \Lambda$ be indecomposable. We proved above that the indecomposable projective functor $(-, C)$ has a unique maximal subfunctor, which coincides thus with $\operatorname{rad}(-, C)$ and is described by

$$\mathrm{rad}(-, C)(X) = \{f : X \to C \mid f \text{ is not a split epimorphism}\}\ .$$

for any X in mod Λ, by Corollary 2.6(b). By duality we obtain that the equality

$$\mathrm{rad}(-, Y)(A) = \{f : A \to Y \mid f \text{ is not a split monomorphism}\}$$

holds for A, Y in mod Λ with A indecomposable.

Next we describe the morphisms in $\mathrm{rad}^2(B, C)$ when B, C are indecomposable modules.

Proposition 4.13. *Let $f : B \to C$ in* mod Λ *with B, C indecomposable. Then the following conditions are equivalent*:

(a) $f \in \mathrm{rad}^2(-, C)(B)$.
(b) $\mathrm{Im}(-, f) \subseteq \mathrm{rad}^2(-, C)$.
(c) *There is L in* mod Λ *such that $f = ht$, where $h : L \to C$ is not a split epimorphism and $t : B \to L$ is not a split monomorphism.*

Proof. The equivalence of (a) and (b) is a direct consequence of Corollary 1.3. Assume that (b) holds, that is, assume that $\mathrm{Im}(-, f) \subseteq \mathrm{rad}^2(-, C)$. Let $(-, h) : (-, L) \to \mathrm{rad}(-, C)$ be a projective cover in (mod Λ^{op}, Ab). The induced morphism $\mathrm{rad}(-, L) \to \mathrm{rad}^2(-, C)$ is an epimorphism and $(-, B)$ is projective, so there is a morphism $t : B \to L$ in mod Λ such that the diagram

$$\begin{array}{ccc} \mathrm{rad}(-, L) & \xrightarrow{(-,h)|_{\mathrm{rad}(-,L)}} & \mathrm{rad}^2(-, C) \\ {\scriptstyle (-,t)}\uparrow & \nearrow {\scriptstyle (-,f)} & \\ (-, B) & & \end{array}$$

commutes. From $(-, f) = (-, h)(-, t)$ we get that $f = ht$. Since $\mathrm{Im}(-, h) \subseteq \mathrm{rad}(-, C)$ we know that h is not a split epimorphism, and since $\mathrm{Im}(-, t) \subseteq \mathrm{rad}(-, L)$ we obtain that t is not a split monomorphism, by the above observations. Thus (c) holds.

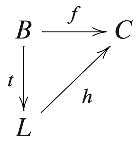
Assume now that (c) holds, and let L in mod Λ such that $f = ht$, where $h : L \to C$ is not a split epimorphism and $t : B \to L$ is not a split monomorphism. Then $t \in \mathrm{rad}(-, L)(B)$ because t is not a split monomorphism (see remark preceding this proposition), and thus $\mathrm{Im}(-, t) \subseteq \mathrm{rad}(-, L)$. On the other hand, since h is not a split epimorphism it follows that $\mathrm{Im}(-, h) \subseteq \mathrm{rad}(-, C)$, so that $(-, h) : (-, L) \to \mathrm{rad}(-, C)$. Therefore $(-, h)(\mathrm{rad}(-, L)) \subseteq \mathrm{rad}^2(-, C)$. Thus $\mathrm{Im}(-, f) = (-, h)((-, t)(-, B)) \subseteq (-, h)(\mathrm{rad}(-, L)) \subseteq \mathrm{rad}^2(-, C)$. This proves that (b) holds and ends the proof of the proposition. □

We are now in a position to describe when a morphism $f : B \to C$ in mod Λ with B, C indecomposable has the property that $\mathrm{Im}\big((-, f) : (-, B) \to (-, C)\big)$ is contained in $\mathrm{rad}(-, C)$ and is not contained in $\mathrm{rad}^2(-, C)$.

Proposition 4.14. *Let $f : B \to C$ be a morphism in* mod Λ, *with B, C indecomposable. Then* $\mathrm{Im}(-, f) \subseteq \mathrm{rad}(-, C)$ *and* $\mathrm{Im}(-, f) \nsubseteq \mathrm{rad}^2(-, C)$ *if and only if f satisfies the following conditions*:

(a) *f is neither a split epimorphism nor a split monomorphism.*

(b) *Given a commutative diagram*



then either t is a split monomorphism or h is a split epimorphism.

Proof. The statement follows from Proposition 4.13 and the remark preceding it. Notice that since B and C are indecomposable, then (a) holds if and only if f is not an isomorphism. □

Definition 4.15. A morphism $f: B \to C$ in $\operatorname{mod} \Lambda$ is called irreducible if it satisfies conditions (a) and (b) in the previous proposition.

Proposition 4.14 can be stated now in the following way.

Proposition 4.16. *Let B, C in* $\operatorname{mod} \Lambda$ *be indecomposable. Then a morphism $f: B \to C$ is irreducible if and only if* $\operatorname{Im}(-, f) \subseteq \operatorname{rad}(-, C)$ *and* $\operatorname{Im}(-, f) \nsubseteq \operatorname{rad}^2(-, C)$.

We have seen that the category $\operatorname{f.p.}(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ of finitely presented contravariant functors from $\operatorname{mod} \Lambda$ to the category of abelian groups has nice properties, and the study of this category can be helpful to study $\operatorname{mod} \Lambda$. An interesting question raised by M. Auslander is: when is this category equivalent to the category of modules over some finite-dimensional algebra Γ? The surprising observation he made is that this is the case if and only if the algebra Λ is of finite representation type, that is, there exist only a finite number of nonisomorphic indecomposable modules.

In fact, if $\operatorname{f.p.}(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ is equivalent to $\operatorname{mod} \Gamma$ for a finite-dimensional algebra Γ, then $\operatorname{f.p.}(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$ has only a finite number of indecomposable nonisomorphic projective objects, because this is the case in $\operatorname{mod} \Gamma$. Since $(-, C)$ is projective and indecomposable for any C indecomposable in $\operatorname{mod} \Lambda$, it follows that there are only a finite number of nonisomorphic indecomposable representable functors $(-, C_1), \ldots, (-, C_n)$. Using Yoneda's Lemma we can conclude that $C_1, \ldots, C_n$ constitute a complete set of nonisomorphic indecomposable Λ-modules, so that Λ is of finite representation type.

The converse follows from the following result, where the evaluation functor at C, $e_C: \operatorname{f.p.}(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab}) \to \operatorname{mod} \operatorname{End}_\Lambda(C)^{\mathrm{op}}$, is defined by $e_C(F) = F(C)$, $e_C(\theta) = \theta_C$, for an object F and a morphism θ in $\operatorname{f.p.}(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab})$.

Proposition 4.17. *Assume that Λ is an algebra of finite representation type, and let $C_1, \ldots, C_n$ be a complete set of nonisomorphic indecomposable Λ-modules, $C = C_1 \oplus \cdots \oplus C_n$. Then the evaluation functor at C, e_C*: $\operatorname{f.p.}(\operatorname{mod} \Lambda^{\mathrm{op}}, \mathrm{Ab}) \to \operatorname{mod} \operatorname{End}_\Lambda(C)^{\mathrm{op}}$ *is an equivalence of categories.*

Proof. We observe first that the functor $(C, -)$ defines an equivalence from $\operatorname{add} C$ to the category of finitely generated projective modules over $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$ [9, Chap. II, Proposition 2.1].

To prove that e_C is a dense functor, let X in $\operatorname{mod}\operatorname{End}_\Lambda(C)^{\mathrm{op}}$, and let $(C, C'') \xrightarrow{(C,f)} (C, C') \to X \to 0$ be a projective presentation of X, with C', C'' in $\operatorname{add} C$. Then $F = \operatorname{Coker}((-, f): (-, C'') \to (-, C'))$ is such that $e_C(F) = F(C) \simeq X$. Thus e_C is dense.

The fact that e_C is faithful follows from the fact that $\operatorname{add} C = \operatorname{mod}\Lambda$ and from the additivity of the functors considered, and is left as an exercise.

Finally we prove that e_C is full. Let F, G in f. p.$(\operatorname{mod}\Lambda^{\mathrm{op}}, \mathrm{Ab})$ and let $h: F(C) \to G(C)$. Since F and G are finitely presented, there are exact sequences $(-, C'') \xrightarrow{f} (-, C') \xrightarrow{g} F \to 0$ and $(-, C_1'') \xrightarrow{f_1} (-, C_1') \xrightarrow{g_1} G \to 0$, with C', C'', C_1', C_1'' in $\operatorname{add} C$. Since $(C, -)$ defines an equivalence from $\operatorname{add} C$ to the category of finitely generated projective modules over $\operatorname{End}_\Lambda(C)^{\mathrm{op}}$, we obtain a commutative diagram in $\operatorname{mod}(\operatorname{End}_\Lambda(C)^{\mathrm{op}})$

$$\begin{array}{ccccccc}
(C, C'') & \xrightarrow{f(C)} & (C, C') & \xrightarrow{g(C)} & F(C) & \longrightarrow & 0 \\
\downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \\
(C, C_1'') & \xrightarrow{f_1(C)} & (C, C_1') & \xrightarrow{g_1(C)} & G(C) & \longrightarrow & 0 .
\end{array}$$

Using again Yoneda's Lemma we conclude that there are maps $u': C' \to C_1'$ and $u'': C'' \to C_1''$ such that $h' = (C, u'), h'' = (C, u'')$. Let $\theta: F \to G$ be such that the diagram

$$\begin{array}{ccccccc}
(-, C'') & \xrightarrow{f} & (-, C') & \xrightarrow{g} & F & \longrightarrow & 0 \\
\downarrow{\scriptstyle (-,u'')} & & \downarrow{\scriptstyle (-,u')} & & \downarrow{\scriptstyle \theta} & & \\
(-, C_1'') & \xrightarrow{f_1} & (-, C_1') & \xrightarrow{g_1} & G & \longrightarrow & 0
\end{array}$$

commutes. Then $e_C(\theta) = \theta_C = h$. This proves that e_C is full and ends the proof of the proposition. □

We recall that the Auslander–Reiten quiver Γ_Δ of a finite-dimensional algebra Δ is the quiver (not necessarily finite) whose vertices are in one-to-one correspondence with the modules in $\operatorname{ind}\Delta$, and there is an arrow in Γ_Δ from the vertex associated to the indecomposable module M to the vertex associated to the indecomposable module N if and only if there is an irreducible morphism from M to N in $\operatorname{mod}\Delta$ (see [9, Chap. VII, 1]).

It follows then from the previous proposition and Proposition 4.16 that when Λ is of finite representation type then the Auslander–Reiten quiver of Λ coincides with the Gabriel quiver of $\operatorname{mod}\operatorname{End}_\Lambda(C)^{\mathrm{op}}$, where C is the sum of all modules in $\operatorname{ind}\Lambda$.

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Auslander–Reiten Theory for Finite-Dimensional Algebras

Piotr Malicki

Dedicated to María Inés Platzeck on the occasion of her 70th birthday

Introduction

This article is based on a course given at the CIMPA School "Homological Methods, Representation Theory and Cluster Algebras," held in March 2016 in Mar del Plata. The aim of the course, consisting of four lectures, was to provide a brief introduction to the notion of an almost split sequence and its use in the representation theory of finite-dimensional algebras. The first two sections are reduced, and the next three sections are extended in comparison with the above-mentioned course.

The representation theory of algebras is one of the most dynamically developing fields of modern mathematics. Its origins date back to the mid-nineteenth century and were inspired by the desire to describe the representations of compact groups which appear naturally in the context of the group theory and Lie algebras. A modern representation theory of algebras has a very wide range of effective tools among which certainly we can rank methods of algebraic geometry, homological algebra, combinatorics of graphs, and the concise language of category theory.

The Auslander–Reiten quiver Γ_A of an algebra A is one of such tools. It is an important combinatorial and homological invariant of the category $\operatorname{mod} A$ of finite-dimensional right A-modules. Frequently, we may recover the structure of $\operatorname{mod} A$ from the combinatorial structure of Γ_A, for example the shape of the connected components of Γ_A. Moreover, very often the behavior and properties of distinguished Auslander–Reiten components of an algebra A in the module category $\operatorname{mod} A$ leads to essential homological data which allow to determine the algebra A and its module category $\operatorname{mod} A$ completely.

A prominent role not only in the Auslander–Reiten theory but also in the representation theory of algebras in general is played by a special type of short exact sequences of modules, called almost split sequences. Basic properties of irreducible morphisms between modules and their connection with almost split sequences have been established in the classical papers by M. Auslander and I. Reiten [10–13] more than 40 years ago. Let us mention that the notion of almost split sequences has proven to be useful in such various fields of

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mathematics as modular group representation, the theory of orders, algebraic singularity theory, and model theory of modules.

We describe now briefly the content of the sections of this paper. In Sect. 1 we present basic definitions and facts concerning the radicals of the module categories of finite-dimensional algebras. In Sect. 2 we recall the Auslander–Reiten theorems playing a fundamental role in the representation theory of finite-dimensional algebras. Section 3 is devoted to the Auslander–Reiten quiver of an algebra. In Sect. 4 we present known results on the hereditary algebras, as well as their Auslander–Reiten quivers. In the final Sect. 5 we discuss the number of terms in the middle of almost split sequences in the module categories of finite-dimensional algebras.

For background on the topics covered in this article we refer to [2, 15, 59, 62, 63, 70]. We also refer the reader to recent books [16] and [61].

1 Basic Facts and Notation

Throughout the article K will denote a fixed field. The term algebra is used for a finite-dimensional K-algebra (associative, with an identity), if not specified otherwise. We also assume (without loss of generality) that the considered algebras are basic and indecomposable. For an algebra A, we denote by A^{op} the opposite algebra of A, and by mod A the category of finite-dimensional (over K) right A-modules. Then mod A^{op} is the category of finite-dimensional (over K) left A-modules, and the functor $D = \mathrm{Hom}_K(-, K)$, called the *standard duality* of mod A, induces the duality of categories

$$\mathrm{mod}\, A \underset{D}{\overset{D}{\rightleftarrows}} \mathrm{mod}\, A^{\mathrm{op}}$$

with $D \circ D \cong 1_{\mathrm{mod}\, A}$ and $D \circ D \cong 1_{\mathrm{mod}\, A^{\mathrm{op}}}$.

Let A be an algebra and $1_A = e_1 + \cdots + e_n$ be a decomposition of the identity 1_A of A into a sum of pairwise orthogonal primitive idempotents. Then

- $P_1 = e_1 A, \dots, P_n = e_n A$ is a complete set of pairwise nonisomorphic indecomposable projective modules in mod A,
- $I_1 = D(Ae_1), \dots, I_n = D(Ae_n)$ is a complete set of pairwise nonisomorphic indecomposable injective modules in mod A,
- $S_1 = \mathrm{top}\, P_1 = e_1 A / e_1 \,\mathrm{rad}\, A, \dots, S_n = \mathrm{top}\, P_n = e_n A / e_n \,\mathrm{rad}\, A$ is a complete set of pairwise nonisomorphic simple modules in mod A,

where rad A denotes the *Jacobson radical* of A, that is, the intersection of all maximal right (equivalently, left) ideals of A. Consequently, rad A is a two-sided ideal of A. Therefore, the epimorphism $P_i \to S_i$ is a projective cover and the monomorphism $S_i \to I_i$ is an injective envelope in mod A, for any $i \in \{1, \dots, n\}$. Moreover, the endomorphism algebras $\mathrm{End}_A(S_1), \mathrm{End}_A(S_2), \dots, \mathrm{End}_A(S_n)$ are division algebras. We denote by Q_A the *valued quiver* of A defined as follows. The vertices of Q_A are the numbers $1, 2, \dots, n$ corresponding to the chosen idempotents $e_1, e_2, \dots, e_n$ of A. Further, there is an arrow from i to j in

Q_A if $\mathrm{Ext}^1_A(S_i, S_j) \neq 0$, and we assign to this arrow the valuation

$$\left(\dim_{\mathrm{End}_A(S_j)} \mathrm{Ext}^1_A(S_i, S_j), \dim_{\mathrm{End}_A(S_i)} \mathrm{Ext}^1_A(S_i, S_j)\right) .$$

We denote by G_A the underlying graph of Q_A, and call it the *valued graph* of A.

We denote by $K_0(A)$ the *Grothendieck group* of A, and by $[M]$ the image of a module M from $\mathrm{mod}\, A$ in $K_0(A)$. Recall that $K_0(A)$ is a free abelian group of rank n with the canonical basis formed by the residue classes of simple modules $S_1, \ldots, S_n$ in $\mathrm{mod}\, A$. Moreover, for two modules M and N in $\mathrm{mod}\, A$, $[M] = [N]$ if and only if M and N have the same (simple) composition factors including the multiplicities.

For an algebra A and a module M in $\mathrm{mod}\, A$ we denote by $\mathrm{pd}_A\ M$, $\mathrm{id}_A\, M$, and $\mathrm{gl.\ dim}\, A$ the *projective dimension*, the *injective dimension* of M in $\mathrm{mod}\, A$, and the *global dimension* of A, respectively. By [68] we know that the global dimension of an algebra A is finite if and only if for every indecomposable module in $\mathrm{mod}\, A$ its projective or injective dimension is finite.

Let A be an algebra of finite global dimension. Then the *Euler form* χ_A of A is the homological integral quadratic form $\chi_A \colon K_0(A) \to \mathbb{Z}$ defined by

$$\chi_A([M]) = \sum_{r=0}^{\infty} (-1)^r \dim_K \mathrm{Ext}^r_A(M, M) .$$

The form χ_A is said to be *positive definite* if $\chi_A(\mathbf{x}) > 0$ for all nonzero vectors $\mathbf{x} \in K_0(A)$, and *positive semidefinite* if $\chi_A(\mathbf{x}) \geq 0$ for all vectors $\mathbf{x} \in K_0(A)$. Let us mention that in [56] (see also [59, 2.4]) C. M. Ringel introduced the Euler quadratic form χ_A on the Grothendieck group $K_0(A)$ of any algebra A of finite global dimension, which is analogous to the Euler characteristic of a topological space. The Euler forms have been successfully applied by D. Happel and C.M. Ringel in study of tilted algebras [40], by C.M. Ringel in study of tubular algebras [59], and many further investigations of algebras of finite global dimension.

Let A be an algebra, and X, Y be modules in $\mathrm{mod}\, A$. The set

$$\mathrm{rad}_A(X, Y) = \{f \in \mathrm{Hom}_A(X, Y) \mid 1_X - gf \textit{ is invertible in } \mathrm{End}_A(X) \text{ for any } g \in \mathrm{Hom}_A(Y, X)\}$$

is said to be the *Jacobson radical* (briefly, *radical*) of $\mathrm{mod}\, A$.

Lemma 1.1. *Let A be an algebra, and X, Y be modules in* $\mathrm{mod}\, A$. *Then we have*

$$\mathrm{rad}_A(X, Y) = \{f \in \mathrm{Hom}_A(X, Y) \mid 1_Y - fg \textit{ is invertible in } \mathrm{End}_A(Y) \text{ for any } g \in \mathrm{Hom}_A(Y, X)\} .$$

Remarks 1.2. Let A be an algebra.

(1) For any module X in $\mathrm{mod}\, A$, $\mathrm{rad}_A(X, X)$ is the radical $\mathrm{rad}\,\mathrm{End}_A(X)$ of endomorphism algebra $\mathrm{End}_A(X)$.

(2) A module X in $\operatorname{mod} A$ is indecomposable if and only if the endomorphism algebra $\operatorname{End}_A(X)$ of X is local.

Recall that an algebra A is called *local* if A has a unique maximal right (equivalently, left) ideal. We note that A is a local algebra if and only if the quotient algebra $A/\operatorname{rad} A$ is a division algebra.

Lemma 1.3. *Let A be an algebra and X, Y be indecomposable modules in* $\operatorname{mod} A$. *Then the following statements hold.*

(1) $\operatorname{rad}_A(X, Y)$ *is the vector space of all nonisomorphisms in* $\operatorname{Hom}_A(X, Y)$.
(2) *If $X \ncong Y$ then* $\operatorname{rad}_A(X, Y) = \operatorname{Hom}_A(X, Y)$.

Let A be an algebra. For each natural number $m \geq 1$, we define the mth power rad_A^m of rad_A such that, for modules X and Y in $\operatorname{mod} A$, $\operatorname{rad}_A^m(X, Y)$ is the subspace of $\operatorname{rad}_A(X, Y)$ consisting of all finite sums of homomorphisms of the form

$$X = X_0 \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \to \cdots \to X_{m-1} \xrightarrow{h_m} X_m = Y \ ,$$

where $h_j \in \operatorname{rad}_A(X_{j-1}, X_j)$ for $j \in \{1, \ldots, m\}$, for some modules $X = X_0, X_1, \ldots, X_m = Y$ in $\operatorname{mod} A$. Moreover, the intersection

$$\operatorname{rad}_A^\infty = \bigcap_{m=1}^{\infty} \operatorname{rad}_A^m$$

of all powers rad_A^m of rad_A is a two-sided ideal of $\operatorname{mod} A$, known as the *infinite radical* of $\operatorname{mod} A$. For arbitrary modules X and Y in $\operatorname{mod} A$, we have the chain of inclusions of K-vector spaces

$$\operatorname{Hom}_A(X, Y) \supseteq \operatorname{rad}_A(X, Y) \supseteq \operatorname{rad}_A^2(X, Y) \supseteq \cdots \supseteq \operatorname{rad}_A^m(X, Y) \supseteq \cdots \supseteq \operatorname{rad}_A^\infty(X, Y) \ .$$

Since $\operatorname{Hom}_A(X, Y)$ is a finite-dimensional K-vector space, as a K-vector subspace of $\operatorname{Hom}_K(X, Y)$, we have the following fact.

Lemma 1.4. *Let A be an algebra and X, Y be modules in* $\operatorname{mod} A$. *Then there exists a natural number $m \geq 1$ such that* $\operatorname{rad}_A^m(X, Y) = \operatorname{rad}_A^\infty(X, Y)$.

Recall that an algebra A is of *finite representation type* if the number of the isomorphism classes of indecomposable A-modules is finite, and A is of *infinite representation type* if A is not of finite representation type. Let us mention that the representation theory of algebras of finite representation type is presently rather well understood (see [19, 21–23, 25]).

Corollary 1.5. *If an algebra A is of finite representation type then there exists a positive natural number m such that* $\operatorname{rad}_A^m = 0$. *In particular, we have* $\operatorname{rad}_A^\infty = 0$.

For the proof of the above corollary see [70, Corollary III.2.2].

The following result due to M. Auslander [7] is very useful.

Theorem 1.6 (Auslander). *An algebra A is of finite representation type if and only if* $\mathrm{rad}_A^{\infty} = 0$.

On the other hand we have the following result proved in [28].

Theorem 1.7 (Coelho–Marcos–Merklen–Skowroński). *Let A be an algebra of infinite representation type. Then* $(\mathrm{rad}_A^{\infty})^2 \neq 0$.

Let us note the following statement, which is known in the representation theory of algebras, as the first Brauer–Thrall conjecture [7, 60].

Proposition 1.8. *Let A be an algebra. Then A is either of finite representation type or admits indecomposable modules with arbitrary large dimension.*

Let A be an algebra. A homomorphism $f: X \to Y$ in $\mathrm{mod}\, A$ is called *irreducible* if f is neither a split epimorphism nor a split monomorphism and, if there is a factorization $f = gh$ of f in $\mathrm{mod}\, A$, then either h is a split epimorphism or g is a split monomorphism. By a result of R. Bautista [17], a homomorphism $f: X \to Y$ in $\mathrm{mod}\, A$ with X and Y indecomposable is irreducible if and only if f belongs to $\mathrm{rad}_A(X,Y) \setminus \mathrm{rad}_A^2(X,Y)$. Let N be a module in $\mathrm{mod}\, A$. Then a *right minimal almost split homomorphism* for N is a homomorphism $g: M \to N$ in $\mathrm{mod}\, A$ such that: (1) g is not a split epimorphism, (2) if $g = gh$ in $\mathrm{mod}\, A$ then h is an automorphism of M, and (3) if $h: M' \to N$ in $\mathrm{mod}\, A$ is not a split epimorphism, then there exists a homomorphism $h': M' \to M$ in $\mathrm{mod}\, A$ such that $h = gh'$. Dually, a *left minimal almost split homomorphism* for N is a homomorphism $f: N \to M$ in $\mathrm{mod}\, A$ such that: (1) f is not a split monomorphism, (2) if $f = hf$ in $\mathrm{mod}\, A$ then h is an automorphism of M, and (3) if $h: N \to M'$ in $\mathrm{mod}\, A$ is not a split monomorphism, then there exists a homomorphism $h': M \to M'$ in $\mathrm{mod}\, A$ such that $h = h'f$.

Lemma 1.9. *Let A be an algebra and $f: X \to Y$ be an irreducible homomorphism in* $\mathrm{mod}\, A$. *Then f is either a proper monomorphism or a proper epimorphism.*

A short exact sequence

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in $\mathrm{mod}\, A$ is called an *almost split sequence* (*Auslander–Reiten sequence*) provided f is a left minimal almost split homomorphism in $\mathrm{mod}\, A$ and g is a right minimal almost split homomorphism in $\mathrm{mod}\, A$.

Remarks 1.10. Let

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{1}$$

be an almost split sequence in $\mathrm{mod}\, A$.

(1) A sequence (1) is never split.
(2) The modules L and N are indecomposable.
(3) The module L is not injective and the module N is not projective.

2 The Auslander–Reiten Theorems

For an algebra A, we denote by proj A the full subcategory of mod A consisting of all projective modules and by inj A the full subcategory of mod A consisting of all injective modules. Following Auslander–Bridger [9], consider the contravariant functor $(-)^{\mathrm{t}} = \mathrm{Hom}_A(-, A)\colon \mathrm{mod}\, A \to \mathrm{mod}\, A^{\mathrm{op}}$. Then the functor $(-)^{\mathrm{t}}$ induces a duality of categories, also denoted by $(-)^{\mathrm{t}}$

$$\mathrm{proj}\, A \underset{(-)^{\mathrm{t}}}{\overset{(-)^{\mathrm{t}}}{\rightleftarrows}} \mathrm{proj}\, A^{\mathrm{op}} .$$

Remarks 2.1. Let e be an idempotent in A.

(1) $(eA)^{\mathrm{t}} = \mathrm{Hom}_A(eA, A) = Ae = eA^{\mathrm{op}}$.
(2) Every module in proj A is a direct sum of the modules of the form eA, where e is primitive.
(3) Every module in proj A^{op} is a direct sum of the modules of the form $Ae = eA^{\mathrm{op}}$, where e is primitive.

Let M be a module in mod A and

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0 \tag{2}$$

be a minimal projective presentation of M in mod A (that is, an exact sequence such that $p_0\colon P_0 \to M$ and $p_1\colon P_1 \to \mathrm{Ker}\, p_0$ are projective covers). Applying to the sequence (2) (left exact, contravariant) functor $(-)^{\mathrm{t}}$, we get an exact sequence in mod A^{op} of the form

$$0 \to M^{\mathrm{t}} \xrightarrow{p_0^{\mathrm{t}}} P_0^{\mathrm{t}} \xrightarrow{p_1^{\mathrm{t}}} P_1^{\mathrm{t}} \to \mathrm{Coker}\, p_1^{\mathrm{t}} \to 0 .$$

Then Coker p_1^{t} we denoted by Tr M and called *transpose* of M.

Remarks 2.2. Let us note the following.

(1) Since projective covers (and hence minimal projective presentations) are uniquely determined up to isomorphism, we get that Tr M is uniquely determined by M, up to isomorphism.
(2) The transpose Tr does not define a duality of categories, because it annihilates the projectives. In order to make this correspondence a duality, we define below two quotient categories, denoted by $\underline{\mathrm{mod}}\, A$ and $\overline{\mathrm{mod}}\, A$.

For two modules M and N in mod A we define two ideals $\mathcal{P}_A$ and $\mathcal{I}_A$ of mod A.

$$\mathcal{P}_A(M, N) = \{ f \in \mathrm{Hom}_A(M, N) \mid f = f_2 f_1, f_1 \in \mathrm{Hom}_A(M, P), f_2 \in \mathrm{Hom}_A(P, N), P \in \mathrm{proj}\, A\} ,$$
$$\mathcal{I}_A(M, N) = \{g \in \mathrm{Hom}_A(M, N) \mid g = g_2 g_1, g_1 \in \mathrm{Hom}_A(M, I), g_2 \in \mathrm{Hom}_A(I, N), I \in \mathrm{inj}\, A\} .$$

The quotient category $\underline{\mathrm{mod}}\, A = \mathrm{mod}\, A/\mathcal{P}_A$ is called the *projectively stable category*, and the quotient category $\overline{\mathrm{mod}}\, A = \mathrm{mod}\, A/\mathcal{I}_A$ is called the *injectively stable category*.

- Objects of $\underline{\text{mod}}\, A$ (respectively, of $\overline{\text{mod}}\, A$) are the same as those of mod A.
- K-vector space of morphisms from M to N in $\underline{\text{mod}}\, A$ (respectively, in $\overline{\text{mod}}\, A$) is the quotient vector space $\underline{\text{Hom}}_A(M,N) = \text{Hom}_A(M,N)/\mathcal{P}_A(M,N)$ (respectively, $\overline{\text{Hom}}_A(M,N) = \text{Hom}_A(M,N)/\mathcal{I}_A(M,N)$).
- The composition of morphisms in $\underline{\text{mod}}\, A$ (respectively, in $\overline{\text{mod}}\, A$) is induced from the composition of homomorphisms in mod A.

Remark 2.3. Let us mention that there are nonzero objects in mod A which become isomorphic to the zero object in $\underline{\text{mod}}\, A$ (respectively, in $\overline{\text{mod}}\, A$).

Proposition 2.4. *Let A be an algebra. Then the transpose* Tr *induces a duality*

$$\underline{\text{mod}}\, A \underset{\text{Tr}}{\overset{\text{Tr}}{\rightleftarrows}} \underline{\text{mod}}\, A^{\text{op}}.$$

Recall that we have the standard duality $D = \text{Hom}_K(-,K)$ between mod A and mod A^{op}. So, D induces a duality between the stable categories $\underline{\text{mod}}\, A$ and $\overline{\text{mod}}\, A^{\text{op}}$. In particular, we have the equivalences of the categories

$$\tau_A = D\,\text{Tr}\colon \underline{\text{mod}}\, A \to \overline{\text{mod}}\, A \quad \text{and} \quad \tau_A^{-1} = \text{Tr}\, D\colon \overline{\text{mod}}\, A \to \underline{\text{mod}}\, A$$

called the *Auslander–Reiten functors*. In fact, for a module M in mod A, we have well-defined modules in mod A

$$\tau_A M = D\,\text{Tr}(M) \quad \text{and} \quad \tau_A^{-1} M = \text{Tr}\, D(M)$$

called the *Auslander–Reiten translations* of M.

The following corollary shows an important property of the Auslander–Reiten translations.

Corollary 2.5. *Let A be an algebra. Then the Auslander–Reiten translation τ_A induces a bijection from the set of isomorphism classes of indecomposable nonprojective modules in* mod A *to the set of isomorphism classes of indecomposable noninjective modules in* mod A, *and τ_A^{-1} is the inverse bijection of τ_A.*

The next proposition gives an easy and useful criterion for a module to have projective, or injective, dimension at most one.

Proposition 2.6. *Let A be an algebra and M be a module in* mod A. *Then the following statements hold.*

(1) $\text{pd}_A M \le 1$ *if and only if* $\text{Hom}_A(D(A), \tau_A M) = 0$.
(2) $\text{id}_A M \le 1$ *if and only if* $\text{Hom}_A(\tau_A^{-1} M, A) = 0$.

We have the following formulas from [10] which allow us to describe the extension spaces between finite-dimensional modules by the corresponding stable homomorphism spaces.

Theorem 2.7 (Auslander–Reiten). *Let A be an algebra and M, N modules in* $\operatorname{mod} A$. *Then there exist isomorphisms of K-vector spaces*

$$D\,\underline{\operatorname{Hom}}_A(\tau_A^{-1}N, M) \cong \operatorname{Ext}_A^1(M, N) \cong D\,\overline{\operatorname{Hom}}_A(N, \tau_A M)\ .$$

Corollary 2.8. *Let A be an algebra and M, N modules in* $\operatorname{mod} A$. *The following statements hold.*

(1) *If* $\operatorname{pd}_A M \leq 1$, *then there exists a K-linear isomorphism*

$$\operatorname{Ext}_A^1(M, N) \cong D\operatorname{Hom}_A(N, \tau_A M)\ .$$

(2) *If* $\operatorname{id}_A M \leq 1$ *then there exists a K-linear isomorphism*

$$\operatorname{Ext}_A^1(M, N) \cong D\operatorname{Hom}_A(\tau_A^{-1}N, M)\ .$$

The following theorem proved by M. Auslander and I. Reiten [10] is fundamental for the representation theory of finite-dimensional algebras.

Theorem 2.9 (Auslander–Reiten). *Let A be an algebra. The following statements hold.*

(1) *For any indecomposable nonprojective module M in* $\operatorname{mod} A$, *there exists a unique (up to isomorphism) almost split sequence* $0 \to L \to E \to M \to 0$ *in* $\operatorname{mod} A$. *Moreover, then L is indecomposable and isomorphic to $\tau_A M$.*
(2) *For any indecomposable noninjective module L in* $\operatorname{mod} A$, *there exists a unique (up to isomorphism) almost split sequence* $0 \to L \to E \to M \to 0$ *in* $\operatorname{mod} A$. *Moreover, then M is indecomposable and isomorphic to $\tau_A^{-1} L$.*
(3) *If P is an indecomposable projective module in* $\operatorname{mod} A$, *then the inclusion homomorphism* $\operatorname{rad} P \to P$ *is a right minimal almost split homomorphism for P in* $\operatorname{mod} A$.
(4) *If I is an indecomposable injective module in* $\operatorname{mod} A$, *then the canonical epimorphism* $I \to I/\operatorname{soc} I$ *is a left minimal almost split homomorphism for I in* $\operatorname{mod} A$.

We have also the following description of irreducible homomorphisms in the module categories with indecomposable domain and codomain, established by M. Auslander and I. Reiten [10].

Theorem 2.10 (Auslander–Reiten). *Let A be an algebra, X an indecomposable module in* $\operatorname{mod} A$, *$g\colon M \to X$ a right minimal almost split homomorphism for X in* $\operatorname{mod} A$, *and $f\colon X \to N$ a left minimal almost split homomorphism for X in* $\operatorname{mod} A$. *The following equivalences hold.*

(1) *A homomorphism $u\colon U \to X$ in* $\operatorname{mod} A$ *is irreducible if and only if $u = gw$ for a split monomorphism $w\colon U \to M$ in* $\operatorname{mod} A$.
(2) *A homomorphism $v\colon X \to V$ in* $\operatorname{mod} A$ *is irreducible if and only if $v = pf$ for a split epimorphism $p\colon N \to V$ in* $\operatorname{mod} A$.

Remark 2.11. Let Q be a finite quiver and K a field. We denote by $\operatorname{rep}_K(Q)$ the category of all finite-dimensional K-linear representations of Q and the morphisms of representations. Let I be an admissible ideal of the path algebra KQ of Q over K. For a bound quiver (Q, I) we denote by $\text{rep}_K(Q, I)$ the full subcategory of $\text{rep}_K(Q)$ consisting of the representations of Q bound by I. It is known that $\text{rep}_K(Q)$ (and so $\text{rep}_K(Q, I)$) is an abelian K-category. Moreover, there exists a K-linear equivalence of categories mod A and $\text{rep}_K(Q, I)$, where $A = KQ/I$.

Example 2.12. Let K be a field and $A = KQ$ the path algebra of Q over K, where Q is the quiver of the form

$$2 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 3 .$$

We identify mod A and $\text{rep}_K(Q)$. Then a complete set of pairwise nonisomorphic indecomposable projective modules in mod A (representations in $\text{rep}_K(Q)$) is of the form

$$P_1 : 0 \to K \leftarrow 0 , \quad P_2 : K \xrightarrow{1} K \leftarrow 0 , \quad P_3 : 0 \to K \xleftarrow{1} K ,$$

and a complete set of pairwise nonisomorphic indecomposable injective modules in mod A (representations in $\text{rep}_K(Q)$) is of the form

$$I_1 : K \xrightarrow{1} K \xleftarrow{1} K , \quad I_2 : K \to 0 \leftarrow 0 , \quad I_3 : 0 \to 0 \leftarrow K$$

Moreover, we observe that $P_1 = S_1$, $I_2 = S_2$, $I_3 = S_3$. Note that rad $P_1 = 0$, rad $P_2 = P_1$, rad $P_3 = P_1$, $I_1/S_1 = S_2 \oplus S_3$, $I_2/S_2 = 0$, $I_3/S_3 = 0$. Since S_1 is simple projective noninjective A-module, we get that the target of each irreducible homomorphism starting with S_1 is projective (see [2, Corollary IV.3.9(a)]). In our case, we have two such homomorphisms, namely $S_1 \to P_2$ and $S_1 \to P_3$ (indeed, $S_1 = \text{rad}\, P_2 = \text{rad}\, P_3$). Because S_1 is not injective, we have in mod A the following almost split sequence:

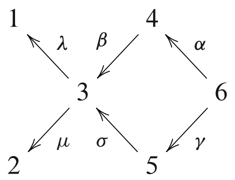
$$0 \to S_1 \to P_2 \oplus P_3 \to \tau_A^{-1} S_1 \to 0 , \quad \text{where } \tau_A^{-1} S_1 = P_2 \oplus P_3/S_1 \cong I_1 .$$

Further, since S_2 and S_3 are simple injective nonprojective A-modules, we get that the source of each irreducible homomorphism ending at S_2 or S_3 is injective (see [2, Corollary IV.3.9(b)]). In our case, we have two such homomorphisms, namely $I_1 \to S_2$ and $I_1 \to S_3$ (indeed, $I_1/\text{soc}\, I_1 \cong I_1/S_1 = S_2 \oplus S_3$). Because S_2 and S_3 are not projective, we have in mod A the following almost split sequences starting with $\tau_A S_2$ and $\tau_A S_3$, respectively:

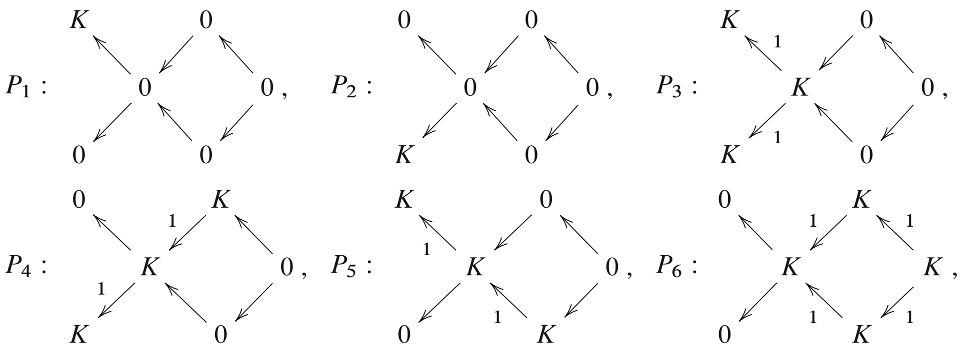
$$0 \to \tau_A S_2 \to I_1 \to S_2 \to 0 , \quad \text{where } \tau_A S_2 = I_1/S_2 \cong P_3 ,$$
$$0 \to \tau_A S_3 \to I_1 \to S_3 \to 0 , \quad \text{where } \tau_A S_3 = I_1/S_3 \cong P_2 .$$

So, in our example we have the following six irreducible homomorphisms: $S_1 \to P_2$, $S_1 \to P_3$, $P_2 \to I_1$, $P_3 \to I_1$, $I_1 \to S_2$, $I_1 \to S_3$. For the Auslander–Reiten quiver Γ_A of A see Example 4.7.

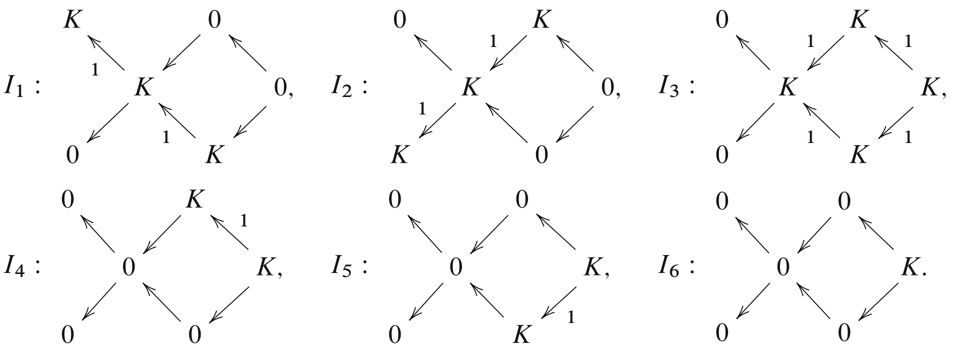
Example 2.13. Let K be a field and $A = KQ/I$ the bound quiver algebra over K, where Q is the quiver of the form



and I the ideal of the path algebra KQ of Q over K generated by the paths $\beta\lambda, \sigma\mu$, $\alpha\beta - \gamma\sigma$. We identify mod A and $\operatorname{rep}_K(Q, I)$. Then a complete set of pairwise nonisomorphic indecomposable projective modules in mod A (representations in $\operatorname{rep}_K(Q, I)$) is of the form



and a complete set of pairwise nonisomorphic indecomposable injective modules in mod A (representations in $\operatorname{rep}_K(Q, I)$) is of the form



We have $P_1 = S_1$, $P_2 = S_2$, $P_4 = I_2$, $P_5 = I_1$, $P_6 = I_3$, and $I_6 = S_6$. Note that rad $P_3 = S_1 \oplus S_2$. Since S_1 and S_2 are simple projective noninjective A-modules, we have in mod A almost split sequences of the forms (see [2, Corollary IV.3.9(a)])

$$0 \to S_1 \to P_3 \to \tau_A^{-1} S_1 \to 0\,, \quad \text{where } \tau_A^{-1} S_1 = P_3/S_1\,,$$
$$0 \to S_2 \to P_3 \to \tau_A^{-1} S_2 \to 0\,, \quad \text{where } \tau_A^{-1} S_2 = P_3/S_2\,.$$

Further, since S_6 is simple injective nonprojective A-module, we have in mod A an almost split sequence of the form (see [2, Corollary IV.3.9(b)])

$$0 \to \tau_A S_6 \to I_4 \oplus I_5 \to S_6 \to 0 , \quad \text{where } \tau_A S_6 = P_6/S_3 .$$

Moreover, $P_6 = I_3$ is projective–injective A-module. Therefore, applying [2, Proposition IV.3.11], we conclude that there is in mod A an almost split sequence of the form

$$0 \to \operatorname{rad} P_6 \to S_4 \oplus S_5 \oplus P_6 \to P_6/S_3 \to 0 , \quad \text{where } S_4 \oplus S_5 \cong \operatorname{rad} P_6/ \operatorname{soc} P_6 .$$

3 The Auslander–Reiten Quiver of an Algebra

In this section we introduce an important combinatorial and homological invariant of a finite-dimensional algebra, called the Auslander–Reiten quiver.

Let A be an algebra. Recall that, by Remarks 1.2(2), we know that for an indecomposable module Z in mod A the endomorphism algebra $\operatorname{End}_A(Z)$ of Z is a local K-algebra. Hence

$$F_Z = \operatorname{End}_A(Z)/ \operatorname{rad} \operatorname{End}_A(Z) = \operatorname{End}_A(Z)/ \operatorname{rad}_A(Z, Z)$$

is a finite-dimensional division K-algebra. For indecomposable modules X and Y in mod A we consider the finite-dimensional K-vector space

$$\operatorname{irr}_A(X, Y) = \operatorname{rad}_A(X, Y)/ \operatorname{rad}_A^2(X, Y)$$

called the *space of irreducible homomorphisms* from X to Y. Note that $\operatorname{irr}_A(X, Y)$ is an F_Y-F_X-bimodule by

$$\big(h + \operatorname{rad}_A(Y, Y)\big)\big(f + \operatorname{rad}_A^2(X, Y)\big) = hf + \operatorname{rad}_A^2(X, Y) ,$$
$$\big(f + \operatorname{rad}_A^2(X, Y)\big)\big(g + \operatorname{rad}_A(X, X)\big) = fg + \operatorname{rad}_A^2(X, Y) ,$$

for $f \in \operatorname{rad}_A(X, Y)$, $g \in \operatorname{End}_A(X)$, $h \in \operatorname{End}_A(Y)$. Denote by

$$d_{XY} = \dim_{F_Y} \operatorname{irr}_A(X, Y) , \quad d'_{XY} = \dim_{F_X} \operatorname{irr}_A(X, Y) .$$

The *Auslander–Reiten quiver* Γ_A of A is the valued translation quiver defined as follows:

(1) The vertices of Γ_A are the isoclasses $\{X\}$ of indecomposable modules X in mod A,
(2) For two vertices $\{X\}$ and $\{Y\}$ in Γ_A, there exists an arrow $\{X\} \to \{Y\}$ if and only if there is an irreducible homomorphism $X \to Y$ in mod A. Moreover, we associate to an arrow $\{X\} \to \{Y\}$ of Γ_A the valued arrow

$$\{X\} \xrightarrow{(d_{XY}, d'_{XY})} \{Y\} ,$$

(3) We have the translation τ_A which assigns to each vertex $\{X\}$ of Γ_A, with X nonprojective module, the vertex

$$\tau_A\{X\} = \{\tau_A X\} = \{D\,\mathrm{Tr}\,X\}\,,$$

(4) We have the translation τ_A^{-1} which assigns to each vertex $\{X\}$ of Γ_A, with X noninjective module, the vertex

$$\tau_A^{-1}\{X\} = \{\tau_A^{-1} X\} = \{\mathrm{Tr}\,DX\}\,.$$

We will usually identify a vertex $\{X\}$ of Γ_A with the indecomposable module X corresponding to it, so we will write

$$X \xrightarrow{(d_{XY},d'_{XY})} Y \quad \text{instead of } \{X\} \xrightarrow{(d_{XY},d'_{XY})} \{Y\}\,.$$

Moreover, instead of an arrow $X \xrightarrow{(1,1)} Y$ of Γ_A we will write $X \to Y$.

Remark 3.1. Let A be an algebra. Then the Auslander–Reiten quiver Γ_A of A has no loops. Indeed, by Lemma 1.9, we know that every irreducible homomorphism $f\colon X \to Y$ is either a proper monomorphism or a proper epimorphism. Moreover, if $X = Y$ then f should be an isomorphism (because X is finite-dimensional as a K-vector space). Therefore, the source and the target of this homomorphism must be distinct.

Remark 3.2. Let A be an algebra. Then the Auslander–Reiten quiver Γ_A of A is locally finite, that is, each vertex of Γ_A is the source (respectively, target) of at most finitely many arrows.

Let X and Y be indecomposable modules in mod A such that there exists an irreducible homomorphism from X to Y in mod A. Then d_{XY} is the multiplicity of Y in the codomain M of a left minimal almost split homomorphism $X \to M$ in mod A with the domain X, that is, $M \cong Y^{d_{XY}} \oplus M'$ with M' without a direct summand isomorphic to Y.

Remarks 3.3. Let A be an algebra.

(1) We know that there exists a left minimal almost split homomorphism $X \to M$ in mod A, namely:
- if X is a nonsimple injective module, then by Theorem 2.9(4) we have $X \to X/\operatorname{soc} X = M$,
- if X is not an injective module, then by [2, Theorem IV.1.13(d)] and Theorem 2.9(2), we have an almost split sequence in mod A of the form

$$0 \to X \to M \to \tau_A^{-1} X \to 0\,.$$

(2) For any irreducible homomorphism $X \to Y^m$ in mod A and positive integer m, we have $m \le d_{XY}$ (see [2, Theorem IV.1.10(a)]).

Similarly, d'_{XY} is the multiplicity of X in the domain N of a right minimal almost split homomorphism $N \to Y$ in mod A with the codomain Y, that is, $N \cong X^{d'_{XY}} \oplus N'$ with N' without a direct summand isomorphic to X.

Remarks 3.4. Let A be an algebra.

(1) We know that there exists a right minimal almost split homomorphism $N \to Y$ in $\mathrm{mod}\, A$, namely:

- if Y is a nonsimple projective module, then by Theorem 2.9(3) we have $N = \mathrm{rad}\, Y \to Y$,
- if Y is not a projective module, then by [2, Theorem IV.1.13(e)] and Theorem 2.9(1), we have an almost split sequence in $\mathrm{mod}\, A$ of the form

$$0 \to \tau_A Y \to N \to Y \to 0\,.$$

(2) For any irreducible homomorphism $X^n \to Y$ in $\mathrm{mod}\, A$ and positive integer n, we have $n \leq d'_{XY}$ (see [2, Theorem IV.1.10(b)]).

Proposition 3.5. *Let A be an algebra and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ be an arrow of Γ_A. Then the following statements hold.*

(1) *If Y is not a projective vertex of Γ_A, then Γ_A admits an arrow $\tau_A Y \xrightarrow{(d_{\tau_A Y,X}, d'_{\tau_A Y,X})} X$ and $d_{\tau_A Y,X} = d'_{XY}$.*

(2) *If X is not an injective vertex of Γ_A, then Γ_A admits an arrow $Y \xrightarrow{(d_{Y,\tau_A^{-1}X}, d'_{Y,\tau_A^{-1}X})} \tau_A^{-1} X$ and $d'_{Y,\tau_A^{-1}X} = d_{XY}$.*

Proof. (1) Assume that Y is not a projective vertex of Γ_A. Then, by Theorem 2.9(1), there exists in $\mathrm{mod}\, A$ an almost split sequence

$$0 \to \tau_A Y \xrightarrow{f} N \xrightarrow{g} Y \to 0\,,$$

where f is a left minimal almost split homomorphism and g is a right minimal almost split homomorphism in $\mathrm{mod}\, A$. It follows from the above remarks that $N \cong X^{d'_{XY}} \oplus N'$ with N' without a direct summand isomorphic to X. Then, by [2, Theorem IV.1.10(a)], there are in $\mathrm{mod}\, A$ irreducible homomorphisms $\tau_A Y \to X$ and $\tau_A Y \to X^{d'_{XY}}$. Hence Γ_A admits a valued arrow $\tau_A Y \xrightarrow{(d_{\tau_A Y,X}, d'_{\tau_A Y,X})} X$ with $d_{\tau_A Y,X} = d'_{XY}$.

The proof of (2) is similar. □

The following immediate consequence of Proposition 3.5 gives more information on the neighbors of simple projective and simple injective modules in Γ_A.

Corollary 3.6. *Let A be an algebra and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ be an arrow of Γ_A. Then the following statements hold.*

(1) *If X is simple projective, then Y is projective.*
(2) *If Y is simple injective, then X is injective.*

Proposition 3.7. *Let A be an algebra, X, Y indecomposable modules in* $\operatorname{mod} A$, *and assume that there exists an irreducible homomorphism from X to Y. Then the following statements hold.*

(1) *If Y is not a projective module, then $d'_{\tau_A Y, X} = d_{XY}$.*
(2) *If X is not an injective module, then $d_{Y, \tau_A^{-1} X} = d'_{XY}$.*

Remarks 3.8. Let us note the following.

(1) If A is an algebra of finite representation type and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow of Γ_A then $d_{XY} = 1$ or $d'_{XY} = 1$.
(2) If A is an algebra over an algebraically closed field K and $X \xrightarrow{(d_{XY}, d'_{XY})} Y$ is an arrow of Γ_A then $d_{XY} = d'_{XY}$ (because $F_X \cong K \cong F_Y$). In particular, $d_{XY} = d'_{XY} = 1$ if A is of finite representation type.

Let A be an algebra and $\mathcal{C}$ be a connected component of the quiver Γ_A. Then $\mathcal{C}$ is said to be *postprojective* if $\mathcal{C}$ is acyclic and each module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective module. Dually, $\mathcal{C}$ is said to be *preinjective* if $\mathcal{C}$ is acyclic and each module in $\mathcal{C}$ belongs to the τ_A-orbit of an injective module. Further, $\mathcal{C}$ is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module. A component $\mathcal{C}$ is called *semiregular* if $\mathcal{C}$ does not contain both a projective module and an injective module. For a component $\mathcal{C}$ of Γ_A, we denote by $\operatorname{ann}_A \mathcal{C}$ the *annihilator* of $\mathcal{C}$ in $\operatorname{mod} A$, that is, the intersection of the annihilators $\operatorname{ann}_A X = \{a \in A \mid Xa = 0\}$ of all modules X in $\mathcal{C}$. Observe that $\operatorname{ann}_A \mathcal{C}$ is a two-sided ideal of A and $\mathcal{C}$ is a component of the Auslander–Reiten quiver $\Gamma_{A/\operatorname{ann}_A \mathcal{C}}$ of the quotient algebra $A/\operatorname{ann}_A \mathcal{C}$. A component $\mathcal{C}$ of Γ_A with $\operatorname{ann}_A \mathcal{C} = 0$ is said to be *faithful*. Therefore, every component $\mathcal{C}$ of Γ_A is a faithful component of $\Gamma_{A/\operatorname{ann}_A \mathcal{C}}$. A component $\mathcal{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$. Two components $\mathcal{C}$ and $\mathcal{D}$ of an Auslander–Reiten quiver Γ_A are said to be *orthogonal* if $\operatorname{Hom}_A(X, Y) = 0$ and $\operatorname{Hom}_A(Y, X) = 0$ for all modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. We also note that if $\mathcal{C}$ and $\mathcal{D}$ are distinct components of Γ_A then $\operatorname{Hom}_A(X, Y) = \operatorname{rad}_A^\infty(X, Y)$ for all modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Hence, the Auslander–Reiten quiver Γ_A of an algebra A describes "only" the quotient category $\operatorname{mod} A/\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$. Nevertheless, the shapes of connected components of Γ_A are the first basic invariants of the module category $\operatorname{mod} A$.

Remark 3.9. Let A be an algebra. If A is of finite representation type then by Corollary 1.5 or Theorem 1.6, we know that $\operatorname{rad}_A^\infty = 0$. Therefore, we may recover the morphisms in $\operatorname{mod} A$ from the quiver Γ_A of A.

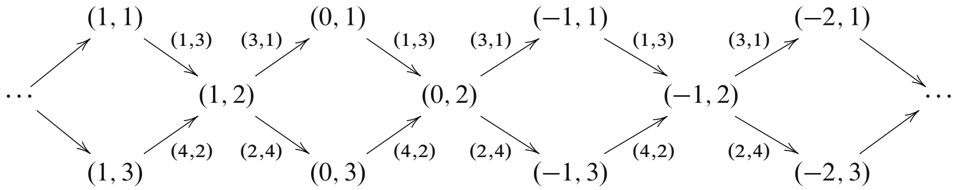
Recall also that, if $\Delta = (\Delta_0, \Delta_1, d, d')$ is a locally finite valued acyclic quiver (without multiple arrows), then $\mathbb{Z}\Delta = \big((\mathbb{Z}\Delta)_0, (\mathbb{Z}\Delta)_1, d, d', \tau\big)$ is a valued translation quiver defined as follows: $(\mathbb{Z}\Delta)_0 = \mathbb{Z} \times \Delta_0 = \{(i, x) \mid i \in \mathbb{Z}, x \in \Delta_0\}$, $(\mathbb{Z}\Delta)_1$ consists of the valued arrows

$$(i, \alpha) : (i, x) \xrightarrow{(d_{xy}, d'_{xy})} (i, y) , \quad (i, \alpha)' : (i + 1, y) \xrightarrow{(d'_{xy}, d_{xy})} (i, x) ,$$

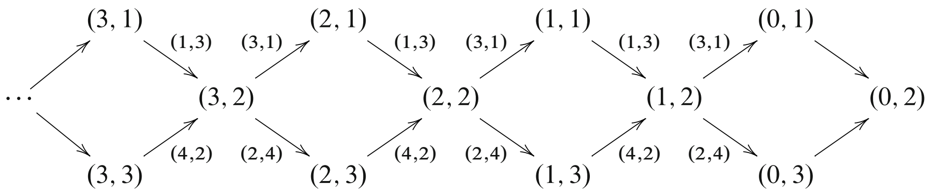
$i \in \mathbb{Z}$, for all arrows $\alpha : x \xrightarrow{(d_{xy}, d'_{xy})} y$ in Δ_1. The translation $\tau : \mathbb{Z}\Delta_0 \to \mathbb{Z}\Delta_0$ is defined by $\tau(i, x) = (i + 1, x)$ for all $i \in \mathbb{Z}$, $x \in \Delta_0$.

If I is a subset of $\mathbb{Z}$, then by $I\Delta$ we denote the full translation subquiver of $\mathbb{Z}\Delta$ with the set of vertices $I \times \Delta_0$. In particular, we have the valued translation subquivers $\mathbb{N}\Delta$ and $(-\mathbb{N})\Delta$ of $\mathbb{Z}\Delta$.

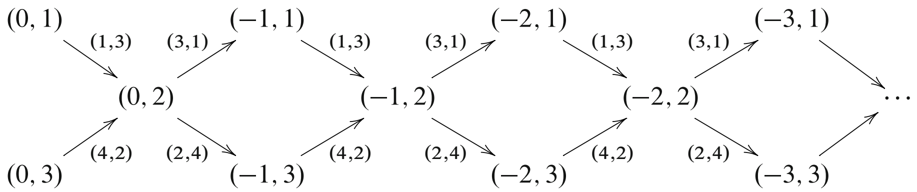
Example 3.10. Let Δ be the valued quiver of the form $1 \xrightarrow{(1,3)} 2 \xleftarrow{(4,2)} 3$. Then $\mathbb{Z}\Delta$ is of the form



$\mathbb{N}\Delta$ is of the form



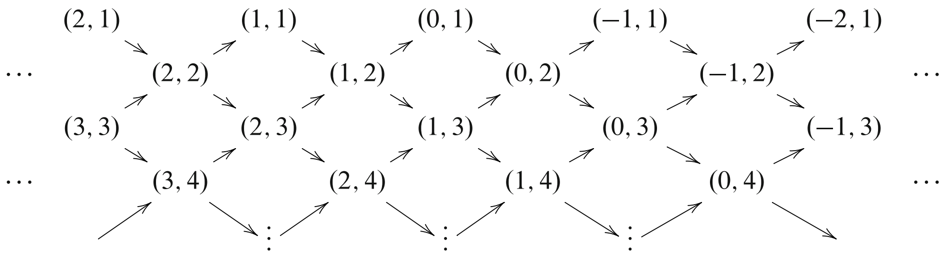
and $(-\mathbb{N})\Delta$ is of the form



If Δ is the infinite quiver

$$\mathbb{A}_\infty : 1 \to 2 \to 3 \to \cdots \to m \to m+1 \to \cdots$$

with trivial valuations $(1,1)$, then $\mathbb{Z}\mathbb{A}_\infty$ is of the form



and its translation τ is defined by $\tau(i, j) = (i+1, j)$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Then, for an integer $r \geq 1$, the orbit quiver $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is called a *stable tube of rank* r. A stable tube of rank one is said to be *homogeneous*.

Remark 3.11. If Δ is a tree then $\mathbb{Z}\Delta$ does not depend on the orientation of Δ.

The following theorem proved independently by S. Liu [43, 45] and Y. Zhang [72] describes the shape of regular components of an Auslander–Reiten quiver.

Theorem 3.12 (Liu, Zhang). *Let A be an algebra and $\mathcal{C}$ be a regular component of Γ_A. The following equivalences hold.*

(1) *$\mathcal{C}$ contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube.*
(2) *$\mathcal{C}$ is acyclic if and only if $\mathcal{C}$ is of the form $\mathbb{Z}\Delta$, for a connected, locally finite, acyclic, valued quiver Δ.*

It follows from the above theorem that a regular component $\mathcal{C}$ of Γ_A is a stable tube if and only if $\mathcal{C}$ contains a τ_A-periodic module (as observed already in [39]), or equivalently, all modules in $\mathcal{C}$ are τ_A-periodic and have the same period with respect to the action of τ_A. Therefore, the stable tubes in Γ_A are exactly the regular periodic components in Γ_A. A component $\mathcal{C}$ of Γ_A is said to be *almost periodic* if all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic. Similarly, the Auslander–Reiten quiver Γ_A of an algebra A is said to be *almost periodic* if all but finitely many τ_A-orbits in Γ_A are periodic.

An important class of almost periodic components with oriented cycles is formed by the *ray tubes*, obtained from stable tubes by a finite number (possibly empty) of ray insertions, and the *coray tubes*, obtained from stable tubes by a finite number (possibly empty) of coray insertions (see [59, 63]).

The following characterizations of ray and coray tubes have been established in [45].

Theorem 3.13 (Liu). *Let A be an algebra and $\mathcal{C}$ be a semiregular component of Γ_A. The following equivalences hold.*

(1) *$\mathcal{C}$ contains an oriented cycle but no injective module if and only if $\mathcal{C}$ is a ray tube.*
(2) *$\mathcal{C}$ contains an oriented cycle but no projective module if and only if $\mathcal{C}$ is a coray tube.*

The following result proved in [64, Proposition 2.6] describes an important property of almost periodic components.

Theorem 3.14 (Skowroński). *Let A be an algebra and $\mathcal{C}$ be an almost periodic component of Γ_A. Then, for each integer $d \geq 1$, $\mathcal{C}$ contains at most finitely many modules of length d.*

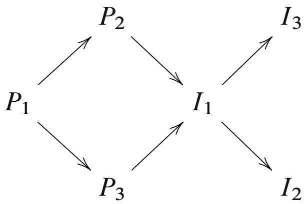
The following theorem proved in [45, Theorem 3.6 and its dual] describes the shape of acyclic semiregular components.

Theorem 3.15 (Liu). *Let A be an algebra and $\mathcal{C}$ be an acyclic semiregular component of Γ_A. The following statements hold.*

(1) *If $\mathcal{C}$ does not contain injective module then there is a connected locally finite valued quiver Δ such that $\mathcal{C}$ is a full translation subquiver of $\mathbb{Z}\Delta$ which is closed under successors.*

(2) *If $\mathcal{C}$ does not contain projective module then there is a connected locally finite valued quiver Δ such that $\mathcal{C}$ is a full translation subquiver of $\mathbb{Z}\Delta$ which is closed under predecessors.*

Example 3.16. Let K be a field and A the path algebra from Example 2.12. Then the Auslander–Reiten quiver Γ_A of A is of the form (see also Example 4.7)



where $P_1 = S_1$, $I_2 = S_3$, and $I_3 = S_2$. We will show that the valuations of all arrows from Γ_A are equal $(1, 1)$ (as in the figure above). We have in Γ_A the valued arrow $P_1 \xrightarrow{(d_{P_1P_2}, d'_{P_1P_2})} P_2$, where $d_{P_1P_2}$ is the multiplicity of P_2 in a left minimal almost split homomorphism $P_1 \to M$ starting at P_1 (see Remarks 3.3). It follows from Example 2.12 that we have in mod A an almost split sequence

$$0 \to P_1 \to P_2 \oplus P_3 \to I_1 \to 0\,, \tag{3}$$

and so $d_{P_1P_2} = 1$. Moreover, $d'_{P_1P_2}$ is the multiplicity of P_1 in a right minimal almost split homomorphism $N \to P_2$ ending in P_2 (see Remarks 3.4), where $N = \operatorname{rad} P_2 = P_1$. Therefore, $d'_{P_1P_2} = 1$ and we get $(d_{P_1P_2}, d'_{P_1P_2}) = (1, 1)$ for the valued arrow $P_1 \xrightarrow{(d_{P_1P_2}, d'_{P_1P_2})} P_2$.

Similarly, using the almost split sequence (3) and the equality $\operatorname{rad} P_3 = P_1$ we receive that $(d_{P_1P_3}, d'_{P_1P_3}) = (1, 1)$ for the valued arrow $P_1 \xrightarrow{(d_{P_1P_3}, d'_{P_1P_3})} P_3$.

Further, consider the valued arrow $I_1 \xrightarrow{(d_{I_1I_3}, d'_{I_1I_3})} I_3$ in Γ_A, where $d_{I_1I_3}$ is the multiplicity of I_3 in a left minimal almost split homomorphism $I_1 \to M'$ starting at I_1 (see Remarks 3.3). Since I_1 is injective, we have

$$I_1 \to I_1/\operatorname{soc} I_1 = I_1/S_1 \cong I_2 \oplus I_3 = M'\,. \tag{4}$$

Therefore, $d_{I_1I_3} = 1$. Moreover, $d'_{I_1I_3}$ is the multiplicity of I_1 in a right minimal almost split homomorphism $N' \to I_3$ ending in I_3 (see Remarks 3.4). It follows from Example 2.12 that we have in mod A an almost split sequence

$$0 \to P_2 \to I_1 \to I_3 \to 0\,, \tag{5}$$

and so $d'_{I_1I_3} = 1$. Hence, we get $(d_{I_1I_3}, d'_{I_1I_3}) = (1, 1)$ for the valued arrow $I_1 \xrightarrow{(d_{I_1I_3}, d'_{I_1I_3})} I_3$.

Similarly, using (4) and the existence in mod A of the almost split sequence of the form

$$0 \to P_3 \to I_1 \to I_2 \to 0\,, \tag{6}$$

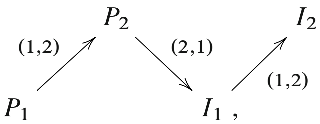
(see Example 2.12), we get $(d_{I_1I_2}, d'_{I_1I_2}) = (1,1)$ for the valued arrow $I_1 \xrightarrow{(d_{I_1I_2}, d'_{I_1I_2})} I_2$.

Since I_1 is not projective, applying Proposition 3.7(1) to the almost split sequence (3), we conclude that $d_{P_2I_1} = d'_{P_1P_2} = 1$ and $d_{P_3I_1} = d'_{P_1P_3} = 1$. Finally, since P_2 (respectively, P_3) is not injective, applying Proposition 3.7(2) to the almost split sequence (5) (respectively, (6)), we get that $d'_{P_2I_1} = d_{I_1I_3} = 1$ (respectively, $d'_{P_3I_1} = d_{I_1I_2} = 1$). Therefore, we have $(d_{P_2I_1}, d'_{P_2I_1}) = (1,1)$ and $(d_{P_3I_1}, d'_{P_3I_1}) = (1,1)$ for the valued arrows $P_2 \xrightarrow{(d_{P_2I_1}, d'_{P_2I_1})} I_1$ and $P_3 \xrightarrow{(d_{P_3I_1}, d'_{P_3I_1})} I_1$ in Γ_A.

Example 3.17. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{C})$:

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{C} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in M_2(\mathbb{C}) \;\middle|\; a \in \mathbb{R}, b, c \in \mathbb{C} \right\}.$$

Then A is 5-dimensional $\mathbb{R}$-algebra of finite representation type and Γ_A is of the form



where $P_1 = e_1A$, $P_2 = e_2A$ are the indecomposable projective modules, $I_1 = D(Ae_1) = \mathrm{Hom}_{\mathbb{R}}(Ae_1, \mathbb{R})$, $I_2 = D(Ae_2) = \mathrm{Hom}_{\mathbb{R}}(Ae_2, \mathbb{R})$ are the indecomposable injective modules and

$$e_1 = \begin{bmatrix} 1_{\mathbb{R}} & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathbb{C}} \end{bmatrix}$$

are the orthogonal primitive idempotents such that $1_A = e_1 \oplus e_2$. Note that $P_1 = S_1 = \left[\begin{smallmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{smallmatrix}\right] \cong \mathbb{R}$ is a simple projective module ($S_1 = e_1A/e_1 \,\mathrm{rad}\, A$ and $\mathrm{rad}\, A = \left[\begin{smallmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{smallmatrix}\right]$), $P_2 = \left[\begin{smallmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{smallmatrix}\right] \cong \mathbb{C} \times \mathbb{C}$, $I_1 \cong \mathbb{R} \times \mathbb{C}$, and $I_2 \cong S_2 \cong \mathbb{C}$ is a simple injective module ($S_2 = e_2A/e_2 \,\mathrm{rad}\, A$). We will show that the valuations of the arrows from Γ_A are the same as shown in the figure above (see also [70, Example III.10.6(a)]).

We have in Γ_A the valued arrow $P_1 \xrightarrow{(d_{P_1P_2}, d'_{P_1P_2})} P_2$, where $d_{P_1P_2}$ is the multiplicity of P_2 in a left minimal almost split homomorphism $P_1 \to M$ starting at P_1 (see Remarks 3.3). One can show that we have in mod A an almost split sequence

$$0 \to P_1 \to P_2 \to I_1 \to 0\,, \tag{7}$$

and so $d_{P_1P_2} = 1$. Moreover, $d'_{P_1P_2}$ is the multiplicity of P_1 in a right minimal almost split homomorphism $N \to P_2$ ending in P_2 (see Remarks 3.4), where

$$N = \mathrm{rad}\; P_2 = \begin{bmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathbb{R} \oplus \mathbb{R}\mathrm{i} & 0 \end{bmatrix} \cong P_1 \oplus P_1\,.$$

Therefore, $d'_{P_1P_2} = 2$ and we get the valued arrow $P_1 \xrightarrow{(1,2)} P_2$.

Similarly, we have in Γ_A the valued arrow $I_1 \xrightarrow{(d_{I_1I_2}, d'_{I_1I_2})} I_2$, where $d_{I_1I_2}$ is the multiplicity of I_2 in a left minimal almost split homomorphism $I_1 \to M'$ starting at I_1 (see Remarks 3.3). Since I_1 is injective, we have

$$I_1 \to I_1/\operatorname{soc} I_1 = I_1/S_1 \cong S_2 \cong I_2 = M' .$$

Therefore, $d_{I_1I_2} = 1$. Moreover, $d'_{I_1I_2}$ is the multiplicity of I_1 in a right minimal almost split homomorphism $N' \to I_2$ ending in I_2 (see Remarks 3.4). One can show that we have in $\operatorname{mod} A$ an almost split sequence

$$0 \to P_2 \to I_1 \oplus I_1 \to I_2 \to 0 , \qquad (8)$$

and so $d'_{I_1I_2} = 2$. Hence, we get the valued arrow $I_1 \xrightarrow{(1,2)} I_2$.

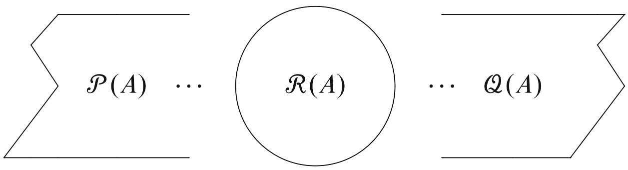
Since I_1 is not projective, applying Proposition 3.7(1) to the almost split sequence (7), we conclude that $d_{P_2I_1} = d'_{P_1P_2} = 2$. Since P_2 is not injective, applying Proposition 3.7(2) to the almost split sequence (8), we get that $d'_{P_2I_1} = d_{I_1I_2} = 1$. Therefore, we have in Γ_A the valued arrow $P_2 \xrightarrow{(2,1)} I_1$.

Finally, note that for the valued quiver Δ of the form $1 \xrightarrow{(1,2)} 2$ we have $\Gamma_A \cong [0,1] \times \Delta$.

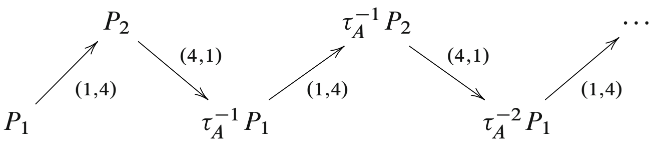
Example 3.18. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_2(\mathbb{H})$:

$$\begin{bmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in M_2(\mathbb{H}) \;\middle|\; a \in \mathbb{R}, b, c \in \mathbb{H} \right\} .$$

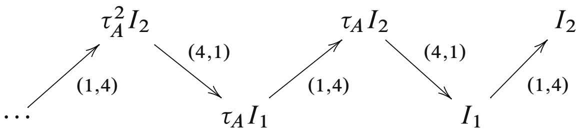
Then A is 9-dimensional $\mathbb{R}$-algebra of infinite representation type and the Auslander–Reiten quiver Γ_A of A is of the form



where $\mathcal{P}(A)$ is a postprojective component containing all indecomposable projective A-modules of the form



$\mathcal{Q}(A)$ is a preinjective component containing all indecomposable injective A-modules of the form



and $\mathcal{R}(A)$ is a family of all regular components containing infinitely many stable tubes of rank one. Moreover, $P_1 = e_1 A$, $P_2 = e_2 A$ are the indecomposable projective modules, $I_1 = D(Ae_1) = \mathrm{Hom}_{\mathbb{R}}(Ae_1, \mathbb{R})$, $I_2 = D(Ae_2) = \mathrm{Hom}_{\mathbb{R}}(Ae_2, \mathbb{R})$ are the indecomposable injective modules and

$$e_1 = \begin{bmatrix} 1_{\mathbb{R}} & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathbb{H}} \end{bmatrix}$$

are the orthogonal primitive idempotents such that $1_A = e_1 \oplus e_2$. Note that $\mathrm{rad}\ A = \left[\begin{smallmatrix} 0 & 0 \\ \mathbb{H} & 0 \end{smallmatrix}\right]$, $\mathrm{rad}^2 A = 0$ and $A/\mathrm{rad}\ A \cong \mathbb{R} \times \mathbb{H}$. Moreover, $P_1 = S_1 = \left[\begin{smallmatrix} \mathbb{R} & 0 \\ 0 & 0 \end{smallmatrix}\right] \cong \mathbb{R}$ is a simple projective module, $P_2 = \left[\begin{smallmatrix} 0 & 0 \\ \mathbb{H} & \mathbb{H} \end{smallmatrix}\right] \cong \mathbb{H} \times \mathbb{H}$, $I_1 \cong \mathbb{R} \times \mathbb{H}$, and $I_2 \cong S_2 \cong \mathbb{H}$ is a simple injective module. Therefore, $\dim_{\mathbb{R}} P_1 = 1$, $\dim_{\mathbb{R}} P_2 = 8$, $\dim_{\mathbb{R}} I_1 = 5$, and $\dim_{\mathbb{R}} I_2 = 4$. In particular, there are no nonzero projective–injective modules in $\mathrm{mod}\, A$. We will show that a postprojective component $\mathcal{P}(A)$ in Γ_A is the same as shown in the figure above. Similarly, one can show that a preinjective component $\mathcal{Q}(A)$ in Γ_A is the same as shown in the figure above (see details in [70, Example III.9.11(c)]).

We have isomorphisms of $\mathbb{R}$-vector spaces

$$\mathrm{Hom}_A(P_1, P_2) \cong e_2 A e_1 \cong \mathbb{H}\ , \quad \mathrm{Hom}_A(P_2, P_1) \cong e_1 A e_2 = 0\ .$$

Moreover, we have isomorphisms of $\mathbb{R}$-algebras

$$\mathrm{End}_A(P_1) \cong e_1 A e_1 \cong \mathbb{R}\ , \quad \mathrm{End}_A(P_2) \cong e_2 A e_2 \cong \mathbb{H}\ ,$$

and hence

$$F_{P_1} = \mathrm{End}_A(P_1)/\,\mathrm{rad}\,\mathrm{End}_A(P_1) \cong \mathbb{R}\ , \quad F_{P_2} = \mathrm{End}_A(P_2)/\,\mathrm{rad}\,\mathrm{End}_A(P_2) \cong \mathbb{H}\ ,$$

since $\mathbb{R}$ and $\mathbb{H}$ are division $\mathbb{R}$-algebras. We have $\mathrm{rad}\ P_1 = 0$, $\mathrm{rad}\ P_2 \cong \mathbb{H}$ and all indecomposable direct summands of $\mathrm{rad}\ P_2$ are isomorphic to P_1. More precisely, we have

$$\mathrm{irr}_A(P_1, P_2) = \mathrm{rad}_A(P_1, P_2)/\,\mathrm{rad}_A^2(P_1, P_2) = \mathrm{Hom}_A(P_1, P_2)\ .$$

Note that $\mathrm{rad}_A^2(P_1, P_2) = 0$. Moreover, $P_1 \ncong P_2$ and by Lemma 1.3(2) we get $\mathrm{rad}_A(P_1, P_2) = \mathrm{Hom}_A(P_1, P_2)$. Now, we can calculate $d_{P_1 P_2}$ and $d'_{P_1 P_2}$. By the definition we have

$$d_{P_1 P_2} = \dim_{F_{P_2}} \mathrm{irr}_A(P_1, P_2) = \dim_{\mathbb{H}} \mathbb{H} = 1$$

and

$$d'_{P_1 P_2} = \dim_{F_{P_1}} \mathrm{irr}_A(P_1, P_2) = \dim_{\mathbb{R}} \mathbb{H} = 4\ .$$

Obviously, $\operatorname{irr}_A(P_2, P_1) = 0$ because $\operatorname{Hom}_A(P_2, P_1) = 0$. In particular, we have in Γ_A the valued arrow $P_1 \xrightarrow{(1,4)} P_2$.

Since P_1 is a simple projective noninjective module in mod A, applying Theorem 2.9(2) and Corollary 3.6(1), we conclude that there exists in mod A an almost split sequence

$$0 \to P_1 \to P_2 \to \tau_A^{-1} P_1 \to 0 .$$

Moreover, we have in mod A a right minimal almost split homomorphism of the form $P_1^4 \to P_2$ (see Theorem 2.9(3)). Further, since P_2 is not an injective module it follows from Propositions 3.5(2) and 3.7(2) that we have in Γ_A the valued arrow $P_2 \xrightarrow{(4,1)} \tau_A^{-1} P_1$. In fact, it is the unique valued arrow in Γ_A with the target $\tau_A^{-1} P_1$, because P_2 is the target of a unique valued arrow in Γ_A with the source P_1 (see Proposition 3.5). Hence we have in mod A an almost split sequence of the form

$$0 \to P_2 \to (\tau_A^{-1} P_1)^4 \to \tau_A^{-1} P_2 \to 0 .$$

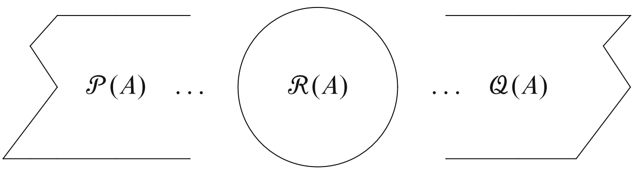
Repeating the arguments we conclude that Γ_A contains an infinite component $\mathcal{P}(A)$ which is shown above.

Finally, note that for the valued quiver Δ of the form $1 \xrightarrow{(1,4)} 2$ we have $\mathcal{P}(A) \cong (-\mathbb{N})\Delta$ and $\mathcal{Q}(A) \cong \mathbb{N}\Delta$.

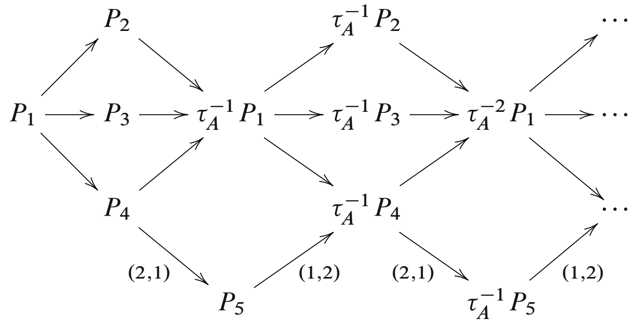
Example 3.19. Let A be the following $\mathbb{R}$-subalgebra of the matrix algebra $M_5(\mathbb{C})$:

$$\begin{bmatrix} \mathbb{C} & 0 & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 & 0 \\ \mathbb{C} & 0 & 0 & \mathbb{C} & 0 \\ \mathbb{C} & 0 & 0 & \mathbb{C} & \mathbb{R} \end{bmatrix} = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_6 & 0 & 0 & 0 \\ a_3 & 0 & a_7 & 0 & 0 \\ a_4 & 0 & 0 & a_8 & 0 \\ a_5 & 0 & 0 & a_9 & b \end{bmatrix} \in M_5(\mathbb{C}) \;\middle|\; a_1, \dots, a_9 \in \mathbb{C}, b \in \mathbb{R} \right\} .$$

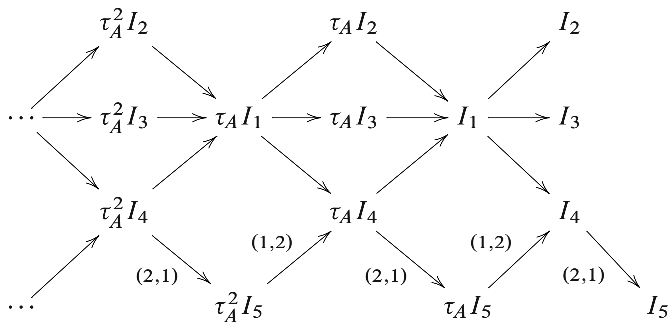
Then A is a 19-dimensional $\mathbb{R}$-algebra of infinite representation type and Γ_A is of the form



where $\mathcal{P}(A)$ is a postprojective component containing all indecomposable projective A-modules of the form

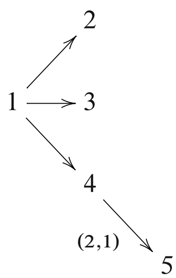


$\mathcal{Q}(A)$ is a preinjective component containing all indecomposable injective A-modules of the form



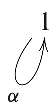
and $\mathcal{R}(A)$ is a family of all regular components containing a stable tube of rank three, a stable tube of rank two and infinitely many stable tubes of rank one.

Finally, note that for the valued quiver Δ of the form



we have $\mathcal{P}(A) \cong (-\mathbb{N})\Delta$ and $\mathcal{Q}(A) \cong \mathbb{N}\Delta$.

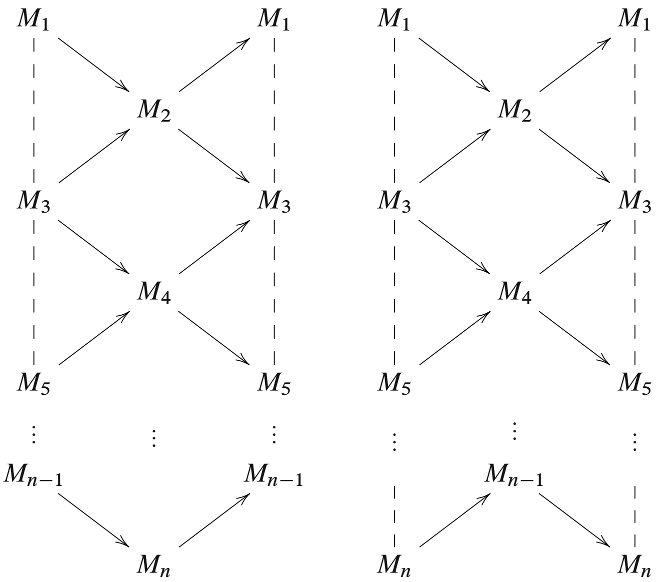
Example 3.20. Let K be a field and $A = KQ/I$ the bound quiver algebra over K, where Q is the quiver of the form



and I the ideal of the path algebra KQ of Q over K generated by the path α^n, where $n \geq 2$. Then $A = KQ/I$ is an n-dimensional K-algebra of finite representation type which is isomorphic to the quotient polynomial algebra $K[x]/(x^n)$. The set of all indecomposable A-modules in mod A is of the form $M_i = K[x]/(x^i)$ for $i \in \{1,\dots,n\}$. Further, for each $i \in \{1,\dots,n-1\}$ we have $\tau_A M_i = M_i$ and $P_1 = I_1 \cong M_n$ is projective–injective A-module. Moreover, $S_1 \cong M_1$, rad $M_n \cong M_{n-1}$, $M_n/\operatorname{soc} M_n \cong M_{n-1}$ and there exist in mod A almost split sequences of the forms

$$0 \to M_1 \to M_2 \to M_1 \to 0,$$
$$0 \to M_i \to M_{i-1} \oplus M_{i+1} \to M_i \to 0\ , \quad \text{where } i \in \{2,\dots,n-1\}\ .$$

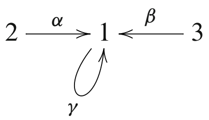
Then the Auslander–Reiten quiver Γ_A of A consists of n vertices M_i, $i \in \{1,\dots,n\}$ and Γ_A is of the form



if n is even if n is odd

where the vertical dashed lines have to be identified.

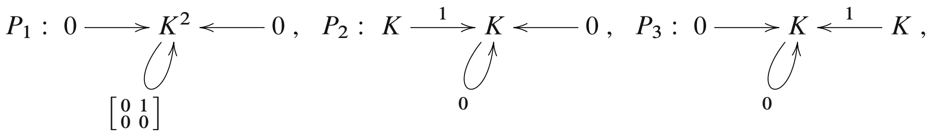
Example 3.21. Let K be a field and $A = KQ/I$ the bound quiver algebra over K, where Q is the quiver of the form



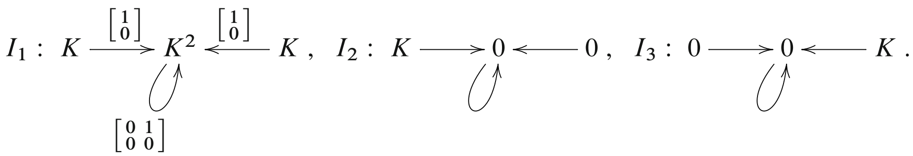
and I the ideal of the path algebra KQ of Q over K generated by the paths $\gamma^2, \alpha\gamma$, and $\beta\gamma$. Then $A = KQ/I$ is isomorphic to the following K-subalgebra of the matrix algebra $M_3(K)$

$$\begin{bmatrix} K[x]/(x^2) & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ b_1 & b_3 & 0 \\ b_2 & 0 & b_4 \end{bmatrix} \in M_3(K) \;\middle|\; a \in K[x]/(x^2), b_1, \dots, b_4 \in K \right\} .$$

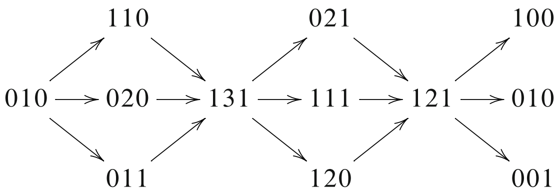
We identify mod A and $\text{rep}_K(Q, I)$. Then a complete set of pairwise nonisomorphic indecomposable projective modules in mod A (representations in $\text{rep}_K(Q, I)$) is of the form



and a complete set of pairwise nonisomorphic indecomposable injective modules in mod A (representations in $\text{rep}_K(Q, I)$) is of the form



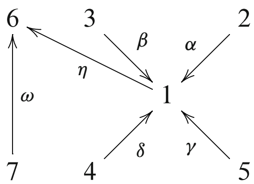
Moreover, A is a 6-dimensional K-algebra of finite representation type and Γ_A is of the form



where all indecomposable modules are represented by their dimension vectors.

We end this section with an example of an almost periodic Auslander–Reiten quiver Γ_A of an algebra A, such that Γ_A contains a coray tube.

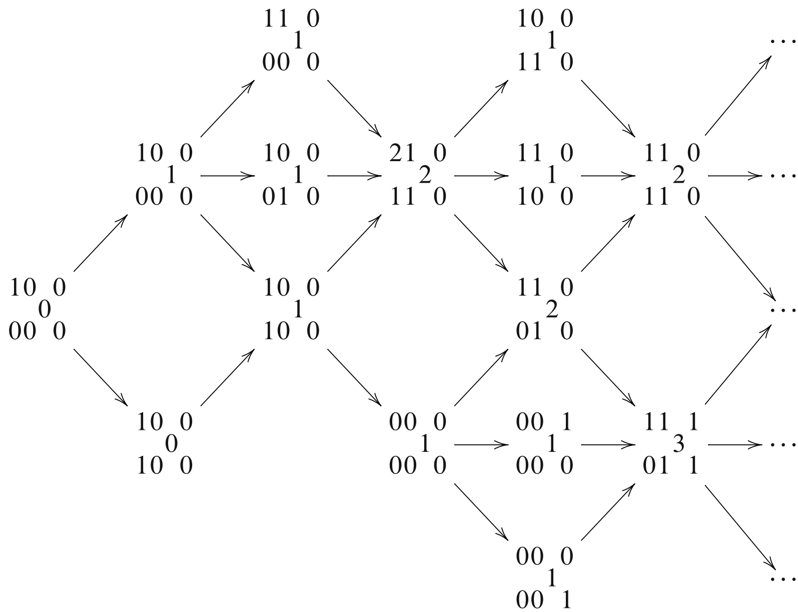
Example 3.22. Let K be a field and $A = KQ/I$ the bound quiver algebra over K, where Q is the quiver of the form



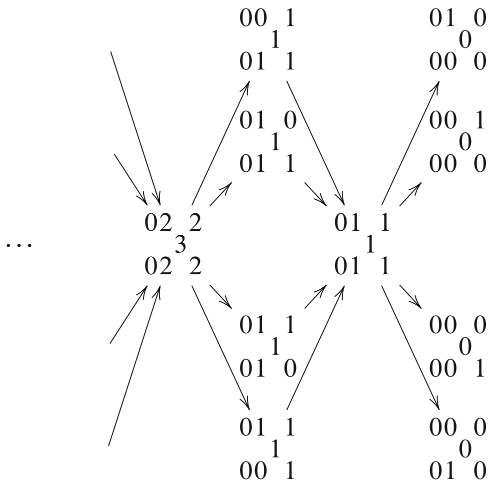
and I the ideal of the path algebra KQ of Q over K generated by the paths $\alpha\eta$ and $\gamma\eta$. Then $A = KQ/I$ is isomorphic to the following K-subalgebra of the matrix algebra $M_7(K)$

$$\begin{bmatrix} K & 0 & 0 & 0 & 0 & K & 0 \\ K & K & 0 & 0 & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 & K & 0 \\ K & 0 & 0 & K & 0 & K & 0 \\ K & 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & 0 & K & K \end{bmatrix} = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & a_{10} & 0 \\ a_2 & a_6 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & a_7 & 0 & 0 & a_{11} & 0 \\ a_4 & 0 & 0 & a_8 & 0 & a_{12} & 0 \\ a_5 & 0 & 0 & 0 & a_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{14} & a_{15} \end{bmatrix} \in M_7(K) \,\middle|\, a_1, \ldots, a_{15} \in K \right\}.$$

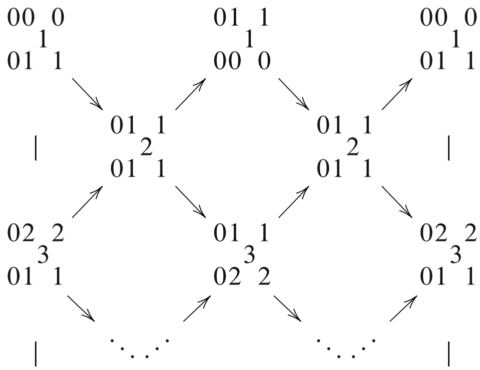
Moreover, A is a 15-dimensional K-algebra of infinite representation type and Γ_A has a disjoint union form $\Gamma_A = \mathcal{P}(A) \cup \mathcal{T}(A) \cup \mathcal{Q}(A)$, where $\mathcal{P}(A)$ is a postprojective component containing all indecomposable projective A-modules of the form



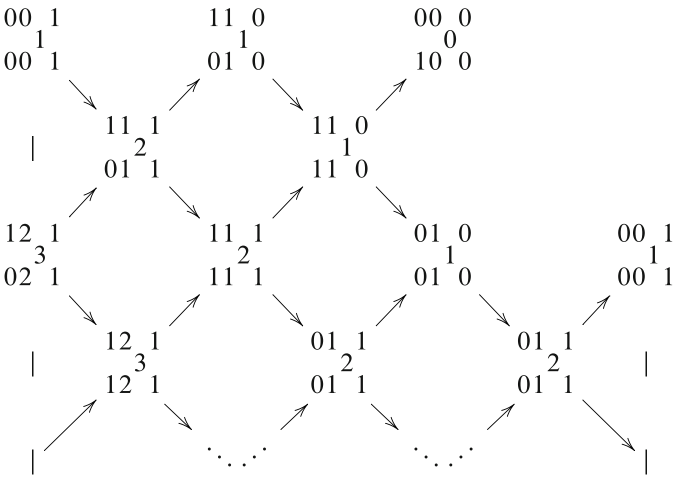
$\mathcal{Q}(A)$ is a preinjective component of the form



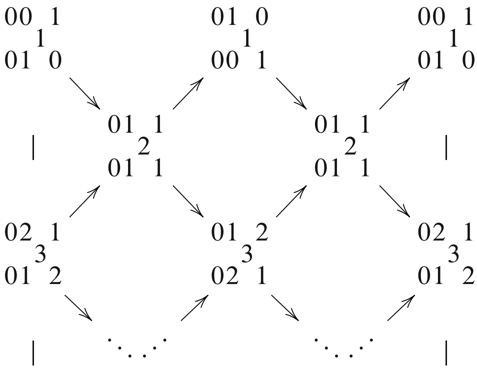
and $\mathcal{T}(A)$ is a family of semiregular components containing one coray tube, two stable tubes of rank two, and infinitely many stable tubes of rank one. More precisely, we have $\mathcal{T}(A) = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$, where, for each $\lambda \in \{0, \infty\}$, $\mathcal{T}_\lambda$ is a stable tube of rank two, $\mathcal{T}_1$ is a coray tube and, for each $\lambda \in \mathbb{P}_1(K) \setminus \{0, 1, \infty\}$, $\mathcal{T}_\lambda$ is a stable tube of rank one.



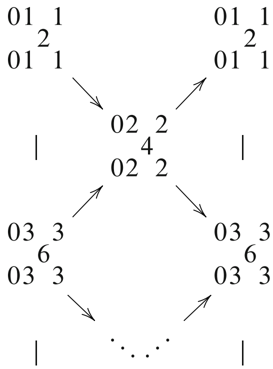
$\mathcal{T}_0$



$\mathcal{T}_1$



$\mathcal{T}_\infty$



$\mathcal{T}_\lambda$

All indecomposable modules are represented by their dimension vectors and one identifies along the vertical dashed lines.

Denote by B the hereditary K-algebra from Example 4.8. Then A is a $\mathcal{T}$-tubular coextension of B (see [63, Chap. XV.2] for the definition), where $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ is a family of stable tubes of Γ_B (see Example 4.8). Note that a coray tube $\mathcal{T}_1$ of Γ_A is obtained from the stable tube $\mathcal{T}_1$ of Γ_B by two coray insertions. One can show that in this case only the stable tube $\mathcal{T}_1$ of Γ_B and the postprojective component of Γ_B may change. The remaining part of Γ_B becomes the components of Γ_A under the standard embeddings.

4 Hereditary Algebras

The aim of this section is to recall the hereditary algebras, which play an important role in the representation theory of algebras. Let us mention that the representation theory of hereditary algebras is one of the most extensively studied and best understood. For the representation theory of hereditary algebras we refer the reader to [2, 8, 15, 20, 29, 30, 32–35, 37, 48, 52, 56–58, 62, 63].

Let A be an algebra. Then A is said to be *right hereditary* if any right ideal of A is a projective right A-module. Similarly, A is said to be *left hereditary* if any left ideal of A is a projective left A-module.

The following characterization of right hereditary algebras and left hereditary algebras is well-known.

Theorem 4.1. *Let A be an algebra. The following conditions are equivalent.*

(1) gl. dim $A \leq 1$.
(2) *A is right hereditary.*
(3) *A is left hereditary.*
(4) *Every right A-submodule of a projective module in* mod A *is projective.*
(5) *Every factor module of an injective module in* mod A *is injective.*
(6) *The radical* rad P *of any indecomposable projective module P in* mod A *is projective.*
(7) *The socle factor $I/\operatorname{soc} I$ of any indecomposable injective module I in* mod A *is injective.*

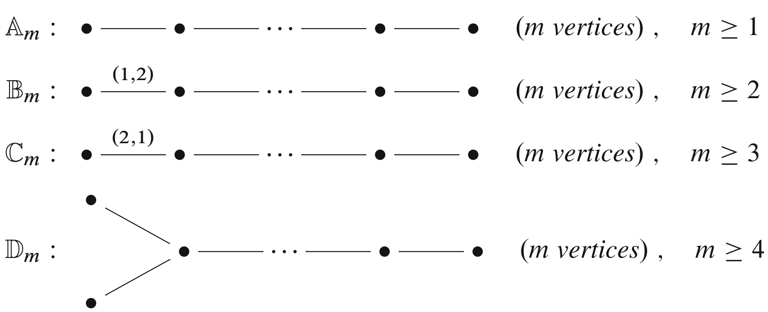
An algebra A is said to be *hereditary* if A is left and right hereditary.

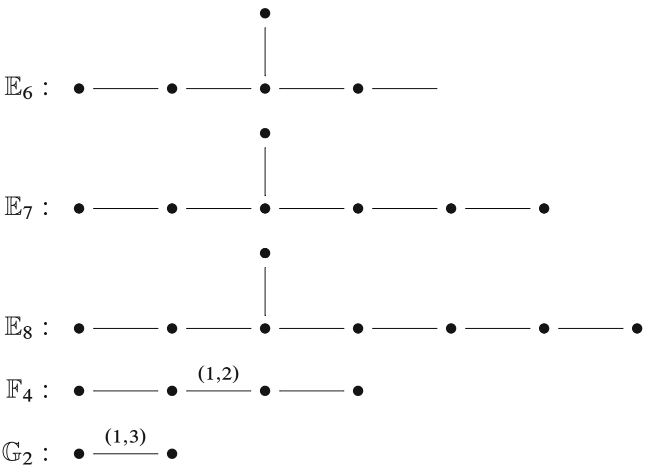
Remark 4.2. We note that A is a hereditary algebra if and only if A^{op} is a hereditary algebra.

Example 4.3. The algebras from Examples 3.17, 3.18 and 3.19 are hereditary $\mathbb{R}$-algebras.

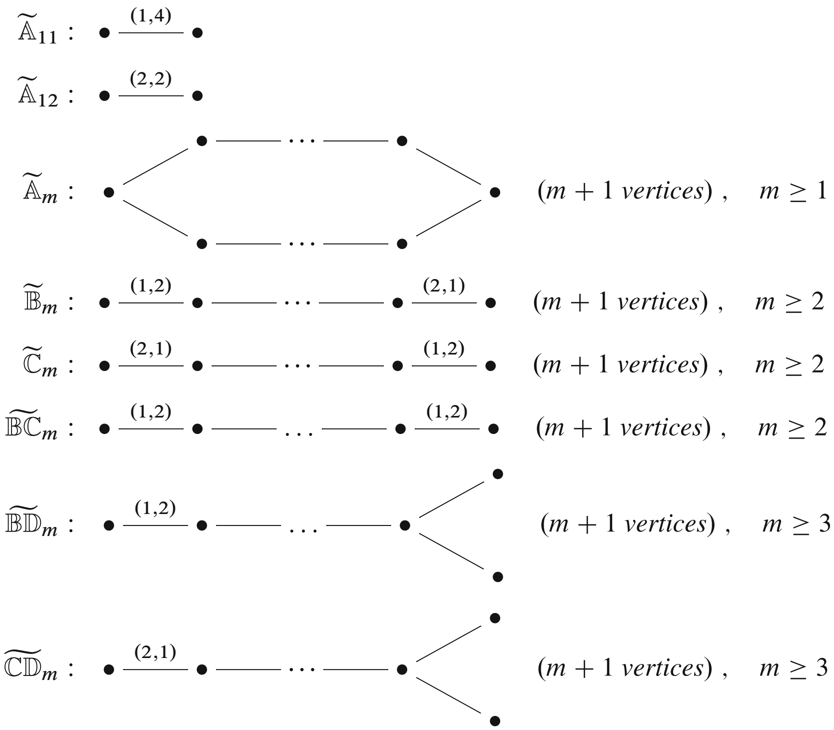
The following known results introduce a natural division of hereditary algebras via behavior of their Euler forms. The Theorem 4.4 is a consequence of results from [33, 37], and Theorem 4.5 is a consequence of results from [33, 35, 48].

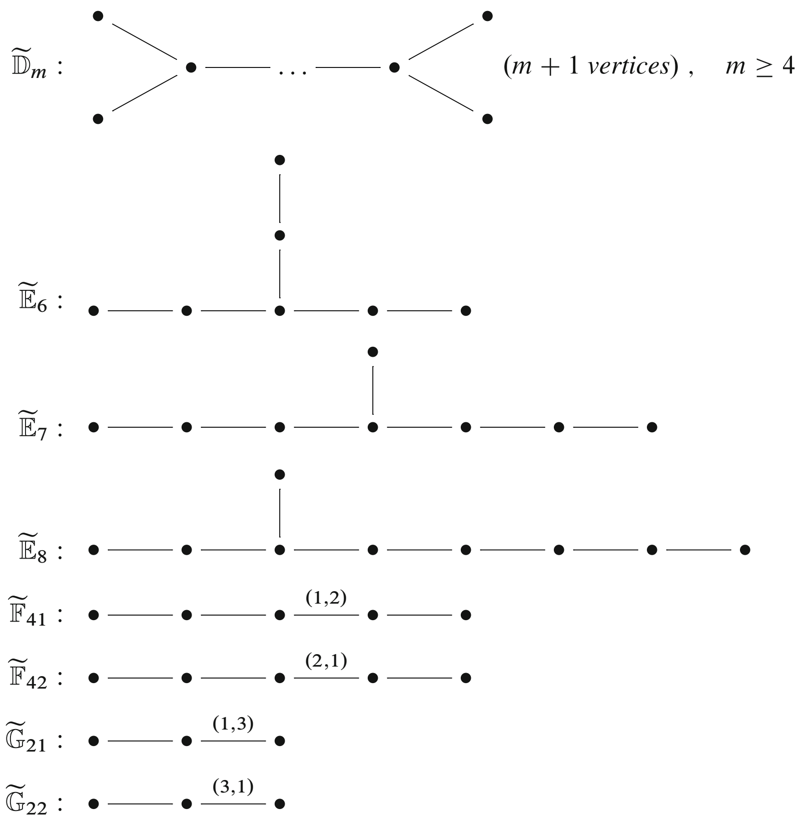
Theorem 4.4 (Gabriel, Dlab–Ringel). *Let A be a hereditary algebra. Then the Euler form χ_A is positive definite if and only if the valued graph G_A of A is one of the following Dynkin graphs*





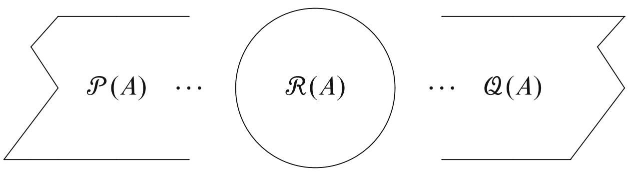
Theorem 4.5 (Donovan–Freislich, Nazarova, Dlab–Ringel). *Let A be a hereditary algebra. Then the Euler form χ_A is positive semidefinite but not positive definite if and only if the valued graph G_A of A is one of the Euclidean graphs*





A hereditary algebra A is said to be a *hereditary algebra of Dynkin* (respectively, *Euclidean*) *type* if the valued graph of A is a Dynkin (respectively, Euclidean) graph. A hereditary algebra A is said to be of *wild type* if A is neither of Dynkin nor Euclidean type. In particular, we have the following theorem describing the structure of the Auslander–Reiten quiver of a hereditary algebra.

Theorem 4.6. *Let A be a hereditary algebra and $Q = Q_A$ be the valued quiver of A. Then the general shape of the Auslander–Reiten quiver Γ_A of A is as follows*



where $\mathcal{P}(A)$ is the postprojective component containing all indecomposable projective A-modules, $\mathcal{Q}(A)$ is the preinjective component containing all indecomposable injective A-modules, and $\mathcal{R}(A)$ is the family of all regular components. More precisely, we have

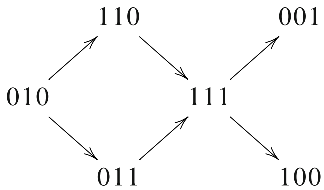
(1) *If A is of Dynkin type, then $\mathcal{R}(A)$ is empty and $\mathcal{P}(A) = \mathcal{Q}(A)$.*

(2) *If A is of Euclidean type, then $\mathcal{P}(A) \cong (-\mathbb{N})Q^{\mathrm{op}}$, $\mathcal{Q}(A) \cong \mathbb{N}Q^{\mathrm{op}}$ and $\mathcal{R}(A)$ is an infinite family of pairwise orthogonal generalized standard faithful stable tubes.*
(3) *If A is of wild type, then $\mathcal{P}(A) \cong (-\mathbb{N})Q^{\mathrm{op}}$, $\mathcal{Q}(A) \cong \mathbb{N}Q^{\mathrm{op}}$ and $\mathcal{R}(A)$ is an infinite family of regular components of type $\mathbb{Z}\mathbb{A}_\infty$.*

Example 4.7. Let K be a field and A the path algebra from Example 2.12. Then A is isomorphic to the following K-subalgebra of the matrix algebra $M_3(K)$:

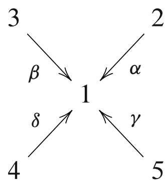
$$\begin{bmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & a_4 & 0 \\ a_3 & 0 & a_5 \end{bmatrix} \in M_3(K) \,\middle|\, a_1, \ldots, a_5 \in K \right\}.$$

Moreover, A is a 5-dimensional K-algebra. Since the valued graph G_A of A is a Dynkin graph, the hereditary K-algebra $A = KQ$ is of finite representation type and Γ_A is of the form



where $\mathcal{R}(A)$ is empty, $\mathcal{P}(A) = \mathcal{Q}(A) = \Gamma_A$, and all indecomposable modules are represented by their dimension vectors.

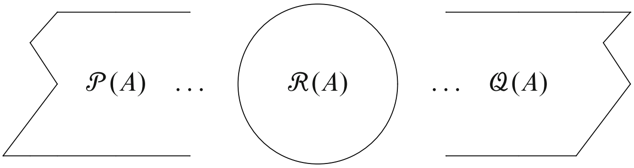
Example 4.8. Let K be a field and A the path algebra of the quiver Q of the form



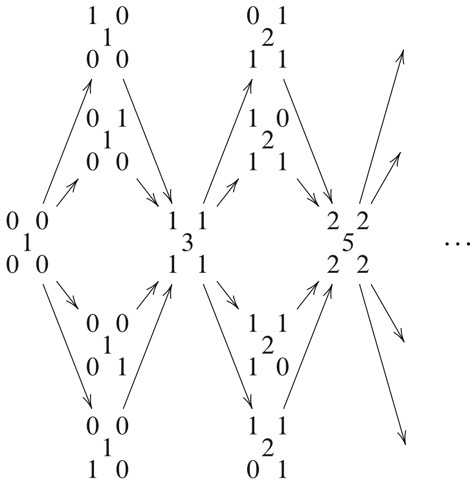
Then $A = KQ$ is isomorphic to the following K-subalgebra of the matrix algebra $M_5(K)$:

$$\begin{bmatrix} K & 0 & 0 & 0 & 0 \\ K & K & 0 & 0 & 0 \\ K & 0 & K & 0 & 0 \\ K & 0 & 0 & K & 0 \\ K & 0 & 0 & 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_6 & 0 & 0 & 0 \\ a_3 & 0 & a_7 & 0 & 0 \\ a_4 & 0 & 0 & a_8 & 0 \\ a_5 & 0 & 0 & 0 & a_9 \end{bmatrix} \in M_5(K) \,\middle|\, a_1, \ldots, a_9 \in K \right\}.$$

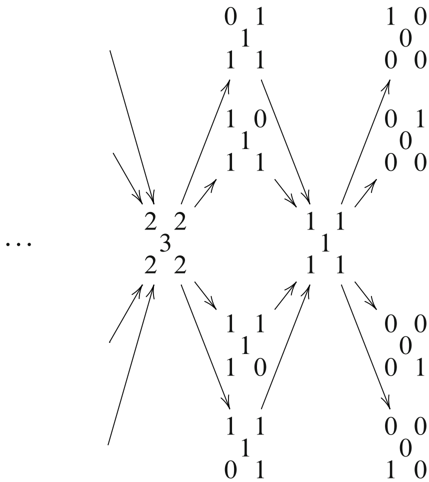
Moreover, A is 9-dimensional K-algebra. Since the valued graph G_A of A is a Euclidean graph (and thus is not a Dynkin graph), the hereditary K-algebra $A = KQ$ is of infinite representation type and Γ_A is of the form



where $\mathcal{P}(A)$ is a postprojective component containing all indecomposable projective A-modules of the form

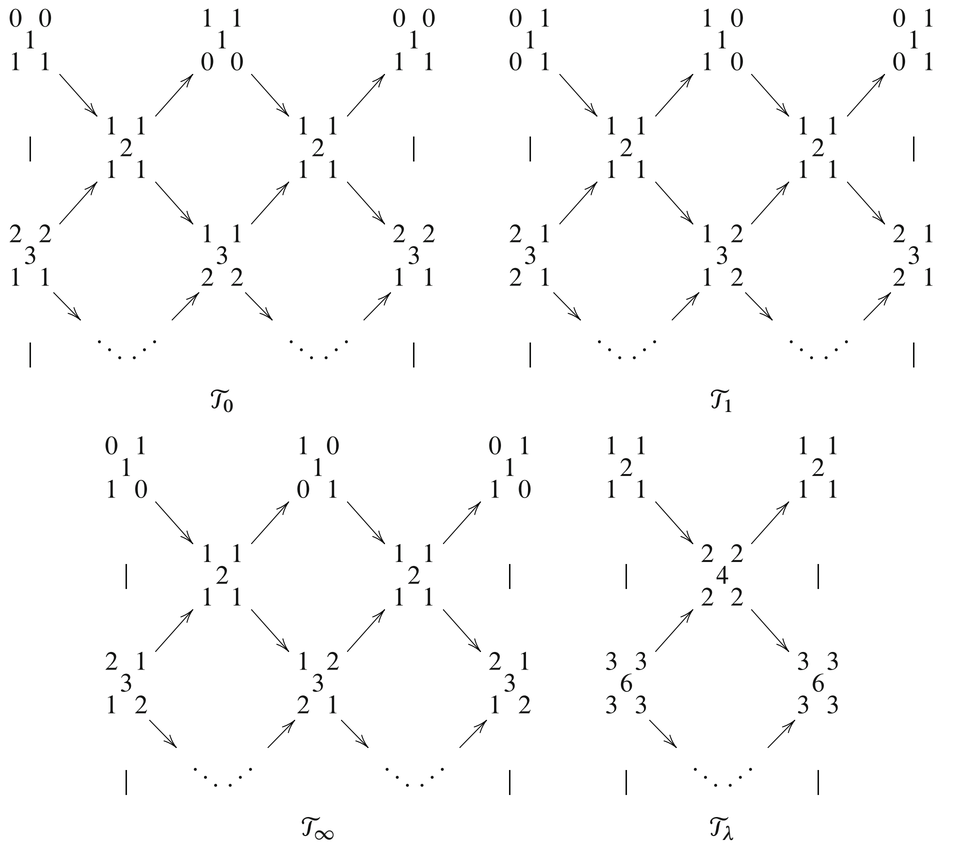


$\mathcal{Q}(A)$ is a preinjective component containing all indecomposable injective A-modules of the form



and $\mathcal{R}(A)$ is a family of all regular components containing three stable tubes of rank two and infinitely many stable tubes of rank one. More precisely, we have $\mathcal{R}(A) =$

$(\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(K)}$, where, for each $\lambda \in \{0, 1, \infty\}$, $\mathcal{T}_\lambda$ is a stable tube of rank two and, for each $\lambda \in \mathbb{P}_1(K) \setminus \{0, 1, \infty\}$, $\mathcal{T}_\lambda$ is a stable tube of rank one.



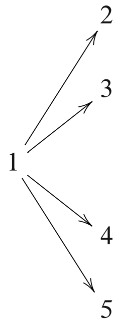
$\mathcal{T}_0$ $\mathcal{T}_1$ $\mathcal{T}_\infty$ $\mathcal{T}_\lambda$

All indecomposable modules are represented by their dimension vectors and one identifies along the vertical dashed lines. Moreover, we have

$$\mathrm{Hom}_A(Y, X) = 0\ , \quad \mathrm{Hom}_A(Z, Y) = 0\ , \quad \mathrm{Hom}_A(Z, X) = 0\ ,$$

for any modules $X \in \mathcal{P}(A)$, $Y \in \mathcal{R}(A)$, and $Z \in \mathcal{Q}(A)$. Additionally, stable tubes $\mathcal{T}_\lambda$, $\lambda \in \mathbb{P}_1(K)$, are generalized standard, faithful, and pairwise orthogonal.

Finally, note that for the valued quiver Δ of the form



we have $\mathscr{P}(A) \cong (-\mathbb{N})\Delta$ and $\mathscr{Q}(A) \cong \mathbb{N}\Delta$.

5 The Number of Terms in the Middle of Almost Split Sequences

As mentioned in the introduction a central role in the representation theory of algebras is played by almost split sequences. For an algebra A and an indecomposable nonprojective module Z in mod A, there is an almost split sequence

$$0 \to X \to \bigoplus_{i=1}^{r} Y_i \to Z \to 0$$

with $X \cong \tau_A Z$ an indecomposable noninjective module in mod A (see Theorem 2.9), and $Y_1, \ldots, Y_r$ indecomposable modules in mod A. So we may associate to Z the numerical homological invariant $r = r(Z)$. Then $r = r(Z)$ measures the complexity of homomorphisms in mod A with domain $\tau_A Z$ and codomain Z. Therefore, it is interesting to study the relation between an algebra A and the values $r = r(Z)$ for all indecomposable modules Z in mod A (we refer to [14, 18, 26, 44, 49, 50, 53, 69, 71] for some results in this direction).

Let A be an algebra, and M be a nonzero module in mod A. It follows from the Jordan–Hölder theorem that the number n of modules in a composition series (M_{i+1}/M_i is simple for $i = 0, \ldots n-1$)

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of M depends only on M. It is called the *length* of M and is denoted by $\ell(M)$.

The next seven facts have been proved by S. Liu [44].

Lemma 5.1. *Let A be an algebra. Moreover, let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in mod A*, $Y_1, \ldots, Y_r$ indecomposable modules, and $\ell(Y_i) < \ell(X)$ for any $i \in \{1, \ldots, r\}$. Then any sectional path in Γ_A ending with Z does not contain a projective module.*

Proof. Let $Z_n \to Z_{n-1} \to \cdots \to Z_1 \to Z_0 = Z$ be a sectional path in Γ_A, where $n \geq 1$. Then we have an irreducible homomorphism $Z_1 \to Z$, and hence $Z_1 \cong Y_s$ for some $s \in \{1, \ldots, r\}$. So, we get an irreducible homomorphism $X \to Z_1$. Using assumption $\ell(X) > \ell(Y_s) = \ell(Z_1)$, we conclude that the module Z_1 is not projective. Indeed, if it is not the case then the module X is a direct summand of rad Z_1, and so $\ell(X) < \ell(Z_1)$, a contradiction. Therefore, by Theorem 2.9(1), we have an almost split sequence in mod A of the form $0 \to \tau_A Z_1 \to E \to Z_1 \to 0$.

If $n \geq 2$ then the module Z_2 (respectively, X) is a direct summand of E, because we have in mod A an irreducible homomorphism $Z_2 \to Z_1$ (respectively, $X \to Z_1$) and $X = \tau_A Z \ncong Z_2$ (we know that the path $Z_2 \to Z_1 \to Z_0 = Z$ is sectional). Therefore, $E \cong X \oplus Z_2 \oplus E'$. Since

$$\ell(E) = \ell(\tau_A Z_1) + \ell(Z_1)\,, \quad \ell(E) = \ell(X) + \ell(Z_2) + \ell(E')\,, \quad \ell(X) > \ell(Z_1)\,,$$

we get that $\ell(\tau_A Z_1) > \ell(Z_2)$. Hence, the module Z_2 is not projective, because we have in mod A an irreducible homomorphism $\tau_A Z_1 \to Z_2$. So, again by Theorem 2.9(1), we have an almost split sequence in mod A of the form $0 \to \tau_A Z_2 \to F \to Z_2 \to 0$.

If $n \geq 3$ then $F \cong \tau_A Z_1 \oplus Z_3 \oplus F'$, because we have in mod A the irreducible homomorphisms $\tau_A Z_1 \to Z_2$, $Z_3 \to Z_2$ and the path $Z_3 \to Z_2 \to Z_1 \to Z_0 = Z$ is sectional (hence $\tau_A Z_1 \ncong Z_3$). Since

$$\ell(F) = \ell(\tau_A Z_2) + \ell(Z_2)\,, \quad \ell(F) = \ell(\tau_A Z_1) + \ell(Z_3) + \ell(F')\,, \quad \ell(\tau_A Z_1) > \ell(Z_2)\,,$$

we get that $\ell(\tau_A Z_2) > \ell(Z_3)$. Hence, the module Z_3 is not projective, because we have in mod A an irreducible homomorphism $\tau_A Z_2 \to Z_3$.

Continuing, we receive that all modules $Z_1, Z_2, \ldots, Z_n$ are nonprojective. □

Proposition 5.2. *Let A be an algebra. Moreover, let $f\colon X \to \bigoplus_{i=1}^4 Y_i$ be an irreducible homomorphism in* mod A, *X an indecomposable module, $Y_1, \ldots, Y_4$ indecomposable non-projective modules, and $\sum_{i=1}^4 \ell(Y_i) \leq 2\ell(X)$. Then X has no projective predecessor in Γ_A.*

Corollary 5.3. *Let A be an algebra. Moreover, let $f\colon X \to \bigoplus_{i=1}^4 Y_i$ be an irreducible epimorphism in* mod A, *X an indecomposable module, and $Y_1, \ldots, Y_4$ indecomposable nonprojective modules. Then X has no projective predecessor in Γ_A.*

Lemma 5.4. *Let A be an algebra. Moreover, let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in mod A, *$Y_1, \ldots, Y_r$ indecomposable modules, and $\ell(Y_i) < \ell(Z)$ for any $i \in \{1, \ldots, r\}$. Then any sectional path in Γ_A starting at X does not contain an injective module.*

Proposition 5.5. *Let A be an algebra. Moreover, let $g\colon \bigoplus_{i=1}^4 Y_i \to Z$ be an irreducible homomorphism in* mod A, *Z an indecomposable module, $Y_1, \ldots, Y_4$ indecomposable non-injective modules, and $\sum_{i=1}^4 \ell(Y_i) \leq 2\ell(Z)$. Then Z has no injective successor in Γ_A.*

Corollary 5.6. *Let A be an algebra. Moreover, let $g\colon \bigoplus_{i=1}^{4} Y_i \to Z$ be an irreducible monomorphism in* mod A, *Z an indecomposable module, and $Y_1, \dots, Y_4$ indecomposable noninjective modules. Then Z has no injective successor in Γ_A.*

Theorem 5.7 (Liu). *Let A be an algebra. Moreover, let*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in mod A, *and $Y_1, \dots, Y_r$ indecomposable modules. Assume that X has a projective predecessor in Γ_A and Z has an injective successor in Γ_A. Then $r \le 4$, and $r = 4$ implies that Y_i is projective–injective for some $i \in \{1, \dots, 4\}$ and Y_j is not projective–injective for any $j \in \{1, \dots, 4\} \setminus \{i\}$.*

As a direct consequence we obtain the well-known theorem proved by Bautista and Brenner in [18, Theorem].

Corollary 5.8 (Bautista–Brenner). *Let A be an algebra of finite representation type, and*

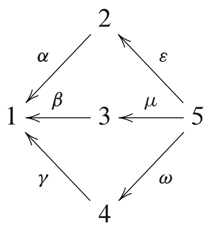
$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

be an almost split sequence in mod A, *where Y_i is the indecomposable module for any $i \in \{1, \dots, r\}$. Then $r \le 4$, and $r = 4$ implies that Y_i is projective–injective for some $i \in \{1, \dots, 4\}$ and Y_j is not projective–injective for any $j \in \{1, \dots, 4\} \setminus \{i\}$.*

Remark 5.9. If an algebra A is of finite representation type, then any indecomposable module has a projective predecessor and an injective successor in Γ_A.

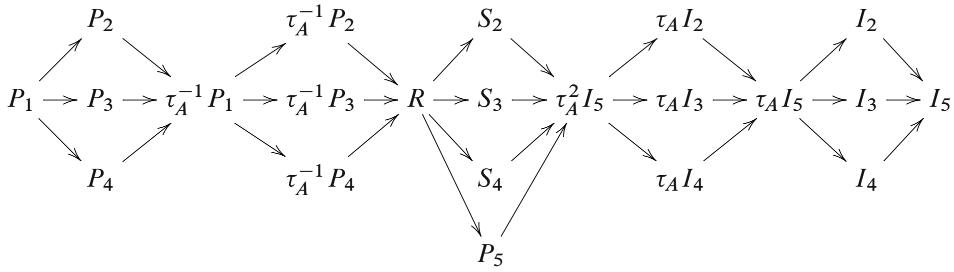
The following example illustrates Corollary 5.8.

Example 5.10. Let K be a field and $A = KQ/I$ the bound quiver algebra over K, where Q is the quiver of the form



and I the ideal of the path algebra KQ of Q over K generated by the paths $\varepsilon\alpha - \mu\beta$, $\mu\beta - \omega\gamma$.

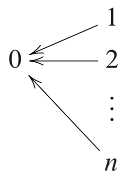
Then A is of finite representation type and Γ_A is of the form



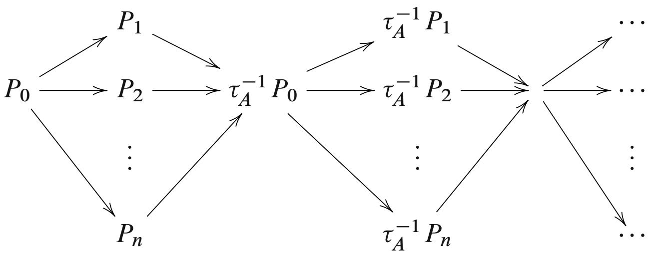
where $P_1 = S_1$, $R = \text{rad } P_5$, $P_5 = I_1$, $I_5 = S_5$.

On the other hand, there are algebras A of infinite representation type such that the middle term of an almost split sequence in mod A has an arbitrary number $n \geq 4$ of indecomposable summands.

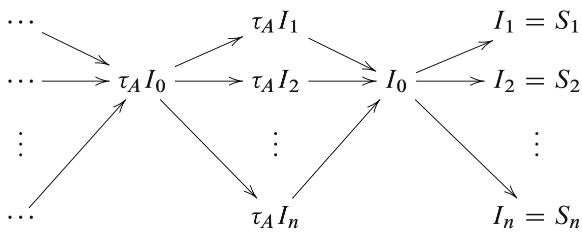
Example 5.11. Let K be a field and $A = KQ$ the path algebra of Q over K, where Q is the quiver of the form



where $n \geq 4$. Then the Auslander–Reiten quiver Γ_A of A contains the postprojective component $\mathcal{P}(A)$ (with all indecomposable projective A-modules and without injective A-modules) of the form



and the preinjective component $\mathcal{Q}(A)$ (with all indecomposable injective A-modules and without projective A-modules) of the form



Note that from Drozd's Tame and Wild Theorem [36] the class of finite-dimensional algebras over an algebraically closed field K may be divided into two disjoint classes (tame and wild dichotomy). One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite-dimensional algebras over K. More precisely, following [36], a finite-dimensional algebra A over an algebraically closed field K is called *tame* if, for any dimension d, there exists a finite number of $K[x]$-A-bimodules M_i, $1 \leq i \leq n_d$, which are free of finite rank as left $K[x]$-modules and all but finitely many isomorphism classes of modules of dimension d in ind A are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \dots, n_d\}$.

We have the following conjecture from [24, Conjecture 1].

Conjecture 5.12 *(Brenner–Butler)*. Let A be a tame finite-dimensional K-algebra over an algebraically closed field K,

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

an almost split sequence in mod A, and Z indecomposable nonprojective module. Then $r \leq 5$.

In the next theorem, proved in [46, Main Theorem], we give the affirmative answer for the above conjecture in the case of cycle-finite algebras.

Let A be an algebra. Recall that a *cycle* of indecomposable modules in mod A is a sequence

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = X_0$$

of nonzero nonisomorphisms in mod A, where X_i is indecomposable for $i \in \{1, \dots, r\}$, and such a cycle is said to be *finite* if the homomorphisms $f_1, \dots, f_r$ do not belong to rad_A^{∞}. Then, following [4, 65], an algebra A is said to be *cycle-finite* if all cycles between indecomposable modules in mod A are finite.

Remark 5.13. The class of cycle-finite algebras contains the following distinguished classes of algebras.

(1) The algebras of finite representation type.
(2) The hereditary algebras of Euclidean type [33, 34].
(3) The tame tilted algebras [40, 41, 59].
(4) The tame double tilted algebras [54] (the tame strict shod algebras in the sense of [27]).
(5) The tame generalized double tilted algebras [55] (the tame laura algebras in the sense of [1]).
(6) The tubular algebras [59].
(7) The iterated tubular algebras [51].
(8) The tame quasi-tilted algebras [42, 67].
(9) The tame coil and multicoil algebras [4–6].
(10) The tame generalized multicoil algebras [47].

(11) The algebras with cycle-finite derived categories [3].
(12) The strongly simply connected algebras of polynomial growth [66].

Theorem 5.14 (Malicki–de la Peña–Skowroński). *Let A be a cycle-finite algebra,*

$$0 \to X \xrightarrow{f} \bigoplus_{i=1}^{r} Y_i \xrightarrow{g} Z \to 0$$

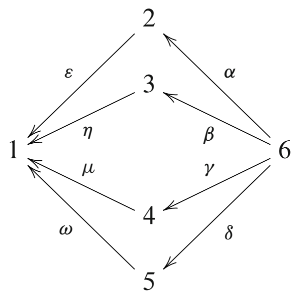
an almost split sequence between indecomposable modules in mod A, *and* Z *nonprojective module. Then* $r \leq 5$, *and* $r = 5$ *implies that* Y_i *is projective–injective for some* $i \in \{1, \ldots, 5\}$ *and* Y_j *is not projective–injective for any* $j \in \{1, \ldots, 5\} \setminus \{i\}$.

Remark 5.15. In the above theorem, if $r = 5$ and Y_i is a projective–injective module for some $i \in \{1, \ldots, 5\}$, then $X \cong \operatorname{rad} Y_i$ and $Z \cong Y_i / \operatorname{soc} Y_i$.

Remark 5.16. For finite-dimensional cycle-finite algebras over an algebraically closed field, the Theorem 5.14 was proved by J.A. de la Peña and M. Takane [50, Theorem 3], by application of spectral properties of Coxeter transformations of algebras and results established in [44].

We end this section with an example of a cycle-finite algebra A, illustrating Theorem 5.14.

Example 5.17. Let K be a field and $A = KQ/I$ the bound quiver algebra over K, where Q is the quiver of the form



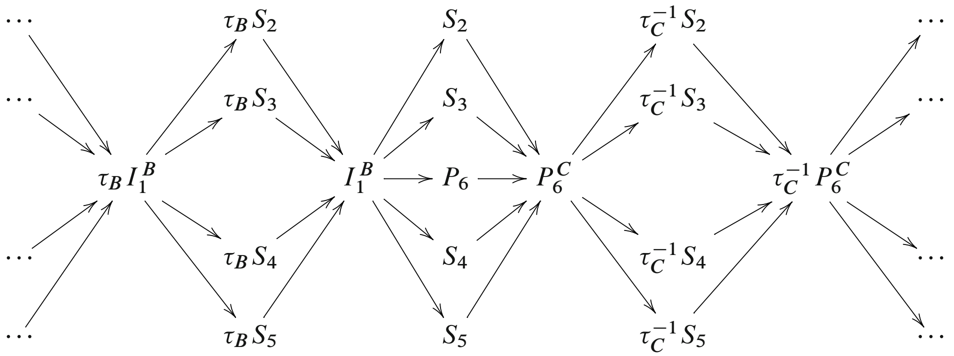
and I the ideal of the path algebra KQ of Q over K generated by the paths $\beta\eta - \alpha\varepsilon$, $\gamma\mu - \alpha\varepsilon$, $\delta\omega - \alpha\varepsilon$.

Denote by B the hereditary algebra given by the full subquiver of Q given by the vertices $1, 2, 3, 4, 5$ and by C the hereditary algebra given by the full subquiver of Q given by the vertices $2, 3, 4, 5, 6$.

Note that $P_6 = I_1$ is a projective–injective A-module. Hence, applying [2, Proposition IV.3.11], we conclude that there is in mod A an almost split sequence of the form

$$0 \to \operatorname{rad} P_6 \to S_2 \oplus S_3 \oplus S_4 \oplus S_5 \oplus P_6 \to P_6/S_1 \to 0 ,$$

where $S_2 \oplus S_3 \oplus S_4 \oplus S_5 \cong \text{rad}\ P_6/S_1$. Moreover, rad P_6 is the indecomposable injective B-module I_1^B, whereas P_6/S_1 is the indecomposable projective C-module P_6^C. The component of Γ_A containing $P_6 = I_1$ is the following gluing of the preinjective component of Γ_B with the postprojective component of Γ_C (see details in [2, Example VIII.5.7(e)]):



where τ_B and τ_C denote the Auslander–Reiten translations in mod B and mod C, respectively.

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Cluster Algebras from Surfaces

Lecture Notes for the CIMPA School Mar del Plata, March 2016

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Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [17] in 2002. Their original motivation was coming from canonical bases in Lie Theory. Today cluster algebras are connected to various fields of mathematics, including

- Combinatorics (polyhedra, frieze patterns, green sequences, snake graphs, T-paths, dimer models, triangulations of surfaces)
- Representation theory of finite dimensional algebras (cluster categories, cluster-tilted algebras, preprojective algebras, tilting theory, 2-Calabi–Yau categories, invariant theory)
- Poisson geometry and algebraic geometry (cluster varieties, Grassmannians, stability conditions, scattering diagrams, Poisson structures on $\mathrm{SL}(n)$)
- Teichmüller theory (lambda-lengths, Penner coordinates, cluster varieties)
- Knot theory (Chern–Simons invariants, volume conjecture, Legendrian knots)
- Dynamical systems (frieze patterns, pentagram map, integrable systems, T-systems, sine-Gordon Y-systems)
- Mathematical Physics (statistical mechanics, Donaldson–Thomas invariants, quantum dilogarithm identities, BPS particles)

The relation between cluster algebras and representation theory has been established by the introduction of cluster categories in [3], and in [5] for type $\mathbb{A}$, as well as their generalizations in [1]. Furthermore, for every cluster of the cluster algebra, a finite-dimensional algebra, called cluster-tilted algebra, was introduced in [5] in type $\mathbb{A}$, and in [4] for all acyclic types. These algebras are endomorphism algebras of cluster-tilting objects in the cluster category.

This development has been very fruitful for both areas. In representation theory, it created the new theory of cluster-tilting which has known a considerable development over the last decade. One of its main subjects is the study of cluster-tilted algebras and their relation to tilted algebras established in [2]. This is the topic of the course by Ibrahim Assem in this volume.

I. Assem and S. Trepode (eds.), *Homological Methods, Representation Theory, and Cluster Algebras*, CRM Short Courses,
https://doi.org/10.1007/978-3-319-74585-5_3

On the other hand, cluster categories provide a categorification of the cluster algebra. This means that there is a map, called the cluster character, from the cluster category to the cluster algebra, which induces bijections between indecomposable rigid objects in the cluster category and cluster variables in the cluster algebra, and the direct sum of two objects in the cluster category is mapped to the product of their images in the cluster algebra. Moreover, the mutation rule in the cluster algebra is recovered by approximations in the cluster category. The cluster character is the subject of the course by Pierre-Guy Plamondon in this volume.

On the other hand, because of an intensive research over the last 15 years, the subject of cluster algebras itself has grown into an independent theory.

In this course, we will focus on cluster algebras from surfaces, a special class of cluster algebras. The first section is a short introduction to cluster algebras, and Sects. 2, 3, and 4 are devoted to cluster algebras from surfaces, especially to the expansion formulas for the cluster variables and the construction of canonical bases in terms of snake and band graphs.

1 Cluster Algebras

The definition of cluster algebras is elementary, but quite complicated. We describe it in this first section. Since these notes are aiming for cluster algebras from surfaces, we do not present the most general definition of cluster algebras, but restrict ourselves to so-called skew-symmetric cluster algebras with principal coefficients. For the general definition and further details we refer to [19].

1.1 *Ground Ring* $\mathbb{Z}\mathbb{P}$

To define a cluster algebra $\mathcal{A}$ we must first fix its ground ring.

Let $(\mathbb{P}, \cdot)$ be a free abelian group (written multiplicatively) on variables $y_1, \ldots, y_n$ and define an addition $\oplus$ in $\mathbb{P}$ by

$$\prod_j y_j^{a_j} \oplus \prod_j y_j^{b_j} = \prod_j y_j^{\min(a_j, b_j)}. \tag{1}$$

For example $y_1^2 y_2^{-3} y_3 \oplus 1 = y_2^{-3}$. Then $(\mathbb{P}, \oplus, \cdot)$ is a semifield.[1], and is called *tropical semifield*

Let $\mathbb{Z}\mathbb{P}$ denote the group ring of $\mathbb{P}$. Then $\mathbb{Z}\mathbb{P}$ is the ring of Laurent polynomials in the variables $y_1, \ldots, y_n$. The ring $\mathbb{Z}\mathbb{P}$ will be the ground ring for the cluster algebra.

Remark 1.1. If this is the first time you see cluster algebras, then you may consider the special case where $\mathbb{P} = 1$, and $\mathbb{Z}\,\mathbb{P} = \mathbb{Z}$ is just the ring of integers. In this case, we say the cluster algebra has *trivial coefficients*.

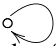
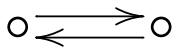
[1] This means that $\oplus$ is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$.

Let $\mathbb{QP}$ denote the field of fractions of $\mathbb{ZP}$ and let $\mathcal{F} = \mathbb{QP}(x_1, \dots, x_n)$ be the field of rational functions in n variables and coefficients in $\mathbb{QP}$.

Remark 1.2. In the case of trivial coefficients, we have $\mathbb{QP} = \mathbb{Q}$.

1.2 *Seeds and Mutations*

The cluster algebra is determined by the choice of an initial seed $(\mathbf{x}, \mathbf{y}, Q)$, which consists of the following data.

- Q is a quiver without loops  and 2-cycles , and with n vertices;
- $\mathbf{y} = (y_1, \dots, y_n)$ is the n-tuple of generators of $\mathbb{P}$, called *initial coefficient tuple*;
- $\mathbf{x} = (x_1, \dots, x_n)$ is the n-tuple of variables of $\mathcal{F}$, called *initial cluster*.

The triple $(\mathbf{x}, \mathbf{y}, Q)$ is called the *initial seed* of the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$.

The cluster algebra is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by so-called *cluster variables*, and these cluster variables are constructed from the initial seed by a recursive method called *mutation*. A mutation transforms a seed $(\mathbf{x}, \mathbf{y}, Q)$ into a new seed $(\mathbf{x}', \mathbf{y}', Q')$. Given any seed there are n different mutations $\mu_1, \dots, \mu_n$, one for each vertex of the quiver, or equivalently, one for each cluster variable in the cluster.

The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, Q)$ into the seed $\mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}', \mathbf{y}', Q')$ defined as follows:

- $\mathbf{x}'$ is obtained from $\mathbf{x}$ by replacing one cluster variable by a new one, $\mathbf{x}' = \mathbf{x} \setminus \{x_k\} \cup \{x_k'\}$, and x_k' is defined by the following *exchange relation*:

$$x_k x_k' = \frac{1}{y_k \oplus 1} \left(y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i \right) \tag{2}$$

  where the first product runs over all arrows in Q that end in k and the second product runs over all arrows that start in k.
- $\mathbf{y}' = (y_1', \dots, y_n')$ is a new coefficient n-tuple, where

$$y_j' = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j \prod\limits_{k \to j} y_k (y_k \oplus 1)^{-1} \prod\limits_{k \leftarrow j} (y_k \oplus 1) & \text{if } j \neq k. \end{cases}$$

  Note that one of the two products is always trivial, hence equal to 1, since Q has no oriented 2-cycles. Also note that $\mathbf{y}'$ depends only on $\mathbf{y}$ and Q.
- The quiver Q' is obtained from Q in three steps:

  (1) for every path $i \to k \to j$ add one arrow $i \to j$,
  (2) reverse all arrows at k,
  (3) delete 2-cycles.

See Fig. 1 for three examples of quiver mutations.

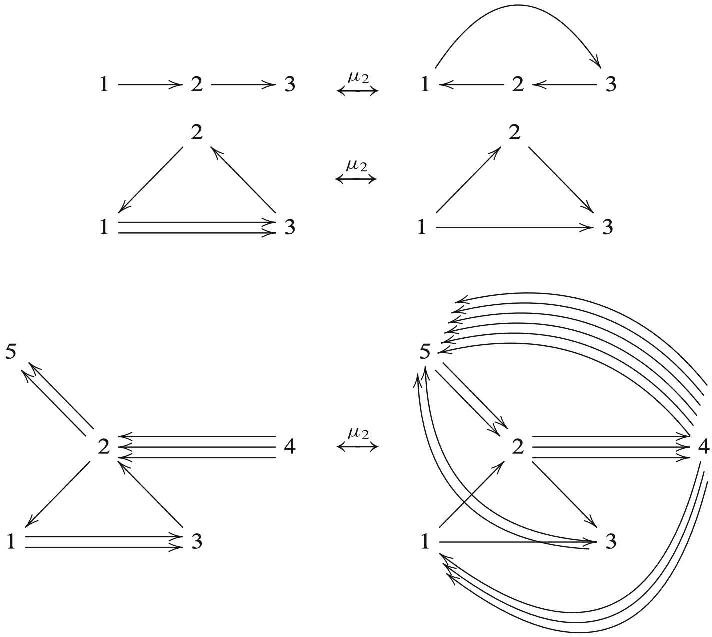


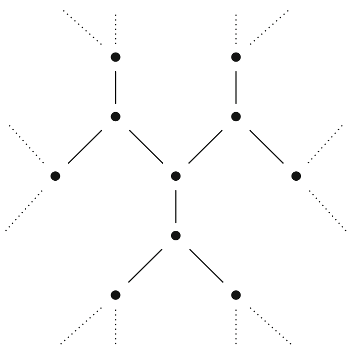
Fig. 1 Examples of quiver mutations

Lemma 1.3. *Mutations are involutions, that is,* $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}, \mathbf{y}, Q)$.

Note that Q' only depends on Q, that $\mathbf{y}'$ depends on $\mathbf{y}$ and Q, and that $\mathbf{x}'$ depends on the whole seed $(\mathbf{x}, \mathbf{y}, Q)$.

It is convenient to picture the mutation procedure in the so-called *exchange graph*. The vertices of this graph are the seeds of the cluster algebra and the edges are the mutations. Since we can always mutate in n directions, each vertex in the exchange graph has exactly n neighbors. See Fig. 2 for an example with $n = 3$.

Fig. 2 A 3-regular graph



The initial seed is one of the vertices in this graph. Applying the first n mutations to this seed, will produce the n neighbors of this vertex in the graph, each of which contains exactly one new cluster variable. So at this stage we have $2n$ cluster variables. Now we can continue mutating these new seeds, and at every step we construct a “new” cluster variable. It may happen, that we obtain a seed that has already appeared previously in this process. In that case we identify the two corresponding vertices in the n-regular graph, and the actual exchange graph is a quotient of the graph in Fig. 2. Such a repetition may happen but it does not have to, and in general the number of seeds is infinite. The whole pattern is determined by the initial seed.

1.3 *Definition*

Let $\mathcal{X}$ be the set of all cluster variables obtained by mutation from $(\mathbf{x}, \mathbf{y}, Q)$. The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$.

By definition, the elements of $\mathcal{A}$ are polynomials in $\mathcal{X}$ with coefficients in $\mathbb{ZP}$, so $\mathcal{A} \subset \mathbb{ZP}[\mathcal{X}]$. On the other hand, $\mathcal{A} \subset \mathcal{F}$, so the elements of $\mathcal{A}$ are also rational functions in $x_1, \ldots, x_n$ with coefficients in $\mathbb{QP}$.

Remark 1.4. *Fomin and Zelevinsky define cluster algebras in a more general setting using skew-symmetrizable matrices instead of quivers. The quiver definition corresponds to the special case where the matrices are skew-symmetric.*

1.4 *Example* 1 → 2

Let $(\mathbf{x}, \mathbf{y}, Q) = ((x_1, x_2), (1, 1), 1 \to 2)$.

Since the coefficients in this example are trivial, the coefficients in any seed will be $\{1, 1\}$. We therefore omit them in the computation below. Start with the initial seed.

$$(x_1, x_2)\ , 1 \to 2\ .$$

Apply mutation μ_1.

$$\left(\frac{x_2 + 1}{x_1}, x_2\right)\ , 1 \leftarrow 2\ .$$

Apply mutation μ_2.

$$\left(\frac{x_2 + 1}{x_1}, \frac{x_2 + 1 + x_1}{x_1 x_2}\right), 1 \to 2$$

Apply mutation μ_1. Let us do this step in detail. We get

$$\frac{\frac{x_2+1+x_1}{x_1x_2} + 1}{\frac{x_2+1}{x_1}} = \frac{(x_2 + 1 + x_1 + x_1x_2)x_1}{x_1x_2(x_2 + 1)} = \frac{(x_2 + 1)(x_1 + 1)}{x_2(x_2 + 1)} = \frac{x_1 + 1}{x_2}\ .$$

Note that the denominator is a monomial. Thus the new seed is

$$\left(\frac{x_1+1}{x_2}, \frac{x_2+1+x_1}{x_1x_2}\right), 1 \leftarrow 2 \ .$$

Apply mutation μ_2.

$$\left(\frac{x_1+1}{x_2}, x_1\right), 1 \rightarrow 2 \ .$$

Apply mutation μ_1.

$$(x_2, x_1) \ , 1 \leftarrow 2 \ .$$

Continuing the process from here will not yield new cluster variables. Thus in this case, there are exactly 5 cluster variables

$$x_1 \ , \quad x_2 \ , \quad \frac{x_2+1}{x_1} \ , \quad \frac{x_2+1+x_1}{x_1x_2} \ , \quad \frac{x_1+1}{x_2} \ .$$

1.5 *Example*

Now consider the quiver $Q = 1 \rightrightarrows 2 \rightrightarrows 3$. Let us rather write it as $1 \xrightarrow{2} 2 \xrightarrow{2} 3$, where the number on the arrow from i to j indicates the number of arrows from i to j. Again we use trivial coefficients, so our initial seed is

$$(x_1, x_2, x_3) \ , \ 1 \xrightarrow{2} 2 \xrightarrow{2} 3 \ .$$

Apply mutation in (2).

$$\left(x_1, \frac{x_1^2+x_3^2}{x_2}, x_3\right) \ , \ \begin{tikzcd} 1 \arrow[rr, bend left, "4"] & 2 \arrow[l, "2"'] & 3 \arrow[l, "2"'] \end{tikzcd} \ .$$

Apply mutation in (1).

$$\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2 + x_3^4}{x_1}, \frac{x_1^2+x_3^2}{x_2}, x_3\right) \ , \ \begin{tikzcd} 1 \arrow[r, "2"] & 2 \arrow[r, "6"] & 3 \arrow[ll, bend right, "4"'] \end{tikzcd} \ .$$

Apply mutation in (3).

$$\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2 + x_3^4}{x_1}, \frac{x_1^2+x_3^2}{x_2}, \frac{\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}\right)^4 + \left(\frac{x_1^2+x_3^2}{x_2}\right)^6}{x_3} \right), \quad 1 \xleftarrow{22} 2 \xleftarrow{6} 3 \ \ (1 \xrightarrow{4} 3).$$

Apply mutation in (2). Then the new variable and the new quiver are

$$\frac{\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}\right)^{22} + \left(\frac{\left(\frac{\left(\frac{x_1^2+x_3^2}{x_2}\right)^2+x_3^4}{x_1}\right)^4 + \left(\frac{x_1^2+x_3^2}{x_2}\right)^6}{x_3}\right)^6}{\frac{x_1^2+x_3^2}{x_2}}, \quad 1 \xrightarrow{22} 2 \xrightarrow{6} 3 \ \ (3 \xrightarrow{128} 1).$$

It is probably clear by now that each new cluster variable we obtain in this example is more complicated than the previous ones. Thus this cluster algebra has infinitely many cluster variables. It is also clear that the quivers we produce will have more and more arrows, and thus there are also infinitely many quivers in this example. Finally, you should be convinced by now that computations in cluster algebras are rather involved in general.

1.6 *Laurent Phenomenon and Positivity*

Theorem 1.5 ([17]). *Let $u \in \mathcal{X}$ be any cluster variable. Then*

$$u = \frac{f(x_1, \ldots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}}$$

where $f \in \mathbb{Z}\mathbb{P}[x_1, \ldots, x_n]$, $d_i \in \mathbb{Z}$.

Remark 1.6. This is a surprising result, since, a priori, the cluster variables are rational functions in the variables $x_1, \ldots, x_n$. The theorem says that the denominators of these rational functions are actually monomials. This means that at each mutation, when we have to divide a binomial of cluster variables by a certain cluster variable x', the numerator of that cluster variable x' is actually a factor of that binomial. Note that the numerator of x' may be a complicated polynomial. We have already observed this phenomenon in the third step of Example 1.4. Try to see this phenomenon in the last step of Example 1.5.

Moreover we have the following positivity result.

Theorem 1.7 ([21]). *The coefficients of the Laurent polynomials in Theorem 1.5 are positive in the sense that* $f \in \mathbb{Z}_{\geq 0}\mathbb{P}[x_1, \ldots, x_n]$.

Remark 1.8. This result is not obvious; and actually 13 years have passed between the proof of Theorem 1.5 and the proof of Theorem 1.7. Although the binomial exchange relations only involve positive terms, one has to make sure that positivity is preserved when reducing the rational functions that one obtains in the mutation procedure to the Laurent polynomials in the theorems. This is not true for arbitrary rational functions as the example $\frac{x^3+1}{x+1} = x^2 - x + 1$ shows.

1.7 *Classifications*

There are several special types of cluster algebras that have been studied using very different methods. We define these types here and show how they overlap.

We say that two quivers Q, Q' are mutation equivalent, and write $Q \sim Q'$, if there exists a finite sequence of mutations transforming Q into Q'.

Definition 1.9. A cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ is said to be of

(a) *finite type* if the number of cluster variables is finite;
(b) *finite mutation type* if the number of quivers Q' that are mutation equivalent to Q is finite;
(c) *acyclic type* if Q is mutation equivalent to a quiver without oriented cycles;
(d) *surface type* if Q is the adjacency quiver of a triangulation of a marked surface (see Sect. 2).

If a cluster algebra is of finite type then it is also of finite mutation type, thus (a) implies (b). We will see below that (a) also implies (c). The relation between all 4 classes are illustrated in Fig. 3.

The cluster algebra of Example 1.4 is of finite type, finite mutation type, acyclic type, and surface type. On the other hand, the cluster algebra of Example 1.5 is not of finite type, not of finite mutation type, not of surface type, but it is of acyclic type.

Fomin and Zelevinsky showed that the finite-type cluster algebras are classified by the Dynkin diagrams. Since we are considering only cluster algebras given by quivers, we only get the simply laced Dynkin diagrams.

Theorem 1.10 ([18]). *The cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ *is of finite type if and only if* Q *is mutation equivalent to a quiver of Dynkin type* $\mathbb{A}, \mathbb{D}$ *or* $\mathbb{E}$.

Finite mutation type is more general than finite type. The finite mutation-type classification is due to Felikson, Shapiro, and Tumarkin.

Theorem 1.11 ([13]). *The cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$ *is of finite mutation type if and only if*

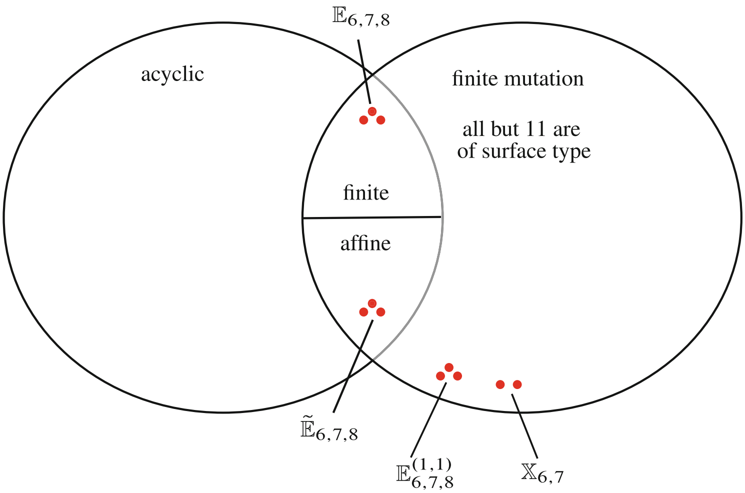


Fig. 3 Different types of cluster algebras of rank $n \geq 3$

- *it is of surface type, or*
- $n \leq 2$*, or*
- *it is one of* 11 *exceptional types* $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8, \mathbb{E}_6^{(1,1)}, \mathbb{E}_7^{(1,1)}, \mathbb{E}_8^{(1,1)}, \mathbb{X}_6, \mathbb{X}_7$.

The overlaps of the various classes of cluster algebras for $n \geq 3$ are illustrated in Fig. 3. The acyclic and the mutation finite types have in common the finite types $\mathbb{A}, \mathbb{D}, \mathbb{E}$ and the tame types corresponding to the extended Dynkin diagrams $\widetilde{\mathbb{A}}, \widetilde{\mathbb{D}}, \widetilde{\mathbb{E}}$, also known as affine Dynkin diagrams. Other acyclic types are called wild. The 11 exceptions in Theorem 1.11 are indicated by dots; 9 of them correspond to root systems of certain $\mathbb{E}$-types. The other 2 types $\mathbb{X}_6, \mathbb{X}_7$ had not appeared elsewhere before this classification.

2 Cluster Algebras of Surface Type

Building on work of Fock and Goncharov [14, 15], and of Gekhtman, Shapiro and Vainshtein [20], Fomin, Shapiro, and Thurston [16] associated a cluster algebra to any *bordered surface with marked points*.

2.1 *Marked Surfaces*

We fix the following notation.

- S is a connected, oriented Riemann surface with (possibly empty) boundary ∂S.
- $M \subset S$ is a finite set of *marked points* with at least one marked point on each connected component of the boundary.

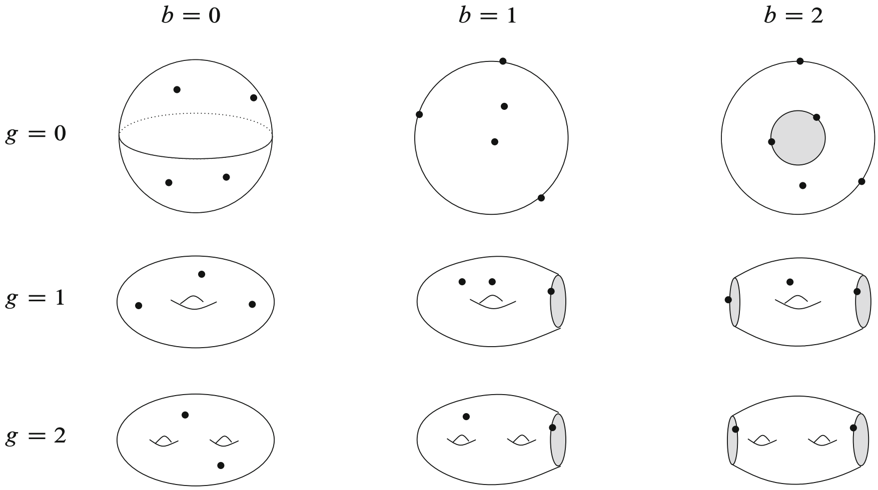


Fig. 4 Examples of surfaces, g is the genus and b is the number of boundary components

We will refer to the pair (S, M) simply as a *surface*. A surface is called *closed* if the boundary is empty. Marked points in the interior of S are called *punctures*. Examples are shown in Fig. 4. For technical reasons, we require that (S, M) is not a sphere with 1, 2, or 3 punctures; a monogon with 0 or 1 puncture; or a bigon or triangle without punctures.

Remark 2.1. In Sects. 3 and 4, we will restrict to surfaces without punctures. The reason for this restriction in Sect. 3 is only for the sake of simplicity, but in Sect. 4 it is necessary. In the appendix, we explain how to modify the results in Sect. 3 in the presence of punctures.

2.2 *Arcs and triangulations*

An *arc* γ in (S, M) is a curve in S, considered up to isotopy,[2] such that

(a) the endpoints of γ are in M;
(b) except for the endpoints, γ is disjoint from M and from ∂S;
(c) γ does not cut out an unpunctured monogon or an unpunctured bigon;
(d) γ does not cross itself, except that its endpoints may coincide.

A *generalized arc* is a curve which satisfies conditions (a), (b), and (c), but it can have self-crossings. Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are called *boundary segments*. By (c),

[2] A *homotopy* between two continuous maps $f, g: X \to Y$ is a continuous map $h: [0,1] \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$. An *isotopy* is a homotopy h such that for all $t \in [0, 1]$ the map $h(t, -): X \to h(t, X)$ is a homeomorphism. In particular an isotopy of curves cannot create self-crossings.

boundary segments are not arcs. A *closed loop* is a closed curve in S which is disjoint from M and the boundary of S.

For any two arcs γ, γ' in S, define

$$e(\gamma, \gamma') = \min\{\text{number of crossings of } \alpha \text{ and } \alpha' \mid \alpha \simeq \gamma, \alpha' \simeq \gamma'\},$$

where α and α' range over all arcs isotopic to γ and γ', respectively. We say that arcs γ and γ' are *compatible* if $e(\gamma, \gamma') = 0$.

An *ideal triangulation* is a maximal collection of pairwise compatible arcs (together with all boundary segments). The arcs of a triangulation cut the surface into *ideal triangles*. Triangles that have only two distinct sides are called *self-folded* triangles. Note that a self-folded triangle consists of a loop ℓ, together with an arc r to an enclosed puncture which we call a *radius*. Examples of ideal triangulations are given in Fig. 5 as well as in Figs. 12 and 13.

Lemma 2.2. *The number of arcs in an ideal triangulation is exactly*

$$n = 6g + 3b + 3p + c - 6\ ,$$

where g is the genus of S, b is the number of boundary components, p is the number of punctures, and $c = |M| - p$ is the number of marked points on the boundary of S. The number n is called the rank *of* (S, M).

For the sake of completeness, we include a proof of this fact, since it is usually omitted in the cited research papers.

Proof. Recall that the Euler characteristic of a surface S is given by $\chi(S) = f - e + v$, where v is the number of vertices, e is the number of edges, and f is the number of faces in any triangulation of S. By induction on the genus, one can show that for a closed surface $\chi(S) = 2 - 2g$. Moreover, if the boundary of S has b connected components then

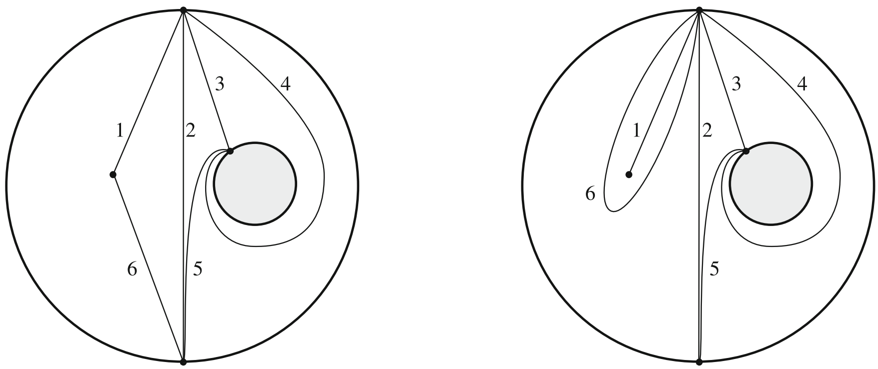


Fig. 5 Two ideal triangulations of a punctured annulus related by a flip of the arc 6. The triangulation on the right-hand side has a self-folded triangle

$$\chi(S) = 2 - 2g - b\,, \tag{3}$$

since removing a disk from S can be thought of reducing the number of faces by one. Now consider a set of marked points M and a triangulation T. Then the number of vertices in T is $|M| = c + p$. The number of edges in T is the number of arcs n plus the number of boundary segments c. Thus

$$e = c + n \quad \text{and} \quad v = c + p\,. \tag{4}$$

We use induction on p. If $p = 0$, then each triangle has 3 distinct sides. Each of the n arcs lies in precisely 2 triangles and each of the c boundary segments lies in precisely 1 triangle. Therefore

$$3f = 2n + c\,. \tag{5}$$

Using (3)–(5) we get

$$\frac{2n+c}{3} - c - n + c = 2 - 2g - b\,,$$

and the statement follows.

Now suppose that $p > 0$. Let T be a triangulation of (S, M), and let us add a puncture x. Then we need to add 3 arcs to complete the triangulation. Indeed, the new puncture x lies in some triangle Δ of the old triangulation T and connecting x with the three vertices of Δ completes the triangulation. Thus adding a puncture increases n by 3. □

Ideal triangulations are connected to each other by sequences of *flips*. Each flip replaces a single arc γ in T by a unique new arc $\gamma' \neq \gamma$ such that

$$T' = (T \setminus \{\gamma\}) \cup \{\gamma'\}$$

is a triangulation. See Fig. 6.

2.3 *Cluster Algebras from Surfaces*

We are now ready to define the cluster algebra associated to the surface. For that purpose, we choose an ideal triangulation $T = \{\tau_1, \tau_2, \ldots, \tau_n\}$ and then define a quiver Q_T without loops or 2-cycles as follows. The vertices of Q_T are in bijection with the arcs of T, and we denote the vertex of Q_T corresponding to the arc τ_i simply by i. The arrows of Q_T

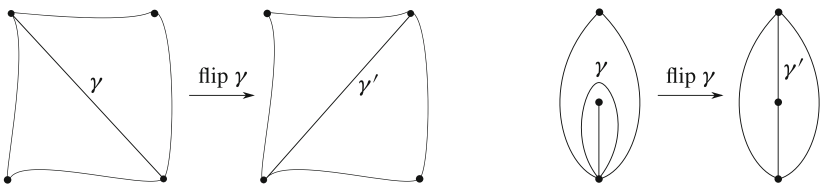


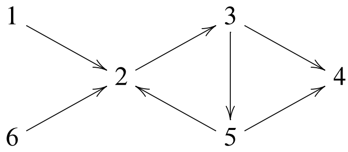
Fig. 6 Two examples of flips

are defined as follows. For any triangle Δ in T which is not self-folded, we add an arrow $i \to j$ whenever

(a) τ_i and τ_j are sides of Δ with τ_j following τ_i in the clockwise order;
(b) τ_j is a radius in a self-folded triangle enclosed by a loop τ_ℓ, and τ_i and τ_ℓ are sides of Δ with τ_ℓ following τ_i in the clockwise order;
(c) τ_i is a radius in a self-folded triangle enclosed by a loop τ_ℓ, and τ_ℓ and τ_j are sides of Δ with τ_j following τ_ℓ in the clockwise order.

Then we remove all 2-cycles.

For example, the quiver corresponding to the triangulation on the right of Fig. 5 is



To define an initial seed, we associate an indeterminate x_i to each $\tau_i \in T$ and set the initial cluster $\mathbf{x}_T = (x_1, \ldots, x_n)$; and we set the initial coefficient tuple $\mathbf{y}_T = (y_1, \ldots, y_n)$ to be the vector of generators of $\mathbb{P}$. Then the cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ is called the *cluster algebra associated to the surface* (S, M) *with principal coefficients in* T.

Fomin, Shapiro, and Thurston showed that, up to a change of coefficients, the cluster algebra does not depend on the choice of the initial triangulation T. Moreover, they proved the following correspondence.

Theorem 2.3 ([16]). *There are bijections*

$$\begin{array}{ccc} \{\text{cluster variables of } \mathcal{A}\} & \longleftrightarrow & \{\text{tagged arcs of } (S,M)\} \\ x_\gamma & & \gamma \\ \{\text{clusters of } \mathcal{A}\} & \longleftrightarrow & \{\text{triangulations of } (S,M)\} \\ \mathbf{x}_T = \{x_{\gamma_1}, \ldots, x_{\gamma_n}\} & & T = \{\gamma_1, \ldots, \gamma_n\} \end{array}$$

Moreover, if γ_k is not the radius of a self-folded triangle in a triangulation T, then the mutation in k corresponds to the flip of the arc γ_k, that is, the cluster

$$\mu_k(\mathbf{x}_T) = (\mathbf{x}_T \setminus \{x_{\gamma_k}\}) \cup \{x'_{\gamma_k}\}$$

corresponds to the triangulation

$$\mu_{\gamma_k}(T) = (T \setminus \{\gamma_k\}) \cup \{\gamma'_k\} .$$

Remark 2.4. For simplicity, we excluded the case where γ_k is the radius of a self-folded triangle because then γ_k cannot be flipped. In [16] the authors solve this problem by introducing tagged arcs and tagged triangulations, replacing the loop of a self-folded triangle by a second radius. In that setup the theorem holds without any restrictions.

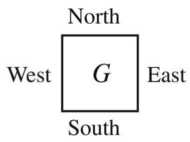
Remark 2.5. If β is a boundary segment, we set $x_\beta = 1$.

3 Snake Graphs and Expansion Formulas

Abstract snake graphs and band graphs were introduced and studied in [7–10] motivated by the snake graphs and band graphs appearing in the combinatorial formulas for cluster algebra elements in [22–24, 26]. Throughout we fix the standard orthonormal basis of the plane.

3.1 *Snake graphs*

A *tile* G is a square in the plane whose sides are parallel or orthogonal to the elements in the fixed basis. All tiles considered will have the same side length.



We consider a tile G as a graph with four vertices and four edges in the obvious way. A *snake graph* $\mathcal{G}$ is a connected planar graph consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$, with $d \geq 1$, such that for each i, the tiles G_i and G_{i+1} share exactly one edge e_i, and this edge is either the north edge of G_i and the south edge of G_{i+1}, or it is the east edge of G_i and the west edge of G_{i+1}. An example of a snake graph with 8 tiles is given in Fig. 7.

The graph consisting of two vertices and one edge joining them is also considered a snake graph. Figure 8 lists all snake graphs with at most 4 tiles.

3.1.1 Some Notation and Terminology

Let $\mathcal{G} = (G_1, G_2, \ldots, G_d)$ be a snake graph.

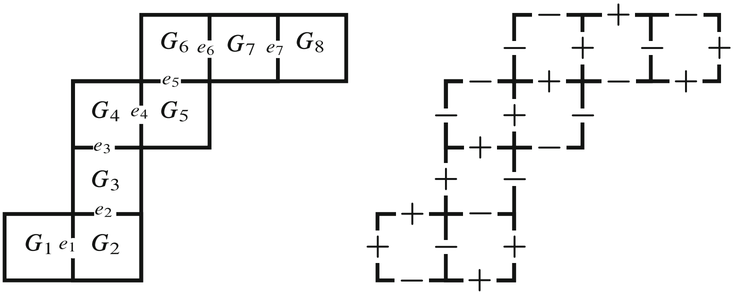


Fig. 7 A snake graph with 8 tiles and 7 interior edges (left); a sign function on the same snake graph (right)

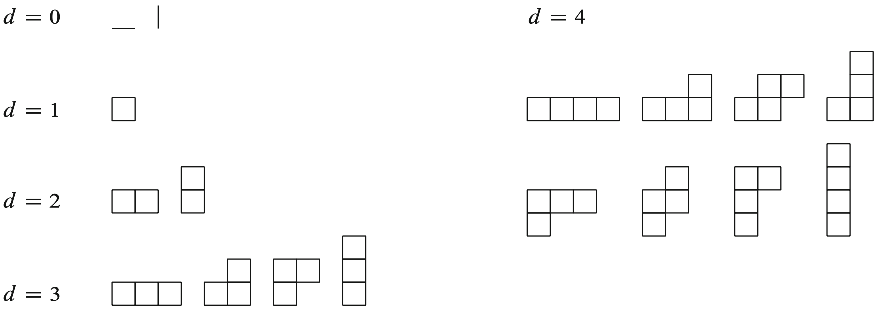


Fig. 8 A list of all snake graphs with at most 4 tiles

- The $d-1$ edges $e_1, e_2, \ldots, e_{d-1}$ which are contained in two tiles are called *interior edges* of $\mathcal{G}$ and the other edges are called *boundary edges*. We will always use the natural ordering of the set of interior edges, so that e_i is the edge shared by the tiles G_i and G_{i+1}.
- Let $\operatorname{Int}\mathcal{G} = \{e_1, \ldots, e_{d-1}\}$ denote the set of all interior edges of $\mathcal{G}$.
- Let ${}_{\mathrm{SW}}\mathcal{G}, \mathcal{G}^{\mathrm{NE}}$ denote the following sets.

$$
\begin{aligned}
{}_{\mathrm{SW}}\mathcal{G} &= \{\text{south edge of } G_1, \text{west edge of } G_1\}\ ;\\
\mathcal{G}^{\mathrm{NE}} &= \{\text{north edge of } G_d, \text{east edge of } G_d\}.
\end{aligned}
$$

  If $\mathcal{G}$ is a single edge, we let ${}_{\mathrm{SW}}\mathcal{G} = \varnothing$ and $\mathcal{G}^{\mathrm{NE}} = \varnothing$.
- We say that two snake graphs are *isomorphic* if they are isomorphic as graphs.

3.1.2 Sign Function

A *sign function* f on a snake graph $\mathcal{G}$ is a map

$$f: \{\text{edges of } \mathcal{G}\} \to \{+, -\}\ ,$$

such that on every tile in $\mathcal{G}$

- the north and the west edge have the same sign,
- the south and the east edge have the same sign,
- the sign on the north edge is opposite to the sign on the south edge.

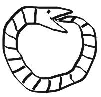
See Fig. 7 for an example. Note that on every snake graph there are exactly two sign functions.

A snake graph is determined (up to a reflection along the diagonal $y = x$ in the plane) by its sequence of tiles together with a sign function f on the interior edges. Indeed, three consecutive tiles G_{i-1}, G_i, G_{i+1} form a straight subsnake graph if and only if on the interior edges e_{i-1}, e_i the sign alternates, thus $f(e_{i-1}) = -f(e_i)$. Similarly, three consecutive tiles G_{i-1}, G_i, G_{i+1} form a zigzag subsnake graph if and only if $f(e_{i-1}) = f(e_i)$.

3.2 *Band Graphs*

Band graphs are obtained from snake graphs by identifying a boundary edge of the first tile with a boundary edge of the last tile, where both edges have the same sign. We use the notation $\mathcal{G}^\circ$ for general band graphs, indicating their circular shape, and we also use the notation $\mathcal{G}^b$ if we know that the band graph is constructed by gluing a snake graph $\mathcal{G}$ along an edge b.

More precisely, to define a band graph $\mathcal{G}^\circ$, we start with an abstract snake graph $\mathcal{G} = (G_1, G_2, \ldots, G_d)$ with $d \geq 1$, and fix a sign function on $\mathcal{G}$. Denote by x the southwest vertex of G_1, let $b \in {}_{\mathrm{SW}}\mathcal{G}$ be the south edge (respectively the west edge) of G_1, and let y denote the other endpoint of b, see Fig. 9. Let b' be the unique edge in $\mathcal{G}^{\mathrm{NE}}$ that has the same sign as b, and let y' be the northeast vertex of G_d and x' the other endpoint of b'.



Let $\mathcal{G}^b$ denote the graph obtained from $\mathcal{G}$ by identifying the edge b with the edge b' and the vertex x with x' and y with y'. The graph $\mathcal{G}^b$ is called a *band graph* or *ouroboros*.[3] Note that non-isomorphic snake graphs can give rise to isomorphic band graphs. See Fig. 9 for examples.

The interior edges of the band graph $\mathcal{G}^b$ are by definition the interior edges of $\mathcal{G}$ plus the gluing edge $b = b'$. A band graph is uniquely determined by its sequence of tiles $G_1, \ldots, G_d$ together with its sign function on the interior edges (including the gluing edge).

In order to be able to formally take sums of band graphs we make the following definition.

Definition 3.1. Let $\mathcal{R}$ denote the free abelian group generated by all isomorphism classes of finite disjoint unions of snake graphs and band graphs. If $\mathcal{G}$ is a snake graph, we also denote its class in $\mathcal{R}$ by $\mathcal{G}$, and we say that $\mathcal{G} \in \mathcal{R}$ is a *positive* snake graph and that its inverse $-\mathcal{G} \in \mathcal{R}$ is a *negative* snake graph.

3.3 *From Snake Graphs to Surfaces*

Given a snake graph $\mathcal{G} = (G_1, G_2, \ldots, G_d)$, we can construct a triangulated polygon as follows.

- In each tile G_i add a diagonal τ_i from the northwest corner to the southeast corner.

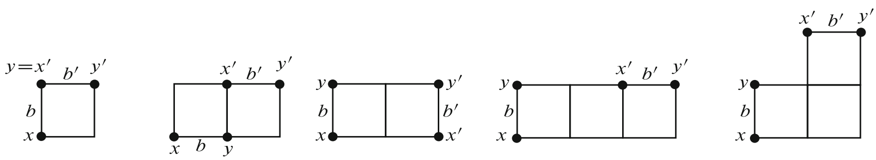


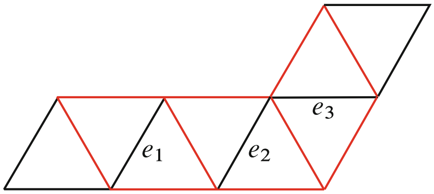
Fig. 9 Examples of small band graphs; the two band graphs with 3 tiles are isomorphic

[3] Ouroboros: a snake devouring its tail.

- Tilt the snake graph such that each tile becomes a parallelogram consisting of two equilateral triangles.
- Fold the snake graph along the interior edges $e_1, \dots, e_{d-1}$, and, at each folding, identify the two triangles on either side of the interior edge.

This produces a surface that is homeomorphic to a triangulated polygon with $d+3$ vertices whose set of boundary segments is precisely ${}_{\mathrm{SW}}\mathcal{G} \cup \mathrm{Int}\ \mathcal{G} \cup \mathcal{G}^{\mathrm{NE}}$ and whose triangulation is given by the diagonals $\tau_1, \dots, \tau_d$.

Exercise 3.2. Cut out the following figure and perform the folding. Interior edges are black.



A *labeled* snake graph is a snake graph in which each edge and each tile carries a label or weight. For example, for snake graphs from cluster algebras of surface type, these labels are cluster variables.

3.4 *Labeled Snake Graphs from Surfaces*

Now we want to go the other way and associate a snake graph to every arc in a triangulated surface.

Let T be an ideal triangulation of a surface (S, M) and let γ be an arc in (S, M) which is not in T. Choose an orientation on γ, let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. Denote by

$$s = p_0, p_1, p_2, \dots, p_{d+1} = t$$

the points of intersection of γ and T in order. For $j = 1, 2, \dots, d$, let τ_{i_j} be the arc of T containing p_j, and let Δ_{j-1} and Δ_j be the two ideal triangles in T on either side of τ_{i_j}. Then, for $j = 1, \dots, d-1$, the arcs τ_{i_j} and $\tau_{i_{j+1}}$ form two sides of the triangle Δ_j in T and we define e_j to be the third arc in this triangle, see Fig. 10.

Let G_j be the quadrilateral in T that contains τ_{i_j} as a diagonal. We will think of G_j as a *tile* as in Sects. 3.1, but now the edges of the tile are arcs in T and thus are labeled edges. We also think of the tile G_j itself being labeled by the diagonal τ_{i_j}.

Define a *sign function* f on the edges $e_1, \dots, e_{d-1}$ by

$$f(e_j) = \begin{cases} +1 & \text{if } e_j \text{ lies on the right of } \gamma \text{ when passing through } \Delta_j \\ -1 & \text{otherwise.} \end{cases}$$

The labeled snake graph $\mathcal{G}_\gamma = (G_1, \dots, G_d)$ with tiles G_i, interior edges $e_1, e_2, \dots, e_{d-1}$, and sign function f is called the *snake graph associated to the arc* γ. Each edge e of

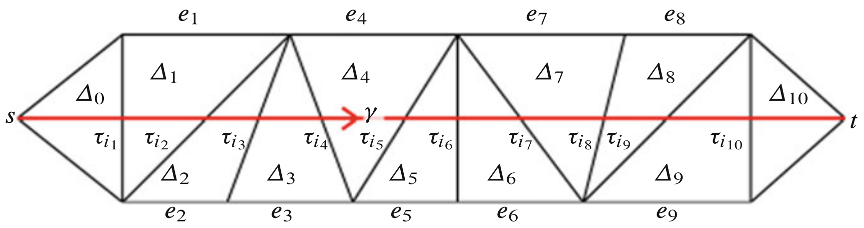


Fig. 10 The arc γ passing through $d + 1$ triangles $\Delta_0, \ldots, \Delta_d$. The corresponding snake graph is given in Fig. 11

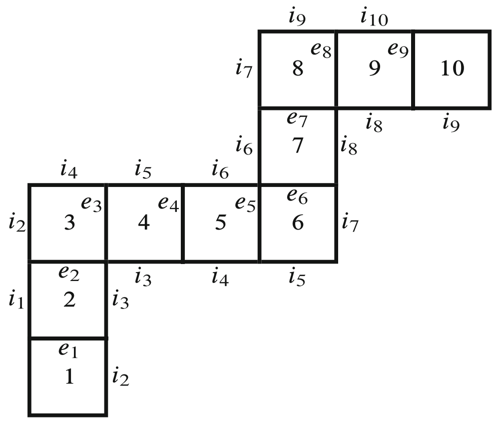


Fig. 11 The snake graph corresponding to the arc γ in Fig. 10

$\mathcal{G}_\gamma$ is labeled by an arc $\tau(e)$ of the triangulation T. We define the *weight* $x(e)$ of the edge e to be the cluster variable associated to the arc $\tau(e)$. Thus $x(e) = x_{\tau(e)}$.

Note that we can define a sign function f in the same way for any closed loop ζ. In that case we define the *band graph* $\mathcal{G}^\circ_\zeta$ of ζ to be the band graph with tiles G_i and sign function f.

3.5 *Perfect Matchings, Height, and Weight*

A *perfect matching* of a graph $\mathcal{G}$ is a subset P of the edges of $\mathcal{G}$ such that each vertex of $\mathcal{G}$ is incident to exactly one edge of P. We define

$$\text{Match } \mathcal{G} = \{\text{perfect matchings of } \mathcal{G}\} .$$

If $\mathcal{G}^\circ = \mathcal{G}^b$ is a band graph, we define Match $\mathcal{G}^\circ$ to be the set of all perfect matchings P of the snake graph $\mathcal{G}$ such that P is a perfect matching of $\mathcal{G}^\circ$, where we identify the two gluing edges.

Each snake graph $\mathcal{G}$ has precisely two perfect matchings P_-, P_+ that contain only boundary edges. We call P_- the minimal matching and P_+ the maximal matching of $\mathcal{G}$.[4]

$P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$ denotes the symmetric difference of an arbitrary perfect matching $P \in \operatorname{Match}\mathcal{G}$ with the minimal matching P_-.

Definition 3.3. Let $P \in \operatorname{Match}\mathcal{G}$. The set $P_- \ominus P$ is the set of boundary edges of a (possibly disconnected) subgraph $\mathcal{G}_P$ of $\mathcal{G}$, and $\mathcal{G}_P$ is a union of tiles

$$\mathcal{G}_P = \bigcup_i G_i \ .$$

We define the *height monomial* of P by

$$y(P) = \prod_{G_i \text{ a tile in } \mathcal{G}_P} y_i \ .$$

Thus $y(P)$ is the product of all y_i for which the tile G_i lies inside in $P \ominus P_-$ with multiplicities.

3.6 *Expansion Formula*

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a triangulation of (S, M) and let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ be the cluster algebra with principal coefficients at T. Thus $\mathbf{x}_T = (x_1, \ldots, x_n)$, $\mathbf{y}_T = (y_1, \ldots, y_n)$ and $\mathbb{P} = \operatorname{Trop}(y_1, \ldots, y_n)$.

For simplicity, we assume here that there are no self-folded triangles in T. For the general case see the appendix.

Theorem 3.4 ([23]). *Let γ be an arc not in the triangulation T. Then the cluster variable x_γ is equal to*

$$x_\gamma = \frac{1}{\operatorname{cross}(\gamma)} \sum_{P \in \operatorname{Match}\mathcal{G}_\gamma} x(P) y(P) \ ,$$

where $x(P) = \prod_{e \in P} x(e)$ is the weight of P, $y(P)$ is the height of P and $\operatorname{cross}(\gamma) = \prod_{j=1}^{d} x_{i_j}$.

Since this theorem gives us a direct formula for the cluster variables, it allows us to redefine the cluster algebra without using mutations as in the following corollary.

Corollary 3.5. *The cluster algebra $\mathcal{A}$ of the surface (S, M) with principal coefficients in the triangulation T is the $\mathbb{Z}\mathbb{P}$- subalgebra of $\mathcal{F}$ generated by all*

[4] There is a choice involved here which of the two is P_- and this will make a difference later when we consider expansion formulas for cluster algebras with nontrivial coefficients. One can determine P_- as follows. If a tile G_j in the snake graph has the same orientation as the corresponding quadrilateral in the surface S, then P_- contains the south and the north edges of G_j if they are boundary edges, and P_- does not contain the east or the west edge of G_j.

$$\frac{1}{\mathrm{cross}(\gamma)} \sum_{P \in \mathrm{Match}\, \mathcal{G}_\gamma} x(P)y(P),$$

where γ runs over all arcs in (S, M).

3.7 *Examples*

In the example in Fig. 12, we compute the snake graph $\mathcal{G}_\gamma$ of an arc γ in a triangulated polygon. The arc γ crosses two arcs of the triangulation, hence the snake graph $\mathcal{G}_\gamma$ has two tiles. The graph $\mathcal{G}_\gamma$ admits exactly 3 perfect matchings (drawn in red), and they form a linear poset in which P_- is the unique minimal element and P_+ is the unique maximal element. The corresponding monomials are listed in the rightmost column. Thus in this example we have

$$x_\gamma = \frac{x_1\, y_1 y_2 + x_3\, y_1 + x_2}{x_1 x_2}\,.$$

In the example in Fig. 13, we compute the snake graph $\mathcal{G}_\gamma$ of an arc γ in a triangulated annulus. The arc γ crosses the triangulation three times, twice in the arc labeled 1 and once in the arc labeled 2. Hence the snake graph $\mathcal{G}_\gamma$ has two tiles labeled 1 and one tile labeled 2. The graph $\mathcal{G}_\gamma$ admits exactly 5 perfect matchings drawn in red in the poset. The corresponding monomials are listed on the right of the poset. Thus in this example we have

$$x_\gamma = \frac{x_1^2\, y_1^2 y_2 + y_1^2 + 2x_2^2\, y_1 + x_2^4}{x_1^2 x_2}\,.$$

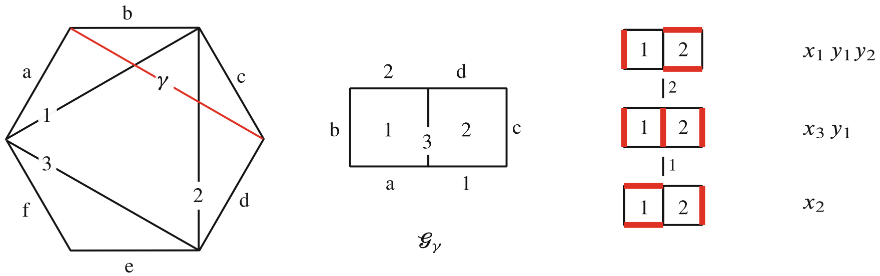


Fig. 12 An arc γ in a triangulated hexagon (left), its snake graph $\mathcal{G}_\gamma$ (center left), and its poset of perfect matchings (center right), and the corresponding monomials (right). The edges labeled a,b,c,d,e,f are boundary edges and their weights are one. The edges labeled 1,2,3 are arcs in the triangulation and their weights are the cluster variables x_1, x_2, x_3, respectively. The perfect matching P_- is the minimal element of the poset and P_+ is the maximal element

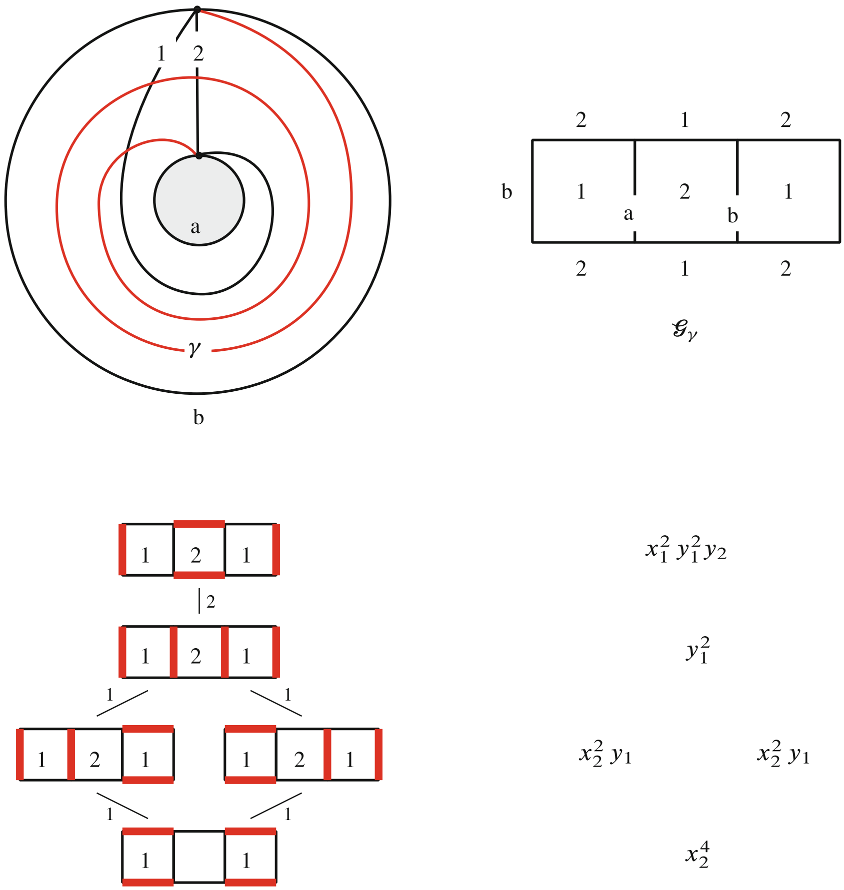


Fig. 13 An arc γ in a triangulated annulus (top left), its snake graph $\mathcal{G}_\gamma$ (top right), the poset of perfect matchings of $\mathcal{G}_\gamma$ (bottom left), and the corresponding monomials (bottom right). The edges labeled a,b are boundary edges and their weights are one. The edges labeled 1,2 are arcs in the triangulation and their weights are the cluster variables x_1, x_2, respectively. The perfect matching P_- is the minimal element of the poset and P_+ is the maximal element

4 Bases for the Cluster Algebra

We would like to have a unique way to write each element of the cluster algebra as a sum of elements of a fixed basis. Recall the mutation exchange relations (2) among the cluster variables

$$(y_k \oplus 1)\, x_k x_k' = y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i \ .$$

This is an example of an element of the cluster algebra that can be written in two different ways. We would like a *unique* way. In this section, we present two solutions to this problem for cluster algebras of surface type.

From now on let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ be a cluster algebra arising from a surface (S, M) with principal coefficients at a fixed triangulation T, and assume that the surface has no punctures.[5] Since the cluster algebra is generated by cluster variables, we need to understand (sums of) products of cluster variables. Since cluster variables are in bijection with arcs, products of cluster variables are in bijection with sets of arcs with multiplicities. We call a set of curves with multiplicities a *multicurve*.

Moreover, to each arc we have associated a snake graph which allows us to compute the cluster variable via the perfect matching formula of Theorem 3.4. Therefore a product of cluster variables can be computed by the same perfect matching formula, replacing the single snake graph by a union of snake graphs. We get the following diagram:

$$\begin{array}{ccc}
\text{product of cluster variables} & \longrightarrow & \text{multicurve} \\
\Vert & & \downarrow \\
\text{Laurent expansion} & \longleftarrow & \text{union of snake graphs}
\end{array}$$

$$\begin{array}{ccc}
\prod_{i=1}^{t} (x_{\gamma_i})^{\epsilon_i} & \longrightarrow & \bigsqcup_{i=1}^{t} \bigsqcup_{j=1}^{\epsilon_i} \{\gamma_i\} \\
\Vert & & \downarrow \\
\frac{1}{\mathrm{cross}(\mathcal{G})} \sum_{P \in \mathrm{Match}\, \mathcal{G}} x(P) y(P) & \longleftarrow & \mathcal{G} = \bigsqcup_{i=1}^{t} \bigsqcup_{j=1}^{\epsilon_i} \mathcal{G}_{\gamma_i}
\end{array}$$

4.1 *Skein Relations*

The relations among the cluster variables can be expressed on the level of arcs using smoothing operations [25] and on the level of snake graphs as resolutions [7, 8, 10].

Let γ_1 and γ_2 be two curves that cross at a point x. Then we define the *smoothing* of $\{\gamma_1, \gamma_2\}$ at x to be the pair of multicurves $\{\gamma_3, \gamma_4\}$ and $\{\gamma_5, \gamma_6\}$ obtained from $\{\gamma_1, \gamma_2\}$ by replacing the crossing $\times$ in a small neighborhood of x with the pair of segments $\asymp$ (respectively $\supset\subset$).

$$\{\gamma_1, \gamma_2\} \xrightarrow{\text{smoothing at } x} \{\gamma_3, \gamma_4\}, \{\gamma_5, \gamma_6\}\ .$$

If γ is a curve with a self-crossing at a point x, we also define the *smoothing* of γ at x to be the pair of curves γ_{34} and γ_{56} obtained from γ by the same local transformation. See Fig. 14 for examples of the smoothing operation.

It is important to notice that performing the smoothing operation on arcs may produce generalized arcs; and performing it on generalized arcs may produce closed loops. For

[5] So far, restricting to the case without punctures has been for the sake of simplicity. But now we really need to make this restriction.

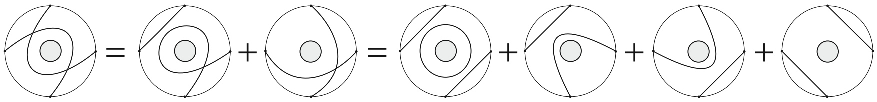


Fig. 14 An example of the smoothing operation. The multicurve on the left has two crossings. Smoothing one of the crossings yields the sum of the two multicurves in the center. Both of them still have one crossing. Smoothing these yields the sum of the 4 multicurves on the right

example, the first smoothing step in Fig. 14 produces a generalized arc, and the second step produces a closed loop.

A closed loop is called *essential* if it is not contractible and it has no self-crossing.

If ζ is an essential loop, we use the perfect matching formula for the band graph $\mathscr{G}^\circ_\zeta$ of ζ to define a Laurent polynomial x_ζ. That is

$$x_\zeta = \frac{1}{\text{cross}(\zeta)} \sum_{P \in \text{Match}\, \mathscr{G}^\circ_\zeta} x(P)y(P) \,.$$

If ζ is a contractible closed loop, we define $x_\zeta = -2$ (the integer -2). If γ is a curve that has a contractible kink then we set $x_\gamma = -x_{\bar{\gamma}}$, where $\bar{\gamma}$ is obtained from γ by contracting the kink to a point. Figure 15 illustrates that this definition is compatible with the smoothing of the self-crossing at the kink.

Theorem 4.1 ([25]).

(a) *If* $\{\gamma_3, \gamma_4\} \cup \{\gamma_5, \gamma_6\}$ *is obtained from* $\{\gamma_1, \gamma_2\}$ *by smoothing a crossing then*

$$x_{\gamma_1} x_{\gamma_2} = y_{34} x_{\gamma_3} x_{\gamma_4} + y_{56} x_{\gamma_5} x_{\gamma_6} \,,$$

for some coefficients $y_{34}, y_{56} \in \mathbb{P}$.

(b) *If the pair* γ_{34}, γ_{56} *is obtained from* γ *by smoothing a self-crossing then*

$$x_\gamma = y_{34} x_{\gamma_{34}} + y_{56} x_{\gamma_{56}} \,,$$

for some coefficients $y_{34}, y_{56} \in \mathbb{P}$.

Remark 4.2. (a) The equations in Theorem 4.1 are called skein relations.

(b) Theorem 4.1 was proved in [25] using hyperbolic geometry. A combinatorial proof in terms of snake and band graphs was given in [7, 8, 10].

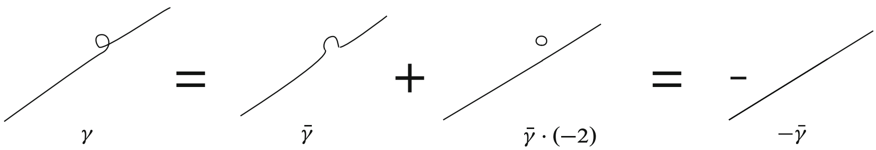


Fig. 15 A curve γ with a contractible kink is equal to the negative of the same curve with the kink removed s$\bar{\gamma}$

4.1.1 Smoothing of Arcs Versus Resolutions of Snake Graphs

The definition of smoothing is very simple. It is defined as a local transformation replacing a crossing $\times$ with the pair of segments $\asymp$ (resp. $\supset\subset$). But once this local transformation is done, one needs to find representatives inside the isotopy classes of the resulting curves which realize the minimal number of crossings with the fixed triangulation. This means that one needs to deform the obtained curves isotopically, and to "unwind" them if possible, in order to see their actual crossing pattern, which is crucial for the applications to cluster algebras. This can be quite complicated especially in a higher genus surface.

The situation for the snake and band graphs is exactly opposite. The definition of the resolution is very complicated because one has to consider many different cases. But once all these cases are worked out, one has a complete list of rules in hand, which one can apply very efficiently in actual computations. The reason for this is that the definitions of the resolutions already take into account the isotopy mentioned above.

For explicit computations in the cluster algebra, one always needs to construct the snake graphs in order to compute the Laurent polynomials. Thus for this purpose it is more efficient to work with resolutions of snake graphs.

We will not give the list of resolutions of snake graphs here. We refer to [7, 8, 10].

4.2 *Definition of the bases $\mathcal{B}^\circ$ and $\mathcal{B}$*

We have seen that in order to define a basis, we need to understand products of cluster variables, and that a product of cluster variables corresponds to a multicurve in the surface. Let k be the number of crossings in this multicurve. Then we can perform a smoothing operation at one of the crossings and obtain two multicurves, each of which will have at most $(k-1)$ crossings. Continuing this way, we can construct a collection of at most 2^k multicurves without crossings.

Now using the skein relations, we can perform each smoothing operation on the level of the Laurent polynomials. Thus we can express our original product of cluster variables as a sum of at most 2^k products $\prod_{\gamma\in C} x_\gamma$ where C is a multicurve without crossings.

We have also seen that the multicurves C that appear may contain arcs, boundary segments, and closed loops. Below, we shall characterize the set $\mathcal{B}^\circ$ of all multicurves C that arise in this way. Then the above argument shows that $\mathcal{B}^\circ$ spans the cluster algebra.

We introduce the following notation. See Fig. 16 for an illustration.

Definition 4.3. Let ζ be an essential loop.

(a) The union of k copies of ζ is called the *k-bangle of ζ* and is denoted by $\mathrm{Bang}_k\,\zeta$.
(b) The closed loop obtained by concatenating ζ with itself k times is called the *k-bracelet of ζ* and is denoted by $\mathrm{Brac}_k\,\zeta$.

Note that the k-bracelet $\mathrm{Brac}_k\,\zeta$ has exactly $k-1$ self-crossings.

Definition 4.4. (a) Let $\mathcal{B}^\circ$ be the set of all products $\prod_{\gamma\in C} x_\gamma$ where C ranges over all multicurves of arcs and essential closed loops without crossings.

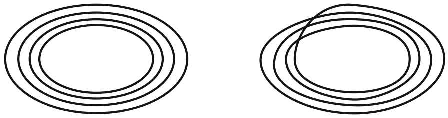


Fig. 16 A bangle $\mathrm{Bang}_4\,\zeta$ on the left and a bracelet $\mathrm{Brac}_4\,\zeta$ on the right

(b) Let $\mathcal{B}$ be the set of all products $\prod_{\gamma\in C} x_\gamma$ where C ranges over all collections of arcs and bracelets such that

- no two elements of C cross, except for the self-crossings of bracelets;
- for every essential loop ζ, if $\mathrm{Brac}_k\,\zeta \in C$ then there is only one copy of it in C and no other bracelet of ζ is in C.

Theorem 4.5 ([24]). *Both $\mathcal{B}$ and $\mathcal{B}^\circ$ are bases for $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$.*

Proof idea: The fact that $\mathcal{B}^\circ$ spans the cluster algebra follows from the skein relations using the method described above. To show that $\mathcal{B}^\circ$ is linearly independent one uses the so- called g-vectors of the cluster algebra elements, which is closely related to the sign functions of the snake graphs. Finally, one needs to show that the Laurent polynomials in $\mathcal{B}^\circ$ actually are elements of the cluster algebra. This is a surprisingly subtle point. In [24] this was proved for unpunctured surfaces which have at least 2 marked points using the smoothing operations on arcs. In [6] the proof was extended to all unpunctured surfaces using snake graph calculus. This shows that $\mathcal{B}^\circ$ is a basis.

To prove that $\mathcal{B}$ is a basis, one needs to replace the bangles by the bracelets. Algebraically this can be done in terms of Chebyshev polynomials, $x_{\mathrm{Brac}_k\,\zeta} = T_k(x_\zeta)$ where T_k is the k-th Chebyshev polynomial of the second kind. These polynomials are defined recursively as

$$T_0(x) = 2\,, \qquad T_1(x) = x\,, \qquad T_k(x) = xT_{k-1}(x) - T_{k-2}(x)\,, \quad \text{for } k \geq 2\,.$$

Thus for the bracelets, we obtain the relation

$$\mathrm{Brac}_k(\zeta) = \zeta\,\mathrm{Brac}_{k-1}(\zeta) - \mathrm{Brac}_{k-2}(\zeta)\ ,$$

which can be seen also directly from the skein relations. In Fig. 17, we illustrate the case $k = 4$, where we are smoothing the top crossing of the 4-bracelet. Note that the rightmost curve in that figure has a contractible kink, which produces the minus sign in the equation.

The basis $\mathcal{B}$ has the following important advantage over the basis $\mathcal{B}^\circ$.

Theorem 4.6 ([27]). *The basis $\mathcal{B}$ has positive structure constants.*

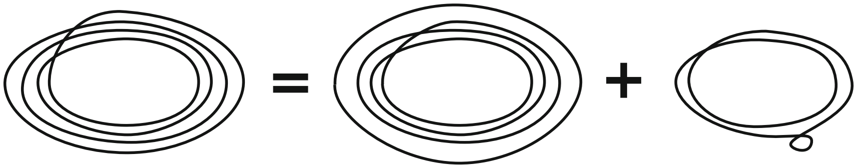


Fig. 17 Skein relation showing that $\mathrm{Brac}_4(\zeta) = \zeta\,\mathrm{Brac}_3(\zeta) - \mathrm{Brac}_2(\zeta)$

This means that if $b, b' \in \mathscr{B}$ are two basis elements, and if we express their product as a linear combination of elements in $\mathscr{B}$ as

$$bb' = \sum_{b'' \in \mathscr{B}} g_{b,b'}^{b''} b'' \,,$$

then the $g_{b,b'}^{b''} \in \mathbb{ZP}$ are called the structure constants of the basis $\mathscr{B}$, and the theorem says that $g_{b,b'}^{b''} \in \mathbb{Z}_{\geq 0}\mathbb{P}$.

It is known that the basis $\mathscr{B}^\circ$ does not have positive structure constants, see [27, Example 4.13].

Remark 4.7. The correspondence between the cluster algebra and a triangulated surface can be generalized to triangulated orbifolds, see [13]. In this setting the cluster algebra does not correspond to a quiver but to a skew-symmetrizable matrix, and, in contrast to the surface, the orbifold is allowed to have singularities. The results in Sects. 3 and 4 have been generalized to this setting in [11, 12].

A Appendix: Generalization to Surfaces with Punctures

A.1 *Tagged arcs*

Note that an arc γ that lies inside a self-folded triangle in T cannot be flipped. In order to rectify this problem, the authors of [16] were led to introduce the slightly more general notion of *tagged arcs*.

A *tagged arc* is obtained by taking an arc that does not cut out a once-punctured monogon and marking ("tagging") each of its ends in one of two ways, *plain* or *notched*, so that the following conditions are satisfied:

- an endpoint lying on the boundary of S must be tagged plain
- both ends of a loop must be tagged in the same way.

Thus there are four ways to tag an arc between two distinct punctures and there are two ways to tag a loop at a puncture, see Fig. 18. The notching is indicated by a bow tie.

One can represent an ordinary arc β by a tagged arc $\iota(\beta)$ as follows. If β does not cut out a once-punctured monogon, then $\iota(\beta)$ is simply β with both ends tagged plain. Otherwise, β is a loop based at some marked point q and cutting out a punctured monogon with the

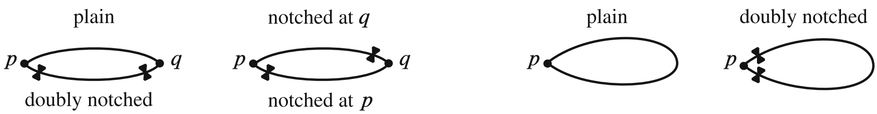


Fig. 18 Four ways to tag an arc between two punctures (left); two ways to tag a loop at a puncture (right)

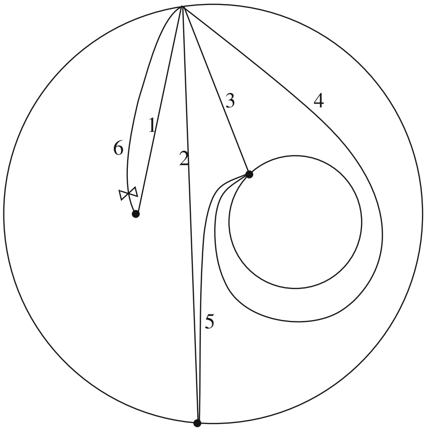


Fig. 19 Tagged triangulation of the punctured annulus corresponding to the ideal triangulation of the right-hand side of Fig. 5

sole puncture p inside it. Let α be the unique arc connecting p and q and compatible with β. Then $\iota(\beta)$ is obtained by tagging α plain at q and notched at p.

Tagged arcs α and β are called *compatible* if and only if the following properties hold:

- the arcs α^0 and β^0 obtained from α and β by forgetting the taggings are compatible;
- if $\alpha^0 = \beta^0$ then one end of α must be tagged in the same way as the corresponding end of β;
- $\alpha^0 \neq \beta^0$ but they share an endpoint a, then the ends of α and β connecting to a must be tagged in the same way.

A maximal collection of pairwise compatible tagged arcs is called a *tagged triangulation*. Figure 19 shows the tagged triangulation corresponding to the triangulation on the right-hand side of Fig. 5.

Given a surface (S, M) with a puncture p and a tagged arc γ, we let $\gamma^{(p)}$ denote the arc obtained from γ by changing its notching at p. If p and q are two punctures, we let $\gamma^{(pq)}$ denote the arc obtained from γ by changing its notching at both p and q.

If ℓ is an unnotched loop with endpoints at q cutting out a once-punctured monogon containing puncture p and radius r, see Fig. 20 then we set

$$x_\ell = x_r x_{r^{(p)}} .$$

Thus the loop is equal to the product of the two radii.

A.2 *Expansion Formula for Plain Arcs in the Presence of Self-Folded Triangles*

If there are self-folded triangles in the triangulation T then we have to modify the y-monomials in the expansion formula of Theorem 3.4 as follows. Recall that we had defined

Fig. 20 A self-folded triangle with loop ℓ and radius r (left); the corresponding tagged arcs r and $r^{(p)}$ (right). In the cluster algebra we have $x_\ell = x_r x_{r^{(p)}}$

$y(P)$ as a monomial in the coefficients $y_1, \ldots, y_n$ and each y_i corresponds to an (untagged) arc y_{τ_i} of T. Now we need to redefine $y(P)$ by replacing every y_i in our previous definition by $\Phi(y_i)$, where Φ is defined below.

$$\Phi(y_i) = \begin{cases} y_i & \text{if } \tau_i \text{ is not a side of a self-folded triangle;} \\ \frac{y_r}{y_{r^{(p)}}} & \text{if } \tau_i \text{ is a radius } r \text{ to puncture } p \text{ in a self-folded triangle;} \\ y_{r^{(p)}} & \text{if } \tau_i \text{ is a loop } \ell \text{ in a self-folded triangle with radius } r \text{ and puncture } p. \end{cases}$$

Then the cluster variable x_γ is equal to

$$x_\gamma = \frac{1}{\text{cross}(\gamma)} \sum_{P \in \text{Match}\, \mathcal{G}_\gamma} x(P) y(P) \,.$$

A.3 *Expansion Formula for Singly Notched Arcs*

Definition A.1. If p is a puncture, and $\gamma^{(p)}$ is a tagged arc with a notch at p but tagged plain at its other end, we define the associated crossing monomial as

$$\text{cross}(\gamma^{(p)}) = \frac{\text{cross}(\ell_p)}{\text{cross}(\gamma)} = \text{cross}(\gamma) \prod_\tau x_\tau,$$

where the product is over all ends of arcs τ of T that are incident to p. If p and q are punctures and $\gamma^{(pq)}$ is a tagged arc with a notch at p and q, we define the associated crossing monomial as

$$\text{cross}(\gamma^{(pq)}) = \frac{\text{cross}(\ell_p)\,\text{cross}(\ell_q)}{\text{cross}(\gamma)^3} = \text{cross}(\gamma) \prod_\tau x_\tau \,,$$

where the product is over all ends of arcs τ that are incident to p or q.

Let p be a puncture and let γ be an arc from a point $q \neq p$ to p. Let $\gamma^{(p)}$ be the tagged arc that is notched at p and plain at q and let ℓ denote the loop at q that cuts out the once-punctured monogon with puncture p and radius γ. Thus $\iota(\ell) = \gamma^{(p)}$. Let $\mathcal{G}_\ell$ be the snake graph of ℓ.

The snake graph $\mathcal{G}_\ell$ contains two disjoint connected subgraphs, one on each end, both of which are isomorphic to $\mathcal{G}_\gamma$. We let $\mathcal{G}_{\gamma_p,1}$ denote the one at the southwest end of $\mathcal{G}_\ell$ and $\mathcal{G}_{\gamma_p,2}$ the one at the northeast end.

We let $\mathcal{H}_{\gamma_p,1}$ be the subgraph of $\mathcal{G}_{\gamma_p,1}$ obtained by deleting the northeast vertex, and $\mathcal{H}_{\gamma_p,2}$ be the subgraph of $\mathcal{G}_{\gamma_p,2}$ obtained by deleting the southwest vertex.

In Fig. 21, the subgraph $\mathcal{G}_{\gamma_p,1}$ is the subgraph of $\mathcal{G}_\ell$ consisting of the first two tiles and $\mathcal{G}_{\gamma_p,2}$ is the subgraph consisting of the last two tiles. In the same figure, the graph $\mathcal{H}_{\gamma_p,1}$ is the graph consisting of the first tile and the south edge of the second tile.

Definition A.2. A perfect matching P of $\mathcal{G}_\ell$ is called γ-*symmetric* if the restrictions of P to the two ends satisfy $P|_{\mathcal{H}_{\gamma_p,1}} \cong P|_{\mathcal{H}_{\gamma_p,2}}$.

If P is γ-symmetric, define

$$\bar{x}(P) = \frac{x(P)}{x(P|_{\mathcal{G}_{\gamma,i}})}, \quad \bar{y}(P) = \frac{y(P)}{y(P|_{\mathcal{G}_{|\gamma,i}})},$$

where $i = 1$ or 2 depending on which subgraph the restriction of P defines a perfect matching.

Let $T = \{\tau_1, \ldots, \tau_n\}$ be a tagged triangulation of (S, M) and let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, Q_T)$ be the cluster algebra with principal coefficients at T. Thus $\mathbf{x}_T = (x_1, \ldots, x_n)$, $\mathbf{y}_T = (y_1, \ldots, y_n)$ and $\mathbb{P} = \mathrm{Trop}(y_1, \ldots, y_n)$.

Let p be a puncture and assume that T contains no arc notched at p. In fact this is not really a restriction, because if the arcs in T are notched at p we can change the tags of all arcs at p to "plain" and obtain the same quiver Q_T.

Let γ be an arc from a point $q \neq p$ to p and let $\gamma^{(p)}$ and ℓ be as above.

Theorem A.3. *If γ is not in the triangulation T, then the cluster variable $x_{\gamma^{(p)}}$ is equal to*

$$x_{\gamma^{(p)}} = \frac{1}{\mathrm{cross}(\gamma^{(p)})} \sum_P \bar{x}(P)\, \bar{y}(P),$$

where the sum is over all γ-symmetric matchings P of $\mathcal{G}_\ell$.

Remark A.4. If γ is in T (so x_γ is an initial cluster variable), then $x_{\gamma^{(p)}} = x_\ell / x_\gamma$, where x_ℓ is computed by the formula in Theorem 3.4.

A.4 *Expansion Formula for Doubly Notched Arcs*

For the case of a tagged arc with notches at both ends, we need two more definitions.

Let p and q be a puncture and let γ be an arc from a point p to q. Let $\gamma^{(p)}$ be the tagged arc that is notched at p and plain at q and let $\gamma^{(q)}$ be the tagged arc that is notched at q and plain at p. Let ℓ_p be the loop at q such that $\iota(\ell_p) = \gamma^{(p)}$, and let ℓ_q be the loop at p such that $\iota(\ell_q) = \gamma^{(q)}$.

Definition A.5. Assume that the tagged triangulation T does not contain either γ, $\gamma^{(p)}$, or $\gamma^{(q)}$. Let P_p and P_q be γ-symmetric matchings of $\mathcal{G}_{\ell_p}$ and $\mathcal{G}_{\ell_q}$, respectively. Then the pair (P_p, P_q) is called γ-*compatible* if at least one of the following two conditions holds.

- The restrictions $P_p|_{\mathcal{G}_{\gamma_p,1}}$, and $P_q|_{\mathcal{G}_{\gamma_q,1}}$ are isomorphic perfect matchings of the subgraph $\mathcal{G}_{\gamma_p,1} \cong \mathcal{G}_{\gamma_q,1}$, or
- the restrictions $P_p|_{\mathcal{G}_{\gamma_p,2}}$, and $P_q|_{\mathcal{G}_{\gamma_q,2}}$ are isomorphic perfect matchings of the subgraph $\mathcal{G}_{\gamma_p,2} \cong \mathcal{G}_{\gamma_q,2}$.

If (P_p, P_q) is a γ-compatible pair of matchings define the weight and height monomial,

$$\bar{\bar{x}}(P_p, P_q) = \frac{x(P_p)\,x(P_q)}{x(P_p|_{\mathcal{G}_{\gamma_p,i}})^3}\,, \qquad \bar{\bar{y}}(P_p, P_q) = \frac{y(P_p)\,y(P_q)}{y(P_p|_{\mathcal{G}_{\gamma_p,i}})^3}\,,$$

where $i = 1$ or 2 depending on the two cases above.

For technical reasons, we require that (S, M) is not a closed surface with exactly 2 marked points for Theorem A.6.

Theorem A.6. *If γ is not in the triangulation T, then the cluster variable $x_{\gamma^{(p)}}$ is equal to*

$$x_{\gamma^{(pq)}} = \frac{1}{\text{cross}(\gamma^{(pq)})} \sum_{(P_p, P_q)} \bar{\bar{x}}(P_p, P_q)\,\bar{\bar{y}}(P_p, P_q)\,,$$

where the sum is over all γ-compatible pairs of matchings (P_p, P_q) of $(\mathcal{G}_{\ell_p}, \mathcal{G}_{\ell_q})$.

A.5 *Example of a Cluster Expansion for a Singly Notched Arc*

To compute the Laurent expansion of $x_{\gamma^{(p)}}$ of the notched arc in top left picture in Fig. 21, we draw the snake graph $\mathcal{G}_\ell$ of the loop ℓ, shown in the top right picture of the same figure. The poset of γ-symmetric matchings of $\mathcal{G}_\ell$ is shown in the bottom left picture. Note that the matchings agree on the subgraphs $\mathcal{H}_{\gamma_p,1}$ and $\mathcal{H}_{\gamma_p,2}$. The corresponding monomials $\bar{x}(P)\bar{y}(P)$ are shown in the bottom right of the figure.

Simplifying and dividing by $\text{cross}(\gamma^{(p)}) = x_1x_2x_3x_4$ we obtain

$$x_{\gamma^{(p)}} = \frac{x_1x_2x_3\, y_1y_2y_3y_4 + x_1x_3\, y_1y_2y_3 + x_4\, y_1y_3 + x_2x_4\, y_3 + x_2x_4\, y_1 + x_2^2x_4}{x_1x_2x_3x_4}\,.$$

Since all the initial variables and coefficients appearing in this sum correspond to ordinary arcs, the specializations of x-weights or y-weights of Sects. A.2 were not necessary in this case.

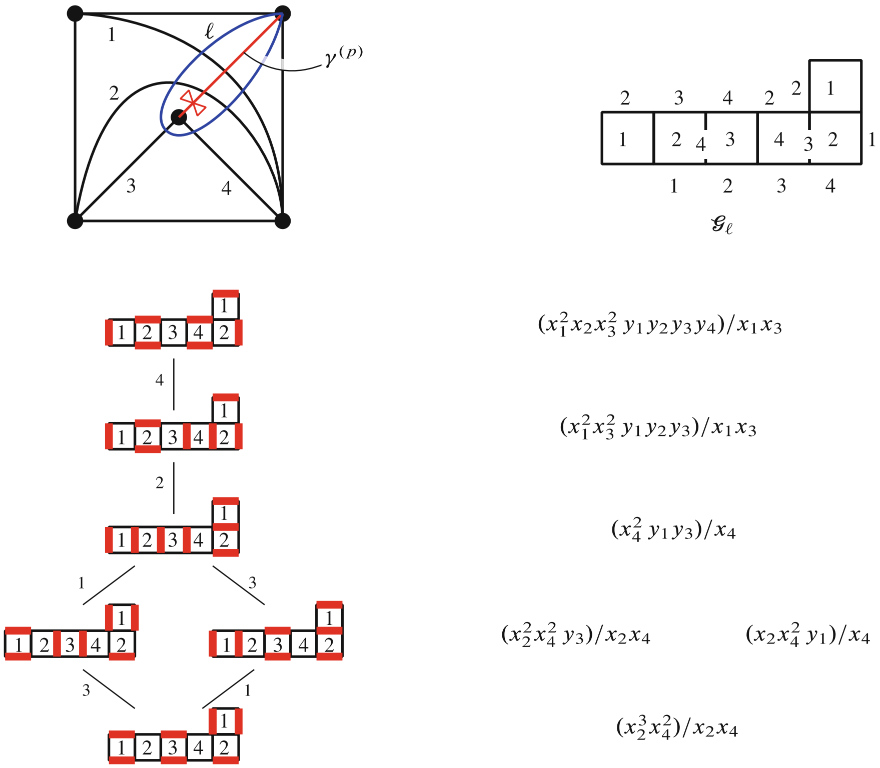


Fig. 21 A notched arc $\gamma^{(p)}$ in a triangulated punctured square (top left), the snake graph $\mathcal{G}_\ell$ of the corresponding loop ℓ (top right) the poset of γ-symmetric matchings of $\mathcal{G}_\ell$ (bottom left) and the corresponding monomials (bottom right)

A.6 *Example of a Laurent Expansion for a Doubly Notched Arc*

We close with an example of a cluster expansion formula for a tagged arc with notches at both endpoints. The top left picture in Fig. 22 shows a triangulation of a sphere with 6 punctures in the shape of an octahedron. The numbers $1,\ldots,12$ in that figure are the labels of the arcs of the triangulation. We compute the Laurent expansion of the red arc $\gamma^{(pq)}$ in the figure. The two snake graphs of the two loops ℓ_p and ℓ_q are shown in the top right of the figure. Note that the two snake graphs have the same shape, but not the same labels. The pictures at the bottom of the figure show the posets of γ-symmetric matchings for both snake graphs. The corresponding monomials $\bar{x}(P)\bar{y}(P)$ are listed below in the shape of the two posets.

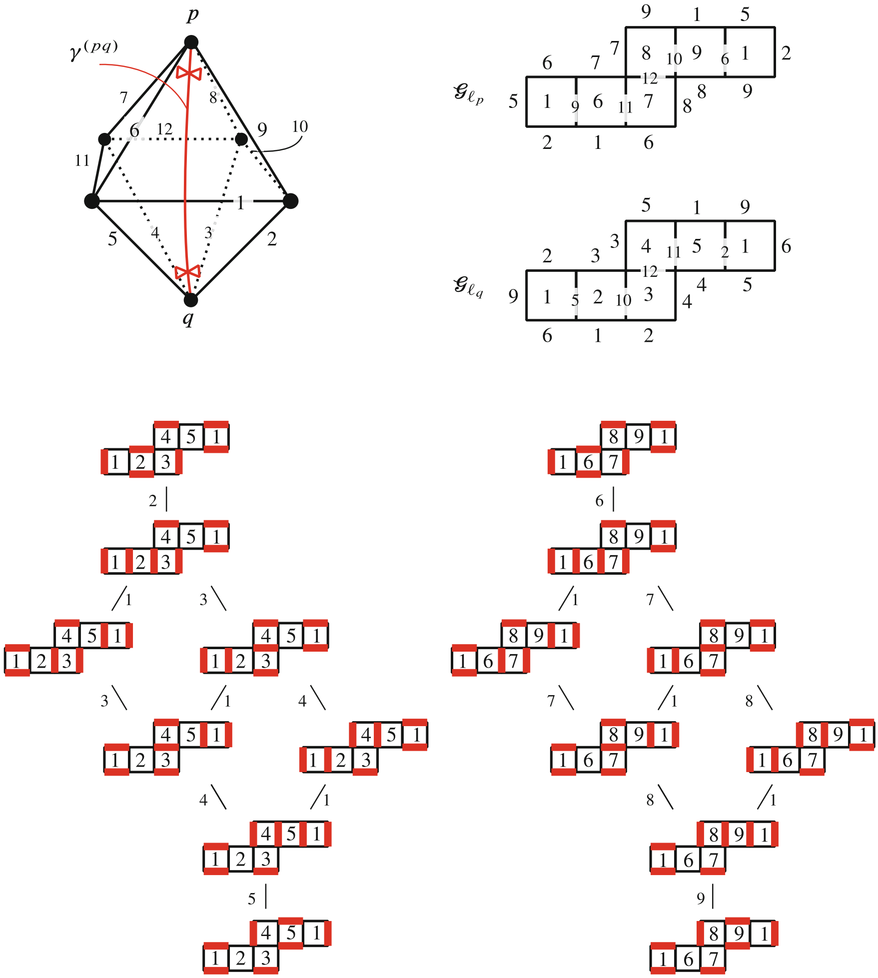


Fig. 22 Ideal triangulation T° and doubly notched arc γ_3

$$x_1x_3x_4x_5x_9\ y_1y_2y_3y_4y_5$$
$$x_4x_5^2x_9x_{10}\ y_1y_3y_4y_5$$
$$x_2x_4x_5x_6x_{10}\ y_3y_4y_5 \qquad x_2x_5^2x_9x_{12}\ y_1y_4y_5$$
$$x_2^2x_5x_6x_{12}\ y_4y_5 \qquad x_2x_3x_5x_9x_{11}\ y_1y_5$$
$$x_2^2x_3x_6x_{11}\ y_5$$
$$x_1x_2x_3x_4x_6$$

$$x_1x_5x_7x_8x_9\ y_1y_6y_7y_8y_9$$
$$x_5x_8x_9^2x_{11}\ y_1y_7y_8y_9$$
$$x_2x_6x_8x_9x_{11}\ y_7y_8y_9 \qquad x_5x_6x_9^2x_{12}\ y_1y_8y_9$$
$$x_2x_6^2x_9x_{12}\ y_8y_9 \qquad x_5x_6x_7x_9x_{10}\ y_1y_9$$
$$x_2x_6^2x_7x_{10}\ y_9$$
$$x_1x_2x_6x_7x_8$$

$$\frac{\begin{array}{c}(x_1^2x_3x_4x_5x_7x_8x_9\, y_1y_2y_3y_4y_5y_6y_7y_8y_9\\ + x_1x_3x_4x_5x_8x_9^2x_{11}\, y_1y_2y_3y_4y_5y_7y_8y_9\\ + x_1x_3x_4x_5x_6x_9^2x_{12}\, y_1y_2y_3y_4y_5y_8y_9\\ + x_1x_3x_4x_5x_6x_7x_9x_{10}\, y_1y_2y_3y_4y_5y_9\\[1em] + x_1x_4x_5^2x_7x_8x_9x_{10}\, y_1y_3y_4y_5y_6y_7y_8y_9\\ + x_4x_5^2x_8x_9^2x_{10}x_{11}\, y_1y_3y_4y_5y_7y_8y_9\\ + x_4x_5^2x_6x_9^2x_{10}x_{12}\, y_1y_3y_4y_5y_8y_9\\ + x_4x_5^2x_6x_7x_9x_{10}^2\, y_1y_3y_4y_5y_9\\[1em] + x_1x_2x_5^2x_7x_8x_9x_{12}\, y_1y_4y_5y_6y_7y_8y_9\\ + x_2x_5^2x_8x_9^2x_{11}x_{12}\, y_1y_4y_5y_7y_8y_9\\ + x_2x_5^2x_6x_9^2x_{12}^2\, y_1y_4y_5y_8y_9\\ + x_2x_5^2x_6x_7x_9x_{10}x_{12}\, y_1y_4y_5y_9\\[1em] + x_1x_2x_3x_5x_7x_8x_9x_{11}\, y_1y_5y_6y_7y_8y_9\\ + x_2x_3x_5x_8x_9^2x_{11}^2\, y_1y_5y_7y_8y_9\\ + x_2x_3x_5x_6x_9^2x_{11}x_{12}\, y_1y_5y_8y_9\\ + x_2x_3x_5x_6x_7x_9x_{10}x_{11}\, y_1y_5y_9\\[1em] + x_2x_4x_5x_6x_8x_9x_{10}x_{11}\, y_3y_4y_5y_7y_8y_9\\ + x_2x_4x_5x_6^2x_9x_{10}x_{12}\, y_3y_4y_5y_8y_9\\ + x_2x_4x_5x_6^2x_7x_{10}^2\, y_3y_4y_5y_9\\ + x_1x_2x_4x_5x_6x_7x_8x_{10}\, y_3y_4y_5\\[1em] + x_2^2x_5x_6x_8x_9x_{11}x_{12}\, y_4y_5y_7y_8y_9\\ + x_2^2x_5x_6^2x_9x_{12}^2\, y_4y_5y_8y_9\\ + x_2^2x_5x_6^2x_7x_{10}x_{12}\, y_4y_5y_9\\ + x_1x_2^2x_5x_6x_7x_8x_{12}\, y_4y_5\\[1em] + x_2^2x_3x_6x_8x_9x_{11}^2\, y_5y_7y_8y_9\\ + x_2^2x_3x_6^2x_9x_{11}x_{12}\, y_5y_8y_9\\ + x_2^2x_3x_6^2x_7x_{10}x_{11}\, y_5y_9\\ + x_1x_2^2x_3x_6x_7x_8x_{11}\, y_5\\[1em] + x_1x_2x_3x_4x_6x_8x_9x_{11}\, y_7y_8y_9\\ + x_1x_2x_3x_4x_6^2x_9x_{12}\, y_8y_9\\ + x_1x_2x_3x_4x_6^2x_7x_{10}\, y_9\\ + x_1^2x_2x_3x_4x_6x_7x_8)\end{array}}{x_1x_2x_3x_4x_5x_6x_7x_8x_9}$$

Fig. 23 The Laurent polynomial of $x_{\gamma^{(pq)}}$

We want to find all γ-compatible pairs. The four perfect matchings on the lower left side of the first poset all have horizontal edges on the first tile. These edges have labels 2 and 6. Therefore, each of these matchings forms a γ-compatible pair with each of the four matchings on the lower left side of the second poset. Similarly, the four perfect matchings on the upper right side of the first poset all have horizontal edges on the last tile. These edges have labels 5 and 9. Therefore, each of these matchings forms a γ-compatible pair with each of the four matchings on the upper right side of the second poset. There are no other γ-compatible pairs, so we have a total of 32 pairs. The Laurent polynomial for $x_{\gamma^{(pq)}}$ is shown in Fig. 23.

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Cluster Characters

Pierre-Guy Plamondon

1 Introduction

Shortly after the introduction of cluster algebras in [19], links with an impressively vast number of fields of mathematics were uncovered. Among these is the representation theory of finite-dimensional algebras, whose links to cluster algebras became apparent in, for instance, [8, 10, 39].

The link between representation theory and cluster algebra has proved itself to be fruitful on both sides: on the one hand, it has allowed an understanding of cluster algebras that has led to the proof of conjectures of Fomin and Zelevinsky: see for instance [13, 16, 22, 25]. On the other hand, it has sparked many developments in representation theory, as illustrated by the introduction of the theory of τ-tilting in [1], the study of cluster-tilted algebras and their representations initiated in [9] and the study of representations of certain quivers with loops in [26], among other examples.

Central in the study of this link are cluster characters. Broadly speaking, they are maps which associate to each module over certain algebras (or object in certain triangulated categories) an element in a certain cluster algebra. They have been introduced in [10], and have been studied, used, and generalized for instance in [11, 12, 22, 40, 43, 47].

The aim of these notes is to introduce cluster characters, present some of their main properties, and show how they can be used to categorify cluster algebras.

The notes are organized as follows. In Sect. 2, we introduce F-polynomials of modules over finite-dimensional algebras. They can be seen as a "homology-free" version of cluster characters. Their definition relies heavily on representation theory of quivers and on projective varieties called submodule Grassmannians; these are introduced first.

In Sect. 3, we introduce the cluster category of an acyclic quiver. We first recall the notion of derived category, and we focus on examples in type A_n.

Section 4 is devoted to the introduction of an abstract setting: that of 2-Calabi–Yau triangulated categories with cluster-tilting objects. This setting contains that of cluster categories, and is the one used in these notes to study cluster characters.

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Finally, cluster characters are introduced in Sect. 5, together with some of their properties leading to a categorification of cluster algebras.

The notes reflect a minicourse I gave at the CIMPA school in Mar del Plata, Argentina, in March 2016. Each section corresponds, more or less, to a one-hour lecture. I take this opportunity to thank the organizers of the CIMPA–ARTA V joint meeting during which this minicourse was given.

2 Quiver Representations and Submodule Grassmannians

In this section, we define in an elementary way the notion of quiver representation, and introduce a projective variety, the *submodule Grassmannian*, whose points parametrize subrepresentations of a given representation.

2.1 *Quiver Representations*

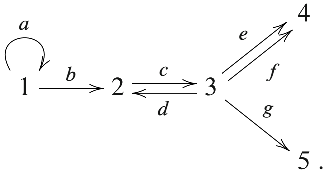
Let k be a field. We are interested in studying modules over k-algebras and their submodules. A convenient setting for this is that of quiver representations. There are many textbooks dealing with the subject, for instance [3, 4, 42, 46, 48].

Definition 2.1. A *quiver* is an oriented graph. More precisely, a quiver Q is given by a 4-tuple (Q_0, Q_1, s, t), where

- Q_0 is a set, whose elements are called *vertices*;
- Q_1 is a set, whose elements are called *arrows*;
- $s, t : Q_1 \to Q_0$ are two maps, which associate to each arrow its *source* or its *target*, respectively.

Quivers are allowed to have multiple edges, oriented cycles, and even loops. Throughout these notes quivers will be assumed to *finite*, that is to say, their sets of vertices and arrows will be finite.

Example 2.2. We will usually number the vertices of a quiver by using natural numbers, and use letters to name the arrows. We will represent quivers as oriented graphs. Here is an example:



A *path* in a quiver is a concatenation of arrows $w = a_m \cdots a_1 a_0$ such that $s(a_{i+1}) = t(a_i)$ for all i from 0 to $m-1$. This means that we compose arrows from right to left. We extend the maps s and t to the set of all paths by putting $s(w) = s(a_0)$ and $t(w) = t(a_m)$.

Additionally, for each vertex i, there is a path of length 0 starting and ending at i and denoted by e_i. We call it either the *trivial path* or the *lazy path* at i. If w is any path, then $w = e_{t(w)}w = we_{s(w)}$.

Definition 2.3. Let Q be a quiver. A *representation of* Q is a tuple $V = (V_i, V_a)_{i \in Q_0, a \in Q_1}$, where

- for each vertex i in Q, V_i is a k-vector space, and
- for each arrow a in Q, $V_a \colon V_{s(a)} \to V_{t(a)}$ is a k-linear map.

A representation V is said to be *finite-dimensional* if all the vector spaces V_i are finite-dimensional; in that case, the *dimension vector* of V is $\underline{\dim}\, V = (\dim V_i)_{i \in Q_0}$. If $w = a_m \cdots a_1 a_0$ is a path in Q, we write $V_w = V_{a_m} \circ \cdots \circ V_{a_1} \circ V_{a_0}$.

In these notes, all representations will be finite-dimensional.

Definition 2.4. Let V and W be two representations of a quiver Q. A *morphism of representations from* V *to* W, denoted by $f \colon V \to W$, is a tuple $f = (f_i)_{i \in Q_0}$, where

- for each vertex i of Q, $f_i \colon V_i \to W_i$ is a k-linear map, and
- for each arrow a of Q, we have that $W_a \circ f_{s(a)} = f_{t(a)} \circ V_a$. In other words, the following diagram commutes:

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{f_{s(a)}} & W_{s(a)} \\ \downarrow{\scriptstyle V_a} & & \downarrow{\scriptstyle W_a} \\ V_{t(a)} & \xrightarrow{f_{t(a)}} & W_{t(a)} \end{array} .$$

Composition of morphisms is defined vertex-wise in the obvious way.

Representations of a quiver Q, together with their morphisms, form a category $\mathrm{Rep}(Q)$. We denote by $\mathrm{rep}(Q)$ its full subcategory whose objects are finite-dimensional representations. These categories are abelian; we can see this by showing that they are equivalent to module categories (see Proposition 2.6).

Definition 2.5. Let Q be a quiver. The *path algebra* of Q is the associative k-algebra kQ defined as follows.

- For all nonnegative integers ℓ, let $(kQ)_\ell$ be the k-vector space with basis the set of paths of length ℓ in Q. Then the underlying vector space of kQ is $\bigoplus_{\ell=0}^{\infty} (kQ)_\ell$.
- Multiplication is defined on paths by

$$w_2 \cdot w_1 = \begin{cases} w_2 w_1 & \text{if } s(w_2) = t(w_1) \\ 0 & \text{otherwise,} \end{cases}$$

  and extended to all of kQ by linearity.

We denote by $\mathfrak{m}$ the two-sided ideal of kQ generated by the arrows of Q. In other words, $\mathfrak{m} = \bigoplus_{\ell=1}^{\infty} (kQ)_\ell$.

If I is any two-sided ideal of kQ, then we denote by $\mathrm{Rep}(Q, I)$ the full subcategory of $\mathrm{Rep}(Q)$ whose objects are representations V "satisfying the relations in I," that is, such that for any linear combination of paths $\sum_i \lambda_i w_i$ lying in I, we have that $\sum_i \lambda_i V_{w_i} = 0$.

One of the main motivations for studying representations of quivers can be summarized in the following results. First, representations of a quiver and modules over its path algebra should be viewed as being the same thing. More precisely:

Proposition 2.6. *Let Q be a quiver and I be a two-sided ideal of kQ. Then the categories* $\mathrm{Mod}\,(kQ/I)$ *and* $\mathrm{Rep}(Q^{\mathrm{op}}, I^{\mathrm{op}})$ *are equivalent. (Here* Mod A *is the category of right(!) modules over A, and Q^{op} is the opposite quiver, obtained by reversing the orientation of all arrows of Q).*

The same is true of $\mathrm{mod}\,(kQ/I)$ *and* $\mathrm{rep}(Q^{\mathrm{op}}, I^{\mathrm{op}})$, *the full subcategories of finite-dimensional modules and representations, respectively.*

We see Q^{op} appearing in the proposition because of our choice of conventions: right modules, and composition of arrows from right to left. The proof of the proposition is straightforward.

Secondly, over an algebraically closed field, the representation theory of *any* finite-dimensional algebra is governed by a quiver with relations. More precisely:

Theorem 2.7. (Gabriel). *Assume that the field k is algebraically closed. For any finite-dimensional associative k-algebra A, there is a unique quiver Q_A and a (nonunique) ideal I of kQ_A such that A and kQ_A/I are Morita equivalent, and $\mathfrak{m}^r \subset I \subset \mathfrak{m}^2$ for some $r \geq 2$.*

An ideal I satisfying $\mathfrak{m}^r \subset I \subset \mathfrak{m}^2$ is called an *admissible ideal*.

2.2 *Submodule Grassmannian*

Let Q be a finite quiver, I be an admissible ideal, and V be a representation of (Q, I).

Definition 2.8. A *subrepresentation* of V is a tuple $(W_i)_{i \in Q_0}$, where

- each W_i is a subspace of V_i, and
- for each arrow a in Q, we have that $V_a(W_{s(a)}) \subset W_{t(a)}$.

In that case, $W = (W_i, V_a|_{W_{s(a)}})_{i \in Q_0, a \in Q_1}$ is a representation of (Q, I), and the canonical inclusion into V is a morphism of representations.

Grassmannians of vector spaces are projective varieties whose points parametrize sub-vector spaces of a given dimension. Submodule Grassmannians of modules generalize this notion: they are projective varieties, whose points parametrize submodules of a given dimension vector.

Definition 2.9. Let $\mathbf{e} \in \mathbb{N}^{Q_0}$ be a dimension vector. The *submodule Grassmannian of* V *with dimension vector* $\mathbf{e}$ is the subset $\mathrm{Gr}_{\mathbf{e}}(V)$ of $\prod_{i \in Q_0} \mathrm{Gr}_{e_i}(V_i)$ of all points $(W_i)_{i \in Q_0}$ defining a subrepresentation of V.

The submodule Grassmannian is in fact a Zariski-closed subset of $\prod_{i \in Q_0} \mathrm{Gr}_{e_i}(V_i)$, so it is a projective variety.

Examples 2.10.(1) If the quiver Q has only one vertex and no arrows, then representations of Q are just vector spaces, and their submodule Grassmannians are just usual Grassmannians.

(2) Let $Q = 1 \rightrightarrows 2$ be the Kronecker quiver. Consider the representation

$$V = k^2 \overset{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)}{\underset{\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)}{\rightrightarrows}} k^2.$$

Then there are six dimension vectors for which the submodule Grassmannian of V is nonempty. The table below lists those dimension vectors and gives a variety to which the corresponding submodule Grassmannian is isomorphic.

$\mathbf{e}$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,2)$	$(2,2)$
$\mathrm{Gr}_{\mathbf{e}}(V)$	point	$\mathbb{P}^1$	point	point	$\mathbb{P}^1$	point

2.3 *F-Polynomials of Modules*

We now define the F-polynomial of a representation of a quiver with relations (Q, I) (or, equivalently, of a module over $A = kQ/I$). Roughly, the F-polynomial can be seen as a generating function for counting submodules of a given module (even though this might not make sense if the base field k is infinite, since a module may have infinitely many submodules). This theory originates from [10], although F-polynomials of modules appeared later in [16]. The general results in this section can be found in [18, Sect. 2].

In the rest of this section, the base field k is the field $\mathbb{C}$ of complex numbers.

Definition 2.11. Let V be an A-module. Its *F-polynomial* is

$$F_V(\mathbf{y}) := \sum_{\mathbf{e} \in \mathbb{N}^{Q_0}} \chi\big(\mathrm{Gr}_{\mathbf{e}}(V)\big)\mathbf{y}^{\mathbf{e}},$$

where

- $\mathbf{y}$ is the tuple of variables $(y_i \mid i \in Q_0)$;
- $\mathbf{y}^{\mathbf{e}} = \prod_{i \in Q_0} y_i^{e_i}$;
- $\mathrm{Gr}_{\mathbf{e}}(V)$ is the submodule Grassmannian (see Definition 2.9); and
- χ is the Euler–Poincaré characteristic.

We give examples of F-polynomials at the end of the section. It is easy to see that the F-polynomial of a module V only depends on the isomorphism class of V.

Remark 2.12. The most difficult part in the computation of an F-polynomial is determining the submodule Grassmannians $\mathrm{Gr}_{\mathbf{e}}(V)$. To compute their Euler–Poincaré characteristic, the following facts (true since we work over $\mathbb{C}$!) are often sufficient (and indeed, suffice to prove all the formulas in these notes):

(1) $\chi(\text{point}) = 1$;
(2) $\chi(\mathbb{A}^n) = 1$, where $\mathbb{A}^n$ is the affine space of dimension n;
(3) $\chi(\mathbb{P}^n) = n + 1$, where $\mathbb{P}^n$ is the projective space of dimension n;
(4) $\chi(\mathcal{U} \times \mathcal{V}) = \chi(\mathcal{U}) \cdot \chi(\mathcal{V})$;
(5) if $\mathcal{U}$ is a disjoint union of two constructible subsets C_1 and C_2, then $\chi(\mathcal{U}) = \chi(C_1) + \chi(C_2)$.
(6) if $f : \mathcal{U} \to \mathcal{V}$ is a surjective morphism of varieties (or even a surjective constructible map) such that all fibers $f^{-1}(x)$ have the same Euler characteristic, say c, then $\chi(\mathcal{U}) = c\chi(\mathcal{V})$.

See [23, Proposition 7.4.1], which itself refers to [17, 37]. On constructible maps, we refer the reader to [31].

The first property of F-polynomials deals with direct sums, or equivalently, with split exact sequences.

Proposition 2.13 ([10, 18])**.** *Let V and W be two modules over $A = kQ/I$. Then $F_V \cdot F_W = F_{V \oplus W}$.*

We outline the proof of this proposition, as it gives the flavor of the methods used to prove the various formulas that appear in these notes. We follow [10, Proposition 3.6].

Proof of Proposition 2.13. Consider the split exact sequence

$$0 \to V \xrightarrow{\iota} V \oplus W \xrightarrow{\pi} W \to 0 \,.$$

To any submodule B of $V \oplus W$ we associate the submodules $\iota^{-1}(B)$ and $\pi(B)$ of V and W, respectively. This defines maps

$$\begin{aligned} \Phi_{\mathbf{e}}\colon \mathrm{Gr}_{\mathbf{e}}(V \oplus W) &\longrightarrow \coprod_{\mathbf{f}+\mathbf{g}=\mathbf{e}} \mathrm{Gr}_{\mathbf{f}}(V) \times \mathrm{Gr}_{\mathbf{g}}(W) \\ B &\longmapsto \big(\iota^{-1}(B), \pi(B)\big) \end{aligned}$$

which are constructible maps. These maps are clearly surjective. Moreover, the fiber of a point (U_1, U_2) can be shown to be an affine space (it is isomorphic to $\mathrm{Hom}_A(U_2, V/U_1)$, see [10, Lemma 3.8]).

Thus, by Remark 2.12, we get

$$\chi\big(\mathrm{Gr}_{\mathbf{e}}(V\oplus W)\big)=\chi\left(\coprod_{\mathbf{f}+\mathbf{g}=\mathbf{e}}\mathrm{Gr}_{\mathbf{f}}(V)\times\mathrm{Gr}_{\mathbf{g}}(W)\right)$$
$$=\sum_{\mathbf{f}+\mathbf{g}=\mathbf{e}}\chi\big(\mathrm{Gr}_{\mathbf{f}}(V)\big)\cdot\chi\big(\mathrm{Gr}_{\mathbf{g}}(W)\big)\,.$$

From there, the proof is a simple computation:

$$\begin{aligned}F_V(\mathbf{y})F_W(\mathbf{y})&=\left(\sum_{\mathbf{f}\in\mathbb{N}^{Q_0}}\chi\big(\mathrm{Gr}_{\mathbf{f}}(V)\big)\mathbf{y}^{\mathbf{f}}\right)\cdot\left(\sum_{\mathbf{g}\in\mathbb{N}^{Q_0}}\chi\big(\mathrm{Gr}_{\mathbf{g}}(W)\big)\mathbf{y}^{\mathbf{g}}\right)\\&=\sum_{\mathbf{f},\mathbf{g}}\chi\big(\mathrm{Gr}_{\mathbf{f}}(V)\big)\chi\big(\mathrm{Gr}_{\mathbf{g}}(W)\big)\mathbf{y}^{\mathbf{f}+\mathbf{g}}\\&=\sum_{\mathbf{e}}\left(\sum_{\mathbf{f}+\mathbf{g}=\mathbf{e}}\chi\big(\mathrm{Gr}_{\mathbf{f}}(V)\big)\chi\big(\mathrm{Gr}_{\mathbf{g}}(W)\big)\right)\mathbf{y}^{\mathbf{e}}\\&=\sum_{\mathbf{e}}\chi\big(\mathrm{Gr}_{\mathbf{e}}(V\oplus W)\big)\mathbf{y}^{\mathbf{e}}\\&=F_{V\oplus W}(\mathbf{y})\,.\end{aligned}$$

□

The second property of F-polynomials, and perhaps the most important one for our purposes, deals with almost-split exact sequences. For the theory of almost-split sequences and the definition of the Auslander–Reiten translation τ, we refer the reader to the notes of the courses [38, 45] in this volume.

Theorem 2.14 **([10, 18]).** *Let* $0\to\tau V\to E\to V\to 0$ *be an almost-split sequence of modules over* $A=kQ/I$*. Then* $F_{\tau V}\cdot F_V=F_E+\mathbf{y}^{\underline{\dim}\,V}$.

The spirit of the proof of this theorem is similar to that of Proposition 2.13. The difference lies in the fact that the morphism $\Phi_{\mathbf{e}}$ is no longer surjective for all $\mathbf{e}$; the term $\mathbf{y}^{\underline{\dim}\,V}$ in the right-hand side of the statement compensates, in some sense, this lack of surjectivity.

2.4 *Examples of F-Polynomials*

2.4.1 .

Let Q be the quiver with one vertex and no arrows. Its path algebra is simply $\mathbb{C}$, and representations of Q are just vector spaces.

Let V be a d-dimensional vector space. Then

$$F_V(y)=\sum_{i=0}^{d}\binom{d}{i}y^i.$$

This can be seen by observing that, for $d = 1$, the F-polynomial is $1 + y$, and then by applying Proposition 2.13. As a corollary, we get a nice proof of the known fact that the Euler–Poincaré characteristic of the (usual) Grassmannian $\mathrm{Gr}_i(\mathbb{C}^d)$ is equal to $\binom{d}{i}$.

2.4.2 .

Let Q be the quiver with one vertex and one loop ℓ, subject to the relation $\ell^2 = 0$. For this quiver, there are only two indecomposable representations (up to isomorphism):

$$V_1 = \mathbb{C} \circlearrowright^{0} \quad \text{and} \quad V_2 = \mathbb{C}^2 \circlearrowright^{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)}$$

and only one almost-split sequence:

$$0 \to V_1 \to V_2 \to V_1 \to 0.$$

The F-polynomials are easily seen to be $F_{V_1}(y) = 1 + y$ and $F_{V_2}(y) = 1 + y + y^2$, and one can check that they satisfy Theorem 2.14.

2.4.3 .

Let Q and V be as in Example 2.10(2). Then $F_V(y_1, y_2) = 1 + 2y_2 + y_2^2 + y_1y_2 + 2y_1y_2^2 + y_1^2y_2^2$.

2.4.4 .

We list a few more examples and properties of F-polynomials.

(1) If V and W are isomorphic, then $F_V = F_W$. The converse is false: consider the Kronecker quiver

$$1 \underset{b}{\overset{a}{\rightrightarrows}} 2 .$$

Then the representations

$$V_1 = \mathbb{C} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{C} \quad \text{and} \quad V_2 = \mathbb{C} \underset{0}{\overset{1}{\rightrightarrows}} \mathbb{C}$$

are not isomorphic, but their F-polynomials are both equal to $1 + y_2 + y_1y_2$.

(2) If F_V is an irreducible polynomial, then V is indecomposable. The converse is false: consider the quiver

$$1 \underset{b}{\overset{a}{\rightleftarrows}} 2 .$$

Then the representation

$$\mathbb{C}^2 \underset{\binom{0}{1}}{\overset{(1\ 0)}{\rightleftarrows}} \mathbb{C}\,.$$

is indecomposable, but its F-polynomial is $1+y_1+y_1y_2+y_1^2y_2 = (1+y_1y_2)(1+y_1)$.

(3) An F-polynomial may have negative coefficients. An example for a quiver with two vertices and four arrows is given in [16, Example 3.6].

3 Cluster Categories

3.1 *Derived Categories*

Derived categories were introduced by J.-L. Verdier in [49, 50]. Their general theory is discussed in numerous books and papers; let us cite [28, 29, 32, 34, 51].

In this section, we only give a brief outline of the theory of derived categories, focusing on aspects that suit the purpose of these notes. Here, k is an arbitrary field.

3.1.1 Generalities

Let $\mathcal{A}$ be an abelian category (for example, the category of modules over a finite-dimensional k-algebra). In particular, every morphism in $\mathcal{A}$ has a kernel and a cokernel.

A *complex* of objects of $\mathcal{A}$ is a sequence of morphisms

$$C = \cdots \xrightarrow{d_{i-2}} C_{i-1} \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} \cdots$$

such that $d_{i+1} \circ d_i = 0$ for all integers i.

Let C and C' be two complexes. A *morphism of complexes* $f: C \to C'$ is an infinite tuple $f = (f_i)_{i \in \mathbb{Z}}$ such that for all integers i, $f_i: C_i \to C'_i$ is a morphism, and the square

$$\begin{array}{ccc} C_i & \xrightarrow{d_i} & C_{i+1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} \\ C'_i & \xrightarrow{d'_i} & C'_{i+1} \end{array}$$

commutes, that is, $f_{i+1} \circ d_i = d'_i \circ f_i$.

We denote by $\mathcal{C}(\mathcal{A})$ the category of complexes of $\mathcal{A}$. It is an abelian category. It admits an automorphism called the *shift functor* and denoted by [1], which is defined by $(C[1])_i = C_{i+1}$, and where the differential δ of $C[1]$ is defined by $\delta_i = -d_{i+1}$.

The *homology* of a complex C at degree i is the object $H_i(C) := \ker(d_i)/\operatorname{im}(d_{i-1})$. It is easy to see that a morphism of complexes $f: C \to C'$ induces in each degree a morphism

$H_i(f)\colon H_i(C) \to H_i(C')$. A *quasi-isomorphism* is a morphism of complexes f such that all induced morphisms in homology are isomorphisms.

The derived category of $\mathcal{A}$ is the category obtained when formally inverting all quasi-isomorphisms in $\mathcal{C}(\mathcal{A})$. A convenient construction of the derived category is given by first defining the *homotopy category* $K(\mathcal{A})$. This category is the quotient of $\mathcal{C}(\mathcal{A})$ by the ideal of all *null-homotopic morphisms*, that is, morphisms of complexes $f\colon C \to C'$ such that there exist morphisms $s_i\colon C_i \to C'_{i-1}$ is $\mathcal{A}$ such that $f_i = d'_{i-1}s_i + s_{i+1}d_i$ for all $i \in \mathbb{Z}$.

The *derived category* $\mathcal{D}(\mathcal{A})$ is then the category obtained from $K(\mathcal{A})$ by formally inverting all quasi-isomorphisms.

If, instead of $\mathcal{C}(\mathcal{A})$, one considers the categories $\mathcal{C}^+(\mathcal{A})$, $\mathcal{C}^-(\mathcal{A})$ and $\mathcal{C}^{\mathrm{b}}(\mathcal{A})$ of complexes bounded on the left, on the right and on both sides, respectively, then one defines derived categories $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^-(\mathcal{A})$, and $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$. Of importance to us in the next section will be $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$, called the *bounded derived category*.

The advantage of defining the derived category by working in $K(\mathcal{A})$ instead of $\mathcal{C}(\mathcal{A})$ is that it allows one to use a notion of "calculus of fractions" of morphisms, see for instance [34, Sect. 2.2]. Another advantage, relevant to our situation, is that if $\mathcal{A}$ is the module category of a finite-dimensional algebra A, and if we denote by proj A the full subcategory of $\mathcal{A}$ whose objects are projective modules, then $\mathcal{D}^-(\mathcal{A})$ is equivalent to $K^-(\text{proj } A)$. This latter category is often easier to work with.

Proposition 3.1. *The functor* $J\colon \mathcal{A} \to \mathcal{D}^*(\mathcal{A})$ *sending an object* M *to the complex* C *with* $C_0 = M$ *and* $C_j = 0$ *if* $j \neq 0$ *is fully faithful. Here,* $*$ *can be* $+, -, \mathrm{b}$ *or an absence of symbol.*

By an abuse of notation, if M is an object of $\mathcal{A}$, then we denote still by M its image by the functor J.

3.1.2 Triangulated Categories

An important property of derived categories is that they are *triangulated categories*. A triangulated category is a k-linear category $\mathcal{T}$ together with a k-linear automorphism $\Sigma\colon \mathcal{T} \to \mathcal{T}$ called the *suspension functor* and with a collection of sequences of morphisms of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \ ,$$

where gf and hg vanish. The sequences belonging to the collection are called *distinguished triangles*, or simply *triangles*. They are required to satisfy several axioms, which are listed below and which can be found in any of the references given at the beginning of the section.

(T1) The class of triangles is closed under isomorphism of complexes of length 4. For any object X, the sequence $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$ is a triangle. Any morphism $X \xrightarrow{f} Y$ can be embedded into a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.

(T2) The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is.

(T3) For any commutative diagram

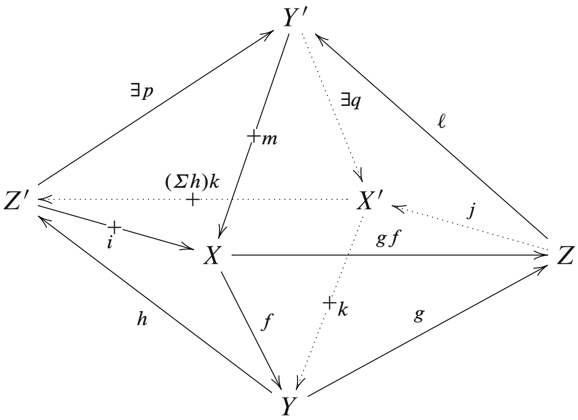
$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle \Sigma u} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

whose rows are triangles, there exists a morphism $w\colon Z \to Z'$ such that the resulting diagram also commutes (that is, $wg = g'v$ and $h'w = (\Sigma u)h$).

(T4) (Octahedral axiom) Assume that

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \xrightarrow{i} \Sigma X\ , \quad Y \xrightarrow{g} Z \xrightarrow{j} X' \xrightarrow{k} \Sigma Y\ ,$$
$$X \xrightarrow{gf} Z \xrightarrow{\ell} Y' \xrightarrow{m} \Sigma X$$

are triangles, and arrange them as in the following picture:



where a "+" on an arrow $A \to B$ means a morphism $A \to \Sigma B$. Then there exist morphisms $p\colon Z' \to Y'$ and $q\colon Y' \to X'$ such that

$$Z' \xrightarrow{p} Y' \xrightarrow{q} X' \xrightarrow{(\Sigma h)k} \Sigma Z'$$

is a triangle, and we have $ph = \ell g$, $(\Sigma f)m = kq$, $i = mp$ and $j = q\ell$.
In other words, the four oriented triangles in the above pictures are triangles of $\mathcal{T}$, the four nonoriented triangles are commutative diagrams, and the two "big squares" containing the top and bottom vertices are commutative diagrams.

A consequence of the axioms is that for any object X of $\mathcal{T}$, the functor $\mathrm{Hom}_{\mathcal{T}}(X, ?)\colon \mathcal{T} \to \mathrm{mod}\ k$ sends triangles to exact sequences.

The derived category $\mathcal{D}(\mathcal{A})$ is a triangulated category whose suspension functor is [1].

3.1.3 Hereditary Case

We now restrict to the case where $\mathcal{A} = \text{mod } kQ$, for some finite quiver Q without oriented cycles. The path algebra kQ is then *hereditary*; in other words, the extension bifunctors $\text{Ext}^i_{kQ}(?, ?)$ vanish for $i \geq 2$.

In this situation, we have a good description of the objects of the bounded derived category $\mathcal{D}^{\text{b}}(\text{mod } kQ)$.

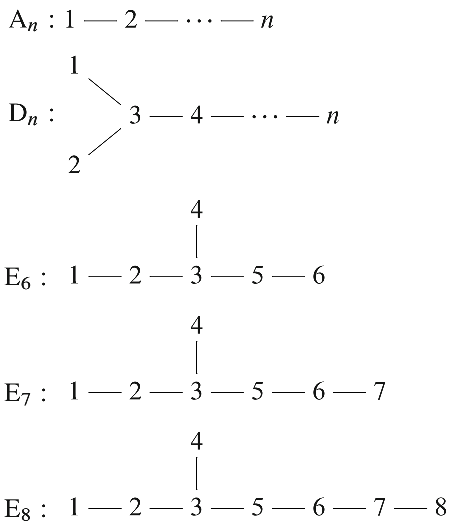
Proposition 3.2 ([28, **Lemma 5.2**]). *All indecomposable objects of* $\mathcal{D}^{\text{b}}(\text{mod } kQ)$ *are isomorphic to indecomposable* stalk complexes, *that is, complexes C for which there is an integer i such that $C_j = 0$ for $j \neq i$, and C_i is an indecomposable kQ-module.*

Thus all indecomposable objects of $\mathcal{D}^{\text{b}}(\text{mod } kQ)$ have the form $M[i]$, for M an indecomposable kQ-module and i an integer.

Another important feature in this case is the existence of an automorphism of the derived category called the *Auslander–Reiten translation* and denoted by τ. We refer the reader to, for instance, [35, Sect. 3] for its definition in the derived category. It is an avatar of the Auslander–Reiten translation in module categories, see [38, 45] in this volume, and also [3, 4].

3.1.4 Dynkin Case

We can say even more about the structure of $\mathcal{D}^{\text{b}}(\text{mod } kQ)$ if Q is an orientation of a simply-laced Dynkin diagram:

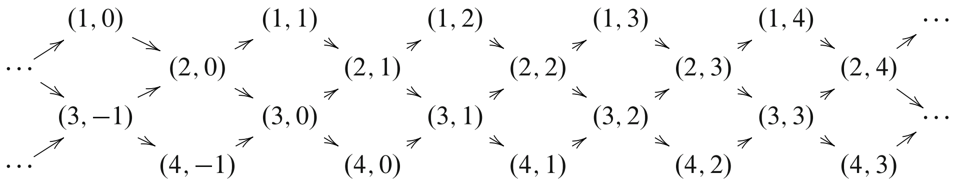


For any quiver Q, define the *repetition quiver* $\mathbb{Z}Q$ as follows:

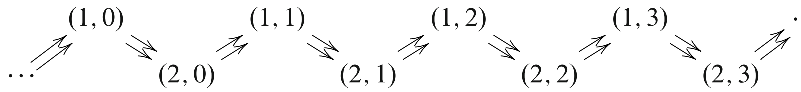
- vertices of $\mathbb{Z}Q$ are elements (i, n) of $Q_0 \times \mathbb{Z}$;

- for every arrow $a: i \to j$ in Q and every integer n, there are arrows $(a, n): (i, n) \to (j, n)$ and $(a^*, n): (j, n) \to (i, n+1)$ in $\mathbb{Z}Q$.

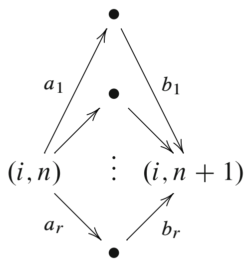
Example 3.3. If $Q = 1 \to 2 \to 3 \to 4$ is a quiver of type A_4, then $\mathbb{Z}Q$ looks like



Example 3.4. If $Q = 1 \rightrightarrows 2$ is the Kronecker quiver, then $\mathbb{Z}Q$ looks like



Define the *mesh category* $k(\mathbb{Z}Q)$ to be the category whose objects are the vertices of $\mathbb{Z}Q$ and whose morphisms are k-linear combinations of paths in $\mathbb{Z}Q$, modulo the *mesh relations*: whenever we have



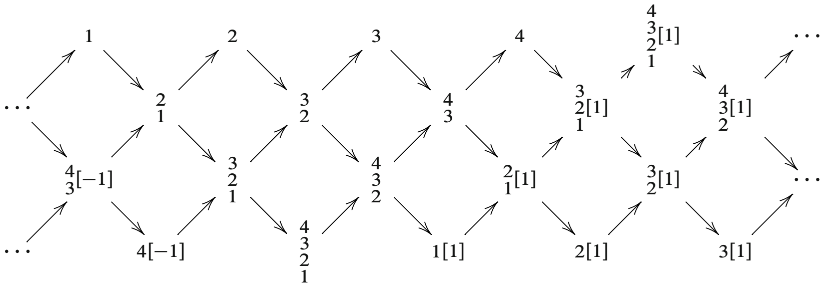
in $\mathbb{Z}Q$, where the a_j are all arrows leaving (i, n) and the b_j are all arrows arriving in $(i, n+1)$, then $\sum_{j=1}^{r} b_j a_j = 0$.

For any category $\mathcal{C}$, let $\operatorname{ind}(\mathcal{C})$ be the full subcategory of indecomposable objects of $\mathcal{C}$.

Theorem 3.5 ([27, **Proposition 4.6**]). *If Q is an orientation of a simply-laced Dynkin diagram, then* $\operatorname{ind}\big(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} kQ)\big)$ *is equivalent to* $k(\mathbb{Z}Q)$.

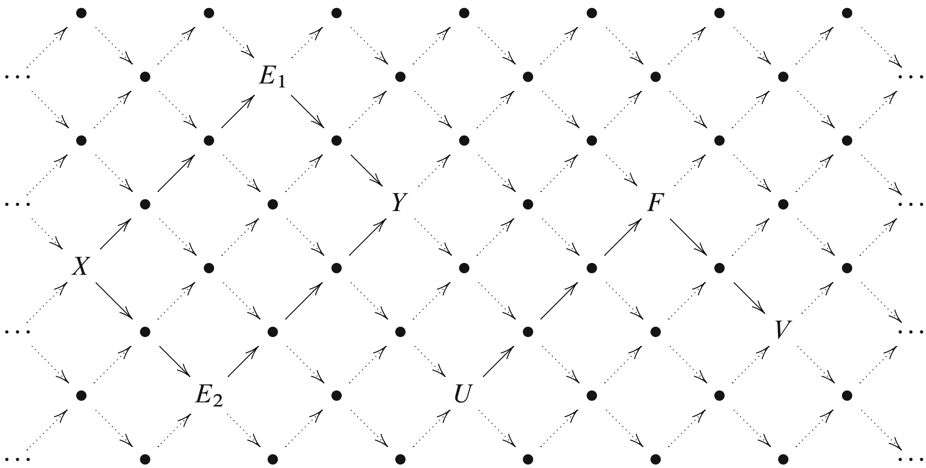
3.1.5 Example: Type A_n

Many computations can be done easily in the derived category of a quiver of type A_n. Let $Q = 1 \to 2 \to \cdots \to n$. Then $\operatorname{ind}\big(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} kQ)\big)$ is equivalent to the mesh category $k(\mathbb{Z}Q)$, and so can be pictured as follows (for $n = 4$):



Here we denoted kQ-modules by their composition series (recall that right kQ-modules are equivalent to representations of Q^{op}!). The action of the shift functor [1] can be seen on the diagram; that of the Auslander–Reiten translation τ is "translation to the left."

Morphism spaces between two indecomposable objects can be completely determined using the mesh relations; in particular, these vector spaces have dimension at most 1. Some triangles can also be derived directly on the picture:



On the left of the picture, we see a "rectangle" of solid arrows; it induces a triangle $X \to E_1 \oplus E_2 \to Y \to \Sigma X$.

On the right of the picture, we see a "hook" of solid arrows, which induces a triangle $U \to F \to V \to \Sigma U$. The rule that "hooks" must obey is the following: the length of the second part of the hook (from F to V on the picture) is one more than the length of the downward path from the first object (here U) to the bottom of the picture. Of course, hooks that are symmetric to the one pictured also yield triangles.

3.2 *Cluster Categories*

Cluster categories are triangulated categories that share many of the combinatorial properties of cluster algebras. They constitute the main setting for the definition of cluster characters (see Sect. 5).

3.2.1 Orbit Categories

Definition 3.6. Let $\mathcal{C}$ be a k-linear category, and let F be an automorphism of $\mathcal{C}$. The *orbit category* $\mathcal{C}/F$ is the k-linear category defined as follows:

- its objects are the objects of $\mathcal{C}$;
- for any objects X and Y, $\mathrm{Hom}_{\mathcal{C}/F}(X, Y) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, F^n Y)$.

As one might expect from the name "orbit category," the objects X and FX become isomorphic in $\mathcal{C}/F$.

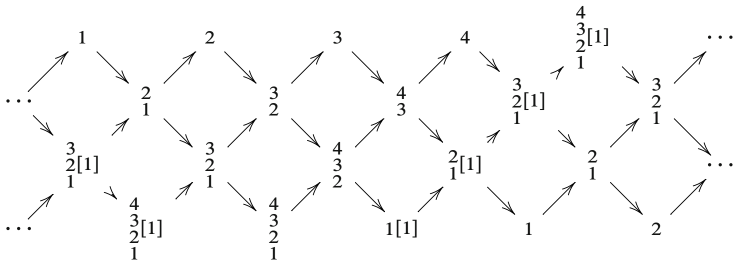
3.2.2 Cluster Categories

Definition 3.7 ([8])**.** Let Q be a quiver without oriented cycles. The *cluster category of* Q is the orbit category

$$\mathcal{C}_Q = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\ kQ)/F\ ,$$

where $F = \tau^{-1} \circ [1]$.

Example 3.8 If $Q = 1 \to 2 \to 3 \to 4$ is a quiver of type A_4, then using Sect. 3.1.5, we get that the cluster category can be depicted as



Notice that the objects repeat in the diagram. What happens is that any object X becomes identified with $FX = \tau^{-1}X[1]$. Morphism spaces and triangles can still be computed as in Sect. 3.1.5.

Let us now list some of the most important properties of the cluster category.

Theorem 3.9 ([33])**.** *The cluster category $\mathcal{C}_Q$ is a triangulated category, and the canonical functor $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\ kQ) \to \mathcal{C}_Q$ is a triangulated functor.*

Proposition 3.10 ([9]). *The functor* $H = \mathrm{Hom}_{\mathcal{C}_Q}(kQ, ?) \colon \mathcal{C}_Q \to \mathrm{mod}\ kQ$ *induces an equivalence of* k*-linear categories*

$$H \colon \mathcal{C}_Q/(kQ[1]) \longrightarrow \mathrm{mod}\ kQ\ ,$$

where $(kQ[1])$ *is the ideal of all morphisms factoring through a direct sum of direct summands of the object* $kQ[1]$.

Proposition 3.11 ([8]). *The cluster category is* 2*-Calabi–Yau, in the sense of Definition* 4.1 *below.*

Proposition 3.12 ([8]). *The cluster category has cluster-tilting objects, in the sense of Definition* 4.6 *below.*

4 2-Calabi–Yau Categories

The properties of the cluster categories listed at the end of the previous section are the ones needed for the theory of cluster characters. For this reason, we will turn to a more abstract setting where these properties are satisfied. In this section, k is an arbitrary field.

4.1 *Definition*

Let $\mathcal{C}$ be a k-linear category. We will assume the following:

- $\mathcal{C}$ is Hom-finite, that is, all morphism spaces in $\mathcal{C}$ are finite-dimensional;
- $\mathcal{C}$ is Krull–Schmidt, that is, every object of $\mathcal{C}$ is isomorphic to a direct sum of indecomposable objects (with local endomorphism rings), and this decomposition is unique up to isomorphism and reordering of the factors;
- $\mathcal{C}$ is triangulated, with shift functor Σ.

Definition 4.1. The category $\mathcal{C}$ is *2-Calabi–Yau* if, for all objects X and Y of $\mathcal{C}$, there is a (bifunctorial) isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y) \to D\ \mathrm{Hom}_{\mathcal{C}}(Y, \Sigma X),$$

where $D = \mathrm{Hom}_k(?, k)$ is the usual vector space duality.

Example 4.2. As seen in the previous section, the cluster category $\mathcal{C}_Q$ of a quiver Q without oriented cycles is a 2-Calabi–Yau category.

Example 4.3. Another family of examples is given by C. Amiot's *generalized cluster category* associated to a quiver with potential. This is developed in [2].

Example 4.4. In [24] and [6], certain subcategories $\mathcal{C}_w$ of the category of modules over a preprojective algebra were studied. These categories are Frobenius categories, and their stable categories are triangulated and 2-Calabi–Yau.

4.2 *Cluster-Tilting Objects*

Keep the notations of Sect. 4.1.

Definition 4.5. Let $\mathcal{C}$ be a triangulated category. An object X of $\mathcal{C}$ is said to be *rigid* if $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma X) = 0$.

Definition 4.6. Let $\mathcal{C}$ be a 2-Calabi–Yau category. An object T of $\mathcal{C}$ is a *cluster-tilting object* if

- T is rigid, and
- for any object X, $\mathrm{Hom}_{\mathcal{C}}(T, \Sigma X) = 0$ only if X is a direct sum of direct summands of T.

We will usually assume that cluster-tilting objects are *basic*, that is, that they can be written as a direct sum of *pairwise nonisomorphic* indecomposable objects.

Example 4.7. In a cluster category $\mathcal{C}_Q$, the object kQ is always a cluster-tilting object.

Example 4.8. An object T is cluster-tilting if and only if ΣT is.

Example 4.9. In Example 3.8, the objects

$$kQ = {\scriptstyle 1} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}, \quad T = {\scriptstyle 1} \oplus {\scriptstyle 3} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix} \quad \text{and} \quad T' = {\scriptstyle 1} \oplus {\scriptstyle 3} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}$$

are cluster-tilting objects.

We will see in Sect. 5.3 how to obtain new cluster-tilting objects from a given one.

The following property is crucial in the definition of cluster characters (Sect. 5): it tells us how to pass from a 2-Calabi–Yau category to a module category.

Proposition 4.10 ([9, 36])**.** *Let $T = T_1 \oplus \ldots \oplus T_n$ be a basic cluster-tilting object of a 2-Calabi–Yau category $\mathcal{C}$. We assume that the T_i are indecomposable. Then the functor*

$$H = \mathrm{Hom}_{\mathcal{C}}(T, \Sigma ?) \colon \mathcal{C} \longrightarrow \mathrm{mod}\ \mathrm{End}_{\mathcal{C}}(T)$$

induces an equivalence of k-linear categories

$$H : \mathcal{C}/(T) \longrightarrow \mathrm{mod}\ \mathrm{End}_{\mathcal{C}}(T).$$

Moreover,

- *$H(\Sigma^{-1}T_i)$ is an indecomposable projective module for all $i \in \{1, 2, \ldots, n\}$;*
- *$H(\Sigma T_i)$ is an indecomposable injective module for all $i \in \{1, 2, \ldots, n\}$;*
- *for any indecomposable object X other than the T_i, $H(\Sigma X) = \tau H(X)$, where τ is the Auslander–Reiten translation;*
- *triangles in $\mathcal{C}$ are sent to long exact sequences in* $\mathrm{mod}\ \mathrm{End}_{\mathcal{C}}(T)$.

4.3 *Index*

In a 2-Calabi–Yau triangulated category, cluster-tilting objects act like generators of the category. To be precise:

Proposition 4.11 **([36]).** *Let $\mathcal{C}$ be a 2-Calabi–Yau category with basic cluster-tilting object $T = \bigoplus_{i=1}^{n} T_i$. Then for any object X of $\mathcal{C}$, there is a triangle*

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X \ ,$$

where $T_0^X = \bigoplus_{i=1}^{n} T_i^{\oplus a_i}$ and $T_1^X = \bigoplus_{i=1}^{n} T_i^{\oplus b_i}$.

Definition 4.12 **([14]).** With the notations of Proposition 4.11, the *index of X* (with respect to T) is the integer vector

$$\mathrm{ind}_T\, X = (a_1 - b_1, \ldots, a_n - b_n) \ .$$

Note that, even though the triangle in Proposition 4.11 is not unique, the index is well-defined.

Remark 4.13. Applying H to the triangle in Proposition 4.11, we get an injective presentation of HX. More precisely, from the triangle

$$T_1^X \to T_0^X \to X \to \Sigma T_1^X \ ,$$

we can deduce another triangle

$$T_0^X \to X \to \Sigma T_1^X \to \Sigma T_0^X \ ,$$

and applying H to this triangle yields the exact sequence

$$0 \to HX \to H(\Sigma T_1^X) \to H(\Sigma T_0^X) \ ,$$

where $H(\Sigma T_0^X)$ and $H(\Sigma T_1^X)$ are injective modules by Proposition 4.10. This can be used to compute indices: if one can compute a minimal injective presentation of HX, then one can deduce the index of X.

Example 4.14. The index of T_i is always the vector with all coordinates zero, except the ith one, which is 1. The index of ΣT_i is the same vector, but replacing 1 by -1. These can be computed from the triangles

$$0 \to T_i \xrightarrow{\mathrm{id}} T_i \to 0$$

and

$$T_i \to 0 \to \Sigma T_i \xrightarrow{\mathrm{id}} \Sigma T_i \ .$$

Example 4.15. Let $Q = 1 \to 2 \to 3 \to 4$, and let $\mathcal{C}$ be the cluster category of Q, as in Example 3.8. Take

$$T = kQ[1] = {\scriptstyle 1}\,[1] \oplus \begin{smallmatrix}2\\1\end{smallmatrix}[1] \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}[1] \oplus \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}[1]\,.$$

Then the choice of name for the objects of $\mathscr{C}$ in the figure of Example 3.8 corresponds to their image by H in mod kQ (except for the summands of T).

We can compute the index of indecomposable objects by computing injective resolutions of modules, as pointed out in Remark 4.13. The injective modules are

$$I_1 = \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix}\,, \quad I_2 = \begin{smallmatrix}4\\3\\2\end{smallmatrix}\,, \quad I_3 = \begin{smallmatrix}4\\3\end{smallmatrix} \quad \text{and} \quad I_4 = {\scriptstyle 4}\,.$$

Here are some minimal injective presentations:

$$0 \to 2 \to \begin{matrix}4\\3\\2\end{matrix} \to \begin{matrix}4\\3\end{matrix}\,,$$

$$0 \to \begin{matrix}3\\2\\1\end{matrix} \to \begin{matrix}4\\3\\2\\1\end{matrix} \to 4\,,$$

$$0 \to 1 \to \begin{matrix}4\\3\\2\\1\end{matrix} \to \begin{matrix}4\\3\\2\end{matrix}\,.$$

Thus $\mathrm{ind}_T(2) = (0, -1, 1, 0)$, $\mathrm{ind}_T\left(\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right) = (-1, 0, 0, 1)$ and $\mathrm{ind}_T(1) = (-1, 1, 0, 0)$.

Here are some properties of indices.

(1) For any objects X and Y, $\mathrm{ind}_T\, X \oplus Y = \mathrm{ind}_T\, X + \mathrm{ind}_T\, Y$.
(2) [14] If X and Y are rigid and $\mathrm{ind}_T\, X = \mathrm{ind}_T\, Y$, then X and Y are isomorphic.
(3) [40] If $X \to Y \to Z \xrightarrow{f} \Sigma X$ is a triangle, and if f lies in (ΣT), then $\mathrm{ind}_T\, Y = \mathrm{ind}_T\, X + \mathrm{ind}_T\, Z$.
(4) [40] For any object X, the vector $(\mathrm{ind}_T\, X + \mathrm{ind}_T\, \Sigma X)$ only depends on the dimension vector of HX.

Notation 4.16. If $\mathbf{e}$ is the dimension vector of HX, then we put $\iota(\mathbf{e}) := (\mathrm{ind}_T\, X + \mathrm{ind}_T\, \Sigma X)$.

5 Cluster Characters

We now come to the main aim of these notes: to define cluster characters and give some of their main properties.

In this section, $\mathcal{C}$ is a 2-Calabi–Yau category and $T = \bigoplus_{i=1}^{n} T_i$ is a basic cluster-tilting object of $\mathcal{C}$. The field k is now assumed to be $\mathbb{C}$.

5.1 *Definition*

Definition 5.1 ([10, 12, 40]). The *cluster character* associated to T is the map CC with values in $\mathbb{Z}[x_1^{\pm 1}, \dots x_n^{\pm 1}]$ defined on objects of $\mathcal{C}$ by the formula

$$CC(X) = \mathbf{x}^{\mathrm{ind}_T X} \sum_{\mathbf{e} \in \mathbb{N}^n} \chi\big(\mathrm{Gr}_{\mathbf{e}}(HX)\big)\mathbf{x}^{-\iota(\mathbf{e})}.$$

Remark 5.2. By computing $\iota(\mathbf{e})$ when $\mathbf{e}$ is the dimension vector of a simple module, and by using the fact that ι is additive, one can show that the above formula is equivalent to

$$CC(X) = \mathbf{x}^{\mathrm{ind}_T X} F_{HX}(\hat{y}_1, \dots, \hat{y}_n),$$

where

- we define a matrix $B = (b_{ij})_{n \times n}$ by $b_{ij} = (\#\text{ arrows } i \to j) - (\#\text{ arrows } j \to i)$, where arrows are taken in the Gabriel quiver of the algebra $\mathrm{End}_{\mathcal{C}}(T)$,
- $\hat{y}_i = \prod_{j=1}^{n} x_j^{b_{ji}}$, and
- F_{HX} is the F-polynomial of HX as defined in Definition 2.11.

An immediate consequence of the above remark is the following.

Proposition 5.3 ([10, 12, 40]). *If X and Y are objects in $\mathcal{C}$, then $CC(X \oplus Y) = CC(X) \cdot CC(Y)$.*

Proof. This is a consequence of Proposition 2.13 and the fact that $\mathrm{ind}_T\, X \oplus Y = \mathrm{ind}_T\, X + \mathrm{ind}_T\, Y$. □

Example 5.4. For any choice of $\mathcal{C}$ and T, we have $CC(0) = 1$ and $CC(T_i) = x_i$.

Example 5.5. In Example 3.8, with $T = kQ[1]$, we have that

$$CC\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{x_1x_2 + x_1x_4 + x_3x_4 + x_2x_3x_4}{x_1x_2x_3}.$$

5.2 *Multiplication Formula*

The main theorem of the theory of cluster characters is the following *multiplication formula*.

Theorem 5.6 ([10, 12, 40]). *Let X and Y be objects of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$ is one-dimensional. Let $\varepsilon \in \mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$ and $\eta \in \mathrm{Hom}_{\mathcal{C}}(Y, \Sigma X)$ be nonzero (they are unique up to a scalar). Let*

$$Y \xrightarrow{i} E \xrightarrow{p} X \xrightarrow{\varepsilon} \Sigma Y$$

and

$$X \xrightarrow{i'} E' \xrightarrow{p'} Y \xrightarrow{\eta} \Sigma X$$

be the corresponding non-split triangles in $\mathcal{C}$. Then

$$CC(X) \cdot CC(Y) = CC(E) + CC(E') \, .$$

This result has the same spirit as Theorem 2.14 for F-polynomials. Its proof relies on the following dichotomy:

Proposition 5.7 (Prop. 4.3 of [40]**).** *Keep the notations of Theorem 5.6. Let U and V be submodules of HX and HY, respectively. Then the two following conditions are equivalent:*

1. *There exists a submodule W of HE such that $Hp(W) = U$ and $(Hi)^{-1}(W) = V$.*
2. *There does not exist any submodule W' of HE' such that $Hp'(W') = V$ and $(Hi')^{-1}(W') = U$.*

This result allows us to compare Euler characteristics of the submodule Grassmannians $\mathrm{Gr}_*(HE)$, $\mathrm{Gr}_*(HE')$, $\mathrm{Gr}_*(HX)$ and $\mathrm{Gr}_*(HY)$, in a way similar to (but more involved than) what we did in the proof of Proposition 2.13. Together with a result concerning the indices [40, Lemma 5.1], it allows to prove Theorem 5.6. We do not recount the proof here, but rather refer the reader to [40].

5.3 *Mutation of Cluster-Tilting Objects*

Assume that $R = R_1 \oplus R_2 \oplus \ldots \oplus R_n$ is a cluster-tilting object of $\mathcal{C}$. Assume that $\mathrm{End}_{\mathcal{C}}(R)$ is written as $\mathbb{C}Q_R/I$, with Q_R a finite quiver without oriented cycles of length 1 or 2, and I an admissible ideal.

Fix $i \in \{1, \ldots, n\}$. Consider the following triangles:

$$R_i \xrightarrow{\alpha} \bigoplus_{a:i \to j \text{ in } Q_R} R_j \to R_i^* \to \Sigma R_i$$

and

$$R_i^{**} \to \bigoplus_{b:h \to i \text{ in } Q_R} R_h \xrightarrow{\beta} R_i \to \Sigma R_i^{**},$$

where α is the direct sum of all morphisms $R_i \to R_j$ corresponding to arrows $a : i \to j$ in Q_R, and β is the direct sum of all morphisms $R_h \to R_i$ corresponding to arrows $b : h \to i$ in Q_R.

Theorem 5.8 ([30])**.**

(1) *The objects* R_i^* *and* R_i^{**} *are isomorphic.*
(2) *The object* $\mu_i(R) := R_1 \oplus \ldots \oplus R_{i-1} \oplus R_i^* \oplus R_{i+1} \oplus \ldots \oplus R_n$ *is a cluster-tilting object of* $\mathcal{C}$.
(3) *The only cluster-tilting objects of* $\mathcal{C}$ *having all* R_j $(j \neq i)$ *as direct summands are* R *and* $\mu_i(R)$.
(4) *The space* $\mathrm{Hom}_{\mathcal{C}}(R_i, \Sigma R_i^*)$ *is one-dimensional.*

Definition 5.9. The object $\mu_i(R)$ of Theorem 5.8 is the *mutation of* R *at* i. Any cluster-tilting object obtained from R by a sequence of mutations is said to be *reachable from* R.

We will see in Sect. 5.4 why this process of mutation, coupled with the multiplication formula of Theorem 5.6, allows for a categorification of cluster algebras.

Example 5.10. In Example 4.9, the cluster-tilting object T is obtained mutating kQ at the second direct summand, and T' is obtained by mutating T at the third direct summand.

An interesting result holds for cluster categories.

Proposition 5.11 ([8, **Proposition 3.5**])**.** *Let* Q *be a quiver without oriented cycles, and let* T *be a cluster-tilting object of the cluster category* $\mathcal{C}_Q$. *Then all cluster-tilting objects of* $\mathcal{C}_Q$ *are reachable from* T.

Remark 5.12. There are 2-Calabi–Yau categories in which cluster-tilting objects are not all reachable from each other. An example is given in [44, Example 4.3].

5.4 *Application*: *Categorification of Cluster Algebras*

The results of the previous sections combine neatly to provide a categorification of cluster algebras.

Corollary 5.13. *Let* T *be a cluster-tilting object of a 2-Calabi–Yau category* $\mathcal{C}$, *and let* R *be as in Theorem 5.8. Then*

$$CC(R_i) \cdot CC(R_i^*) = \prod_{a:i \to j \text{ in } Q_R} CC(R_j) + \prod_{b:h \to i \text{ in } Q_R} CC(R_h).$$

Proof. This follows directly from Theorem 5.8(4) and from Theorem 5.6, and from the fact that $CC(X \oplus Y) = CC(X) \cdot CC(Y)$ for all objects X and Y. □

The point of this corollary is that it writes down exactly an exchange relation in a cluster algebra:

Definition 5.14 ([5, 19–21])**.** Let Q be a quiver with n vertices without oriented cycles of length 1 or 2, and let $\mathbf{u} = (u_1, \ldots, u_n)$ be a free generating set of the field $\mathbb{Q}(x_1, \ldots, x_n)$. Call $(Q, \mathbf{u})$ a *seed*.

Then the mutation of $(Q, \mathbf{u})$ at i is a new seed $(Q', \mathbf{u}')$ $(u_1, \ldots, u_{i-1}, u'_i, u_{i+1}, \ldots, u_n)$, where

$$u_i \cdot u'_i = \prod_{a:i \to j \text{ in } Q} u_j + \prod_{b:h \to i \text{ in } Q} u_h.$$

and Q' is the quiver obtained from Q by changing the orientation of all arrows adjacent to i, adding an arrow $h \to j$ for every path $h \to i \to j$, and removing cycles of length 2.

Now, the mutation of quivers can also be interpreted inside the cluster category:

Theorem 5.15 ([7, **Theorem 5.2]**). *Let R be as in Theorem 5.8. Assume that the endomorphism algebra of R is the Jacobian algebra of a quiver with potential (Q_R, W_R) (see [15]). Then the endomorphism algebra of $\mu_i(R)$ is the Jacobian algebra of the mutated quiver with potential $\mu_i(Q_R, W_R)$. In particular, $Q_{\mu_i(R)} = \mu_i(Q_R)$.*

Thus we get:

Corollary 5.16 ([10, 12, 40, 43]**...**). *If $\mathscr{C}$ is a cluster category or a generalized cluster category (see [2]), then the cluster character sends reachable indecomposable objects of $\mathscr{C}$ to cluster variables in the cluster algebra of Q, where Q is the Gabriel quiver of* $\mathrm{End}_{\mathscr{C}}(T)$.

Remark 5.17.(1) The multiplication formula of Theorem 5.6 can be generalized to the case when the dimension of the space $\mathrm{Hom}_{\mathscr{C}}(X, \Sigma Y)$ is greater than 1, see [23, 41].

(2) Cluster characters can also be defined in the setting of stably 2-Calabi–Yau Frobenius categories, see [22] (and also [6, 24]).

(3) If we work over finite fields instead of $\mathbb{C}$, then we can define cluster characters by counting points in submodule Grassmannians. This leads to a categorification of quantum cluster algebras, see [47].

(4) It is possible to study cluster characters without the assumption that $\mathscr{C}$ is Hom-finite, provided $\mathscr{C}$ is the generalized cluster category of a quiver with potential, see [43].

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A Course on Cluster Tilted Algebras

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Introduction

These notes are an expanded version of a mini-course given in the CIMPA School 'Homological Methods, Representation Theory and Cluster Algebras', held from the 7th to the 18th of March 2016 in Mar del Plata (Argentina). The aim of the course was to introduce the participants to cluster tilted algebras and their applications in the representation theory of algebras.

Cluster tilted algebras were defined in [38] and also, independently, in [43] for type $\mathbb{A}$. This class of finite-dimensional algebras appeared as a byproduct of the now extensive theory of cluster algebras of Fomin and Zelevinsky [46]. They are the endomorphism algebras of tilting objects in the cluster category of [36]. Since their introduction, they have been the subject of several investigations, which we survey in this course.

For reasons of space, it was not possible to be encyclopaedic. Thus, we have chosen to concentrate on the representation theoretical aspects and to ignore other aspects of the theory like, for instance, the relations between cluster tilted algebras and cluster algebras arising from surfaces, or the combinatorics of cluster variables. In keeping with the nature of the course, we tried to make these notes as self-contained as we could, providing examples for most results and proofs whenever possible.

1 Tilting in the Cluster Category

1.1 *Notation*

Throughout these notes, k denotes an algebraically closed field and algebras are, unless otherwise specified, basic and connected finite-dimensional k-algebras. For such an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\operatorname{ind} A$ a full subcategory containing exactly one representative from each isoclass (= isomorphism

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class) of indecomposable A-modules. When we speak about an A-module or an indecomposable A-module, we always mean implicitly that it belongs to mod A or to ind A, respectively. Given a module M, we denote by pd M and id M its projective and injective dimension, respectively.The global dimension of A is denoted by gl. dim A. Given an additive category $\mathcal{C}$, we sometimes write $M \in \mathcal{C}_0$ to express that M is an object in $\mathcal{C}$. We denote by add M the full additive subcategory of $\mathcal{C}$ consisting of the finite direct sums of indecomposable summands of M.

We recall that any algebra A can be written in the form $A \cong \mathsf{k}Q_A/I$ where $\mathsf{k}Q_A$ is the path algebra of the quiver Q_A of A, and I is an ideal generated by finitely many relations.

We recall that a quiver Q is defined by two sets: the set Q_0 of its *points* (also called *vertices*) and the set Q_1 of its *arrows*. We follow the notation of [17], thus, if α, β are arrows such that the target of α coincides with the source of β, we write their composition as $\alpha\beta$. A *relation* on a quiver is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where the λ_i are nonzero scalars and the w_i are paths of length at least two all having the same source and the same target. It is a *zero-relation* if it is a path, and a *commutativity relation* if it is of the form $\rho = w_1 - w_2$. Following [33], we sometimes consider equivalently an algebra A as a k-category, in which the object class A_0 is a complete set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$ in A and the set of morphisms $A(i, j)$ from e_i to e_j is the vector space $e_i A e_j$. We denote by P_i, I_i and S_i, respectively, the indecomposable projective, the indecomposable injective and the simple A-module corresponding to e_i.

For unexplained notions and results of representation theory, we refer the reader to [17, 20]. For tilting theory, we refer to [9, 17].

1.2 *The Derived Category of a Hereditary Algebra*

Once an exotic mathematical object, the derived category is now an indispensable tool of homological algebra. For its definition and properties, we refer the reader to [49, 50, 68]. Here we are only interested in derived categories of hereditary algebras.

Let Q be a finite, connected and acyclic quiver. Its path algebra $\mathsf{k}Q$ is hereditary, see [17, 20]. Let $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\mathsf{k}Q)$ denote the bounded derived category of mod $\mathsf{k}Q$, that is, the derived category of the category of bounded complexes of finitely generated $\mathsf{k}Q$-modules. It is well known that $\mathcal{D}$ is a triangulated category with almost split triangles, see [51]. We denote by $[1]_{\mathcal{D}}$ the shift of $\mathcal{D}$ and by $\tau_{\mathcal{D}}$ its Auslander–Reiten translation, or simply by [1] and τ, respectively, if no ambiguity may arise. Both are automorphisms of $\mathcal{D}$.

Because $\mathcal{D}$ is a Krull–Schmidt category, every object in $\mathcal{D}$ decomposes as a finite direct sum of objects having local endomorphisms rings. Let ind $\mathcal{D}$ denote a full subcategory of $\mathcal{D}$ consisting of exactly one representative from each isoclass of indecomposable objects in $\mathcal{D}$. Because $\mathsf{k}Q$ is hereditary, these indecomposable objects have a particularly simple form: they can be written as $M[i]$, that is, stalk complexes with M an indecomposable $\mathsf{k}Q$-module concentrated in degree i, for $i \in \mathbb{Z}$, see [51, p. 49]. Morphisms between indecomposable objects in $\mathcal{D}$ are computed according to the rule:

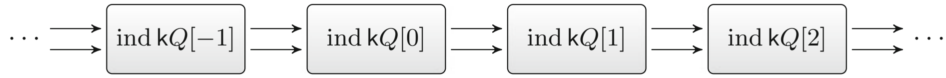
$$\mathrm{Hom}_{\mathcal{D}}(M[i], N[j]) = \begin{cases} \mathrm{Hom}_{\mathsf{k}Q}(M, N) & \text{if } j = i \\ \mathrm{Ext}^1_{\mathsf{k}Q}(M, N) & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

As is usual when dealing with triangulated categories, morphisms from an object to a shift of another are called *extensions* and we write $\mathrm{Ext}^i_{\mathcal{D}}(M, N) = \mathrm{Hom}_{\mathcal{D}}(M, N[i])$. Denoting by $\mathrm{D} = \mathrm{Hom}_{\mathsf{k}}(-, \mathsf{k})$ the usual vector space duality, the shift and the Auslander–Reiten translation of $\mathcal{D}$ are related by the following bifunctorial isomorphism, known as *Serre duality*:

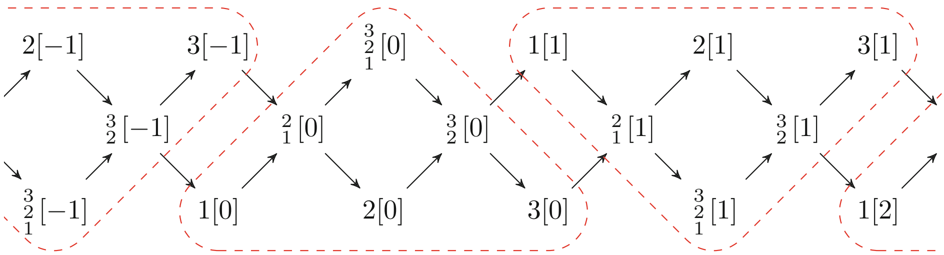
$$\mathrm{Hom}_{\mathcal{D}}(M, N[1]) \cong \mathrm{D}\,\mathrm{Hom}_{\mathcal{D}}(N, \tau M)$$

which is the analogue inside $\mathcal{D}$ of the well-known Auslander–Reiten formula in a module category, see [17, p. 118].

We now describe the Auslander–Reiten quiver $\Gamma(\mathcal{D})$ of $\mathcal{D}$. Let $\Gamma(\mathrm{mod}\,\mathsf{k}Q)$ denote the Auslander–Reiten quiver of $\mathsf{k}Q$. For each $i \in \mathbb{Z}$, denote by Γ_i a copy of $\Gamma(\mathrm{mod}\,\mathsf{k}Q)$. Then $\Gamma(\mathcal{D})$ is the translation quiver obtained from the disjoint union $\coprod_{i\in\mathbb{Z}} \Gamma_i$ by adding, for each $i \in \mathbb{Z}$ and each arrow $x \to y$ in Q, an arrow from I_x in Γ_i to P_y in Γ_{i+1} and by setting $\tau P_x[1] = I_x$ for each point $x \in Q_0$, see [51, p. 52]. Identifying Γ_i with $\mathrm{ind}\,\mathsf{k}Q[i]$, one can then think of $\mathrm{ind}\,\mathcal{D}$ as being formed by copies of $\mathrm{ind}\,\mathsf{k}Q$ joined together by extra arrows.

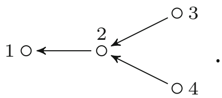


Examples 1.1. (a) Let Q be the quiver $\overset{3}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{1}{\circ}$.
Denoting the indecomposable $\mathsf{k}Q$-modules by their Loewy series, the Auslander–Reiten quiver of $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\mathsf{k}Q)$ is

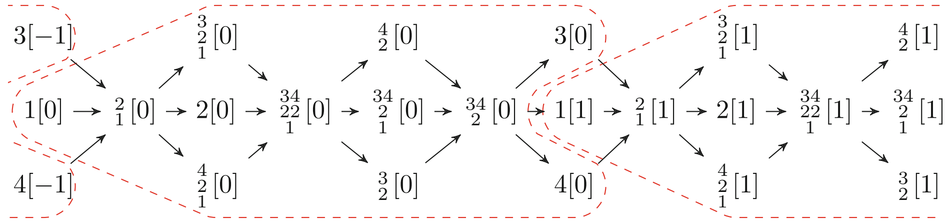


Distinct copies of $\Gamma(\mathrm{mod}\,\mathsf{k}Q)$ are indicated by dotted lines.

(b) Let Q be the quiver

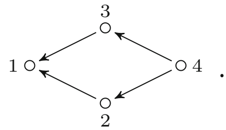


The Auslander–Reiten quiver of $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\mathsf{k}Q)$ is

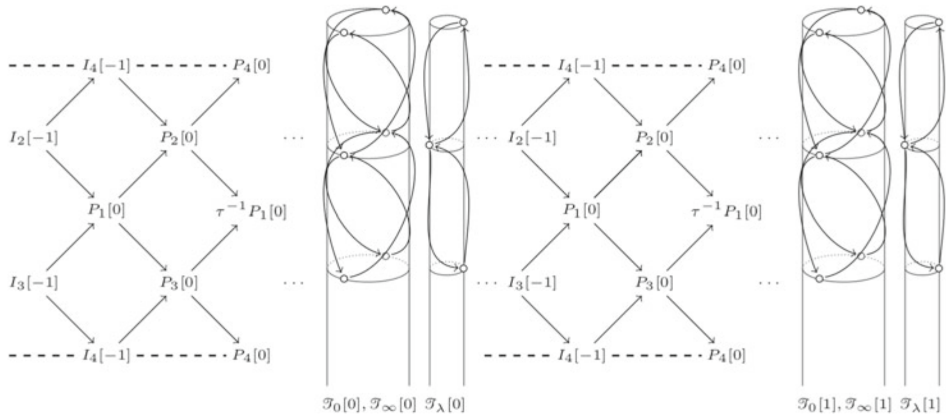


where again, distinct copies of $\Gamma(\operatorname{mod} \mathsf{k}Q)$ are indicated by dashed lines.

(c) Let Q be the quiver



Then $\mathsf{k}Q$ is tame hereditary, $\Gamma(\operatorname{mod} \mathsf{k}Q)$ consists of an infinite postprojective component, an infinite preinjective component, two exceptional tubes of rank two and an infinite family of tubes of rank one. Thus the Auslander–Reiten quiver of $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} \mathsf{k}Q)$ is



where one identifies the horizontal dashed lines.

This is the general shape: if Q is Dynkin, then $\Gamma\big(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} \mathsf{k}Q)\big)$ has a unique component of the form $\mathbb{Z}Q$ while, if Q is Euclidean or wild, it has infinitely many such components, any two of which are separated by infinitely many tubes, or components of type $\mathbb{Z}\mathbb{A}_\infty$, respectively. The components of the form $\mathbb{Z}Q$ are called *transjective*, and the others are called *regular*.

1.3 *The Cluster Category*

Let Q be a finite, connected and acyclic quiver and $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\operatorname{mod} \mathsf{k}Q)$ its derived category. Because each of the shift [1] and the Auslander–Reiten translation τ is an automorphism of $\mathcal{D}$, so is the composition $F = \tau^{-1}[1]$. One may thus define the *orbit category* $\mathcal{D}/F$. Its objects are the F-orbits of the objects in $\mathcal{D}$. For each $X \in \mathcal{D}_0$, we denote by $\widetilde{X} = (F^i X)_{i\in\mathbb{Z}}$ its F-orbit. Then, for two objects $\widetilde{X}, \widetilde{Y}$ in $\mathcal{D}/F$, we define

$$\operatorname{Hom}_{\mathcal{D}/F}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(X, F^i Y).$$

One shows easily that this vector space does not depend on the choices of the objects X and Y in their respective orbits and thus, the Hom-spaces of $\mathcal{D}/F$ are well defined.

Definition 1.2 ([36]). The *cluster category* of the quiver Q is the orbit category $\mathcal{D}/F$. It is denoted by $\mathcal{C}_Q$, or simply $\mathcal{C}$, if no ambiguity may arise.

It follows directly from the definition that $\mathcal{C}$ is a k-linear category and that there exists a canonical projection functor $\pi: \mathcal{D} \to \mathcal{C}$ which sends each $X \in \mathcal{D}_0$ to its F-orbit $\widetilde{X}$ in $\mathcal{C}$ and acts in the obvious way on morphisms.

The next theorem summarises the elementary properties of $\mathcal{C}$ and π.

Theorem 1.3 ([36, 57]). *With the above notations*

(a) *$\mathcal{C}$ is a Krull–Schmidt category and $\pi: \mathcal{D} \to \mathcal{C}$ preserves indecomposability,*
(b) *$\mathcal{C}$ is a triangulated category and $\pi: \mathcal{D} \to \mathcal{C}$ is a triangle functor,*
(c) *$\mathcal{C}$ has almost split triangles and $\pi: \mathcal{D} \to \mathcal{C}$ preserves almost split triangles.*

We derive some consequences. Because of (b) and (c) above, the shift $[1]_{\mathcal{C}}$ of $\mathcal{C}$ and its Auslander–Reiten translation $\tau_{\mathcal{C}}$ are induced by $[1]_{\mathcal{D}}$ and $\tau_{\mathcal{D}}$, respectively. Thus, for each $X \in \mathcal{D}_0$, we have

$$\widetilde{X}[1]_{\mathcal{C}} = \widetilde{X[1]_{\mathcal{D}}} \quad \text{and} \quad \tau_{\mathcal{C}}\widetilde{X} = \widetilde{\tau_{\mathcal{D}}X} \ .$$

As a direct consequence, we have, for each $\widetilde{X} \in \mathcal{C}_0$,

$$\tau_{\mathcal{C}}\widetilde{X} = \widetilde{X}[1]_{\mathcal{C}} \ .$$

Indeed, we have $\widetilde{X} = \widetilde{FX} = \widetilde{\tau_{\mathcal{D}}^{-1}X[1]} = \tau_{\mathcal{C}}^{-1}\widetilde{X}[1]$, which establishes our claim.

Again, we denote briefly $[1]_{\mathcal{C}} = \tau_{\mathcal{C}}$ by [1], or by τ, if no ambiguity may arise.

Another easy consequence is that, if Q, Q' are quivers such that there is a triangle equivalence $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} \mathsf{k}Q) \cong \mathcal{D}^{\mathrm{b}}(\operatorname{mod} \mathsf{k}Q')$, then this equivalence induces another triangle equivalence $\mathcal{C}_Q \cong \mathcal{C}_{Q'}$. This is expressed by saying that $\mathcal{C}_Q$ is invariant under derived equivalence.

Denoting $\operatorname{Ext}^1_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{C}}(X, Y[1])$, we get the following formula.

Lemma 1.4 ([36, **(1.4)**]). *Let $\widetilde{X}, \widetilde{Y} \in \mathcal{C}_0$, then we have a bifunctorial isomorphism*

$$\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \cong \mathrm{D}\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{Y}, \widetilde{X}) \ .$$

Proof. Using that $F = \tau_{\mathcal{D}}^{-1}[1]_{\mathcal{D}}$ (and thus $\tau_{\mathcal{D}} = F^{-1}[1]_{\mathcal{D}}$), we have

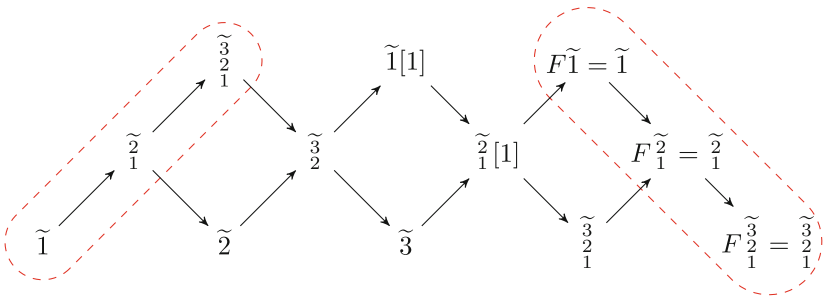
$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) &= \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}\big(X, F^i Y[1]_{\mathcal{D}}\big)\\ &\cong \bigoplus_{i\in\mathbb{Z}} \operatorname{D}\operatorname{Hom}_{\mathcal{D}}\big(F^i Y, \tau_{\mathcal{D}} X\big)\\ &\cong \bigoplus_{i\in\mathbb{Z}} \operatorname{D}\operatorname{Hom}_{\mathcal{D}}\big(F^i Y, F^{-1} X[1]_{\mathcal{D}}\big)\\ &\cong \bigoplus_{i\in\mathbb{Z}} \operatorname{D}\operatorname{Hom}_{\mathcal{D}}\big(F^i Y, X[1]_{\mathcal{D}}\big) \cong \operatorname{D}\operatorname{Ext}^1_{\mathcal{C}}(\widetilde{Y}, \widetilde{X})\ .\end{aligned}$$

□

The previous formula says that $\mathcal{C}$ is what is called a *2-Calabi–Yau* category. Reading the formula as $\operatorname{Ext}^1_{\mathcal{C}}\big(\widetilde{X}, \widetilde{Y}\big) \cong \operatorname{D}\operatorname{Hom}_{\mathcal{C}}\big(\widetilde{Y}, \tau\widetilde{X}\big)$, we see that it can be interpreted as the Auslander–Reiten formula in $\mathcal{C}$.

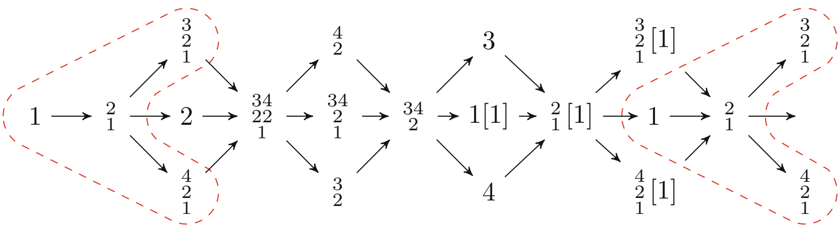
We now show how to compute the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ of $\mathcal{C}$.

Examples 1.5. (a) Let Q be as in Example 1.1(a). The cluster category is constructed from the derived category by identifying the objects which lie in the same F-orbit, hence each X with the corresponding $FX = \tau^{-1}X[1]$. Thus $\Gamma(\mathcal{C})$ is obtained by identifying the dotted sections in the figure below, so that $\Gamma(\mathcal{C})$ lies on a Möbius strip.



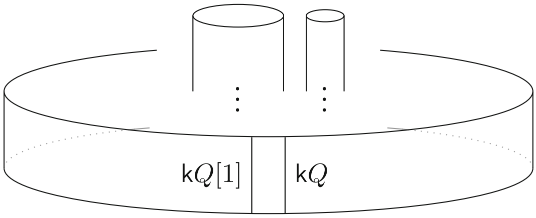
From now on, in examples, we drop the $\sim$ denoting the orbit of an object.

(b) Let Q be as in Example 1.1(b). Applying the same recipe, we get that $\Gamma(\mathcal{C})$ lies on the cylinder obtained by identifying the dotted sections.



(c) This procedure is general: in order to construct $\Gamma(\mathcal{C}_Q)$, we must identify the sections in $\Gamma\big(\mathcal{D}^{\mathrm{b}}(\operatorname{mod} \mathsf{k}Q)\big)$ corresponding to $\mathsf{k}Q_{\mathsf{k}Q}$, that is, the indecomposable projective $\mathsf{k}Q$-

modules, and to $\tau^{-1}\mathsf{k}Q[1]$. So, if Q is as in example 1.2(c), then $\Gamma(\mathcal{C}_Q)$ is of the form



Thus, the Auslander–Reiten quiver of the cluster category always admits a transjective component, which is the whole quiver if Q is Dynkin, and is of the form $\mathbb{Z}Q$ otherwise. In this latter case, $\Gamma(\mathcal{C}_Q)$ also admits tubes if Q is Euclidean, or components of the form $\mathbb{Z}\mathbb{A}_\infty$ if Q is wild.

1.4 *Tilting Objects*

The tilting objects in the cluster category are the analogues of the tilting modules over a hereditary algebra, see [17, Chap. VI]. Let Q be a finite, connected and acyclic quiver and $\mathcal{C} = \mathcal{C}_Q$ the corresponding cluster category.

Definition 1.6 ([36, **(3.3)**])**.** An object T in $\mathcal{C}$ is called *rigid* if $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$. It is called *tilting* if it is rigid and has a maximal number of isoclasses of indecomposable direct summands.

Actually, the maximality in the definition may be replaced by the following condition, easier to verify.

Proposition 1.7 ([36, **(3.3)**])**.** *Let T be a rigid object in $\mathcal{C}_Q$. Then T is tilting if and only if it has $|Q_0|$ isoclasses of indecomposable direct summands.*

Examples 1.8. (a) Let T be a tilting $\mathsf{k}Q$-module. Denoting by

$$i\colon \mathrm{mod}\,\mathsf{k}Q \hookrightarrow \mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\mathsf{k}Q)$$

the canonical embedding $X \mapsto X[0]$, then the image of T under the composition of functors

$$\mathrm{mod}\,\mathsf{k}Q \xhookrightarrow{i} \mathcal{D} \xrightarrow{\pi} \mathcal{C}_Q = \mathcal{C}$$

is rigid. Indeed, T embeds as a rigid object in $\mathcal{D}$, and thus is projected on a rigid object in $\mathcal{C}$ (using that $\mathcal{C}$ is a 2-Calabi–Yau category). It has obviously as many isoclasses of indecomposable summands as T has in $\mathrm{mod}\,\mathsf{k}Q$. Therefore, it is a tilting object in $\mathcal{C}$. Such an object is said to be *induced* from the tilting module T. For instance, in Example 1.5(a), the tilting $\mathsf{k}Q$-module $T = 1 \oplus {\begin{smallmatrix}3\\2\\1\end{smallmatrix}} \oplus 3$ induces a tilting object in $\mathcal{C}$.

Of course, there exist tilting objects which are not induced from tilting modules. Also in the algebra of Example 1.5(a), the tilting object $2 \oplus 1[1] \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}[1]$ is not induced.

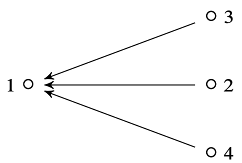
(b) In the algebra of Example 1.5(b), the object

$$T = 2 \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus 1[1]$$

is not induced. We check that it is a tilting object. Because it has obviously $4 = |Q_0|$ isoclasses of indecomposable summands, we just have to check its rigidity. As an example, we check here that $\mathrm{Ext}^1_{\mathcal{C}}(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}, 1[1]) = 0$. Because of Lemma 1.4, it is equivalent to prove that $\mathrm{Ext}^1_{\mathcal{C}}(1[1], \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}) = 0$. Now

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{C}}(1[1], \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}) &= \mathrm{Hom}_{\mathcal{C}}(1[1], \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}[1]) \\ &= \mathrm{Hom}_{\mathcal{C}}(1, \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}) \\ &= \mathrm{Hom}_{\mathcal{D}}(1, \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}) \oplus \mathrm{Hom}_{\mathcal{D}}(1, \tau^{-1} \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}[1]) \\ &= \mathrm{Hom}_{\mathsf{k}Q}(1, \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}) \oplus \mathrm{Ext}^1_{\mathsf{k}Q}(1, 3) \\ &= 0\,. \end{aligned}$$

However, one can look at this example from another point of view. Indeed $\mathsf{k}Q$ is derived equivalent to $\mathsf{k}Q'$ where Q' is the quiver



and, under this triangle equivalence, T corresponds to the tilting $\mathsf{k}Q'$-module $T' = 1 \oplus \begin{smallmatrix} 3\,2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3\,4 \\ 1 \end{smallmatrix} \oplus 2$, that is, T may be considered as induced from a tilting $\mathsf{k}Q'$-module.

This change of quiver is actually always possible.

Proposition 1.9 ([36, **(3.3)**])**.** *Let T be a tilting object in $\mathcal{C}_Q$. Then there exists a quiver Q' such that $\mathsf{k}Q$ and $\mathsf{k}Q'$ are derived equivalent, and T is induced from a tilting $\mathsf{k}Q'$-module.*

In one important aspect, tilting objects behave better than tilting modules. Indeed, let A be a hereditary algebra, a rigid A-module T is called an *almost complete tilting module* if it has $|Q_0| - 1$ isoclasses of indecomposable summands. Because of Bongartz' lemma [17, p. 196], there always exists an indecomposable module M such that $T \oplus M$ is a tilting module. Such an M is called a *complement* of T. It is known that an almost complete tilting module has at most two nonisomorphic complements and it has two if and only if it is sincere, see [55]. We now look at the corresponding result inside the cluster category.

Definition 1.10. Let $\mathcal{C}$ be a Krull–Schmidt category, and $X \in \mathcal{C}_0$. For $U \in \mathcal{C}_0$, a morphism

$$f_X\colon\ U_X \to U$$

with $U_X \in (\operatorname{add} X)_0$ is called a *right X-approximation* for U if, for every object $X' \in (\operatorname{add} X)_0$ and morphism $f' \colon X' \to U$, there exists $g \colon X' \to U_X$ such that $f' = f_X g$

$$\begin{array}{ccc} U_X & \xrightarrow{f_X} & U \, . \\ {\scriptstyle g}\big\uparrow & \nearrow{\scriptstyle f'} & \\ X' & & \end{array}$$

Such a right approximation f_X is called *right minimal* if, for a morphism $h \colon U_X \to U_X$, the relation $f_X h = f_X$ implies that h is an automorphism

$$\begin{array}{ccc} U_X & \xrightarrow{f_X} & U \\ {\scriptstyle h}\big\downarrow & & \big\| \\ U_X & \xrightarrow{f_X} & U \, . \end{array}$$

One defines dually *left X-approximations* and *left minimal X-approximations*.

Let now T be a rigid object in the cluster category $\mathcal{C}$. In analogy with the situation for modules, T is called an *almost complete tilting object* if it has $|Q_0| - 1$ isoclasses of indecomposable summands. Again, because of Bongartz' lemma and Proposition 1.9 above, there exists at least one indecomposable object M in $\mathcal{C}$ such that $T \oplus M$ is a tilting object. Then M is called a *complement* to T.

Theorem 1.11 ([36, **(6.8)**]). *An almost complete tilting object T in $\mathcal{C}$ has exactly two isoclasses of indecomposable complements M_1, M_2 and moreover there exist triangles*

$$M_2 \xrightarrow{g_1} T_1 \xrightarrow{f_1} M_1 \longrightarrow M_2[1]$$

and

$$M_1 \xrightarrow{g_2} T_2 \xrightarrow{f_2} M_2 \longrightarrow M_1[1]$$

where f_1, f_2 are right minimal T-approximations and g_1, g_2 are left minimal T-approximations.

Example 1.12. In the cluster category of Example 1.1(b), the object $T = \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$ is almost complete. It has exactly two (isoclasses of) complements, namely $M_1 = 2$ and $M_2 = \begin{smallmatrix}3\,4\\2\end{smallmatrix}$. We also have triangles

$$2 \longrightarrow \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\,4\\2\end{smallmatrix} \longrightarrow 2[1]$$

$$\begin{smallmatrix}3\,4\\2\end{smallmatrix} \longrightarrow 1[1] \longrightarrow 2 \longrightarrow \begin{smallmatrix}3\,4\\2\end{smallmatrix}[1]$$

where the morphisms are minimal approximations.

2 Cluster Tilted Algebras

2.1 *The Definition and Examples*

In classical tilting theory, the endomorphism algebra of a tilting module over a hereditary algebra is called a *tilted algebra*. Due to its proximity to hereditary algebras, this class of algebras was heavily investigated and is by now considered to be well understood. Moreover, it turned out to play an important rôle in representation theory, see [9, 17]. The corresponding notion in the cluster category is that of cluster tilted algebras.

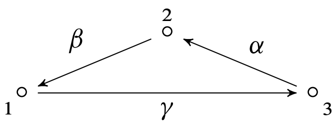
Definition 2.1 ([38])**.** Let Q be a finite, connected and acyclic quiver. An algebra B is called *cluster tilted of type* Q if there exists a tilting object T in the cluster category $\mathcal{C}_Q$ such that $B = \mathrm{End}_{\mathcal{C}_Q} T$.

Because, from the representation theoretic point of view, we may restrict ourselves to basic algebras, we assume, from now on and without loss of generality, that the indecomposable summands of a tilting object are pairwise nonisomorphic. This ensures that the endomorphism algebra is basic. Such a tilting object is then called *basic*.

Any hereditary algebra A is cluster tilted: let indeed $A = \mathsf{k}Q$ and consider the tilting object in $\mathcal{C}_Q$ induced by $T = A_A$, its endomorphism algebra in $\mathcal{C}_Q$ is A.

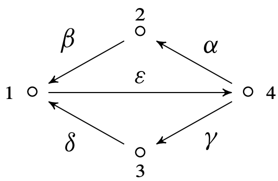
Actually, as we see in Sect. 3.2 below, a cluster tilted algebra is either hereditary or it has infinite global dimension.

Examples 2.2. (a) Let Q be as in Example 1.5(a) and T the tilting object in $\mathcal{C}_Q$ induced by the tilting module $1 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix} \oplus 3$. Its endomorphism algebra is given by the quiver



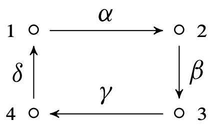
bound by $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$.

(b) Let Q be as in Example 1.5(b), and $T = 2 \oplus \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$. Its endomorphism algebra is given by the quiver



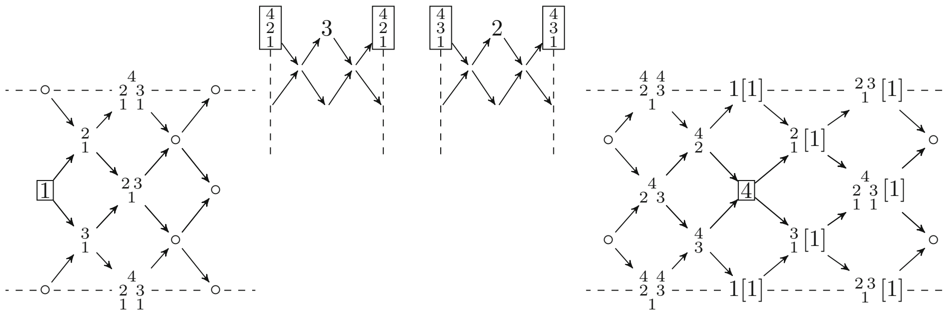
bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\beta\varepsilon = 0$, $\delta\varepsilon = 0$.

(c) For the same Q of Example 1.5(b), and $T = 2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1] \oplus \begin{smallmatrix}4\\2\\1\end{smallmatrix}[1]$, the endomorphism algebra is given by the quiver

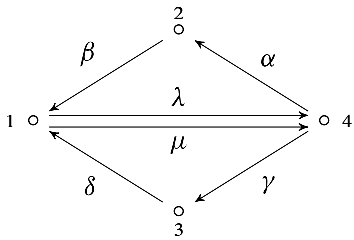


bound by $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$.

(d) Let Q be as in Example 1.5(c), and $T = 1 \oplus \begin{smallmatrix}4\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix} \oplus 4$. Here, $\Gamma(\mathcal{C}_Q)$ is as follows:



where we have only drawn the tubes of rank two. One has to identify the horizontal dashed lines to get the transjective component and the vertical dashed lines to get the tubes. The direct summands of T are indicated by squares. The endomorphism algebra of T is given by the quiver



bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$.

2.2 *Relation with Mutations*

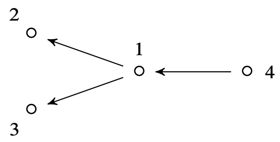
Mutation is an essential tool in the construction of cluster algebras. Let Q be a quiver having neither loops ($\circ \circlearrowleft$) nor 2-cycles ($\circ \rightleftarrows \circ$) and x a point in Q. The *mutation* μ_x at the point x transforms Q into another quiver $Q' = \mu_x Q$ constructed as follows:

(a) The points of Q' are the same as those of Q.
(b) If, in Q, there are r_{ij} paths of length two of the form $i \longrightarrow x \longrightarrow j$, then we add r_{ij} arrows from i to j in Q'.

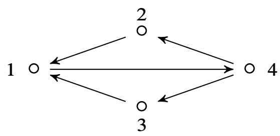
(c) We reverse the direction of all arrows incident to x.
(d) All other arrows remain the same.
(e) We successively delete all pairs of 2-cycles thus obtained until Q' has no more 2-cycles.

It is well known and easy to prove that mutation is an involutive process, that is, μ_x^2 is the identity transformation on Q.

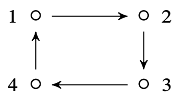
Example 2.3. (a) Let Q be the Dynkin quiver



then $Q' = \mu_1 Q$ is the quiver



which is the quiver of the cluster tilted algebra of Example 2.2(b). Repeating, and mutating this time at 2, we get $Q'' = \mu_2\mu_1 Q$ which is the quiver



of Example 2.2(c).

Now, recall from Theorem 1.11 that any almost complete tilting object T_0 of the cluster category $\mathcal{C}$ has exactly two nonisomorphic complements M_1 and M_2, giving rise to two tilting objects $T_1 = T_0 \oplus M_1$ and $T_2 = T_0 \oplus M_2$. To these correspond in turn two cluster tilted algebras $B_1 = \operatorname{End}_{\mathcal{C}} T_1$ and $B_2 = \operatorname{End}_{\mathcal{C}} T_2$ with respective quivers Q_{B_1} and Q_{B_2}. It turns out that we can pass from one to the other using mutation.

Theorem 2.4 ([39]). *With the previous notation, let x be the point in Q_{B_1} corresponding to the summand M_1 of T_1, then $Q_{B_2} = \mu_x Q_{B_1}$.*

Example 2.3. (b) As seen in Example 1.12(c), the almost complete tilting object $T_0 = \begin{smallmatrix}4\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 1[1]$ in the cluster category of Example 1.5(c) has exactly two complements, $M_1 = 2$ and $M_2 = \begin{smallmatrix}34\\2\end{smallmatrix}$, The endomorphism algebra of $T_2 = T_0 \oplus M_1$ is the algebra of Example 2.2(b), while that of $T_2 = T_0 \oplus M_2$ is that of Example 2.2(c). We have just seen in example (a) that mutating the quiver of End T_1 gives the quiver of End T_2.

Because of part (e) of the definition of mutation, it creates neither loops nor 2-cycles, we deduce the following corollary.

Corollary 2.5. *The quiver of a cluster tilted algebra contains neither loops nor 2-cycles.*

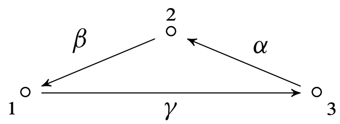
Moreover, we can obtain all the quivers of cluster tilted algebras of type Q by repeatedly mutating the quiver Q itself. This indeed follows from the fact that, if T, T' are two tilting objects in $\mathcal{C}_Q$, then there exists a sequence $T = T_0, T_1, \dots, T_n = T'$ such that, for each i with $0 \leq i < n$, we have that T_i and T_{i+1} are as in Theorem 1.11, that is, they have all but one indecomposable summand in common. This is sometimes expressed by saying that the exchange graph of the tilting objects is connected, see [36, (3.5)].

Corollary 2.6. *Let Q be a finite, connected and acyclic quiver. The class of quivers obtained from Q by successive mutations coincides with the class of quivers of cluster tilted algebras of type Q.*

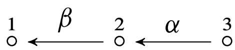
2.3 *Relation Extensions Algebras*

So far, in order to know whether a given algebra is cluster tilted or not, we need to identify a tilting object in some cluster category and verify whether the given algebra is its endomorphism algebra or not. This is clearly a difficult process in general (however, in the representation-finite case, we refer the reader to [30]). It is thus reasonable to ask for an intrinsic characterisation of cluster tilted algebras.

In order to motivate the next definition, consider the cluster tilted algebra B of Example 2.2(a). It is given by the quiver



bound by $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$. Deleting the arrow γ, we get the quiver



bound by $\alpha\beta = 0$: this is the bound quiver of a tilted algebra, which we call C. The two-sided ideal $E = B\gamma B$ has a natural structure of C-C-bimodule and $C = B/E$. As a k-vector space, $B = C \oplus E$. This is actually a classical construction.

Definition 2.7. Let C be an algebra, and E a C-C-bimodule. The *trivial extension* $B = C \ltimes E$ is the k-vector space

$$B = C \oplus E = \{(c, e) \mid c \in C, e \in E\}$$

with the multiplication induced from the bimodule structure of E, that is

$$(c,e)(c',e') = (cc', ce' + ec')$$

for $c, c' \in C$ and $e, e' \in E$.

Equivalently, we may describe B as being the algebra of 2×2-matrices

$$B = \left\{ \begin{pmatrix} c & 0 \\ e & c \end{pmatrix} \;\middle|\; c \in C, e \in E \right\}$$

with the usual matrix addition and the multiplication induced from the bimodule structure of E.

If $B = C \ltimes E$, then there exists a short exact sequence of C-C-bimodules

$$0 \longrightarrow E \xrightarrow{\;i\;} B \xrightarrow{\;p\;} C \longrightarrow 0$$

where $i : e \mapsto (0, e)$ (for $e \in E$) is the canonical inclusion and the projection $p : (c, e) \mapsto c$ (for $(c, e) \in B$) is an algebra morphism with another algebra morphism as section $q : c \mapsto (c, 0)$ (for $c \in C$). Thus, this sequence splits as a sequence of C-C-bimodules. Moreover, $E^2 = 0$, so that $E \subseteq \text{rad } B$. This implies that $\text{rad } B = \text{rad } C \oplus E$, as vector spaces. We now show how to compute the quiver of a trivial extension.

Lemma 2.8 ([5])**.** *Let C be an algebra, and E a C-C-bimodule. The quiver Q_B of $B = C \ltimes E$ is constructed as follows*:

(a) $(Q_B)_0 = (Q_C)_0$,
(b) *for $x, y \in (Q_C)_0$, the set of arrows in Q_B from x to y equals the set of arrows in Q_C from x to y plus*

$$\dim_k \frac{e_x E e_y}{e_x(\text{rad } C)Ee_y + e_x E(\text{rad } C)e_y}$$

additional arrows.

Proof. (a) This follows from the fact that $E \subseteq \text{rad } B$.

(b) The arrows in Q_B from x to y are in bijection with a basis of $e_x \left(\frac{\text{rad } B}{\text{rad}^2 B}\right) e_y$. Now $\text{rad } B = \text{rad } C \oplus E$ and $E^2 = 0$ imply that

$$\text{rad}^2 B = \text{rad}^2 C \oplus \big((\text{rad } C)E + E(\text{rad } C)\big).$$

The statement follows from the facts that $\text{rad}^2 C \subseteq \text{rad}\, C$ and $(\text{rad } C)E + E(\text{rad } C) \subseteq E$.

□

Recall that an algebra whose quiver is acyclic is called *triangular*.

Definition 2.9 ([5])**.** Let C be a triangular algebra of global dimension at most two. Its *relation extension* is the algebra $\widetilde{C} = C \ltimes E$, where $E = \text{Ext}^2_C(\text{D}\, C, C)$ is considered as a C-C-bimodule with the natural actions.

If C is hereditary, then $E = 0$ and $\widetilde{C} = C$ is its own relation extension. On the other hand, if gl. dim $C = 2$, then there exist simple C-modules S, S' such that $\mathrm{Ext}^2_C(S, S') \neq 0$. Let I be the injective envelope of S and P' the projective cover of S', then the short exact sequences $0 \to \mathrm{rad}\ P' \to P' \to S' \to 0$ and $0 \to S \to I \to I/S \to 0$ induce an epimorphism $\mathrm{Ext}^2_C(I, P') \to \mathrm{Ext}^2_C(S, S')$. Therefore $\mathrm{Ext}^2_C(\mathrm{D}\,C, C) \neq 0$.

Following [32], we define a *system of relations* for an algebra $C = \mathsf{k}Q_C/I$ to be a subset R of $\bigcup_{x,y\in(Q_C)_0} e_x I e_y$ such that R, but no proper subset of R, generates I as a two-sided ideal.

Theorem 2.10 ([5, **(2.6)**]). *Let $C = \mathsf{k}Q_C/I$ be a triangular algebra of global dimension at most two, and R a system of relations for C. The quiver of the relation extension $\widetilde{C}$ is constructed as follows:*

(a) $\left(Q_{\widetilde{C}}\right)_0 = (Q_C)_0$
(b) *For $x, y \in \left(Q_{\widetilde{C}}\right)_0$, the set of arrows in $Q_{\widetilde{C}}$ from x to y equals the set of arrows in Q_C from x to y plus $|R \cap (e_y I e_x)|$ additional arrows.*

"Proof". Let S be the direct sum of a complete set of representatives of the isoclasses of simple C-modules. Because C is basic, we have $S = \mathrm{top}\, C_C = \mathrm{soc}(\mathrm{D}\,C)_C$. Because of [32, (1.2)], the relations in R correspond to a k-basis of $\mathrm{Ext}^2_{\mathcal{C}}(S, S)$. Because of [5, (2.4)] $\mathrm{Ext}^2_{\mathcal{C}}(S, S) \cong \mathrm{top}\,\mathrm{Ext}^2_{\mathcal{C}}(\mathrm{D}\,C, C)$. Lemma 2.8 implies that the number of additional arrows is $\dim_{\mathsf{k}} e_x\, \mathrm{Ext}^2_{\mathcal{C}}(S, S) e_y = \dim_{\mathsf{k}} \mathrm{Ext}^2_{\mathcal{C}}(S_y, S_x)$, hence the result. □

In view of the theorem, we sometimes refer to the arrows of Q_C as the 'old' arrows in $Q_{\widetilde{C}}$, the remaining being called the 'new' arrows

As a consequence of the theorem, the quiver of a nonhereditary relation extension always has oriented cycles. Indeed, this follows from the facts that the relation extension of a hereditary algebra is itself and a nonhereditary algebra is bound by at least one relation. We still have to describe the relations occurring in the quiver of a relation extension algebra. This is done in the next subsection. For the time being we establish the relation between cluster tilted algebras and relation extensions.

Theorem 2.11 ([5, **(3.4)**]). *An algebra B is cluster tilted of type Q if and only if there exists a tilted algebra C of type Q such that $B = \widetilde{C}$.*

Proof. Assume that B is cluster tilted of type Q. Then there exists a tilting object T in the cluster category $\mathcal{C}_Q$ such that $B = \mathrm{End}_{\mathcal{C}_Q} T$. Because of Proposition 1.9, we may assume that T is induced from a tilting $\mathsf{k}Q$-module. Let $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\mathsf{k}Q)$. We have

$$B = \mathrm{End}_{\mathcal{C}_Q} T = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(T, F^i T)$$

as k-spaces. Because T is a $\mathsf{k}Q$-module, $\mathrm{Hom}_{\mathcal{D}}(T, F^i T) = 0$ for $i \geq 2$. Moreover $C = \mathrm{End}\, T_{\mathsf{k}Q}$ is tilted and, as k-vector spaces

$$B \cong \mathrm{End}\, T_{\mathsf{k}Q} \oplus \mathrm{Hom}_{\mathcal{D}}(T, FT) = C \oplus \mathrm{Hom}_{\mathcal{D}}(T, FT)\ .$$

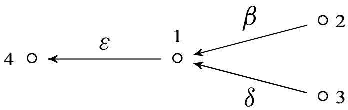
Because of Happel's theorem, see [51, p. 109], $\mathcal{D}$ is triangle equivalent to $\mathcal{D}' = \mathcal{D}^{\mathrm{b}}(\operatorname{mod} C)$. Setting $F' = \tau^{-1}[1]$ in $\mathcal{D}'$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}}(T, FT) &\cong \operatorname{Hom}_{\mathcal{D}'}(C, F'C)\\ &\cong \operatorname{Hom}_{\mathcal{D}'}(\tau C[1], C[2])\\ &\cong \operatorname{Hom}_{\mathcal{D}'}(\mathrm{D}\,C, C[2])\\ &\cong \operatorname{Ext}^2_C(\mathrm{D}\,C, C)\ ,\end{aligned}$$

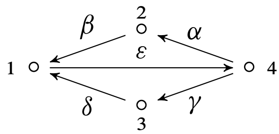
where we have used that, in $\mathcal{D}^{\mathrm{b}}(\operatorname{mod} C)$, the Auslander–Reiten translate of $C[1]$ is $\mathrm{D}\,C$ (see [51, Proof of Theorem I.4.6]). We leave to the reader the verification that the multiplicative structure of B is the same as that of $\widetilde{C}$. This proves necessity. Sufficiency is proved in the same way. □

Thus, there exists a surjective map from the class of tilted algebras to the class of cluster tilted algebras, given by $C \mapsto \widetilde{C}$. However, this map is not injective as we see in example (a) below. It is therefore an interesting question to find all the tilted algebras which lie in the fibre of a given cluster tilted algebra. We return to this question in Sect. 3.4 below.

Examples 2.12. (a) Let C be the algebra given by the quiver



bound by $\beta\varepsilon = 0$, $\delta\varepsilon = 0$. It is tilted of type $\mathbb{D}_4$. Because of Theorem 2.10, the quiver of its relation extension is



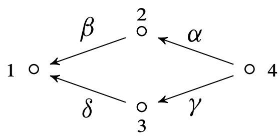
with α, γ new arrows. In order to compute a system of relations, we use the following observation. Let P_x, $\widetilde{P}_x$ denote, respectively, the indecomposable projective C and $\widetilde{C}$-modules corresponding to x.
Because P_x and $\widetilde{P}_x$ admit both S_x as simple top, there exists a projective cover morphism $p_x \colon \widetilde{P}_x \to P_x$ in $\operatorname{mod} \widetilde{C}$. On the other hand, we have an isomorphism of vector spaces $\widetilde{P}_x = e_x\widetilde{C} = e_xC \oplus e_x \operatorname{Ext}^2_C(\mathrm{D}\,C, C) = P_x \oplus \operatorname{Ext}^2_C(\mathrm{D}\,C, P_x)$ hence we have another exact sequence

$$0 \longrightarrow \operatorname{Ext}^2_C(\mathrm{D}\,C, P_x) \longrightarrow \widetilde{P}_x \xrightarrow{p_x} P_x \longrightarrow 0$$

where p_x is a projective cover. Now, in this example, it is easily seen that $\operatorname{Ext}^2_C(I_2, P_4) \cong \operatorname{Ext}^2_C(I_3, P_4) \cong \operatorname{Ext}^2_C(I_1, P_4) \cong \mathsf{k}$ and all other $\operatorname{Ext}^2_C(I_i, P_j) = 0$. Thus

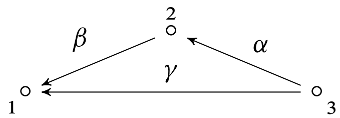
$$\widetilde{C}_{\widetilde{C}} = \begin{smallmatrix}1\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\2&&3\\&1&\end{smallmatrix}$$

and so the quiver above is bound by $\alpha\beta = \gamma\delta$, $\beta\varepsilon = 0$, $\varepsilon\gamma = 0$, $\varepsilon\delta = 0$, $\alpha\varepsilon = 0$. This is the bound quiver of Example 2.2(b)
Now let C' be the tilted algebra of type $\mathbb{D}_4$ given by the quiver

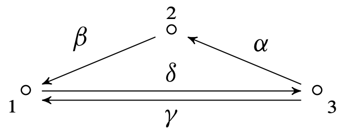


bound by $\alpha\beta = \gamma\delta$, then a similar calculation yields $\widetilde{C'} \cong \widetilde{C}$. This shows that the mapping $C \mapsto \widetilde{C}$ is not injective.

(b) Let C be the triangular algebra of global dimension 2 given by



bound by $\alpha\beta = 0$. Here, C is not tilted because its Auslander–Reiten quiver contains an oriented cycle. Applying Theorem 2.10, a calculation similar to that of example (a) above shows that $\widetilde{C}$ is given by



bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$, $\delta\gamma\delta = 0$. We see that $\widetilde{C}$ is not cluster tilted because its quiver contains a 2-cycle, contrary to Corollary 2.5.

2.4 *The Relations on a Cluster Tilted Algebra*

Starting from a tilted algebra C, Theorem 2.10 allows to construct easily the quiver of its relation extension $\widetilde{C}$ which is cluster tilted, thanks to Theorem 2.11. Now we show how to compute as easily a system of relations for $\widetilde{C}$.

Let $C = \mathsf{k}Q_C/I$ be a triangular algebra of global dimension at most 2, and $R = \{\rho_1, \dots, \rho_t\}$ a system of relations for C. To the relation ρ_i from x_i to y_i, say, there corresponds in $\widetilde{C}$ a new arrow $\alpha_i \colon y_i \to x_i$, as in Theorem 2.10. The *Keller potential* on $\widetilde{C}$ is the element

$$w = \sum_{i=1}^{t} \rho_i \alpha_i$$

of $\mathsf{k}Q_{\widetilde{C}}$. This element is considered up to cyclic equivalence: two potentials are called *cyclically equivalent* if their difference is a linear combination of elements of the form $\gamma_1\gamma_2\cdots\gamma_m - \gamma_m\gamma_1\cdots\gamma_{m-1}$, where $\gamma_1\gamma_2\cdots\gamma_m$ is a cycle in the quiver $Q_{\widetilde{C}}$. For a given arrow γ, the *cyclic partial derivative* of this cycle with respect to γ is defined to be

$$\partial_\gamma(\gamma_1\cdots\gamma_m) = \sum_{\gamma_i=\gamma} \gamma_{i+1}\cdots\gamma_m\gamma_1\cdots\gamma_{i-1}.$$

In particular, the cyclic partial derivative is invariant under cyclic permutations. The *Jacobian algebra* $\mathcal{J}(Q_{\widetilde{C}}, w)$ is the quotient of $\mathsf{k}Q_{\widetilde{C}}$ by the ideal generated by all cyclic partial derivatives $\partial_\gamma w$ of the Keller potential w with respect to all the arrows γ in $Q_{\widetilde{C}}$, see [58].

Proposition 2.13 ([14, **(5.2)**]). *Let C be a triangular algebra of global dimension at most two, and w the Keller potential on $\widetilde{C}$. Then*

$$\widetilde{C} \cong \mathcal{J}(Q_{\widetilde{C}}, w)/J$$

where J is the square of the ideal generated by the new arrows.

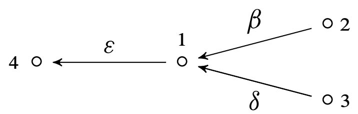
"Proof". It was shown in [58, (6.12a)] that $\mathcal{J}(Q_{\widetilde{C}}, w)$ is isomorphic to the endomorphism algebra of the tilting object C in Amiot's generalised cluster category associated with C. Because of [1, (1.7)], this endomorphism algebra is isomorphic to the tensor algebra of the bimodule ${}_CE_C$ and its quiver is isomorphic to $Q_{\widetilde{C}}$, which is also the quiver of $\mathcal{J}(Q_{\widetilde{C}}, w)$. Taking the quotient of the tensor algebra by the ideal J generated by all tensor powers $E^{\otimes_C i}$ with $i \geq 2$, we get exactly $\widetilde{C}$. But now J is the square of the ideal generated by the new arrows. □

The next result is proven, for instance, in [35], see also [22, (4.22)] or [2, p. 17].

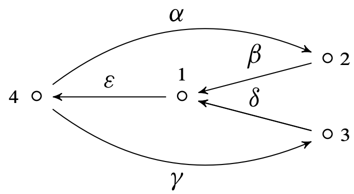
Proposition 2.14. *Let C be a tilted algebra, and w the Keller potential on $\widetilde{C}$, then $\widetilde{C} = \mathcal{J}(Q_{\widetilde{C}}, w)$.*

That is, if C is tilted, then the square J of the ideal generated by the new arrows is contained in the ideal generated by all cyclic partial derivatives of the Keller potential. This gives a system of relations on a cluster tilted algebra.

Examples 2.15. (a) Let C be the tilted algebra of Example 2.12(a), given by the quiver



bound by $\beta\varepsilon = 0$, $\delta\varepsilon = 0$. Applying Theorem 2.10 yields the quiver

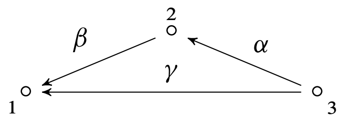


with new arrows α, γ. The Keller potential is then $w = \beta\varepsilon\alpha + \delta\varepsilon\gamma$. We compute its cyclic partial derivatives.

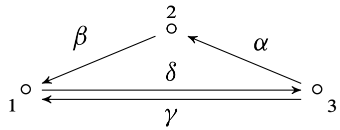
$$\partial_\alpha(w) = \beta\varepsilon\ , \quad \partial_\beta(w) = \varepsilon\alpha\ , \quad \partial_\gamma(w) = \delta\varepsilon\ , \quad \partial_\delta(w) = \varepsilon\gamma\ , \quad \partial_\varepsilon(w) = \alpha\beta + \gamma\delta\ .$$

Thus, besides the 'old' relations $\beta\varepsilon = 0$, $\delta\varepsilon = 0$, we also have 'new' relations $\varepsilon\alpha = \varepsilon\gamma = 0$ and $\alpha\beta + \gamma\delta = 0$. Moreover $J = \langle\alpha, \gamma\rangle^2 = 0$ so that we get the cluster tilted algebra of Example 2.2(b).

(b) Let C be the (non tilted) triangular algebra of global dimension two given by the quiver



bound by $\alpha\beta = 0$. Here, $\widetilde{C}$ is given by the quiver



and $w = \alpha\beta\delta$. Thus, the Jacobian algebra $\mathcal{J}(Q_{\widetilde{C}}, w)$ is given by the previous quiver bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$. Here, $J = \langle\delta\rangle^2 = \langle\delta\gamma\delta\rangle$ is nonzero. Therefore $\widetilde{C}$ is given by the above quiver bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$, $\delta\gamma\delta = 0$.

Now, recall from Theorem 2.4 that a mutation transforms the quiver of a cluster tilted algebra into the quiver of another. There is also a recipe for transforming the potential, thus obtaining a system of relations on the new cluster tilted algebra. We refer the reader to [35] where it is also proven that cluster tilted algebras are uniquely determined by their quivers.

Theorem 2.16. *Let B_1, B_2 be two cluster tilted algebras. If the quivers of B_1 and B_2 are isomorphic, then we have an isomorphism of algebras $B_1 \cong B_2$.*

The set of relations given by the cyclic partial derivatives of the Keller potential is generally not a system of minimal relations. Following [37], we say that a relation ρ is *minimal* if, whenever $\rho = \sum_i \beta_i \rho_i \gamma_i$, where ρ_i is a relation for each i, then there is an index i such that both β_i and γ_i are scalars, that is, a minimal relation in a bound quiver (Q, I) is any element of I not lying in $(\mathsf{k}Q^+)I + I(\mathsf{k}Q^+)$, where $\mathsf{k}Q^+$ denotes the two-sided ideal of

$\mathsf{k}Q$ generated by all the arrows of Q. There is however one particular case in which we have minimal relations. We need the following definitions.

Definition 2.17. Let Q be a quiver with neither loops nor 2-cycles.

(a) [23] A full subquiver of Q is a *chordless cycle* if it is induced by a set of points $\{x_1, x_2, \dots, x_p\}$ which is topologically a cycle, that is, the edges on it are precisely the edges $x_i — x_{i+1}$ (where we set $x_{p+1} = x_1$).

(b) A path γ which is antiparallel to an arrow ζ in Q is a *shortest path* if the full subquiver generated by the cycle $\zeta\gamma$ is chordless.

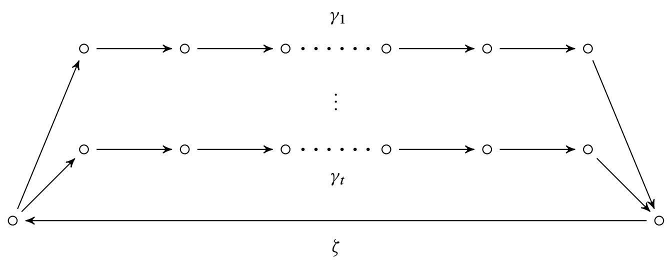
(c) [24] The quiver Q is called *cyclically oriented* if each chordless cycle is an oriented cycle.

For instance, any tree is trivially cyclically oriented. The easiest nontrivial cyclically oriented quiver is a single oriented cycle. Note that the definition of cyclically oriented excludes the existence of multiple arrows. It is also easy to see that the quiver of a cluster tilted algebra of Dynkin type is cyclically oriented.

Theorem 2.18 ([24]). *Let B be cluster tilted with a cyclically oriented quiver. Then*:

(a) *The arrows in Q_B which occur in some chordless cycle are in bijection with the minimal relations in any presentation of B.*

(b) *Let $\zeta \in (Q_B)_1$ occur in a chordless cycle, and $\gamma_1 \cdots \gamma_t$ be all the shortest paths antiparallel to ζ. Then the minimal relation corresponding to ζ is of the form $\sum_{i=1}^{t} \lambda_i \gamma_i$, where the λ_i are nonzero scalars. Also, the quiver restricted to the γ_i is of the form*

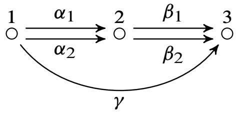


In particular, the γ_i only share their endpoints.

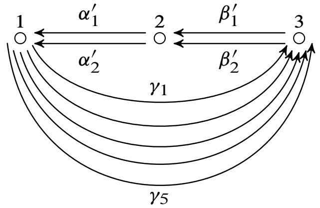
In particular, let B be a representation-finite cluster tilted algebra. As we see in Sect. 3.1 below, B is of Dynkin type, therefore its quiver is cyclically oriented and the previous theorem yields a system of minimal relations for B. Actually, in this case, for any arrow ζ, the number t of shortest antiparallel paths is 1 or 2. If there is one shortest path γ, we choose γ as a generator and, if there are two, γ_1 and γ_2, we choose $\gamma_1 - \gamma_2$ as a generator. Then the ideal generated by these relations is a system of minimal relations [37].

If Q_B is not cyclically oriented, then the assertion of the theorem does not necessarily hold true, as we now see.

Example 2.15. (c) [24] Let A be the path algebra of the quiver



Mutating at 2 yields the quiver



All four paths from 3 to 1, namely the $\beta'_i\alpha'_j$ are zero. Hence there are 4 relations from 3 to 1, but there are 5 arrows antiparallel to them.

Besides cluster tilted algebras with cyclically oriented quiver, minimal relations are only known for cluster tilted algebras of type $\tilde{\mathbb{A}}$, see Sect. 2.5 below. We may formulate the following problem.

Problem. Give systems of minimal relations for any cluster tilted algebra.

2.5 *Gentle Cluster Tilted Algebras*

There were several attempts to classify classes of cluster tilted algebras, see, for instance, [21, 34, 43, 48, 62], or to classify algebras derived equivalent to certain cluster tilted algebras, see, for instance [25–27, 31, 40]. We refrain from quoting all these results and concentrate rather on gentle algebras, introduced in [18].

Definition 2.19. An algebra B is *gentle* if there exists a presentation $B \cong \mathsf{k}Q/I$ such that

(a) every point of Q is the source, or the target, of at most two arrows;
(b) I is generated by paths of length 2;
(c) for every $\alpha \in Q_1$, there is at most one $\beta \in Q_1$ such that $\alpha\beta \notin I$ and at most one $\gamma \in Q_1$ such that $\gamma\alpha \notin I$;
(d) for every $\alpha \in Q_1$, there is at most one $\xi \in Q_1$ such that $\alpha\xi \in I$ and at most one $\zeta \in Q_1$ such that $\zeta\alpha \in I$.

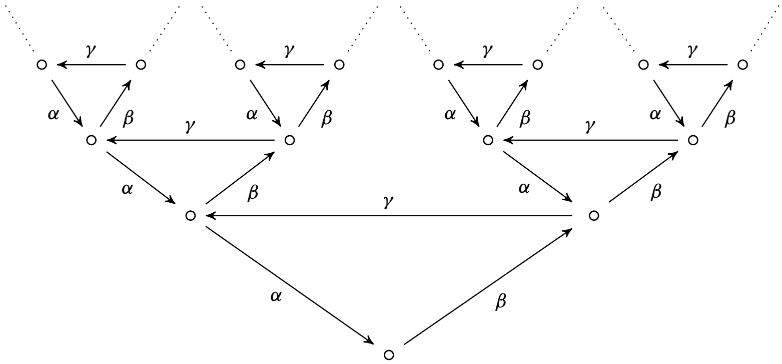
Gentle algebras are string algebras [42], so we can describe all their indecomposable modules and all their almost split sequences. Gentle algebras are also tame and this class is stable under tilting [65]. We characterise gentle cluster tilted algebras.

Theorem 2.20 ([3]). *Let C be a tilted algebra, the following conditions are equivalent*

(a) *C is gentle*
(b) *$\widetilde{C}$ is gentle*
(c) *C is of Dynkin type $\mathbb{A}$ or of Euclidean type $\tilde{\mathbb{A}}$.*

Moreover, the set of relations induced from the Keller potential is a system of minimal relations in these two cases.

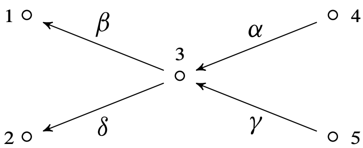
Cluster tilted algebras of type $\mathbb{A}$ are particularly easy to describe.Their quivers are full connected subquivers of the following infinite quiver



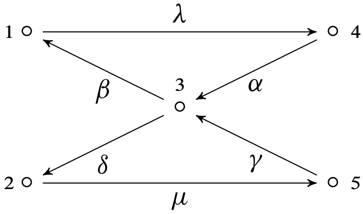
bound by all possible relations of the forms $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\alpha = 0$.

Examples 2.21. Clearly, the algebra of Example 2.2(a) is gentle of type $\mathbb{A}_3$. We give two more examples.

(a) Let C be the tilted algebra of type $\mathbb{A}_5$ given by the quiver

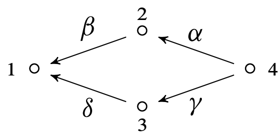


bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Its relation extension $\widetilde{C}$ is given by the quiver

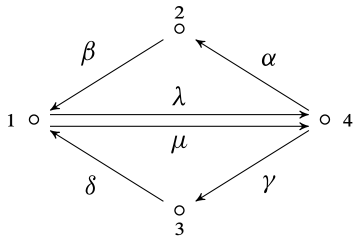


bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$ and $\mu\gamma = 0$. Both are gentle.

(b) Let C be the tilted algebra of type $\tilde{\mathbb{A}}_3$ given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, which is gentle.
Then its relation extension $\widetilde{C}$ is given by the quiver



bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$. It is also gentle.
Note that, while C is representation-finite, $\widetilde{C}$ is representation-infinite.

3 The Module Category of a Cluster Tilted Algebra

3.1 *Recovering the Module Category from the Cluster Category*

Let T be a tilting object in a cluster category $\mathcal{C}$ and $B = \mathrm{End}_{\mathcal{C}}\, T$ the corresponding cluster tilted algebra. Then there is an obvious functor

$$\mathrm{Hom}_{\mathcal{C}}(T, -) \colon \mathcal{C} \to \mathrm{mod}\, B$$

which *projectivises* T, that is, which induces an equivalence between $\mathrm{add}\, T$ and the full subcategory of $\mathrm{mod}\, B$ consisting of the projective B-modules, see [20, p. 32]. We claim that $\mathrm{Hom}_{\mathcal{C}}(T, -)$ is full and dense.

Indeed, let M be a B-module, and take a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

in $\mathrm{mod}\, B$. Because P_0, P_1 are projective, there exist T_0, T_1 in $\mathrm{add}\, T$ and a morphism $g \colon T_1 \to T_0$ such that $\mathrm{Hom}_{\mathcal{C}}(T, T_i) \cong P_i$ for $i = 0, 1$ and $\mathrm{Hom}_{\mathcal{C}}(T, g) = f$. Then there exists a triangle

$$T_1 \xrightarrow{g} T_0 \longrightarrow X \longrightarrow T_1[1]$$

in $\mathscr{C}$. Applying $\mathrm{Hom}_{\mathscr{C}}(T,-)$ yields an exact sequence

$$\mathrm{Hom}_{\mathscr{C}}(T,T_1) \xrightarrow{\mathrm{Hom}_{\mathscr{C}}(T,g)} \mathrm{Hom}_{\mathscr{C}}(T,T_0) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(T,X) \longrightarrow 0$$

because $\mathrm{Hom}_{\mathscr{C}}(T,T_1[1]) = \mathrm{Ext}^1_{\mathscr{C}}(T,T_1) = 0$. Therefore $M \cong \mathrm{Hom}_{\mathscr{C}}(T,X)$ and our functor is dense. One proves its fullness in exactly the same way.

On the other hand, it is certainly not faithful, because

$$\mathrm{Hom}_{\mathscr{C}}(T,\tau T) = \mathrm{Ext}^1_{\mathscr{C}}(T,T) = 0$$

and hence the image of any object in $\mathrm{add}\,\tau T$ is zero.

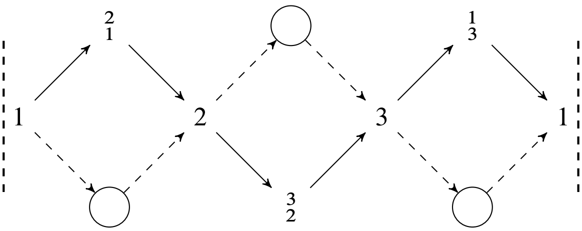
Let $\mathscr{C}/\langle \mathrm{add}\,\tau T\rangle$ denote the quotient of $\mathscr{C}$ by the ideal $\langle \mathrm{add}\,\tau T\rangle$ consisting of all the morphisms in $\mathscr{C}$ which factor through an object in $\mathrm{add}\,\tau T$. The objects in this quotient category are the same as those of $\mathscr{C}$ and the set of morphisms from X to Y, say, equals $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ modulo the subspace consisting of those lying in $\langle \mathrm{add}\,\tau T\rangle$. In $\mathscr{C}/\langle \mathrm{add}\,\tau T\rangle$, the objects of $\mathrm{add}\,\tau T$ are isomorphic to zero.

Because $\mathrm{Hom}_{\mathscr{C}}(T,-)\colon \mathscr{C} \to \mathrm{mod}\,B$ is a full and dense functor which vanishes on $\langle \mathrm{add}\,\tau T\rangle$, it induces a full and dense functor from $\mathscr{C}/\langle \mathrm{add}\,\tau T\rangle$ to $\mathrm{mod}\,B$. It turns out that this induced functor is also faithful, and thus is an equivalence.

Theorem 3.1 ([38, **(2.2)**]). *The functor* $\mathrm{Hom}_{\mathscr{C}}(T,-)\colon \mathscr{C} \to \mathrm{mod}\,B$ *induces an equivalence between* $\mathscr{C}/\langle \mathrm{add}\,\tau T\rangle$ *and* $\mathrm{mod}\,B$.

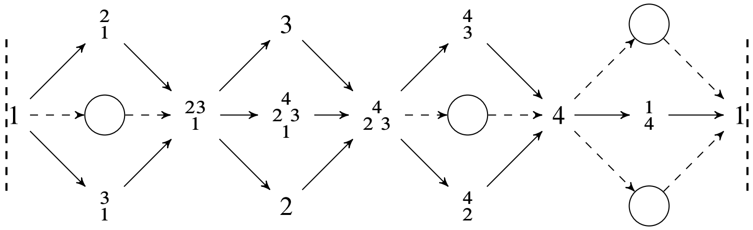
An immediate consequence is the shape of the Auslander–Reiten quiver of B. Indeed, starting from $\Gamma(\mathscr{C})$, the theorem says that one gets $\Gamma(\mathrm{mod}\,B)$ by setting equal to zero all the indecomposable summands of τT, thus by deleting the corresponding points from $\Gamma(\mathscr{C})$. In particular, $\Gamma(\mathrm{mod}\,B)$ has the same type of components as $\Gamma(\mathscr{C})$, that is, transjective and regular, from which are deleted each time finitely many points.

Examples 3.2. (a) If B is as in Example 2.2(a), then $\Gamma(\mathrm{mod}\,B)$ is



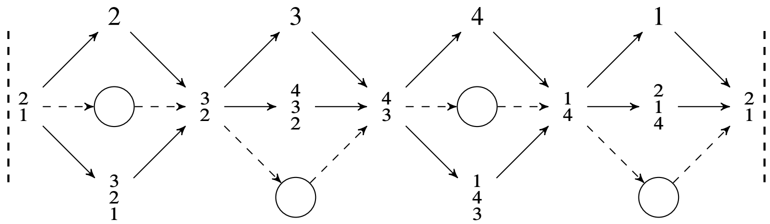
where we identify the vertical dashed lines. If we add the indecomposable summands of τT, denoted by $\bigcirc$, we get exactly $\Gamma(\mathscr{C})$.

(b) Let B be as in Example 2.2(b), then $\Gamma(\mathrm{mod}\,B)$ is



where we identify the vertical dashed lines. Adding the points denoted by ◯, we get again $\Gamma(\mathcal{C})$.

(c) Let B be as in Example 2.2(c), then $\Gamma(\operatorname{mod} B)$ is



where we identify the vertical dashed lines.

Corollary 3.3 ([38, **(2.4)**]). *A cluster tilted algebra B of type Q is representation-finite if and only if Q is a Dynkin quiver. In this case, the numbers of isoclasses of indecomposable B-modules and $\mathsf{k}Q$-modules are equal.*

Proof. The first statement follows easily from Theorem 3.1. Let $n = |Q_0|$ and m be the number of isoclasses of indecomposable $\mathsf{k}Q$-modules. The cluster category $\mathcal{C}_Q$ has exactly $n + m$ isoclasses of indecomposable objects. To get the number of isoclasses of indecomposable B-modules, we subtract the number n of indecomposable summands of τT, getting $(n + m) - n = m$, as required. □

The examples also show that the Auslander–Reiten translation is preserved by the equivalence of Theorem 3.1.

Proposition 3.4 ([38, **(3.2)**]). *The almost split sequences in* mod B *are induced from the almost split triangles of $\mathcal{C}$.*

3.2 *Global Dimension*

As an easy application of Theorem 3.1, we compute the global dimension of a cluster tilted algebra. Let B be cluster tilted of type Q and T a tilting object in $\mathcal{C} = \mathcal{C}_Q$ such that $B = \operatorname{End}_{\mathcal{C}} T$. For $x \in (Q_B)_0$ we denote by $\widetilde{P}_x$, $\tilde{I}_x$, respectively, the corresponding indecomposable projective and injective B-modules. It follows easily from [20, p. 33] that $\widetilde{P_x} = \mathrm{Hom}_{\mathcal{C}}(T, T_x)$, where T_x is the summand of T corresponding to x. We now compute $\tilde{I}_x$.

Lemma 3.5. *With this notation,* $\tilde{I}_x = \mathrm{Hom}_{\mathcal{C}}(T, \tau^2 T_x)$.

Proof. Because of [20, p. 33], we have $\tilde{I}_x = \mathrm{D}\,\mathrm{Hom}_{\mathcal{C}}(T_x, T)$. Setting $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\mathsf{k}Q)$ we have functorial isomorphisms

$$\begin{aligned}\tilde{I}_x &= \mathrm{D}\,\mathrm{Hom}_{\mathcal{D}}(T_x, T) \oplus \mathrm{D}\,\mathrm{Hom}_{\mathcal{D}}(T_x, \tau^{-1}T[1]) \\ &= \mathrm{Ext}^1_{\mathcal{D}}(T, \tau T_x) \oplus \mathrm{D}\,\mathrm{Ext}^1_{\mathcal{D}}(T_x, \tau^{-1}T) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(T, \tau T_x[1]) \oplus \mathrm{Hom}_{\mathcal{D}}(T, \tau^2 T_x) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(T, \tau^2 T_x)\ .\end{aligned}$$

□

Recall from [19] that an algebra B is *Iwanaga–Gorenstein* if both $\mathrm{id}\, B_B < \infty$ and $\mathrm{pd}(\mathrm{D}\,B)_B < \infty$. Actually, if both dimensions are finite then they are equal. Letting $d = \mathrm{id}\, B_B = \mathrm{pd}(\mathrm{D}\,B)_B$, we then say that B is *Iwanaga–Gorenstein of injective dimension* d.

Theorem 3.6 ([59])**.** *Any cluster tilted algebra B is Iwanaga–Gorenstein of injective dimension* 1. *In particular*, $\mathrm{gl.\,dim}\, B \in \{1, \infty\}$.

Proof. Let, as above, $B = \mathrm{End}_{\mathcal{C}}\, T$, with T a tilting object in the cluster category $\mathcal{C}$. In order to prove that $\mathrm{pd}(\mathrm{D}\,B)_B \le 1$, we must show that, for any injective B-module $\tilde{I}$, we have

$$\mathrm{Hom}_B(\mathrm{D}\,B, \tau_B \tilde{I}) = 0\ ,$$

(see [17, Lemma IV.2.7 p. 115]). Because of Lemma 3.5, we have $\tilde{I} = \mathrm{Hom}_{\mathcal{C}}(T, \tau^2 T_0)$, for some T_0 in $\mathrm{add}\, T$. Because $\mathrm{Hom}_B(-, ?)$ is a quotient of $\mathrm{Hom}_{\mathcal{C}}(-, ?)$ when both are evaluated as groups, it suffices to prove that $\mathrm{Hom}_{\mathcal{C}}(\tau^2 T, \tau^3 T_0) = 0$. But this follows from $\mathrm{Hom}_{\mathcal{C}}(\tau^2 T, \tau^3 T_0) \cong \mathrm{Hom}_{\mathcal{C}}(T, \tau T_0) \cong \mathrm{Ext}^1_{\mathcal{C}}(T, T_0) = 0$. Thus, $\mathrm{pd}(\mathrm{D}\,B)_B \le 1$. Similarly, $\mathrm{id}\, B_B \le 1$.

We now prove that, for any B-module M, the finiteness of $\mathrm{id}\, M$ implies $\mathrm{pd}\, M \le 1$. Indeed, if $\mathrm{id}\, M = m < \infty$, then we have a minimal injective coresolution

$$\begin{array}{ccccccccccccccccc} 0 \to M \to I^0 & & \longrightarrow & & I^1 & & \longrightarrow & & I^2 \to \cdots & & \longrightarrow & & I^{m-1} & \longrightarrow & I^m \to 0 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\ & & L^0 & & & & L^1 & & & & L^{m-2} & & & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\ 0 & & & & 0 & & & & 0 \quad 0 & & & & 0 & & . \end{array}$$

The short exact sequence $0 \to L^{m-2} \to I^{m-1} \to I^m \to 0$ and the argument above yield $\mathrm{pd}\, L^{m-2} \le 1$. An easy induction gives $\mathrm{pd}\, M \le 1$.

Thus, if $\mathrm{gl.\,dim}\, B > 1$, then there exists a module M such that $\mathrm{pd}\, M > 1$. But then $\mathrm{id}\, M = \infty$ and so $\mathrm{gl.\,dim}\, B = \infty$. This proves the second statement. □

3.3 *The Cluster Repetitive Algebra*

Let C be an algebra and E a C-C-bimodule. We construct a Galois covering of the trivial extension $C \ltimes E$. Consider the following locally finite-dimensional algebra without identity:

$$\breve{C} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & C_{-1} & & & \\ & E_0 & C_0 & & \\ & & E_1 & C_1 & \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

where matrices have only finitely many nonzero coefficients, $C_i = C$ are all terms on the main diagonal, $E_i = E$ are all terms below it, for all $i \in \mathbb{Z}$, and all the remaining coefficients are zero. Addition is the usual matrix addition, while multiplication is induced from the bimodule structure of E and the map $E \otimes_C E \to 0$. The identity maps $C_i \to C_{i-1}$, $E_i \to E_{i-1}$, induce an automorphism φ of $\breve{C}$. The orbit category $\breve{C}/\langle\varphi\rangle$ inherits from $\breve{C}$ an algebra structure, which is easily seen to be isomorphic to that of $C \ltimes E$. The projection functor $G: \breve{C} \to C \ltimes E$ induces a Galois covering with the infinite cyclic group generated by φ, see [47]. In view of Theorem 2.11, we are mostly interested in the case where C is tilted and $E = \mathrm{Ext}^2_C(\mathrm{D}\,C, C)$ so that $C \ltimes E = \widetilde{C}$. In this case, $\breve{C}$ is called the *cluster repetitive algebra*. Its quiver follows easily from Theorem 2.10.

Lemma 3.7 ([6, **(1.3)**]). *Let $C = \mathsf{k}Q_C/I$ be a tilted algebra and R a system of relations for I. The quiver of $\breve{C}$ is constructed as follows:*

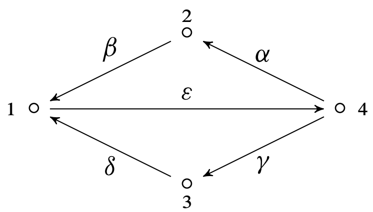
(a) $\left(Q_{\breve{C}}\right)_0 = (Q_C)_0 \times \mathbb{Z} = \{(x,i) \mid x \in (Q_C)_0, i \in \mathbb{Z}\}$
(b) *for $(x,i),(y,j) \in \left(Q_{\breve{C}}\right)_0$, the set of arrows in $\left(Q_{\breve{C}}\right)_0$ from (x,i) to (y,j) equals:*

 (i) *the set of arrows from x to y if $i = j$, or*
 (ii) *$|R \cap e_y I e_x|$ new arrows if $i = j+1$*

 and is empty otherwise.

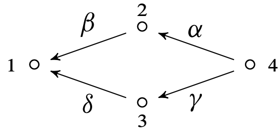
Because the relations are just lifted from those of $\widetilde{C}$, this allows to compute without difficulty the bound quiver of $\breve{C}$.

Examples 3.8. (a) Let B be the cluster tilted algebra of Example 2.2(b), given by the quiver

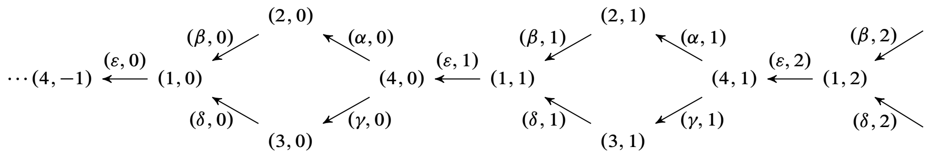


bound by $\alpha\beta = \gamma\delta$, $\beta\varepsilon = 0$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$.

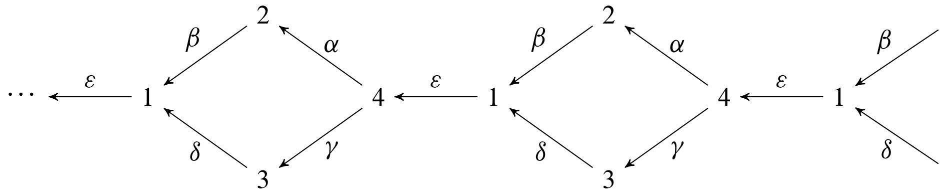
Then $B = \widetilde{C}$ where C is given by the quiver



bound by $\alpha\beta = \gamma\delta$. The quiver of the cluster repetitive algebra $\breve{C}$ is the infinite quiver

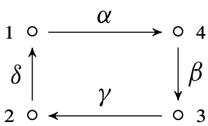


bound by all the lifted relations: $(\alpha, i)(\beta, i) = (\gamma, i)(\delta, i)$, $(\beta, i)(\varepsilon, i) = 0$, $(\delta, i)(\varepsilon, i) = 0$, $(\varepsilon, i+1)(\alpha, i) = 0$, $(\varepsilon, i+1)(\gamma, i) = 0$ for all $i \in \mathbb{Z}$.
In practice, one drops the index $i \in \mathbb{Z}$ so that the quiver of $\breve{C}$ looks as follows:

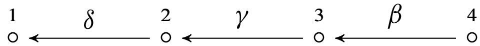


and the relations read exactly as those of $\widetilde{C}$.

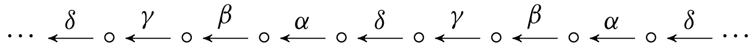
(b) Let B be the cluster tilted algebra of Example 2.2(c) given by the quiver



bound by $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$.
Then $B = \widetilde{C}$ where C is given by the quiver



bound by $\beta\gamma\delta = 0$.
Then $\breve{C}$ is given by the quiver



bound by all possible relations of the forms $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$.

Assume that C is a tilted algebra of type Q and that T is a tilting object in $\mathcal{C}_Q$ such that $\widetilde{C} = \operatorname{End}_{\mathcal{C}_Q} T$. Because of Proposition 1.9, we may assume without loss of generality that T is a tilting $\mathsf{k}Q$-module so that $C = \operatorname{End} T_{\mathsf{k}Q}$.

Theorem 3.9 ([6, **(1.2), (2.1)**]). *Let T be a tilting $\mathsf{k}Q$-module and $C = \operatorname{End} T_{\mathsf{k}Q}$. Then we have*

(a) $\breve{C} = \operatorname{End}_{\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q)}\left(\bigoplus_{i\in\mathbb{Z}} F^i T\right)$
(b) $\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q)}\left(\bigoplus_{i\in\mathbb{Z}} F^i T, -\right) : \mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q) \to \operatorname{mod}\breve{C}$ *induces an equivalence*

$$\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q) \Big/ \left\langle \operatorname{add}\left(\bigoplus_{i\in\mathbb{Z}} \tau F^i(T)\right)\right\rangle \cong \operatorname{mod}\breve{C}\ .$$

Proof. (a) Set $\mathcal{D} = \mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q)$. As k-vector spaces, we have

$$\operatorname{End}_{\mathcal{D}}\left(\bigoplus_{i\in\mathbb{Z}} F^i T\right) \cong \bigoplus_{i,j} \operatorname{Hom}_{\mathcal{D}}\left(F^i T, F^j T\right)\ .$$

Because T is a $\mathsf{k}Q$-module, all the summands on the right-hand side vanish except when $j \in \{i, i+1\}$. If $j = i$, then the corresponding summand is $\operatorname{Hom}_{\mathcal{D}}(T,T) = \operatorname{Hom}_{\mathsf{k}Q}(T,T) \cong C$, while, if $j = i+1$, it is $\operatorname{Hom}_{\mathcal{D}}(T, FT) \cong \operatorname{Ext}^2_C(\mathrm{D}C, C)$ as seen in the proof of Theorem 2.11. □

Associated to the Galois covering $G\colon \breve{C} \to \widetilde{C}$, there is a pushdown functor $G_\lambda\colon \operatorname{mod}\breve{C} \to \operatorname{mod}\widetilde{C}$ defined on the objects by

$$G_\lambda \breve{M}(a) = \bigoplus_{x\in G^{-1}(a)} \breve{M}(x)$$

where $\breve{M}$ is a $\breve{C}$-module and $a \in \left(Q_{\widetilde{C}}\right)_0$, see [47]. We now state the main result of this subsection

Theorem 3.10 ([6, **(2.4)**]). *Let T be a tilting $\mathsf{k}Q$-module and $C = \operatorname{End} T_{\mathsf{k}Q}$. Then, for the cluster repetitive algebra $\breve{C}$ and the relation extension $\widetilde{C}$, there is a commutative diagram of dense functors*

$$\begin{array}{ccc}
\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q) & \xrightarrow{\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(\operatorname{mod}\mathsf{k}Q)}\left(\bigoplus_{i\in\mathbb{Z}} F^i T, -\right)} & \operatorname{mod}\breve{C} \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle G_\lambda} \\
\mathcal{C}_Q & \xrightarrow{\operatorname{Hom}_{\mathcal{C}_Q}(\pi T, -)} & \operatorname{mod}\widetilde{C}\ .
\end{array}$$

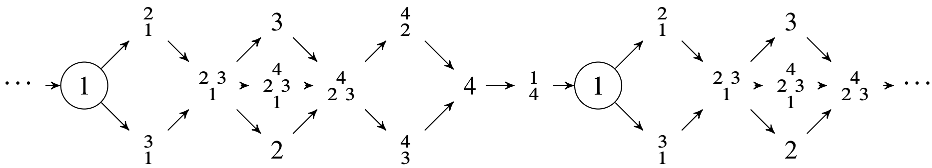
As an immediate consequence of the density of G_λ, we have the following corollary.

Corollary 3.11 ([6, **(2.5)**]).

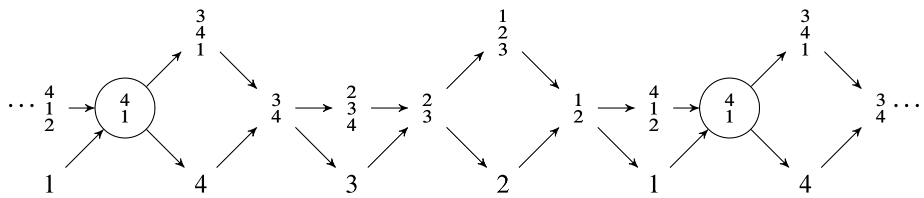
(a) *The pushdown of an almost split sequence in* $\operatorname{mod}\breve{C}$ *is an almost split sequence in* $\operatorname{mod}\widetilde{C}$.
(b) *The pushdown functor* G_λ *induces a quiver isomorphism between the orbit quiver* $\Gamma(\operatorname{mod}\breve{C})/\mathbb{Z}$ *of* $\Gamma(\operatorname{mod}\breve{C})$ *under the action of* $\mathbb{Z} \cong \langle\varphi\rangle$, *and* $\Gamma(\operatorname{mod}\widetilde{C})$.

Thus, in order to construct the Auslander–Reiten quiver $\Gamma(\operatorname{mod}\widetilde{C})$, it suffices to compute $\Gamma(\operatorname{mod}\breve{C})$ and then do the identifications required by passing to the orbit quiver.

Examples 3.8. (c) Let $C, \widetilde{C}, \breve{C}$ be as in example (a) above, then $\Gamma(\operatorname{mod}\breve{C})$ is



In order to get $\Gamma(\operatorname{mod}\widetilde{C})$, it suffices to identify the two encircled copies of 1.
(d) Let $C, \widetilde{C}, \breve{C}$ be as in example (b) above, then $\Gamma(\operatorname{mod}\breve{C})$ is



In order to get $\Gamma(\operatorname{mod}\widetilde{C})$, it suffices to identify the two encircled copies of ${}^{4}_{1}$.

3.4 *Cluster Tilted Algebras and Slices*

We recall that the map $C \mapsto \widetilde{C}$ from tilted algebras to cluster tilted is surjective, but generally not injective. We then ask, given a cluster tilted algebra B, how to find all the tilted algebras C such that $B = \widetilde{C}$. We answer this question by means of slices. Indeed, tilted algebras are characterised by the existence of complete slices, see, for instance [17, p. 320]. The corresponding notion in our situation is the following.

Definition 3.12. Let B be an algebra. A *local slice* in $\Gamma(\operatorname{mod}B)$ is a full connected subquiver Σ in a component Γ of $\Gamma(\operatorname{mod}B)$ such that:

(a) Σ is a *presection*, that is, if $X \to Y$ is an arrow in Γ, then

(i) $X \in \Sigma_0$ implies either $Y \in \Sigma_0$ or $\tau Y \in \Sigma_0$, and
(ii) $Y \in \Sigma_0$ implies either $X \in \Sigma_0$ or $\tau^{-1}X \in \Sigma_0$.

(b) Σ is *sectionally convex*, that is, if $X = X_0 \to X_1 \to \cdots \to X_t = Y$ is a sectional path of irreducible morphisms between indecomposable modules, then $X, Y \in \Sigma_0$ implies that $X_i \in \Sigma_0$ for all i.
(c) $|\Sigma_0| = \operatorname{rk} K_0(C)$ (that is, equals the number of isoclasses of simple C-modules).

For instance, if C is a tilted algebra, then it is easily seen that any complete slice in $\Gamma(\operatorname{mod} C)$ is a local slice. For cluster tilted algebras, in Examples 3.2(a)–(c), the sets $\left\{ {\begin{smallmatrix}2\\1\end{smallmatrix}}, 2, {\begin{smallmatrix}3\\2\end{smallmatrix}} \right\}$, $\left\{ {\begin{smallmatrix}2\\1\end{smallmatrix}}, {\begin{smallmatrix}2\ 3\\1\end{smallmatrix}}, {\begin{smallmatrix}&4&\\2&&3\\&1&\end{smallmatrix}}, 2 \right\}$ and $\left\{ 2, {\begin{smallmatrix}3\\2\\1\end{smallmatrix}}, {\begin{smallmatrix}3\\2\end{smallmatrix}}, {\begin{smallmatrix}4\\3\\2\end{smallmatrix}} \right\}$ are local slices, respectively. We shall now see that cluster tilted algebras always have (a lot of) local slices. Assume that C is a tilted algebra, and Σ a complete slice in $\Gamma(\operatorname{mod} C)$. Then there exist a hereditary algebra A and a tilting module T_A such that $C = \operatorname{End} T_A$ and $\Sigma = \operatorname{add} \operatorname{Hom}_A(T_A, \mathrm{D}\, A)$, see [17, Theorem VIII.5.6, p. 342]. On the other hand $\widetilde{C} = C \ltimes \operatorname{Ext}^2_C(\mathrm{D}\, C, C)$ is cluster tilted, and the surjective algebra morphism $p\colon \widetilde{C} \to C$ of Sect. 2.4 induces an embedding $\operatorname{mod} C \hookrightarrow \operatorname{mod} \widetilde{C}$.

Proposition 3.13 ([4])**.** *With the above notation, Σ embeds in $\Gamma(\operatorname{mod} \widetilde{C})$ as a local slice in the transjective component. Moreover, every local slice in $\Gamma(\operatorname{mod} \widetilde{C})$ occurs in this way.*

The above embedding turns out to preserve the Auslander–Reiten translates.

Lemma 3.14 ([4])**.** *With the above notation, if $M \in \Sigma_0$ then*

(a) $\tau_C M \cong \tau_{\widetilde{C}} M$ *and*
(b) $\tau_C^{-1} M \cong \tau_{\widetilde{C}}^{-1} M$.

Consider Σ as embedded in $\Gamma(\operatorname{mod} \widetilde{C})$. Its *annihilator* $\operatorname{Ann}_{\widetilde{C}} \Sigma$, namely, the intersection of all the annihilators $\bigcap_{M \in \Sigma_0} \operatorname{Ann}_{\widetilde{C}} M$ of the modules $M \in \Sigma_0$, is equal to $\operatorname{Ext}^2_C(\mathrm{D}\, C, C)$. This is the main step in the proof of the main theorem of this subsection, which answers the question asked at its beginning.

Theorem 3.15 ([4])**.** *Let B be a cluster tilted algebra. Then there exists a tilted algebra C such that $B = \widetilde{C}$ if and only if there exists a local slice Σ in $\Gamma(\operatorname{mod} B)$ such that $C = B/\operatorname{Ann}_B \Sigma$.*

Cluster tilted algebras have usually a lot of local slices.

Proposition 3.16 ([4])**.** *Let B be cluster tilted of tree type and M an indecomposable B-module lying in its transjective component. Then there exists a local slice Σ such that $M \in \Sigma_0$.*

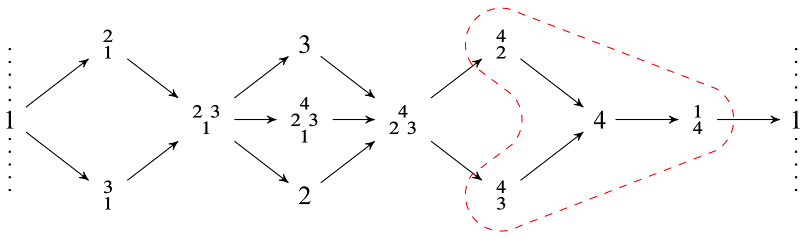
In particular, if B is representation-finite, then any indecomposable B-module lies on some local slice.

The following remark, which is an immediate consequence of [12, (1.3)], is particularly useful in calculations.

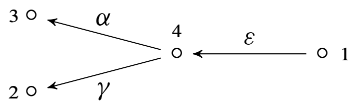
Proposition 3.17 ([4])**.** *Let B be a cluster tilted algebra, and Σ a local slice in $\Gamma(\operatorname{mod} B)$. Then $\operatorname{Ann}_B \Sigma$ is generated, as a two-sided ideal, by arrows in the quiver of B.*

For another approach to find all tilted algebras whose relation extension is a given cluster tilted algebra, we refer the reader to [29].

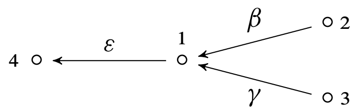
Examples 3.18. (a) Let B be the cluster tilted algebra of Example 2.2(b). We illustrate a local slice Σ in $\Gamma(\text{mod}\,B)$ by a dotted line.



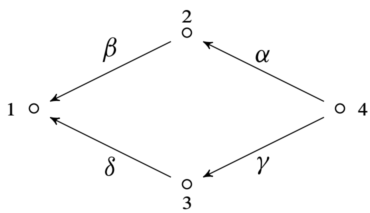
where we identify the two copies of 1. Here, $\text{Ann}_B\,\Sigma = \langle\beta,\gamma\rangle$ so that C is the quiver containing the remaining arrows



bound by $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$. There are only two other algebras which arise in this way from local slices. Namely, the algebra C_1 given by the quiver

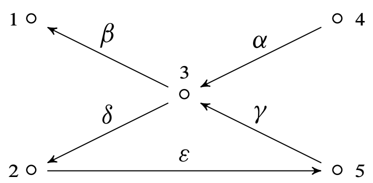


bound by $\beta\varepsilon = 0$, $\gamma\varepsilon = 0$, and C_2 given by the quiver

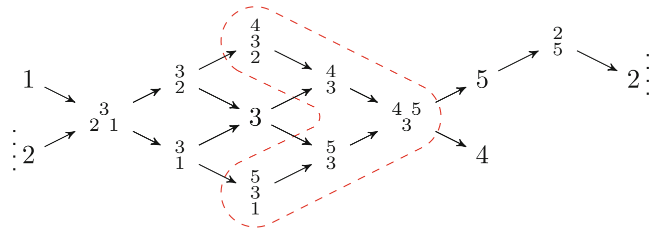


bound by $\alpha\beta = \gamma\delta$. Thus, we have $\widetilde{C} = \widetilde{C_1} = \widetilde{C_2}$.

(b) In contrast to tilted algebras, local slices do not characterise cluster tilted algebras. We give an example of an algebra which is not cluster tilted but has a local slice. Let A be given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\delta\varepsilon = 0$, $\varepsilon\gamma = 0$. We show a local slice in $\Gamma(\mathrm{mod}\, A)$



where we identify the two copies of 2.

In view of example (b), we may formulate the following problem.

Problem. Identify the class of algebras having local slices.

A partial solution is presented in [7].

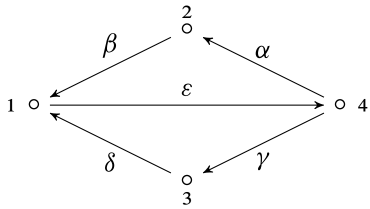
3.5 *Smaller and Larger Cluster Tilted Algebras*

Let C be a tilted algebra and $e \in C$ an idempotent, then it is known that eCe is tilted, see [51, Corollary III.6.5, p. 146]. This is not the case for cluster tilted algebras. On the other hand, factoring out the two-sided ideal generated by an idempotent, we obtain a smaller cluster tilted algebra.

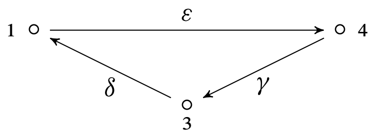
Theorem 3.19 ([39])**.** *Let B be a cluster tilted algebra, and $e \in B$ an idempotent. Then B/BeB is cluster tilted.*

If B is given as a bound quiver algebra and e is the sum of primitive idempotents corresponding to points in the quiver, then the bound quiver of B/BeB is obtained from that of B by deleting the points appearing in e, and all arrows incident to these points, with the inherited relations.

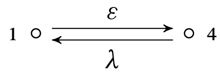
Examples 3.20. (a) If B is as in Example 2.2(b), thus given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$, $\beta\varepsilon = 0$, and e_2 is the primitive idempotent corresponding to the point 2, then B/Be_2B is given by the quiver

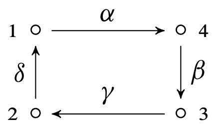


bound by $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$ and also $\gamma\delta = 0$ (because in B, we have $\gamma\delta = \alpha\beta$ and both α, β are set equal to zero when passing to the quotient B/Be_2B). This is the algebra in Example 2.2(a).
We also give an example of a full subcategory of a cluster tilted algebra which is not cluster tilted. In the previous example, let $e = e_1 + e_4$, then eBe is given by the quiver

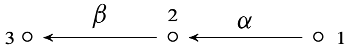


bound by $\varepsilon\lambda = 0$, $\lambda\varepsilon = 0$. This is not a cluster tilted algebra because its quiver contains a 2-cycle, see Corollary 2.5.

(b) If B is as in Example 2.2(c), given by the quiver



bound by $\alpha\beta\gamma = 0$, $\beta\gamma\delta = 0$, $\gamma\delta\alpha = 0$, $\delta\alpha\beta = 0$, then B/Be_4B is hereditary with quiver



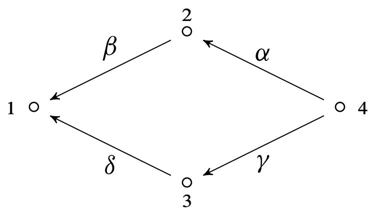
We may ask whether the above procedure can be reversed, that is, given a cluster tilted algebra B, whether there exists a (larger) cluster tilted algebra B' and an idempotent $e' \in B'$ such that $B \cong B'/B'e'B'$.

Let B be cluster tilted, and Σ a local slice in $\Gamma(\text{mod}\, B)$. Let $C = B/\operatorname{Ann}_B \Sigma$. Then Σ embeds in $\Gamma(\text{mod}\, C)$ as a complete slice, because of Proposition 3.13. Let M be a, not necessarily indecomposable, B-module all of whose indecomposable summands lie on Σ. In particular, M is a C-module. It is then known, and easy to prove, that the one-point extension $C' = C[M]$ is tilted. Let $B' = C' \ltimes \mathrm{Ext}^2_{C'}(\mathrm{D}\,C', C')$ be the relation extension of C'. We have the following theorem which can be seen as the reverse procedure to Theorem 3.19 above.

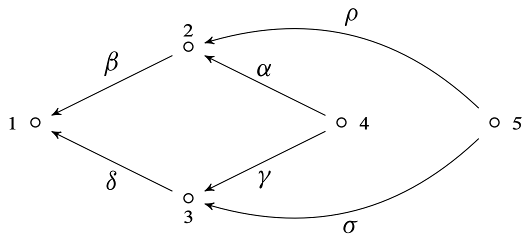
Theorem 3.21 ([60]). *With the above notation, B' is cluster tilted and, if e' is the primitive idempotent corresponding to the extension point, then*

$$B'/B'e'B' \cong B\ .$$

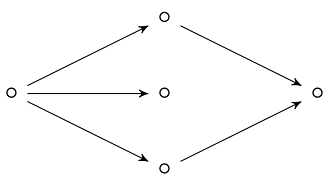
Examples 3.20. (c) Let B be as in example (a), with C given by the quiver



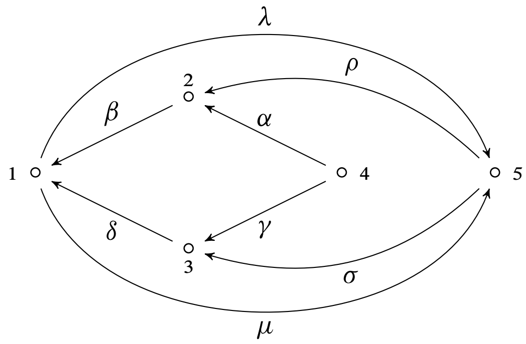
bound by $\alpha\beta = \gamma\delta$. Let $M = 2 \oplus 3$. Then both summands of M lie on a complete slice and $C' = C[M]$ is the tilted algebra given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\rho\beta = 0$, $\sigma\delta = 0$. It is of wild type



The relation extension B' of C' is given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$, $\beta\varepsilon = 0$, $\delta\varepsilon = 0$, $\lambda\rho = 0$, $\rho\beta = 0$, $\beta\lambda = 0$, $\mu\sigma = 0$, $\sigma\delta = 0$, $\delta\mu = 0$.
According to Theorem 3.21, B' is cluster tilted and moreover $B'/B'e_5B' = B$.

4 Particular Modules over Cluster Tilted Algebras

4.1 *The Left Part of a Cluster Tilted Algebra*

Because tilting theory lies at the heart of the study of cluster tilted algebras, it is natural to ask what are the tilting modules over these algebras. We first see that they correspond to tilting objects in the cluster category. Indeed, recall from Theorem 3.1 that given a tilting object T in a cluster category $\mathcal{C}$ and $B = \mathrm{End}_{\mathcal{C}}\, T$ then the functor $\mathrm{Hom}_{\mathcal{C}}(T,-)\colon \mathcal{C} \to \mathrm{mod}\, B$ induces an equivalence $\mathcal{C}/\langle \mathrm{add}\, \tau T\rangle \cong \mathrm{mod}\, B$. We wish to see what are the preimages under $\mathrm{Hom}_{\mathcal{C}}(T,-)$ of partial tilting and tilting B-modules (we call them their *lifts*).

Theorem 4.1 ([67]). *Let Q be a finite acyclic quiver, $\mathcal{C}_Q$ the corresponding cluster category, T a tilting object in $\mathcal{C}_Q$ and $B = \mathrm{End}_{\mathcal{C}_Q} T$. Then*:

(a) *any partial tilting B-module lifts to a rigid object in $\mathcal{C}_Q$, and*
(b) *any tilting B-module lifts to a tilting object in $\mathcal{C}_Q$.*

An immediate consequence of this theorem and Theorem 3.1 is that, if U is a tilting B-module with lift $\overline{U}$, then $\mathrm{End}_B\, U$ is a quotient of $\mathrm{End}_{\mathcal{C}_Q} \overline{U}$.

Recall that, for an algebra A, the *left part* $\mathcal{L}_A$ of mod A is the full subcategory of ind A consisting of all the M such that, for any L in ind A such that there exists a path of nonzero morphisms between indecomposables $L = L_0 \to L_1 \to \cdots \to L_t = M$ we have $\mathrm{pd}\, L \leq 1$. The *right part* $\mathcal{R}_A$ is defined dually, see [54]. We want to study the left and right parts of a cluster tilted algebra. We need one lemma.

Lemma 4.2 ([67, **(5.1)**]). *Let B be a nonhereditary cluster tilted algebra. Then any connected component of $\Gamma(\mathrm{mod}\, B)$ either contains no projectives and no injectives, or it contains both projectives and injectives.*

Proof. Let P be an indecomposable projective lying in a component Γ of $\Gamma(\text{mod}\, B)$. Let Σ be the maximal full, connected convex subquiver of Γ containing only indecomposable projectives, including P. Because B is not hereditary, the number of points of Σ is strictly less than the number of τ-orbits in Γ. Therefore there exist $P' \in \Sigma_0$ and $M \notin \Sigma_0$ such that there is an irreducible morphism $M \to P'$: indeed, if this is not the case, then there is an irreducible morphism $P' \to N$ with $N \notin \Sigma_0$ and N projective, a contradiction. Let T' be the indecomposable summand of the tilting object T in $\mathcal{C}_Q$ corresponding to P'. Because M is nonprojective, there is in $\mathcal{C}_Q$ an arrow $\tau^2 T' \to \overline{M}$, where $\overline{M}$ denotes the lift of M. This corresponds in Γ to an irreducible morphism from an indecomposable injective B-module to τM. Hence Γ contains at least one injective. Dually, if Γ contains an injective, then it also contains a projective. □

Proposition 4.3 ([67, **(5.2)**]). *Let B be a nonhereditary cluster tilted algebra. Then $\mathcal{L}_B$ and $\mathcal{R}_B$ are finite.*

Proof. Assume $\mathcal{L}_B \neq \varnothing$. Because $\mathcal{L}_B$ is closed under predecessors in ind B, it contains at least one indecomposable projective B-module P. Because of [45, (1.1)] and Lemma 4.2 above, there exists $m \geq 0$ such that $\tau^{-m} P$ is a successor of an injective module. We may assume m to be minimal for this property. Because of [10, (1.6)], we have $\tau^{-m} P \notin \mathcal{L}_B$ and so $\tau^{-m} P$ is Ext-injective in $\mathcal{L}_B$. Because this holds for any indecomposable projective in $\mathcal{L}_B$, it follows from [11, (5.4)] that $\mathcal{L}_B$ is finite. Dually, $\mathcal{R}_B$ is finite. □

As easy consequences, any cluster tilted algebra is left and right supported in the sense of [11], and it is laura [10] if and only if it is hereditary or representation-finite.

Given an algebra A, its *left support* A_λ is the endomorphism algebra of the direct sum of all indecomposable projective A-modules lying in $\mathcal{L}_A$. The dual notion is the *right support* algebra A_ρ. It is shown in [11, (2.3)] that A_λ, A_ρ are always products of quasi-tilted algebras. We show that they are, for cluster tilted algebras, products of hereditary algebras.

Proposition 4.4 ([67, **(5.4)**]). *Let B be Iwanaga–Gorenstein of injective dimension* 1, *then B_λ, B_ρ are direct products of hereditary algebras.*

Proof. Because $\mathcal{L}_B \subseteq \text{ind}\, B_\lambda$, see [11], it suffices to prove that, if P is a projective indecomposable B-module lying in $\mathcal{L}_B$ and $M \to P$ is an irreducible morphism with M indecomposable, then M is projective. Assume not, then $\tau M \neq 0$ and $\text{Hom}_B(\tau^{-1}(\tau M), P) \neq 0$ implies $\text{id}(\tau M)_B > 1$, because of [17, p. 115]. Because B is Iwanaga–Gorenstein of injective dimension 1, we infer that $\text{pd}(\tau M) > 1$, contradicting the fact that $\tau M \in \mathcal{L}_B$. Therefore M is projective. This shows that B_λ is a direct product of hereditary algebras. Dually, B_ρ is also a direct product of hereditary algebras. □

Actually, one can show, see [67, (5.5)], that $\mathcal{L}_B$ contains no indecomposable injective B-module. Therefore $\mathcal{L}_B$ can be characterised as the set of those indecomposable modules which are not successors of an injective (by a sequence of nonzero morphisms between indecomposable modules). Dually, $\mathcal{R}_B$ consists of those indecomposables which are not predecessors of a projective.

We also refer the reader to [28] for modules of projective dimension one over cluster tilted algebras.

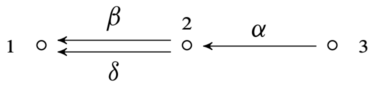
4.2 *Modules Determined by Their Composition Factors*

It is a standard question in representation theory to identify those indecomposable modules over a given algebra which are uniquely determined by their composition factors or, equivalently, by their dimension vectors. We have seen in (3.1) that the Auslander–Reiten quiver of a cluster tilted algebra contains a unique transjective component, and this is the only component containing local slices, see Sect. 3.4. If the cluster tilted algebra is of Dynkin type, then the transjective component is the whole Auslander–Reiten quiver. We have the following theorem.

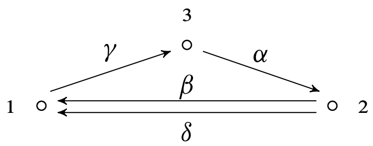
Theorem 4.5 ([13]). *Let B be a cluster tilted algebra and M, N indecomposable B-modules lying in the transjective component. Then $M \cong N$ if and only if M and N have the same composition factors.*

As a consequence, over a representation-finite cluster tilted algebra, all indecomposables are uniquely determined by their composition factors.

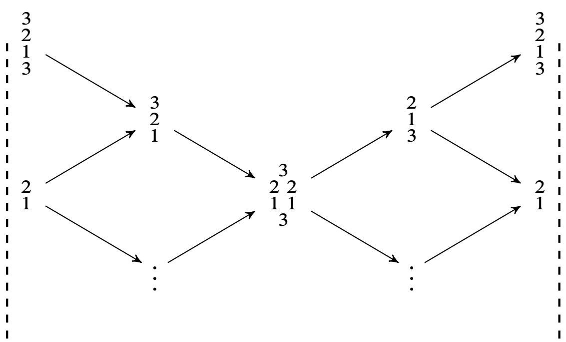
Example 4.6. The statement of the theorem does not hold true if M, N are not transjective. Let indeed C be the tilted algebra of type $\tilde{\mathbb{A}}_2$ given by the quiver



bound by $\alpha\beta = 0$. Its relation extension $B = \widetilde{C}$ is given by the quiver



bound by $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\alpha = 0$. Then $\Gamma(\text{mod } B)$ contains exactly one tube of rank 2, all others being of rank 1. This tube is of the form



where we identify along the vertical dashed lines. Clearly the modules rad $P_3 = \begin{smallmatrix}2\\1\\3\end{smallmatrix}$ and $P_3/\operatorname{soc} P_3 = \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ are nonisomorphic but have the same composition factors.

The situation is slightly better for cluster concealed algebras of Euclidean type, see [63]: these are the relation extensions of concealed algebras, that is, of tilted algebras which are endomorphism algebras of a postprojective (or a preinjective) tilting module over a hereditary algebra.

Proposition 4.7 ([13]). *Let B be a cluster concealed algebra of Euclidean type, and M, N two rigid indecomposable modules. Then $M \cong N$ if and only if M and N have the same composition factors.*

If B is cluster concealed of wild type and M, N are not only rigid but also lift to rigid objects in the cluster category, then the statement holds true: $M \cong N$ if and only if M and N have the same composition factors.

4.3 *Induced and Coinduced Modules*

Another successful approach for studying modules over cluster tilted algebras is by considering them as induced or coinduced from modules over an underlying tilted algebra. A similar approach is used extensively in the representation theory of finite groups. Indeed, let C be tilted, $E = \operatorname{Ext}^2_C(\mathrm{D}\,C, C)$ and $B = C \ltimes E$ be its relation extension. There are two change of rings functors allowing to pass from $\operatorname{mod} C$ to $\operatorname{mod} B$; these are as follows:

(i) the *induction functor* $- \otimes_C B_B \colon \operatorname{mod} C \to \operatorname{mod} B$, and
(ii) the *coinduction functor* $\operatorname{Hom}_B({}_B B_C, -) \colon \operatorname{mod} C \to \operatorname{mod} B$.

A B-module is said to be *induced* (or *coinduced*) if it lies in the image of the induction functor (or the coinduction functor, respectively).

Lemma 4.8 ([64, **(4.2)**]). *Let M be a C-module, then*

(a) $\operatorname{id} M_C \leq 1$ *if and only if* $M \otimes_C B \cong M$.
(b) $\operatorname{pd} M_C \leq 1$ *if and only if* $\operatorname{Hom}_C(B, M) \cong M$.

Proof. We only prove (a), because the proof of (b) is similar. Recall that, as left C-modules, we have ${}_C B \cong {}_C C \oplus {}_C E$. Therefore $M \otimes_C B \cong M \oplus (M \otimes_C E)$. Thus, $M \otimes_C B \cong M$ if and only if $M \otimes_C E = 0$. Now we have

$$E = \operatorname{Ext}^2_C(\mathrm{D}\,C, C) \cong \operatorname{Ext}^1_C(\Omega\,\mathrm{D}\,C, C) \cong \mathrm{D}\operatorname{Hom}_C(C, \tau\Omega\,\mathrm{D}\,C) \cong \mathrm{D}(\tau\Omega\,\mathrm{D}\,C)$$

where we used that $\operatorname{pd}(\Omega\,\mathrm{D}\,C) \leq 1$ because gl. $\dim C \leq 2$. Therefore

$$\begin{aligned} M \otimes_C E &\cong M \otimes_C \mathrm{D}(\tau\Omega\,\mathrm{D}\,C) \cong \mathrm{D}\operatorname{Hom}_C(M, \tau\Omega\,\mathrm{D}\,C) \\ &\cong \operatorname{Ext}^1_C(\Omega\,\mathrm{D}\,C, M) \cong \operatorname{Ext}^2_C(\mathrm{D}\,C, M) \\ &\cong \operatorname{Ext}^1_C(\mathrm{D}\,C, \Omega^{-1}M) \cong \mathrm{D}\operatorname{Hom}_C(\tau^{-1}\Omega^{-1}M, \mathrm{D}\,C) \\ &\cong \tau^{-1}\Omega^{-1}M \end{aligned}$$

where we used that $\mathrm{pd}(\Omega\,\mathrm{D}\,C) \le 1$ and also that $\mathrm{id}(\Omega^{-1}M) \le 1$. Now, $\tau^{-1}\Omega^{-1}M$ vanishes if and only if $\Omega^{-1}M$ is injective, that is, if and only if $\mathrm{id}\,M_C \le 1$. □

We recall some notation associated with the tilting theorem, see [17, p. 205]. Let $\mathsf{k}Q$ be the path algebra of a quiver Q, $T_{\mathsf{k}Q}$ a tilting module and $C = \mathrm{End}\,T_{\mathsf{k}Q}$. Then every indecomposable C-module belongs to one of the classes

$$\mathcal{X}(T) = \{M \mid M \otimes_C T = 0\}$$

and

$$\mathcal{Y}(T) = \left\{M \mid \mathrm{Tor}_1^C(M,T) = 0\right\} .$$

Let $\mathcal{C}_Q$ denote the cluster category.

Lemma 4.9 ([64, **(6.2), (6.4)**]). *Let M be an indecomposable C-module, then*

(a)
$$M \otimes_C B \cong \begin{cases} \mathrm{Hom}_{\mathcal{C}_Q}(T, M \otimes_C T) & \textit{if } M \in \mathcal{Y}(T) \\ M & \textit{if } M \in \mathcal{X}(T), \end{cases}$$

(b)
$$\mathrm{Hom}_C(B,M) \cong \begin{cases} \mathrm{Ext}^1_{\mathcal{C}_Q}\left(T, \mathrm{Tor}_1^C(M,T)\right) & \textit{if } M \in \mathcal{X}(T) \\ M & \textit{if } M \in \mathcal{Y}(T). \end{cases}$$

Proof. We only sketch the proof of (a). We know that M either lies in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$. If $M \in \mathcal{X}(T)$, then $\mathrm{id}\,M_C \le 1$ (see [17, (VIII.3)]). Because of Lemma 4.8, we have $M \otimes_C B \cong M$. If, on the other hand, $M \in \mathcal{Y}(T)$, then, because of the tilting theorem, we have $M \cong \mathrm{Hom}_{\mathsf{k}Q}(T, M \otimes_C T)$. One can then prove that $M \otimes_C B \cong \mathrm{Hom}_{\mathsf{k}Q}(T, M \otimes_C T) \otimes_C B \cong \mathrm{Hom}_{\mathcal{C}_Q}(T, M \otimes_C T)$, see [64, (6.1)]. □

We can now state the main result of this subsection.

Theorem 4.10 ([64, **(7.2), (7.4), (7.5)**]). *Let B be a cluster tilted algebra.*

(a) *If B is representation-finite and M is an indecomposable B-module, then there exists a tilted algebra C such that $B = \widetilde{C}$ and M is a C-module. In particular, M is both induced and coinduced from a C-module.*

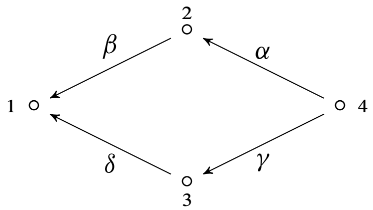
(b) *If B is arbitrary, and M is an indecomposable B -module lying in the transjective component, then there exists a tilted algebra C such that $B = \widetilde{C}$ and M is a C-module. In particular, M is induced or coinduced from a C-module.*

(c) *If B is cluster concealed, and M is an indecomposable B-module, then there exists a tilted algebra C such that $B \cong \widetilde{C}$ and M is a C-module. In particular, M is induced or coinduced from a C-module.*

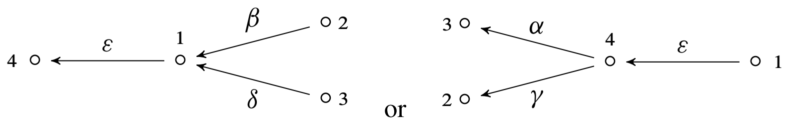
Proof of (a). Because B is representation-finite, C is of tree type. Because of Proposition 3.16, there exists a local slice Σ in $\Gamma(\mathrm{mod}\,B)$ on which M lies. Let $C = B/\mathrm{Ann}_B\,\Sigma$. Then B is the relation extension of the tilted algebra C and M lies on the complete slice Σ in $\Gamma(\mathrm{mod}\,C)$. But then $\mathrm{id}\,M_C \le 1$ and $\mathrm{pd}\,M_C \le 1$. Because of Lemmata 4.8 and 4.9 above, we have both $M \cong M \otimes_C B$ and $M \cong \mathrm{Hom}_C(B,M)$. This completes the proof. □

It is important to observe that the tilted algebra C depends essentially on the choice of M.

Example 4.11. Let B be given by the bound quiver of Example 2.2(b). Choosing $M = \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$, we get that C is given by the quiver



bound by $\alpha\beta = \gamma\delta$. Then $M \cong M \otimes_C B \cong \mathrm{Hom}_C(B, M)$, so is induced and coinduced. On the other hand, if we choose $M' = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix}$, we get that C is given by one of the quivers



bound, respectively, by $\beta\varepsilon = 0$, $\delta\varepsilon = 0$ and $\varepsilon\alpha = 0$, $\varepsilon\gamma = 0$. This indeed depends on the chosen local slice containing M'. In each case, M' is again both induced and coinduced.

5 Hochschild Cohomology of Cluster Tilted Algebras

5.1 *The Hochschild Projection Morphism*

The Hochschild cohomology groups were introduced by G. Hochschild in 1945, see [44]. Let C be an algebra and ${}_CE_C$ a bimodule. Denoting by $C^{\otimes_k i}$ the ith tensor power of C over k, we have a complex

$$0 \longrightarrow E \xrightarrow{b^1} \mathrm{Hom}_k(C, E) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(C^{\otimes_k i}, E) \xrightarrow{b^{i+1}} \mathrm{Hom}_k(C^{\otimes_k i+1}, E) \longrightarrow \cdots$$

where $b^1 \colon E \to \mathrm{Hom}_k(C, E)$ is defined for $x \in E$, $c \in C$ by

$$b^1(x)(c) = cx - xc$$

while $b^{i+1} \colon \mathrm{Hom}_k(C^{\otimes_k i}, E) \to \mathrm{Hom}_k(C^{\otimes_k i+1}, E)$ maps $f \colon C^{\otimes_k i} \to E$ to the map $b^{i+1}(f)$ from $C^{\otimes_k i+1}$ to E defined on the generators by

$$b^{i+1}(f)(c_1 \otimes \cdots \otimes c_{i+1}) = c_1 f(c_2 \otimes \cdots \otimes c_{i+1}) + \sum_{j=1}^{i}(-1)^j f(c_1 \otimes \cdots \otimes c_j c_{j+1} \otimes \cdots \otimes c_{i+1}) + (-1)^{i+1} f(c_1 \otimes \cdots \otimes c_i)c_{i+1}$$

where all $c_j \in C$.

The ith *Hochschild cohomology group* is the ith cohomology group of this complex, that is,

$$\mathrm{H}^i(C, E) = \frac{\mathrm{Ker}\, b^{i+1}}{\mathrm{Im}\, b^i}.$$

If $E = {}_C C_C$, then we write

$$\mathrm{H}^i(C, C) = \mathrm{HH}^i(C)\ .$$

The lower index groups have concrete interpretations. For instance,

$$\mathrm{H}^0(C, E) = \{c \in C \mid cx = xc \text{ for all } x \in E\}\ .$$

In particular, $\mathrm{HH}^0(C)$ is the centre of the algebra C. For the first group $\mathrm{H}^1(C, E)$, let $\mathrm{Der}(C, E)$ denote the subspace of $\mathrm{Hom}_k(C, E)$ consisting of all $d: C \to E$ such that

$$d(cc') = d(c)\, c' + c\, d(c')$$

for all $c, c' \in C$. Such maps are called *derivations*. For instance, to each $x \in E$ corresponds a derivation d_x defined by $d_x(c) = cx - xc$ (for $c \in C$). The d_x are called *inner* (or *interior*) *derivations*, and we denote their set by $\mathrm{IDer}(C, E)$. Then, clearly

$$\mathrm{H}^1(C, E) = \frac{\mathrm{Der}(C, E)}{\mathrm{IDer}(C, E)}\ .$$

The Hochschild groups are not only invariants of the algebra, they are also derived invariants, that is, if $\mathcal{D}^b(\mathrm{mod}\, C) \cong \mathcal{D}^b(\mathrm{mod}\, C')$ is a triangle equivalence, then $\mathrm{HH}^i(C) \cong \mathrm{HH}^i(C')$ for all i, see [52, 56].

Moreover $\mathrm{HH}^*(C) = \bigoplus_{i \geq 0} \mathrm{HH}^i(C)$ carries a natural ring structure with the so-called cup product: if $\xi = [f] \in \mathrm{HH}^i(C)$ and $\zeta = [g] \in \mathrm{HH}^j(C)$, then we define $f \times g: C^{\otimes_k i} \otimes_k C^{\otimes_k j} \to C$ by

$$(f \times g)(c_1 \otimes \cdots \otimes c_i \otimes c_{i+1} \otimes \cdots \otimes c_j) = f(c_1 \otimes \cdots \otimes c_i) g(c_{i+1} \otimes \cdots \otimes c_j),$$

where all $c_k \in C$. One verifies that this defines unambiguously a product. We set $\xi \cup \zeta = [f \times g]$ and call it the *cup product* of ξ and ζ. With this product $\mathrm{HH}^*(C)$ is a graded commutative ring, that is, if ξ, ζ are as above, then

$$\xi \cup \zeta = (-1)^{ij} \zeta \cup \xi\ .$$

We now let C be triangular of global dimension two and $E = \mathrm{Ext}^2_C(\mathrm{D}C, C)$. Denoting by $B = C \ltimes E$ the relation extension of C, we have a short exact sequence

$$0 \longrightarrow E \xrightarrow{i} B \underset{q}{\overset{p}{\rightleftarrows}} C \longrightarrow 0$$

as in Sect. 2.3. Let $[f] \in \mathrm{HH}^i(B)$, then we have a diagram

$$\begin{array}{ccc} B^{\otimes_k i} & \xrightarrow{f} & B \\ {\scriptstyle q^{\otimes i}}\uparrow & & \downarrow{\scriptstyle p} \\ C^{\otimes_k i} & \overset{pfq^{\otimes i}}{\dashrightarrow} & C. \end{array}$$

We set $\varphi^i[f] = [pfq^{\otimes i}]$. It is easily checked that this gives rise to a well-defined k-linear map $\varphi^i : \mathrm{HH}^i(B) \to \mathrm{HH}^i(C)$, which we call the *ith Hochschild projection morphism*, see [14, (2.2)].

Theorem 5.1 ([14, **(2.3)**]). *Considering* $\mathrm{HH}^*(B)$ *and* $\mathrm{HH}^*(C)$ *as associative algebras with the cup product, the* φ^i *induce an algebra morphism*

$$\varphi^* : \mathrm{HH}^*(B) \to \mathrm{HH}^*(C)\ .$$

Note that φ^* is only a morphism of associative algebras: the Hochschild cohomology ring also carries a natural Lie algebra structure, but φ^* is not in general a morphism of Lie algebras. For a counterexample, see [14, (2.5)].

Consider the short exact sequence of B-B-bimodules

$$0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$$

and apply to it the functor $\mathrm{Hom}_{B\text{-}B}(B, -)$ (we denote by $\mathrm{Hom}_{B\text{-}B}$ the morphisms of B-B-bimodules). We get a long exact cohomology sequence

$$0 \longrightarrow \mathrm{H}^0(B,E) \longrightarrow \mathrm{HH}^0(B) \longrightarrow \mathrm{H}^0(B,C) \xrightarrow{\delta^0} \mathrm{H}^1(B,E) \longrightarrow \mathrm{HH}^1(B) \longrightarrow \mathrm{H}(B,C) \xrightarrow{\delta^1} \cdots$$

where δ^i denotes the i^{th} connecting morphism.

It is easy to prove that $\mathrm{H}^0(B,C) \cong \mathrm{HH}^0(C)$, see, for instance [14, (2.7)], and thus the composition of this isomorphism with the map $\mathrm{HH}^0(B) \to \mathrm{H}^0(B,C)$ of the previous long exact sequence is just $\varphi^0 : \mathrm{HH}^0(B) \to \mathrm{HH}^0(C)$. Now C is triangular, and $\mathrm{HH}^0(C)$ is its centre, hence $\mathrm{HH}^0(C) = \mathsf{k}$. On the other hand $\varphi^0 \neq 0$ because it maps the identity of B to that of C. Therefore, we have a short exact sequence

$$0 \longrightarrow \mathrm{H}^0(B,E) \longrightarrow \mathrm{HH}^0(B) \xrightarrow{\varphi^0} \mathrm{HH}^0(C) \longrightarrow 0\ .$$

We also have the following theorem.

Theorem 5.2 ([14, **(5.7)**]). *Let C be triangular of global dimension at most two, and B its relation extension. Then we have a short exact sequence*

$$0 \longrightarrow \mathrm{H}^1(B,E) \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi^1} \mathrm{HH}^1(C) \longrightarrow 0 \quad .$$

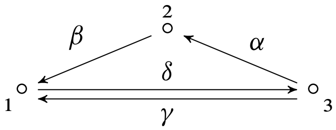
If, in particular, C is tilted so that B is cluster tilted, then φ^1 is surjective.

In actual computations, one uses the fact that, for C triangular of global dimension two, one can prove that

$$\mathrm{H}^1(B,E) \cong \mathrm{H}^1(C,E) \oplus \mathrm{End}_{C\text{-}C}\, E$$

see [14, (5.9)].

Example 5.3. Let B be the algebra of Example 2.12(b), given by the quiver



bound by $\alpha\beta = 0$, $\beta\delta = 0$, $\delta\alpha = 0$, $\delta\gamma\delta = 0$. This is the relation extension of the (non-tilted) algebra $C = B/B\delta B$. Then one can prove that $\mathrm{H}^1(C,E) = 0$ while $\mathrm{End}_{C\text{-}C}\, E = \mathsf{k}$ (indeed, E has simple top generated by the arrow δ). Because $\mathrm{HH}^1(C) = \mathsf{k}$, we get that $\mathrm{HH}^1(B) = \mathsf{k}^2$.

In this example, the higher φ^i are not surjective: indeed, one can prove that $\varphi^2 = 0$, while $\mathrm{HH}^2(C) \neq 0$, see [14, (5.12)].

Corollary 5.4 ([14, **(5.8)**]). *Let B be cluster tilted and C tilted such that $B = \widetilde{C}$, then there is a short exact sequence*

$$0 \longrightarrow \mathrm{H}^0(B,E) \oplus \mathrm{H}^1(B,E) \longrightarrow \mathrm{HH}^*(B) \xrightarrow{\varphi^*} \mathrm{HH}^*(C) \longrightarrow 0 \quad .$$

Proof. Because C is tilted, it follows from [53] that $\mathrm{HH}^i(C) = 0$ for all $i \geq 2$. □

5.2 *The Tame and Representation-Finite cases*

Now we consider cluster tilted algebras of Dynkin or Euclidean type. Let C be tilted and $B = \widetilde{C}$. We need to define an invariant $n_{B,C}$ depending on the choice of C.

Let $\rho = \sum_{i=1}^{m} \lambda_i w_i$ be a relation in a bound quiver (Q, I), where each w_i is a path of length at least two from x to y, say, and each λ_i is a nonzero scalar. Then ρ is called *strongly minimal* if, for every nonempty proper subset J of $\{1, 2, \ldots, m\}$ and every family $(\mu_j)_{j\in J}$ of nonzero scalars, we have $\sum_{j\in J} \mu_j w_j \notin I$. It is proved in [16, (2.2)] that, if B is cluster tilted, then it has a presentation consisting of strongly minimal relations.

Let now $C = \mathsf{k}Q/I$ be a tilted algebra and $B = \widetilde{C} = \mathsf{k}\widetilde{Q}/\tilde{I}$ be its relation extension, where $\tilde{I}$ is generated by the partial derivatives of the Keller potential, see Sect. 2.4. Let

$\rho = \sum_{i=1}^{m} \lambda_i w_i$ be a strongly minimal relation in $\tilde{I}$, then either ρ is a relation in I, or there exist exactly m new arrows $\alpha_1, \dots, \alpha_m$ such that $w_i = u_i \alpha_i v_i$, with u_i, v_i paths consisting entirely of old arrows [16, (3.1)]. Moreover, each new arrow α_i must appear in this way.

We define a relation $\sim$ on the set $\widetilde{Q}_1 \setminus Q_1$ of new arrows. For every $\alpha \in \widetilde{Q}_1 \setminus Q_1$, we set $\alpha \sim \alpha$. If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a strongly minimal relation in $\tilde{I}$ and the α_i are as above, then we set $\alpha_i \sim \alpha_j$ for all i, j such that $1 \le i, j \le m$.

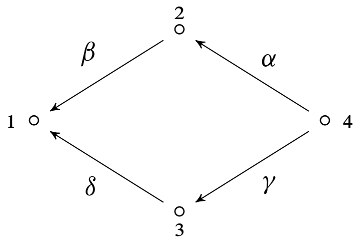
One can show that $\sim$ is unambiguously defined. It is clearly reflexive and symmetric. We let $\approx$ be the least equivalence relation on $\widetilde{Q}_1 \setminus Q_1$ such that $\alpha \sim \beta$ implies $\alpha \approx \beta$ (that is, $\approx$ is the transitive closure of $\sim$).

We define the *relation invariant* of B, relative to C, to be the number $n_{B,C}$ of equivalence classes of new arrows under the relation $\approx$.

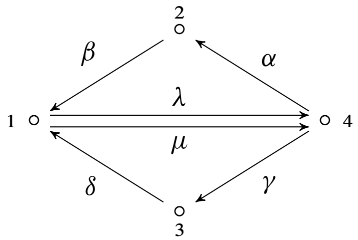
This equivalence is related to the direct sum decomposition of the C-C-bimodule $E = \mathrm{Ext}^2_C(\mathrm{D}\,C, C)$. Indeed, E is generated, as C-C-bimodule, by the new arrows. If two new arrows occur in a strongly minimal relation, this means that they are somehow yoked together in E. It is shown in [8, (4.3)] that E decomposes, as C-C-bimodule, into the direct sum of $n_{B,C}$ summands.

Theorem 5.5 ([16, **(5.3)**])**.** *Let $B = C \ltimes E$ be a cluster tilted algebra. If B is of Dynkin or of Euclidean type, then* $\mathrm{H\,H}^1(B) = \mathrm{H\,H}^1(C) \oplus \mathsf{k}^{n_{B,C}}$.

Example 5.6. Let C be the (representation-finite) tilted algebra of Euclidean type $\tilde{\mathbb{A}}_3$ given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Its relation extension B is as in Example 2.2(d), that is, given by the quiver

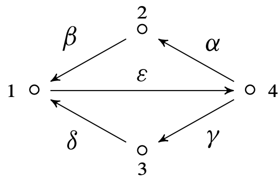


bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$. There are two equivalence classes of new arrows, namely $\{\lambda\}$ and $\{\mu\}$. Therefore $n_{B,C} = 2$. Because of the theorem, we have $\mathrm{H\,H}^1(B) \cong \mathrm{H\,H}^1(C) \oplus \mathsf{k}^2 \cong \mathsf{k}^3$.

There is a better result in the representation-finite case. If B is representation-finite, then C is tilted of Dynkin type. Because Dynkin quivers are trees, and the Hochschild groups are invariant under tilting (see Sect. 5.1 above), we have $\mathrm{HH}^1(C) = 0$. Therefore $\mathrm{HH}^1(B) = \mathsf{k}^{n_{B,C}}$ and so the invariant $n_{B,C}$ does not depend on the choice of C. We therefore denote it by n and give an easy way to compute it. We have defined chordless cycles in Sect. 2.4. An arrow in the quiver of a cluster tilted algebra is called *inner* if it belongs to two chordless cycles.

Theorem 5.7 ([16, **(6.4)**]). *If B is a representation-finite cluster tilted algebra, then the dimension n of $\mathrm{HH}^1(B)$ equals the number of chordless cycles minus the number of inner arrows in the quiver of B.*

Example 5.8. Let B be as in Example 2.2(b) given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\varepsilon\alpha = 0$, $\beta\varepsilon = 0$, $\varepsilon\gamma = 0$, $\delta\varepsilon = 0$. There are two chordless cycles and just one inner arrow so $n = 2 - 1 = 1$ and $\mathrm{HH}^1(B) = \mathsf{k}$.

We get a characterisation of the fundamental group of B.

Corollary 5.9 ([15, **(4.1)**]). *If $B = \mathsf{k}\widetilde{Q}/\tilde{I}$ is a representation-finite cluster tilted algebra, then $\pi_1(\widetilde{Q}, \tilde{I})$ is free on n generators.*

For instance, in the above example, $\pi_1(\widetilde{Q}, \tilde{I}) \cong \mathbb{Z}$. This is a particular case of the following problem.

Problem. Let $B = \mathsf{k}\widetilde{Q}/\tilde{I}$ be a cluster tilted algebra, with the presentation induced from the Keller potential. Prove that $\pi_1(\widetilde{Q}, \tilde{I})$ is free.

Finally, we refer the reader to [41] for the study of the Hochschild groups as derived invariants of (an overclass of) cluster tilted algebras of type $\mathbb{A}$.

5.3 *Simply Connected Cluster Tilted Algebras*

In [66], Skowroński asked for which algebras the vanishing of the first Hochschild cohomology group is equivalent to simple connectedness. We prove that this is the case for cluster tilted algebras.

Theorem 5.10 ([16, **(5.11)**]). *Let B be cluster tilted. The following conditions are equivalent:*

(a) $\mathrm{HH}^1(B) = 0$.
(b) *B is simply connected.*
(c) *B is hereditary and its quiver is a tree.*

Proof. (b) implies (c). If B is simply connected, then it is triangular and hence it is hereditary. Moreover its quiver must be a tree.

(c) implies (a). This is trivial, see [52].

(a) implies (c). If B is not hereditary, and C is tilted such that $B = \widetilde{C}$, then because of Lemma 3.7, we have a connected Galois covering $\breve{C} \to \widetilde{C} = B$ with group $\mathbb{Z}$. The universal property of Galois coverings yields a group epimorphism $\pi_1(\widetilde{Q}, \tilde{I}) \to \mathbb{Z}$ where $B = \mathsf{k}\widetilde{Q}/\tilde{I}$. This epimorphism induces a monomorphism of abelian groups $\mathrm{Hom}(\mathbb{Z}, \mathsf{k}^+) \to \mathrm{Hom}\big(\pi_1(\widetilde{Q}, \tilde{I}), \mathsf{k}^+\big)$. Because of a well-known result of [61], we have a monomorphism

$$\mathrm{Hom}\big(\pi_1(\widetilde{Q}, \tilde{I}), \mathsf{k}^+\big) \to \mathrm{HH}^1(B)\ .$$

Therefore the composed monomorphism $\mathrm{Hom}(\mathbb{Z}, \mathsf{k}^+) \to \mathrm{HH}^1(B)$ gives $\mathrm{HH}^1(B) \neq 0$. Thus $\mathrm{HH}^1(B) = 0$ implies that B is hereditary. Applying [52] we get that $\widetilde{Q}$ is a tree. □

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Brauer Graph Algebras

A Survey on Brauer Graph Algebras, Associated Gentle Algebras and Their Connections to Cluster Theory

Sibylle Schroll

Introduction

Brauer graph algebras originate in the modular representation theory of finite groups where they first appear in the form of Brauer tree algebras in the work of Janusz [53] based on work of Dade [23]. Brauer graph algebras in general, are defined by Donovan and Freislich in [30]. In particular, they classify the indecomposable representations of a Brauer graph algebra in terms of canonical modules of the first and second kind (string and band modules respectively). This classification is based on the work of Ringel [77] on indecomposable representations of dihedral 2-groups and the work of Gel′fand and Ponomarev [42] on indecomposable representations of the Lorentz group.

Brauer graph algebras are defined by combinatorial data based on graphs: Underlying every Brauer graph algebra is a finite graph with a cyclic orientation of the edges at every vertex and a multiplicity function. This combinatorial data encodes much of the representation theory of Brauer graph algebras and is part of the reason for the ongoing interest in this class of algebras.

In [66] the point of view of interpreting Brauer graphs as ribbon graphs has been introduced adding a geometric perspective to the representation theory of Brauer graph algebras and relating Brauer graph algebras with surface cluster theory. The idea of relating Brauer graphs to surfaces was already suggested in [30] where a description of Brauer graphs is given as graphs embedded in oriented surfaces. In [1] the geometric approach based on ribbon graphs has been used to classify two-term tilting complexes over Brauer graph algebras.

The class of Brauer graph algebras coincides with the class of symmetric special biserial algebras [79, 82]. This connection has introduced string combinatorics to the subject as well as a large body of literature on biserial and special biserial algebras, for an overview see for example, the webpages maintained by Julian Külshammer[1] and by Jan Schröer.[2]

[1] http://www.iaz.uni-stuttgart.de/LstAGeoAlg/Kuelshammer/topics/biserial.html

[2] http://www.math.uni-bonn.de/people/schroer/fd-atlas.html

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https://doi.org/10.1007/978-3-319-74585-5_6

In recent years there has been a renewed interest in Brauer graph algebras. In addition to the results mentioned in these lecture notes, there are new results 2-term tilting complexes of Brauer graph algebras [1, 90, 91], Brauer graph algebras associated to partial triangulations [25], coverings of Brauer graphs [47], the finite generation of the Yoneda (or Ext) algebra of a Brauer graph algebra [5, 48] and the generalised Koszulity of Brauer graph algebras [48], as well as new results on the non-periodicity of modules of finite complexity over weakly symmetric Brauer graph algebras [37]. Note also that Brauer graph algebras play a central role in the recent survey on the connection between the representation theory of finite groups and the theory of cluster algebras [64].

On the other hand from the point of view of modular representation theory of finite groups, there has been much recent work to identify specific Brauer trees and Brauer graphs arising in that context, see for example [21, 32–34].

In these lecture notes, after motivating the study of Brauer graph algebras from the perspective of symmetric special biserial algebras in Sect. 1, in Sect. 2 we give a detailed definition of Brauer graph algebras. Section 3 focuses on the connection of Brauer graph algebras with gentle algebras and the connection of Brauer graph algebras with surface cluster theory. In Sect. 4, mutation of Brauer graph algebras and derived equivalences are discussed. Section 5 describes the Auslander–Reiten theory of Brauer graph algebras.

1 Motivation and Connections

1.1 *Special Biserial Algebras*

Let K be an algebraically closed field. A quiver $Q = (Q_0, Q_1, s, t)$ is given by a set of vertices Q_0, a set of arrows Q_1, a function $s\colon Q_1 \to Q_0$ denoting the start of an arrow and $t\colon Q_1 \to Q_0$ denoting the end of an arrow. A path in Q is a sequence of arrows $p = \alpha_0\alpha_1 \cdots \alpha_n$ such that $t(\alpha_i) = s(\alpha_{i+1})$, for all $0 \leq i \leq n-1$. We set $s(p) = s(\alpha_0)$ and $t(p) = t(\alpha_n)$. Furthermore, we denote by $\ell(p)$ the length of p, that is $\ell(p) = n$.

The *path algebra* of a quiver KQ is the algebra whose underlying vector space has a basis given by all possible paths in the quiver Q. This includes a trivial path e_v for every vertex $v \in Q_0$. The multiplication is given by linearly extending the multiplication of two paths p and q in Q, which is given by concatenation if possible and zero otherwise. That is,

$$p \cdot q = \begin{cases} pq & \text{if } t(p) = s(q) \\ 0 & \text{otherwise.} \end{cases}$$

The algebra KQ has been extensively studied and we refer the reader to classical representation theory textbooks, see for example [8, 9, 11, 12, 80]. It follows directly from the definition, that if the quiver Q has a loop or an oriented cycle, the algebra KQ is infinite-dimensional.

Denote by $\mathcal{R}$ the two-sided ideal of KQ generated by the arrows in Q. We call a two-sided ideal I of KQ *admissible* if there exists a strictly positive integer $n \geq 2$ such that

$$\mathcal{R}^n \subseteq I \subseteq \mathcal{R}^2 .$$

If I is an admissible ideal of KQ, then the algebra KQ/I is a finite-dimensional algebra and we say that KQ/I is given by quiver and relations. The algebra KQ/I is indecomposable if and only if the quiver Q is connected.

The motivation behind studying the representation theory of algebras by studying algebras given by quiver and relations is the following theorem due to Gabriel.

Theorem 1.1 ([41]). *Every connected finite-dimensional K-algebra is Morita equivalent to an algebra KQ/I for a unique quiver Q and where I is an admissible ideal of KQ.*

The algebras of the form KQ/I, I admissible, are still arbitrarily complicated and therefore subclasses of these algebras are often considered. For example, to restrict the class of finite-dimensional algebras considered, further restrictions on Q and on I can be imposed.

One example is the restriction of the number of arrows starting and ending at each vertex in the quiver. If at each vertex $v \in Q_0$ there is at most one arrow starting and at most one arrow ending at v, then KQ/I is monomial and of *finite representation type* that is, up to isomorphism, there are only finitely many distinct indecomposable KQ/I-modules. These algebras are the so-called *Nakayama algebras* [8, 9, 67]. An algebra KQ/I is *monomial*, if the ideal I is generated by paths. Note that this definition depends on the presentation of the algebra, that is on the choice of the ideal I. In general, there might be many different ideals J, such that $KQ/I \simeq KQ/J$. It is an open question, raised by Auslander, to find a homological characterisation that implies that a given algebra is monomial. Nevertheless, monomial algebras have been extensively studied and many open problems have been answered in the case of monomial algebras.

The next level of complexity is to allow at most two arrows to begin and end at every vertex in Q:

(S0) At every vertex v in Q there are at most two arrows starting at v and there are at most two arrows ending at v.

While this is a very strong restriction on the algebras one can consider, most algebras whose quiver satisfies condition (S0) are of *wild representation type* (see Sect. 5 for the definition of the representation type of an algebra).

To pose a further restriction on the class of algebras we consider, we set

(S1) For every arrow α in Q there exists at most one arrow β such that $\alpha\beta \notin I$ and there exists at most one arrow γ such that $\gamma\alpha \notin I$.

Almost all algebras satisfying condition (S1) (and not necessarily condition (S0)) are of wild representation type. In fact, algebras satisfying (S1) are called *special multiserial algebras*. They were first defined in [52] and their representation theory has been studied in [43–46, 52].

Together conditions (S0) and (S1) are very strong and the corresponding class of algebras has many special properties.

Definition 1.2. A finite-dimensional K-algebra A is called *special biserial* if there is a quiver Q and an admissible ideal I in KQ such that A is Morita equivalent to KQ/I and such that KQ/I satisfies conditions (S0) and (S1).

Examples of special biserial algebras include the algebras appearing in the work of Gel′fand and Ponomarev [42] on the classification of Harish-Chandra modules over the Lorentz group and they are closely linked to the modular representation theory of finite groups, see for example, Ringel's classification of the indecomposable modules over dihedral 2-groups, see, e.g. [77]. Other examples of special biserial algebras are string algebras (monomial special biserial algebras) see for example [18], discrete derived algebras (classified by Vossieck in [87]), Jacobian algebras of triangulations of marked oriented surfaces with boundary where all marked points lie in the boundary [7] and Brauer graph algebras (see Theorem 2.8).

Special biserial algebras have been intensely studied. We now give a list of some of their most important properties and results.

Theorem 1.3 ([88]). *Special biserial algebras are of tame representation type.*

Wald and Waschbüsch show that special biserial algebras are of tame representation type by classifying their indecomposable representations. These are given by the so-called *string and band modules* (which first appear in [42] as modules of the first and second kind, see also [30, 77] and also [88]). Based on strings and bands a nice combinatorial description of the morphism between string and band modules is given in Crawley-Boevey [22] and Krause [58].

Furthermore, Wald and Waschbüsch [88] and Butler and Ringel [18] give a combinatorial description of the irreducible maps between string modules and between band modules in terms of string combinatorics giving a description of Auslander–Reiten sequences.

Definition 1.4. A finite-dimensional K-algebra A is called *biserial* if every indecomposable projective left and right A-module P is such that $\mathrm{rad}(P) = U + V$, where U and V are uniserial modules and $U \cap V$ is either a simple A-module or zero.

Theorem 1.5 ([86]). *A special biserial algebra is biserial.*

Pogorzały and Skowroński prove a partial converse in case that the algebra is *standard self-injective*, that is if it admits a simply connected Galois covering, see, e.g. [85].

Theorem 1.6 ([70]). *Let A be a representation-infinite self-injective K-algebra. Then A is standard biserial if and only if A is special biserial.*

Recall that a linear form $f : A \mapsto K$ is *symmetric* if $f(ab) = f(ba)$ for all $a, b \in A$ and that a finite-dimensional K-algebra A is *symmetric* if there exists a symmetric linear form $f : A \mapsto K$ such that the kernel of f contains no non-zero left or right ideal. An equivalent formulation of this is that A considered as an A-A-bimodule is isomorphic to its K-linear dual $D(A) = \mathrm{Hom}_K(A, K)$ as an A-A-bimodule. More details on equivalent definitions can be found, for example, in [75].

Every finite-dimensional K-algebra A is a quotient of a symmetric K-algebra, for example, A is a quotient of its trivial extension $T(A)$, where $T(A)$ is given by the semidirect product of A with its minimal injective co-generator $D(A)$ (see Sect. 3 for more details on trivial extensions). However, the class of special biserial algebras is special in that every special biserial algebra is a quotient of a symmetric algebra, such that the symmetric algebra is again special biserial.

Theorem 1.7 ([88]). *Every special biserial algebra is a quotient of a symmetric special biserial algebra.*

Symmetric special biserial algebras are well understood and are ubiquitous in the modular representation theory of finite groups where they have appeared under a different name, that of Brauer graph algebras. While we will define Brauer graph algebras in Sect. 2, we state the connection of special biserial algebras and Brauer graph algebras here.

Theorem 1.8 ([79, 82]). *Let KQ/I be a finite-dimensional K-algebra. Then KQ/I is a symmetric special biserial algebra if and only if KQ/I is a Brauer graph algebra.*

This theorem was proved in [79] for the case that the algebra KQ/I is such that the quiver Q contains no parallel arrows. In [82] it was proved that the result holds for all quivers. The following directly follows from Theorems 1.7 and 1.8:

Corollary 1.9. *Every special biserial algebra is a quotient of a Brauer graph algebra.*

2 Brauer Graph Algebras

Recall from the introduction that Brauer graph algebras have their origin in the modular representation theory of finite groups. We will start by briefly recalling how they appear in this context.

2.1 *Brauer Graph Algebras and Modular Representation Theory of Finite Groups*

Let G be a finite group. Suppose that the characteristic of K is equal to some prime number p and suppose that p divides the order of G. Then it follows from Maschke's Theorem that the group algebra KG is not semi-simple. Instead it decomposes into a direct sum of indecomposable two-sided ideals B_i such that $KG = B_1 \oplus \cdots \oplus B_n$. The identity element e of G decomposes as $e = e_1 + \cdots + e_n$ where the e_i are orthogonal central idempotents of KG such that $e_i \in B_i$. Each B_i is called a *p-block of G* (or of KG) and it is a symmetric finite-dimensional K-algebra with identity element e_i. Each p-block has an associated invariant in the form of a p-group D_i called the *defect group of B_i*. Dade showed that if the defect group D of a p-block B of some finite group G is cyclic then B is a Brauer tree algebra [23] and Donovan showed that if the characteristic K is two and if D is a dihedral 2-group then B is a Brauer graph algebra [29]. We note that the blocks with cyclic defect groups are precisely the p-blocks of G which are of finite representation type.

2.2 *Definition of Brauer Graphs*

Definition 2.1. A *Brauer graph* G is a tuple $G = (G_0, G_1, m, \mathfrak{o})$ where

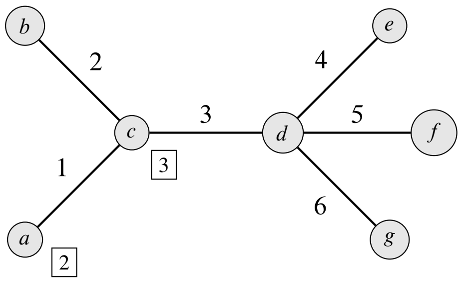
- (G_0, G_1) is a finite (unoriented) connected graph with vertex set G_0 and edge set G_1,
- $m: G_0 \to \mathbb{Z}_{>0}$ is a function, called the *multiplicity* or *multiplicity function* of G,
- $\mathfrak{o}$ is called the *orientation* of G and is given, for every vertex $v \in G_0$, by a cyclic ordering of the edges incident with v such that if v is a vertex incident to a single edge i then if $m(v) = 1$, the cyclic ordering at v is given by i and if $m(v) > 1$ the cyclic ordering at v is given by $i < i$.

A *Brauer tree* is a Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$ such that (G_0, G_1) is a tree and $m(v) = 1$, for all but at most one $v \in G_0$.

We note that the graph (G_0, G_1) which (by abuse of notation) we will also simply refer to as G may contain loops and multiple edges. Denote by $\mathrm{val}(v)$ the *valency* of the vertex $v \in G_0$. It is defined to be the number of edges in G incident to v, with the convention that a loop is counted twice (see Example 2.2(2)). Equivalently we can define $\mathrm{val}(v)$ to be the number of half-edges incident with vertex v. We call the edge i with vertex v *truncated at* v if $m(v)\,\mathrm{val}(v) = 1$. Note that if i is truncated at both vertices v and w, that is if both $m(v)\,\mathrm{val}(v) = 1$ and $m(w)\,\mathrm{val}(w) = 1$ then G is the Brauer graph given by a single edge with both vertices v and w of multiplicity 1 and the corresponding Brauer graph algebra is defined to be $k[x]/(x^2)$.

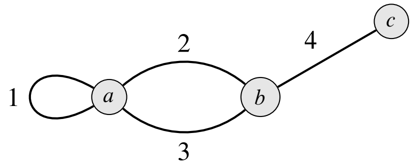
Examples 2.2. In all four examples below, the orientation of the Brauer graph is given by locally embedding each vertex of the Brauer graph into the clockwise oriented plane. If for some $v \in G_0$, we have $m(v) > 1$, then we record the value of $m(v)$ in a square box next to the corresponding vertex on the graph.

(1) This is an example of what is called a *generalised Brauer tree*, that is the underlying graph is a tree with at least two vertices with multiplicity greater than one. For example, let $H_1 = (G_0, G_1, m, \mathfrak{o})$ be given by



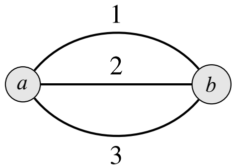
We set $m(i) = 1$ for all $i \in G_0, i \neq a, c$ and $m(a) = 2$ and $m(c) = 3$. The cyclic ordering of the edges incident with vertex a is given by $1 < 1$, with vertex b it is given by 2, with vertex c it is given by $1 < 2 < 3 < 1$, with vertex d it is given by $3 < 4 < 5 < 6 < 3$, etc. We have the following valencies for vertices of H_1: $\mathrm{val}(a) = \mathrm{val}(b) = \mathrm{val}(e) = \mathrm{val}(f) = \mathrm{val}(g) = 1$, $\mathrm{val}(c) = 3$ and $\mathrm{val}(d) = 4$.

(2) Let the Brauer graph $H_2 = (G_0, G_1, m, \mathfrak{o})$ be given by



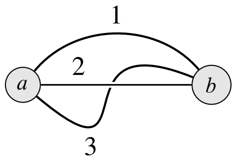
We set $m(i) = 1$ for $i = a, b, c$. The cyclic ordering of the edges incident with vertex a is given by $1 < 1 < 2 < 3 < 1$, with vertex b is given by $2 < 4 < 3 < 2$ and with c is given by 4. We have $\text{val}(a) = 4$, $\text{val}(b) = 3$ and $\text{val}(c) = 1$. Note that the edge 4 is a truncated edge since the vertex c is such that $m(c)\,\text{val}(c) = 1$.

(3) Let the Brauer graph $H_3 = (G_0, G_1, m, \mathfrak{o})$ be given by



We set $m(a) = m(b) = 1$. The cyclic ordering at vertex a is given by $1 < 2 < 3 < 1$ and at vertex b by $1 < 3 < 2 < 1$ and $\text{val}(a) = \text{val}(b) = 3$.

(4) Let the Brauer graph $H_4 = (G_0, G_1, m, \mathfrak{o})$ be given by



We set $m(a) = m(b) = 1$. The cyclic ordering at vertex a is given by $1 < 2 < 3 < 1$ and at vertex b by $1 < 2 < 3 < 1$ and $\text{val}(a) = \text{val}(b) = 3$.

2.3 *Quivers from Brauer Graphs*

Let a be a vertex of some Brauer graph G, and let $i_1, i_2, \ldots, i_n$ be the edges in G incident with a (note that we might have $i_j = i_k$, for some j, k, if the corresponding edge is a loop). Suppose that the cyclic ordering at a is given by $i_1 < i_2 < \cdots < i_n < i_1$. We call $i_1, i_2, \ldots, i_n$ the *successor* sequence at a and i_{j+1} is the successor of i_j, for $1 \leq j \leq n-1$ and i_1 is the successor of i_n, and i_{k-1} is the *predecessor* of i_k for $2 \leq k \leq n$ and i_n is the predecessor of i_1.

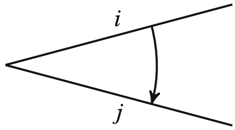


Fig. 1 Arrow α defined by the successor relation (i, j)

Note that if v is a vertex at edge i with $\text{val}(v) = 1$ and if $m(v) > 1$ then $i > i$ and the successor (and predecessor) of i is i, if $m(v) = 1$ then i does not have a successor or predecessor.

In order to specify more precisely the cyclic ordering at a given vertex, especially if there is a loop at that vertex, half-edges are often used. In particular, the language of *ribbon graphs*, see Sect. 2.7, introduced to Brauer graph algebras in [66], has the advantage of making the notion of half-edge and their cyclic orderings very precise.

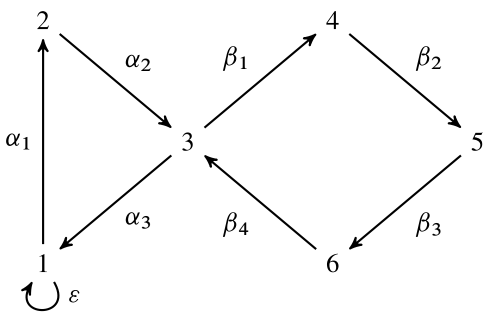
Given a Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$ we define a quiver $Q_G = (Q_0, Q_1)$ in the following way. The set of vertices Q_0 is given by the set of edges G_1 of G, denoting the vertex in Q_0 corresponding to the edge i in G_1 also by i. The arrows in Q are induced by the orientation $\mathfrak{o}$. More precisely, let i and j be two edges in G_0 incident with a common vertex v and such that j is a direct successor of i in the cyclic ordering of the edges at v. Then there is an arrow $\alpha: i \to j$ in Q_G.

We say that α is given by the *successor relation* (i, j). Since every arrow of Q_G starts and ends at an edge of G, there are at most two arrows starting and ending at every vertex of Q_G. Every vertex $v \in G_0$ such that $m(v)\,\text{val}(v) \geq 2$, gives rise to an oriented cycle C_v in Q_G, which is unique up to cyclic permutation. We call C_v a *special cycle at* v. Let C_v be such a special cycle at v. Then if C_v is a representative in its cyclic permutation class such that $s(C_v) = i = t(C_v)$, $i \in Q_0$, we say that C_v is a special i*-cycle at* v. To simplify notation we will simply write C_v for the special i-cycle at v and specify if necessary that $s(C_v) = i$ or if more detail is needed, we will specify the first or last arrow in C_v.

For $i \in Q_0$ and $v \in G_0$, a special i-cycle at v is not necessarily unique, however, there are at most two special i-cycles at v for any $i \in G_1$ and $v \in G_0$ and this happens exactly when i is a loop. An example of this is given in Example 2.3(2).

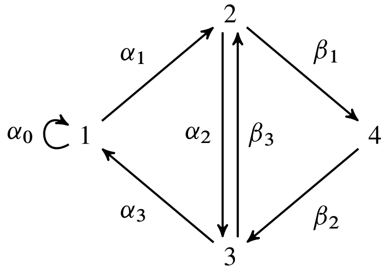
Examples 2.3. In this example we give the quivers of the Brauer graphs in Example 2.2 and list some of the special cycles.

(1) Quiver Q_{H_1} of H_1.



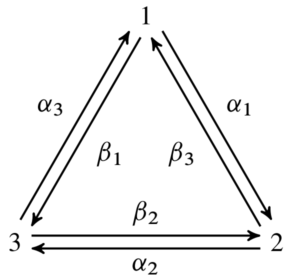
The special cycles in Q_{H_1} are given by $C_a = \epsilon$, the special 1-cycle at c given by $\alpha_1\alpha_2\alpha_3$, the special 2-cycle at c given by $\alpha_2\alpha_3\alpha_1$, the special 3-cycle at c given by $\alpha_3\alpha_1\alpha_2$, and the special 3-cycle at d given by $\beta_1\beta_2\beta_3\beta_4$, etc.

(2) Quiver Q_{H_2} of H_2.



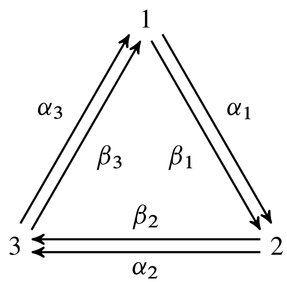
The special cycles in Q_{H_2} are the special 1-cycles at a corresponding to $\alpha_0\alpha_1\alpha_2\alpha_3$, and $\alpha_1\alpha_2\alpha_3\alpha_0$, the special 2-cycle at a given by $\alpha_2\alpha_3\alpha_0\alpha_1$, the special 2-cycle at b given by $\beta_1\beta_2\beta_3$, etc. Note that there are two distinct 1-cycles at the vertex a, namely $\alpha_0\alpha_1\alpha_2\alpha_3$ and $\alpha_1\alpha_2\alpha_3\alpha_0$.

(3) Quiver Q_{H_3} of H_3.



The special 1-cycle at a is given by $\alpha_1\alpha_2\alpha_3$, the special 1-cycle at b is given by $\beta_1\beta_2\beta_3$, etc.

(4) Quiver Q_{H_4} of H_4.



The special 1-cycle at a is given by $\alpha_1\alpha_2\alpha_3$, the special 1-cycle at b is given by $\beta_1\beta_2\beta_3$, etc.

2.4 *Set of Relations and Definition of Brauer Graph Algebras*

We define an ideal of relations I_G in KQ_G generated by three types of relations. For this recall that we identify the set of edges G_1 of a Brauer graph G with the set of vertices Q_0 of the corresponding quiver Q_G and that we denote the set of vertices of the Brauer graph by G_0.

Relations of type I:

$$C_v^{m(v)} - C_{v'}^{m(v')} ,$$

for any $i \in Q_0$ and for any special i-cycles C_v and $C_{v'}$ at v and v' such that both v and v' are not truncated (i.e. $\mathrm{val}(v)m(v) \neq 1$ and $\mathrm{val}(v')m(v') \neq 1$).

Relations of type II:

$$C_v^{m(v)}\alpha_1 ,$$

for any $i \in Q_0$, any $v \in G_0$ and where $C_v = \alpha_1\alpha_2\cdots\alpha_n$ is any special i-cycle C_v.

Relations of type III:

$$\alpha\beta ,$$

for any $\alpha, \beta \in Q_1$ such that $\alpha\beta$ is not a subpath of any special cycle except if $\alpha = \beta$ is a loop associated to a vertex v of valency one and multiplicity $m(v) > 1$.

The algebra $A_G = KQ_G/I_G$ is called the *Brauer graph algebra* associated to the Brauer graph G.

We note that the relations generating I_G do not constitute a minimal set of relations. Many of the relations, in particular many of the relations of type II, are redundant. In [48] for every Brauer graph algebra a minimal set of relations is determined: The relations of type I and III are always minimal and the only relations of type II appearing in a minimal generating set of relations are those corresponding to an edge i with vertices v and w, such that i is truncated at vertex v and such that the immediate successor of i in the cyclic ordering at w is also truncated at its other endpoint.

Examples 2.4. Sets of relations for the examples in Example 2.2.

(1) Set of relations of the three types of the Brauer graph algebra $B_1 = KQ_{H_1}/I_{H_1}$:

Type I: $(\alpha_1\alpha_2\alpha_3)^3 - \epsilon^2, (\alpha_3\alpha_1\alpha_2)^3 - \beta_1\beta_2\beta_3\beta_4$
Type II: $(\alpha_1\alpha_2\alpha_3)^3\alpha_1, (\alpha_2\alpha_3\alpha_1)^3\alpha_2, (\alpha_3\alpha_1\alpha_2)^3\alpha_3, \beta_1\beta_2\beta_3\beta_4\beta_1, \beta_2\beta_3\beta_4\beta_1\beta_2,$ $\beta_3\beta_4\beta_1\beta_2\beta_3, \beta_4\beta_1\beta_2\beta_3\beta_4, \epsilon^3$
Type III: $\epsilon\alpha_1, \alpha_3\epsilon, \alpha_2\beta_1, \beta_4\alpha_3$

A minimal set of relations is given by all relations of types I and III and the relations $\beta_2\beta_3\beta_4\beta_1\beta_2$ and $\beta_3\beta_4\beta_1\beta_2\beta_3$.
In the following, by abuse of notation, we will abbreviate the relations of type II as follows, for $\beta_i\beta_{i+1}\beta_{i+2}\beta_{i+3}\beta_{i+4}$ we write β^5, etc.

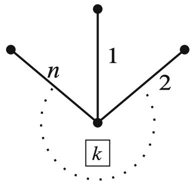


Fig. 2 Brauer graph of a symmetric Nakayama algebra with n arrows $\alpha_1, \ldots, \alpha_n$ and relations of the form $(\alpha_{i_1} \ldots \alpha_{i_n})^k \alpha_{i_1}$

(2) Set of relations of the three types of the Brauer graph algebra $B_2 = KQ_{H_2}/I_{H_2}$:

Type I: $\alpha_0\alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2\alpha_3\alpha_0, \alpha_2\alpha_3\alpha_0\alpha_1 - \beta_1\beta_2\beta_3, \alpha_3\alpha_0\alpha_1\alpha_2 - \beta_3\beta_1\beta_2$
Type II: α^5, β^4
Type III: $\alpha_0^2, \alpha_1\beta_1, \alpha_2\beta_3, \alpha_3\alpha_1, \beta_2\alpha_3, \beta_3\alpha_2$

A minimal set of relations is given by all relations of types I and III.

(3) Set of relations of the three types of the Brauer graph algebra $B_3 = KQ_{H_3}/I_{H_3}$:

Type I: $\alpha_1\alpha_2\alpha_3 - \beta_1\beta_2\beta_3, \alpha_2\alpha_3\alpha_1 - \beta_3\beta_1\beta_2, \alpha_3\alpha_1\alpha_2 - \beta_2\beta_3\beta_1$
Type II: α^4, β^4
Type III: $\alpha_1\beta_3, \beta_3\alpha_1, \alpha_2\beta_2, \beta_2\alpha_2, \alpha_3\beta_1, \beta_1\alpha_3$

A minimal set of relations is given by all relations of types I and III.

(4) Set of relations of the three types of the Brauer graph algebra $B_4 = KQ_{H_4}/I_{H_4}$:

Type I: $\alpha_1\alpha_2\alpha_3 - \beta_1\beta_2\beta_3, \alpha_2\alpha_3\alpha_1 - \beta_2\beta_3\beta_1, \alpha_3\alpha_1\alpha_2 - \beta_3\beta_1\beta_2$
Type II: α^4, β^4
Type III: $\alpha_i\beta_{i+1}, \beta_i\alpha_{i+1}$ for $i = 1, 2$ and $\alpha_3\beta_1, \beta_3\alpha_1$

A minimal set of relations is given by all relations of types I and III.

Remark 2.5. Any symmetric Nakayama algebra is a Brauer graph algebra. Let Q be the quiver given by a non-zero cycle $\alpha_1\alpha_2 \cdots \alpha_n$ with $s(\alpha_1) = t(\alpha_n)$. Let $I = (\alpha^{kn+1})$ for any $k \in \mathbb{Z}_{>0}$. Then $A = KQ/I$ is a Brauer tree algebra with Brauer tree T given by a star with n edges and multiplicity 1 everywhere, except for the central vertex whose multiplicity is equal to k, see Fig. 2.

2.5 *First Properties of Brauer Graph Algebras*

Theorem 2.6. *Given a Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$, the associated Brauer graph algebra $A_G = K_G/I_G$ is a finite-dimensional symmetric algebra.*

Proof. We start by showing that I_G is admissible. By the definition of a special cycle C_v, we have that either $\ell(C_v) > 1$ or $m(v) > 1$. Thus $I_G \subset KQ^2$. It follows from the relations of type I, II and III that, for $N = \max_{v \in G_0} \ell(C_v^{m(v)})$, any path p in Q with $\ell(p) \geq N + 1$ is such that $p \in I_G$.

To show that A_G is symmetric, define the following symmetric linear function $f : A_G \to K$ by setting

$$f(p) = \begin{cases} 1 & \text{if } p = C_v^{m(v)} \text{ for some } v \in G_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then it has no non-zero left-ideal in the kernel and thus A_G is symmetric. □

Remark 2.7. A Brauer graph algebra is indecomposable if and only if its Brauer graph is connected as a graph.

The following theorem is well-known. It also follows from recent work on special multiserial and Brauer configuration algebras in [44, 45].

Theorem 2.8. *Brauer graph algebras are special biserial.*

Corollary 2.9. *Brauer graph algebras are of tame representation type. A Brauer graph algebra is of finite representation type if and only if it is a Brauer tree algebra.*

Remark 2.10. It is possible to introduce scalars in the relations of a Brauer graph algebra. In this case, the resulting Brauer graph algebra is no longer necessarily symmetric, but instead it is weakly symmetric. The case of a weakly symmetric Brauer graph algebra is treated in detail, for example, in [47] and [48]. We note that many interesting examples, such as the existence of non-periodic modules with complexity one appear precisely in the weakly symmetric (non-symmetric) case, see [37]. However, in these notes we will focus on the case of symmetric Brauer graph algebras and refer for a rigorous definition of weakly symmetric Brauer graph algebras to [47].

2.6 *Indecomposable Projective Modules of Brauer Graph Algebras*

We now list the Loewy series of the projective indecomposable modules of the Brauer graph algebras in the examples above. Since Brauer graph algebras are special biserial, they are biserial and the composition factors of the maximal uniserial submodules of the indecomposable projectives can conveniently be read off the Brauer graph. In the following the projective indecomposable module at vertex i of Q_0 will be denoted by P_i and we denote by i the simple module at vertex i.

Examples 2.11. Let $B_1, \ldots, B_4$ be the Brauer graph algebras with Brauer graphs $H_1, \ldots, H_4$ respectively, given in Example 2.2. Then indecomposable projective modules are given by

(1) Indecomposable projective modules over B_1

$$P_1 : \begin{array}{ccc} & 1 & \\ & & 2 \\ & & 3 \\ & & 1 \\ 1 & & 2 \\ & & 3 \\ & & 1 \\ & & 2 \\ & & 3 \\ & 1 & \end{array} \qquad P_2 : \begin{array}{c} 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{array} \qquad P_3 : \begin{array}{ccc} & 3 & \\ 1 & & \\ 2 & & \\ 3 & & 4 \\ 1 & & 5 \\ 2 & & 6 \\ 3 & & \\ 1 & & \\ 2 & & \\ & 3 & \end{array} \qquad P_4 : \begin{array}{c} 4 \\ 5 \\ 6 \\ 3 \\ 4 \end{array} \qquad P_5 : \begin{array}{c} 5 \\ 6 \\ 3 \\ 4 \\ 5 \end{array} \qquad P_6 : \begin{array}{c} 6 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

(2) Indecomposable projective modules over B_2

$$P_1: \begin{array}{ccc} & 1 & \\ \begin{array}{c}1\\2\\3\end{array} & & \begin{array}{c}2\\3\\1\end{array} \\ & 1 & \end{array} \qquad P_2: \begin{array}{ccc} & 2 & \\ \begin{array}{c}3\\1\\1\end{array} & & \begin{array}{c}4\\3\end{array} \\ & 2 & \end{array} \qquad P_3: \begin{array}{ccc} & 3 & \\ \begin{array}{c}1\\1\\2\end{array} & & \begin{array}{c}2\\4\end{array} \\ & 3 & \end{array} \qquad P_4: \begin{array}{c}4\\3\\2\\4\end{array}$$

(3) Indecomposable projective modules over B_3

$$P_1: \begin{array}{ccc} & 1 & \\ \begin{array}{c}2\\3\end{array} & & \begin{array}{c}3\\2\end{array} \\ & 1 & \end{array} \qquad P_2: \begin{array}{ccc} & 2 & \\ \begin{array}{c}3\\1\end{array} & & \begin{array}{c}1\\3\end{array} \\ & 2 & \end{array} \qquad P_3: \begin{array}{ccc} & 3 & \\ \begin{array}{c}1\\2\end{array} & & \begin{array}{c}2\\1\end{array} \\ & 3 & \end{array}$$

(4) Indecomposable projective modules over B_1

$$P_1: \begin{array}{ccc} & 1 & \\ \begin{array}{c}2\\3\end{array} & & \begin{array}{c}2\\3\end{array} \\ & 1 & \end{array} \qquad P_2: \begin{array}{ccc} & 2 & \\ \begin{array}{c}3\\1\end{array} & & \begin{array}{c}3\\1\end{array} \\ & 2 & \end{array} \qquad P_3: \begin{array}{ccc} & 3 & \\ \begin{array}{c}1\\2\end{array} & & \begin{array}{c}1\\2\end{array} \\ & 3 & \end{array}$$

2.7 *Brauer Graphs as Ribbon Graphs*

In [66] the point of view of interpreting a Brauer graph as a ribbon graph has been introduced. A ribbon graph uniquely defines a compact oriented surface and connects the representation theory of Brauer graph algebras to combinatorial surface geometry. We recall the details of this construction. More details on ribbon graphs and associated surfaces, as well as the proofs of the below statements can be found, for example, in [61].

In order to define ribbon graphs, we start by giving a more formal definition of a graph.

A *graph* is a tuple $\Gamma = (V, E, s, \iota)$ where

(1) V is a finite set, called the set of vertices,
(2) E is finite set, called the set of half-edges
(3) $s: E \to V$ is a function,
(4) $\iota: E \to E$ is an involution without fixed points.

We think of s as the function sending a half-edge to the vertex it is incident with and for an edge that is not a loop, we think of ι as the function sending a half-edge incident with a vertex to the half-edge incident with the other vertex of the same edge. In the case of a loop, ι sends a half-edge incident with the only vertex of the corresponding edge to the other half-edge of the same edge. That is, a loop in G corresponds to a pair of half-edges $e, e' \in E$ such that $\iota(e) = e'$ and $s(e) = s(\iota(e'))$.

A *ribbon graph* is a graph $\Gamma = (V, E, s, \iota)$ together with a permutation $\sigma: E \to E$ such that the cycles of σ correspond to the sets $s^{-1}(v)$, for all $v \in V$. In other words a ribbon graph is a graph together with, for every vertex $v \in V$, a cyclic ordering of the half-edges incident with v.

Remark 2.12. (1) It follows from the above definitions that a Brauer graph can be regarded as a (vertex) weighted ribbon graph where the weights are given by the multiplicity function.

(2) The language of ribbon graphs in the context of Brauer graph algebras is particularly useful in the treatment of loops in the Brauer graph.

Examples of ribbon graphs are planar graphs and graphs locally embedded in the (oriented) plane such as the Brauer graphs in Example 2.2. Note that as abstract graphs G_3 and G_4 in Example 2.2 are isomorphic, but that as ribbon graphs they are not isomorphic.

Any embedding of a graph into an oriented surface gives a ribbon graph structure on the graph, where the cyclic orderings are induced from the embedding of the edges around each vertex and the orientation of the surface.

On the other hand, we obtain surfaces from ribbon graphs. For this we first define the geometric realisation of a ribbon graph and then we embed it into an open oriented surface.

The *geometric realisation* $|\Gamma|$ of a ribbon graph Γ is the topological space

$$|\Gamma| = (E \times [0,1])/\sim$$

where $\sim$ is the equivalence relation defined by $(e,t) \sim (\iota(e), 1-t)$ for all $t \in [0,1]$ and $(e,0) \sim (f,0)$ if $s(e) = f(e)$ and $(e,1) \sim (f,1)$ if $s\big(\iota(e)\big) = f\big(\iota(e)\big)$, for all $e, f, \in E$.

Lemma 2.13 ([61, **2.2.4]**). *Every ribbon graph Γ can be embedded in an open oriented surface with boundary in such a way that the cyclic orderings around each of its vertices arise from the orientation of the surface.*

An example of such a surface is the *ribbon surface* S_Γ° of Γ which is constructed in the following way: Firstly, an oriented surface is associated to each vertex and edge of Γ as in Fig. 3. Then the structure and orientation of the ribbon graph determine a gluing of the corresponding surfaces giving an oriented surface S_Γ° together with an embedding of Γ into S_Γ°.

Note that S_Γ° has a number of boundary components corresponding to the faces of Γ. A *face* of Γ is an equivalence class, up to cyclic permutation, of n-tuples $(e_1, e_2, \ldots, e_n)$ of half-edges satisfying

$$e_{p+1} = \begin{cases} \iota(e_p) & \text{if } s(e_p) = s(e_{p-1}) \\ \sigma(e_p) & \text{if } s(e_p) \neq s(e_{p-1}) \end{cases}$$

for all p with $1 \leq p \leq n$ (where subscripts are taken modulo n).

Now let S be a compact oriented surface. An embedding $\Gamma \hookrightarrow S$ is said to be a *filling embedding* if $S \setminus |\Gamma| = \bigsqcup_{f \in F} D_f$, where each D_f is a disc, and F is a finite set. That is, the complement of the embedding is a disjoint union of finitely many discs.

Proposition 2.14 ([61, **2.2.7]**). *Every ribbon graph has a filling embedding into a compact oriented surface such that the connected components of $S \setminus |\Gamma|$ are in bijection with the faces of Γ in the above sense.*

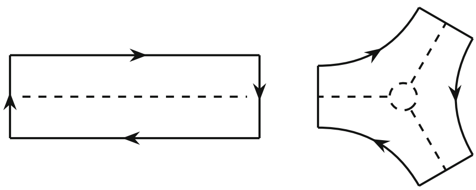


Fig. 3 Oriented surfaces corresponding to vertices and edges in Γ

This is proved by gluing discs onto the ribbon surface of the ribbon graph to fill in the boundary components. Such an embedding has the following uniqueness property. We first recall that morphisms and isomorphisms of ribbon graphs are defined in the natural way, that is, they are maps of graphs respecting the permutations.

Proposition 2.15 ([61, **2.2.10]**). *Let $\Gamma \to S$ and $\Gamma' \to S'$ be filling embeddings of ribbon graphs Γ and Γ' into compact oriented surfaces S and S' and let $f : \Gamma \to \Gamma'$ be an isomorphism of ribbon graphs. Then f induces an orientation-preserving homeomorphism $f : |\Gamma| \to |\Gamma'|$ extending to a homeomorphism from S to S'.*

Corollary 2.16. *If Γ is a ribbon graph, then there is a compact oriented surface S_Γ together with a filling embedding $\Gamma \to S_\Gamma$, unique up to homeomorphism.*

Thus we see that there is a filling embedding of an arbitrary Brauer graph (without, or ignoring, multiplicity) into a compact oriented surface, unique up to homeomorphism, in such a way that the cyclic ordering around each vertex arises from the orientation of the surface. Conversely, we have the following:

Proposition 2.17 ([61, **2.2.12]**). *Every compact oriented surface admits a filling ribbon graph.*

Remark 2.18. (1) Note that in [5] there also is a surface associated to a Brauer graph. This surface arises from a CW-complex associated to the corresponding Brauer graph algebra. Comparing the definitions, it can be seen that this gives rise to the same surface as the ribbon graph construction in (Corollary 2.16) above. In particular, the G-cycles in [5, Sect. 2] correspond to the faces of the Brauer graph as a ribbon graph.

(2) In Sects. 3 and 4 (ideal) triangulations of marked oriented surfaces play an important role (we refer the reader to Sect. 3.3 or to [40] for the definition of an ideal triangulation), these are connected to ribbon graphs in the following way. Let T be a triangulation of a marked oriented surface (S, M) where S might have a boundary and punctures. Then T (including the boundary edges) can be considered as a ribbon graph with boundary where the cyclic ordering of the edges around each vertex is induced by the orientation of S and where the boundary of S induces the set of boundary faces of T. Then there is an embedding $T \hookrightarrow S$ mapping the boundary faces to the boundaries of the boundary components, for more details see for example [66, Sect. 2.2].

3 Brauer Graph Algebras and Gentle Algebras

In this section we give an explicit construction to show how gentle algebras and Brauer graph algebras are related via trivial extensions. We do this by constructing a ribbon graph for every gentle algebra.

Definition 3.1. A finite-dimensional algebra is *gentle* if it is Morita equivalent to a special biserial algebra $A = KQ/I$, that is, (S0) and (S1), hold as well as

(S2) For every arrow α in Q there exists at most one arrow β such that $\alpha\beta \in I$ where $t(\alpha) = s(\beta)$ and there exists at most one arrow γ such that $\gamma\alpha \in I$ where $t(\gamma) = s(\alpha)$.

(S3) The ideal I is generated by paths of length 2.

Example 3.2. Examples of gentle algebras:

(1) Tilted algebras of type A, see for example, [8, Chap. VIII] for a definition of tilted algebras.
(2) Gentle algebras arising as Jacobian algebras associated to marked oriented surfaces where all marked points lie in the boundary (see [7, 60]).
(3) Discrete derived algebras as classified in [87].

The *trivial extension* $T(A) = A \ltimes D(A)$ of an algebra A by its injective co-generator $D(A) = \mathrm{Hom}_K(A, K)$ is the algebra whose underlying K-vector space is given by $A \oplus D(A)$ and where the multiplication is given as follows:

$$(a, f)(b, g) = (ab, ag + fb) \quad \text{for all } a, b \in A \text{ and } g, f \in D(A)\ .$$

The trivial extension $T(A)$ is symmetric (see for example [80]).

The following result shows the connection between gentle algebras and Brauer graph algebras.

Theorem 3.3 ([70])**.** *Let A be a finite-dimensional K-algebra. Then A is gentle if and only if $T(A)$ is special biserial.*

Therefore if A is gentle, its trivial extension $T(A)$ is a symmetric special biserial algebra, that is, it is a Brauer graph algebra.

Question 3.4. Given a gentle algebra, what is the Brauer graph of its trivial extension?

We will answer this question in Theorem 3.7.

3.1 *Graph of a Gentle Algebra*

In [82], to any gentle algebra A is associated a ribbon graph Γ_A whose underlying graph structure is determined by the set of maximal paths in A. The ribbon graph structure in the form of the cyclic orderings of the edges around the vertices is also induced by the maximal paths.

Let $A = KQ/I$ and $\mathcal{M}$ be the set of maximal paths in A. That is $\mathcal{M}$ consists of all images in KQ/I of paths $p \in KQ$ such that $p \notin I$, but $\alpha p \in I$ and $p\alpha \in I$, for all non-trivial $\alpha \in KQ$.

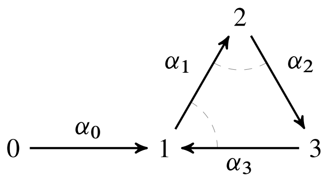
Set $\mathcal{M}_0 = \{e_v, v \in V_0\}$ where V_0 is the set of vertices $v \in Q_0$ such that one of the following holds:

(1) There exist unique arrows α and β such that $t(\alpha) = v = s(\beta)$ and, furthermore, $\alpha\beta \notin I$.

(2) v is a source and there is a single arrow starting at v.
(3) v is a sink and there is a single arrow ending at v.

Set $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}_0$.

Example 3.5. Let A be the gentle algebra with quiver



(The dashed lines indicate the relations of length 2.) and relations $\alpha_1\alpha_2$ and $\alpha_3\alpha_1$.
Then $V_0 = \{0, 3\}$ and $\overline{\mathcal{M}} = \{e_0, \alpha_0\alpha_1, \alpha_2\alpha_3, e_3\}$.

Construction of the Ribbon Graph Γ_A of a Gentle Algebra

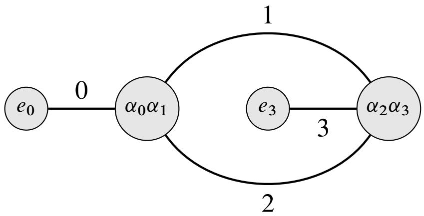
Let $A = KQ/I$ be a gentle algebra. The vertices of Γ_A are in bijection with the elements in $\overline{\mathcal{M}}$ and the edges of Γ_A are in bijection with the vertices of Q_0. Furthermore, there exists an edge E_v for $v \in Q_0$ between the vertices $m, m' \in \overline{\mathcal{M}}$ if v as a vertex in Q_0 lies in both the paths m and m'. That is if there exists paths p, q, p', q' such that $m = pe_vq$ and $m' = p'e_vq$ where e_v is the trivial path at v and where p, q, p', q' might also possibly be trivial paths at v.

We note that every vertex v in Q_0 lies in exactly two elements of $\overline{\mathcal{M}}$ which are not necessarily distinct. If they are not distinct the corresponding edge E_v in Γ_A is a loop.

Note that if a vertex in Γ_A corresponds to a trivial path in Q then as a vertex in Γ_A it has valency 1, that is, there is only one edge of Γ_A incident with that vertex. In all other cases there are at least two edges incident with every vertex of Γ_A.

Let v be a vertex of Γ_A such that there are at least two edges of Γ_A incident at v and let $a_1 \xrightarrow{\alpha_1} a_2 \xrightarrow{\alpha_2} \cdots a_{k-1} \xrightarrow{\alpha_{k-1}} \longrightarrow a_k$ be the element in $\overline{\mathcal{M}}$ corresponding to v. Let $E_1, E_2, \ldots, E_k$ be the edges in Γ_A corresponding to the vertices $a_1, a_2, \ldots, a_k$ respectively. Note that they all have the vertex v in common. Then we linearly order $E_1, E_2, \ldots, E_k$ as follows: $E_1 < E_2 < \cdots < E_k$. We complete this linear order to a cyclic order by adding $E_k < E_1$ and we do this at every vertex of Γ_A. This equips the graph Γ_A with the structure of a ribbon graph.

Example 3.6. For the algebra in Example 3.5 above, we get the following ribbon graph.



The cyclic ordering here corresponds to the ordering given by the embedding into the clockwise oriented plane induced by the maximal paths containing the given vertex. That is at vertex $\alpha_0\alpha_1$ we have cyclic ordering $0 < 1 < 2 < 0$, at vertex $\alpha_2\alpha_3$ we have cyclic ordering $1 < 2 < 3 < 1$, at vertex e_0 it is 0 and at vertex e_3 it is 3.

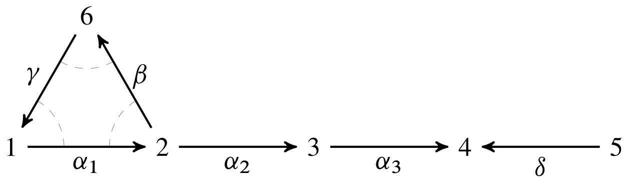
Theorem 3.7 ([82]). *Let $A = KQ/I$ be a gentle algebra with ribbon graph Γ_A. Then $T(A)$ is the Brauer graph algebra with Brauer graph Γ_A (and with multiplicity function identically equal to one).*

There are many situations when there are naturally associated graphs to gentle algebras, given for example by triangulations or partial triangulations of marked unpunctured surfaces. These graphs often coincide with the ribbon graphs defined above. More precisely, using the admissible cuts in the next section we get the following corollary to Theorem 3.7.

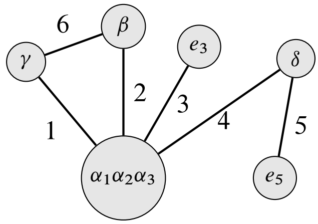
Corollary 3.8. (1) *If A is a gentle algebra arising from a triangulation T of an oriented marked surface with marked points in the boundary as defined in* [7, 60] *then $\Gamma_A = T$ as ribbon graphs.*
(2) *If A is a surface algebra as defined in* [24] *of a partial triangulation T of an oriented marked surface with marked points in the boundary then $\Gamma_A = T$ as ribbon graphs.*
(3) *If A is a tiling algebra as defined in* [20] *with partial triangulation T then $\Gamma_A = T$.*

We will give three more examples of the graphs of gentle algebras. In particular, the first two examples illustrate each of the statements in Corollary 3.8(1) and (3), respectively. In all three examples the orientation of the graph of the corresponding gentle algebra (induced by the maximal paths) is given by the clockwise orientation of the plane.

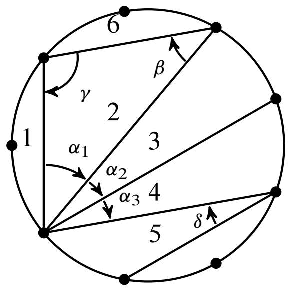
Example 3.9 (*Example illustrating Corollary* 3.8(1)). Let $A = KQ/(\alpha_1\beta, \beta\gamma, \gamma\alpha_1)$ be the gentle algebra with quiver Q given by



Then the graph Γ_A of A is given by



The graph Γ_A coincides with the following triangulation of the 9-gon and the gentle algebra A corresponds to the associated Jacobian algebra arising from the associated quiver with potential [7, 60].



Example 3.10 (*Example illustrating Corollary* 3.8(3)). Example of a tiling algebra as defined in [20] (and using the notation of [20]). In Fig. 4 is the algebra A_{10} corresponding to a tiling of $P_{10,1}$ as in Fig. 6. The algebra A_{10} is defined to be the algebra KQ_{10}/I where Q_{10} is the quiver in Fig. 4 and where $I = (\alpha_1\beta_3, \alpha_2\gamma_1, \gamma_1\delta_2, \delta_2\beta_2, \beta_2\alpha_2, \delta_1\gamma_2, \epsilon\delta_1)$.

Example 3.11. Kalck [54] has shown that the algebras $A_1 = KQ_1/(\alpha_1\alpha_2, \beta_1\beta_2)$ and $A_2 = KQ_2/(\alpha_1\alpha_2, \gamma\alpha_1, \alpha_2\beta)$ are not derived equivalent. However, both algebras have the same Avella–Geiss invariants [10] (these are invariants of gentle algebras such that if two gentle algebras are derived equivalent then they have the same Avella–Geiss invariants) and they also have the same graph as gentle algebras. This graph is the ribbon graph given in Fig. 9 (compare also with the Brauer graph in Example 2.2(4). Note that the associated Brauer graph algebra is the trivial extension of both A_1 and A_2).

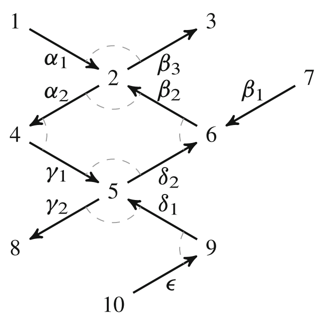


Fig. 4 The algebra A_{10} of a tiling of $P_{10,1}$ (see [20])

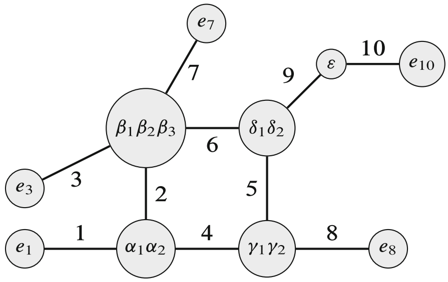


Fig. 5 The graph $\Gamma_{A_{10}}$ of the algebra A_{10} in Example 3.10

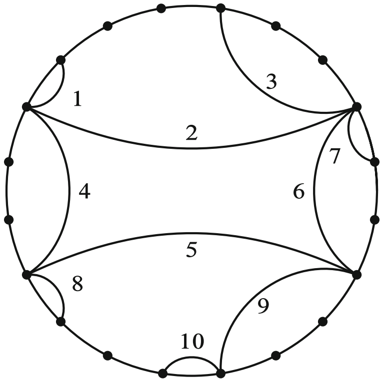


Fig. 6 The tiling of $P_{10,1}$ (see [20]) giving rise to the gentle algebra A_{10}. The graph $\Gamma_{A_{10}}$ of A_{10} gives the internal arcs of the tiling of $P_{10,1}$

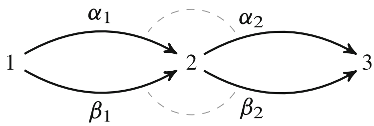


Fig. 7 Quiver Q_1 of the algebra A_1 in Example 3.11

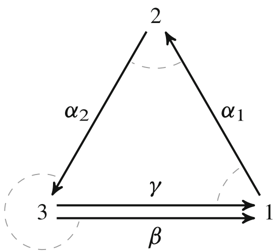


Fig. 8 Quiver Q_2 of the algebra A_2 in Example 3.11

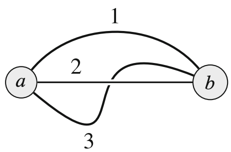


Fig. 9 The graph $\Gamma_{A_1} = \Gamma_{A_2}$ of the algebras A_1 and A_2 in Example 3.10

3.2 *Gentle Algebras from Admissible cuts of Brauer Graph Algebras*

Admissible cuts first appear in the PhD thesis of Fernández [39] where they were defined in the context of showing that two Schurian algebras Λ and Λ' have isomorphic trivial extensions if and only if Λ' is an admissible cut of $T(\Lambda)$. Recall that Λ is a *Schurian algebra* if for each pair of primitive idempotents e and f in Λ, we have $\dim_K(e\Lambda f) \leq 1$.

In [82], *admissible cuts of Brauer graph algebras* are defined as a slight modification of Fernández notion.

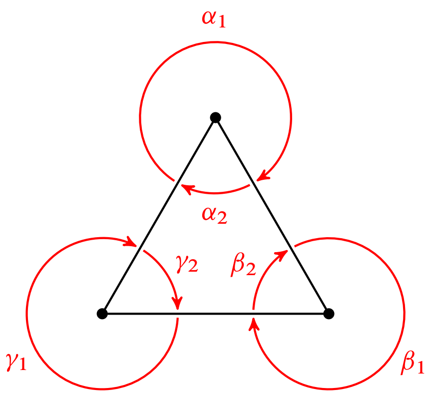
Definition 3.12 ([82])**.** Let $A = KQ/I$ be a Brauer graph algebra with Brauer graph G and multiplicity function m identically equal to one. An *admissible cut* Δ of Q is a set of arrows containing exactly one arrow from every special cycle C_v (up to rotation), where v is not of valency one as a vertex of G.

We define the *cut algebra with cut* Δ to be the algebra $A_\Delta = KQ/\langle I \cup \Delta\rangle$ where $\langle I \cup \Delta\rangle$ is the ideal of KQ generated by $I \cup \Delta$.

Theorem 3.13 ([82])**.** *Let $A = KQ/I$ be a Brauer graph algebra with Brauer graph G with multiplicity function identically equal to one. Let Δ be an admissible cut of Q. Then A_Δ is a gentle algebra and $\Gamma_{A_\Delta} = G$.*

Corollary 3.14 ([82])**.** *Every gentle algebra is the cut algebra of a unique Brauer graph algebra, given by its trivial extension, and conversely, every Brauer graph algebra with multiplicity function identically equal to one is the trivial extension of a (not necessarily unique) gentle algebra.*

Example 3.15. Let $\Lambda = KQ/I$ be the Brauer graph algebra with Brauer graph given by



Let $\Delta_1 = \{\alpha_1, \beta_1, \gamma_1\}$ and $\Delta_2 = \{\alpha_1, \beta_1, \gamma_2\}$ be admissible cuts of Q. Then the cut algebras are isomorphic to the following algebras $A_{\Delta_1} \simeq K(Q \setminus \Delta_1)/(\beta_2\alpha_2, \gamma_2\beta_2, \alpha_2\gamma_2)$ and $A_{\Delta_2} \simeq K(Q \setminus \Delta_2)/(\beta_2\alpha_2)$ and it is easy to see that they are gentle.

Remark 3.16. Let $A = KQ/I$ be a Brauer graph algebra and Δ and Δ' be two distinct admissible cuts of Q. Then A_Δ and $A_{\Delta'}$ have the same number of simple modules but they are not necessarily isomorphic nor derived equivalent. It is possible that A_Δ is of finite global dimension and $A_{\Delta'}$ is of infinite global dimension as is the case in Example 3.15 where A_{Δ_1} is of infinite global dimension whereas A_{Δ_2} is of finite global dimension.

3.3 *Gentle Surface Algebras and Quiver Mutation*

In [62, 63] Ladkani studies when two mutation-equivalent Jacobian algebras arising from quivers with potential associated to marked surfaces with all marked points on the boundary are derived equivalent. Note that in this case the Jacobian algebras are gentle algebras [7, 60]. We will start by recalling Ladkani's result and relate it to the corresponding Brauer graph algebras (given by the trivial extensions of the Jacobian algebras).

For this, recall the Fomin–Zelevinsky quiver mutation.

Definition 3.17. Let Q be a quiver without loops or two cycles and let k be a vertex of Q. The *mutation of* Q *at* k is the quiver $\mu_k(Q)$ obtained from Q as follows:

(1) for each subquiver $i \to k \to j$, add a new arrow $i \to j$;
(2) all arrows with source or target k are reversed
(3) remove all newly-created 2-cycles.

Quiver mutation is an involution, that is $\mu_k(\mu_k(Q)) = Q$.

Let S be an oriented surface and let M be a set of marked points in S. An *arc* is a curve without self-intersections with endpoints in M, considered up to homotopy, and such that it does not cut out a monogon or bigon and such that it is disjoint from the boundary except for the endpoints. An *ideal triangulation* is a maximal collection of non-crossing arcs. Let T be an ideal triangulation of the *marked surface* (S, M). We associate to T a quiver $Q(T)$ such that the vertices of $Q(T)$ correspond to the internal (=non-boundary) edges of T and where the arrows are given by the immediate successor relations of two edges incident with a common vertex induced by the orientation of the surface. For more details on triangulations and the associated quivers we refer the reader to [40]. In what follows we will often refer to an ideal triangulation as a triangulation.

Let s be an internal edge in T. Then the *flip of* T *at* s is defined by removing s and replacing it by s' such that we obtain the triangulation $T' = (T \setminus s) \cup s'$, where $s' \neq s$ is the unique edge completing $T \setminus s$ to a triangulation of (S, M).

Proposition 3.18 ([40])**.** *Let T be a triangulation of a marked surface (S, M) and let T' be the triangulation obtained by flipping an internal s edge in T. Then $Q(T') = \mu_s(Q(T))$.*

From now on let (S, M) be such that S is a surface with boundary ∂S and M is a set of marked points such that $M \subset \partial S$. Let T be a triangulation of (S, M) and A_T the gentle algebra associated to (S, M, T) [7, 60] with quiver $Q(T)$. Let g be the genus of S and b the number of boundary components in S.

Definition 3.19. (1) An (ideal) triangle in T is a *boundary triangle* if exactly two of its sides are boundary segments of S.

(2) Let $B_1, \ldots, B_b$ be the boundary components in S. Let n_i be the number of marked points on B_i and let d_i be the number of boundary triangles incident with B_i, that is triangles where exactly two sides lie in the boundary component B_i. Note that the d_i depend on T. We call the sequence

$$g, b, (n_1, d_1), \ldots, (n_b, d_b)$$

the *parameters* of T.

For a finite-dimensional algebra A denote by $\mathcal{D}^b(A)$ the bounded derived category of finitely generated A-modules. For more details on the definition of bounded derived categories and their properties we refer the reader to App. A and to standard textbooks such as [50, 89].

Theorem 3.20 ([62])**.** *Let (S, M) be a marked surface with all marked points in the boundary of (S, M). Let T and T' be triangulations of (S, M) having the same parameters. Let A_T and $A_{T'}$ be the associated gentle algebras. Then*

$$\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'}) \ .$$

Remark 3.21. (1) If (S, M) is a marked disc and T and T' are two triangulations of (S, M) then $\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'})$ if and only if T and T' have the same number of boundary triangles. Note that by [7] the algebras A_T and $A_{T'}$ are cluster-tilted algebras of Dynkin type A and their derived equivalence classification was already done in [16]. For the notion of a cluster-tilted algebras, we refer, for example, to [80].

(2) If T' is obtained from T by flipping one arc then $\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'})$ if and only if T and T' have the same number of boundary triangles.

(3) In [62] a boundary triangle is called a *dome*.

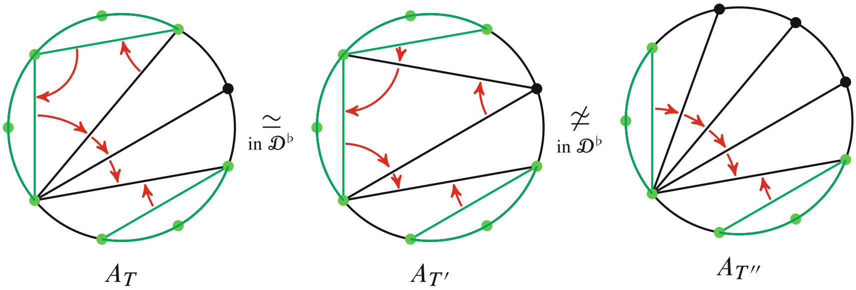
Corollary 3.22. *Let A_T and $A_{T'}$ be gentle algebras associated to triangulations T and T' of (S, M) such that T and T' have the same parameters and let Λ_T and $\Lambda_{T'}$ be the associated Brauer graph algebras, that is $\Lambda_T = A_T \ltimes D(A_T)$ and $\Lambda_{T'} = A_{T'} \ltimes D(A_{T'})$. Then*

$$\mathcal{D}^b(\Lambda_T) \simeq \mathcal{D}^b(\Lambda_{T'}) \ .$$

Proof. This immediately follows from Theorem 3.7 showing that $\Lambda_T = T(A_T)$ and $\Lambda_{T'} = T(A_{T'})$ and the result by Rickard [74] that if two finite-dimensional K-algebras A and B are derived equivalent then their trivial extensions $T(A)$ and $T(B)$ are derived equivalent. □

In particular, in the setting of Corollary 3.22, the quiver $Q_{T'}$ of the Brauer graph algebra $\Lambda_{T'}$ is obtained from the quiver Q_T of the Brauer graph algebra Λ_T by successive Fomin–Zelevinsky quiver mutations.

Example 3.23. Let T, T' and T'' be the following triangulations of the 9-gon giving rise to the associated gentle algebras A_T, $A_{T'}$ and $A_{T''}$.



Then T and T' have the same number of boundary triangles and A_T and $A_{T'}$ are derived equivalent. Furthermore, the corresponding Brauer graph algebras $\Lambda_T = T(A_T)$ and $\Lambda_{T'} = T(A_{T'})$ are derived equivalent. But A_T and $A_{T''}$ are not derived equivalent. Note that here we also have that the corresponding Brauer graph algebras Λ_T and $\Lambda_{T''} = T(A_{T''})$ are not derived equivalent (since T is a graph and T'' is a tree so that Λ_T is of tame representation type whereas $\Lambda_{T''}$ is of finite representation type). However, this does not always hold, see for example in Example 3.15 where the gentle algebras A_{Δ_1} and A_{Δ_2} are not derived equivalent but have isomorphic trivial extensions, i.e. they give rise to the same Brauer graph algebra.

4 Derived Equivalences and Mutation of Brauer Graph Algebras

Gentle algebras have a remarkable property. Namely their class is closed under derived equivalence, that is any algebra derived equivalent to a gentle algebra is again a gentle algebra [81].

The same is expected to hold for Brauer graph algebras [6].

4.1 *Mutation of Brauer Graph Algebras*

In [55] Kauer defined tilting complexes for Brauer graph algebras and showed that they give rise to derived equivalences in such a way that the endomorphism algebra of the tilting complex is again a Brauer graph algebra where the Brauer graph is obtained from the original one by a simple geometric move on the Brauer graph. We note that the proof at the time of writing was not complete, in that Kauer assumed, but omitted to show, that the tilted algebra has again the structure of a Brauer graph algebra. However, it follows from [72] that this holds for Brauer tree algebras, see also [13]. In the general case of a Brauer graph algebra that is not necessarily a Brauer tree algebra, this has been shown to hold, for example, in [25] and also in [1].

We will describe Kauer's tilting complex construction as well as the corresponding geometric moves on the Brauer graphs below. We note that Kauer's construction corresponds to a more general construction of two-term tilting complexes by Okuyama [68] (based on a manuscript by Rickard Linckelmann [71]). We will state a reformulation of this result due to Linckelmann [65]. For a module M, denote by $P(M)$ the projective cover of M. The definition of a tilting complex is recalled in App. A.

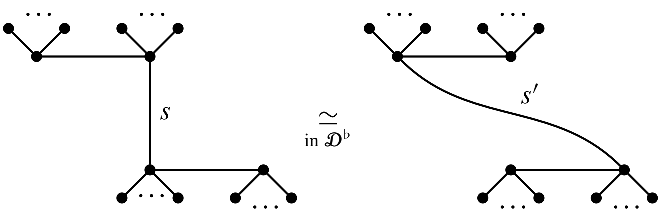
Theorem 4.1 ([65, 68, 71]). *Let A be a symmetric K-algebra. Let I, I' be disjoint sets of simple A-modules such that $I \cup I'$ is a complete set of representatives of the isomorphism classes of simple A-modules such that for any $S, U \in I$, we have $\mathrm{Ext}^1_A(S, U) = \{0\}$.*

For any $S \in I$, let T_S be the complex $P\left(\mathrm{rad}\big(P(S)\big)\right) \xrightarrow{\pi_S} P(S)$ with non-zero terms in degrees zero and one, and such that $\mathrm{Im}\,\pi_S = \mathrm{rad}\big(P(S)\big)$. For any $S' \in I'$ consider $P(S')$ as a complex concentrated in degree zero.

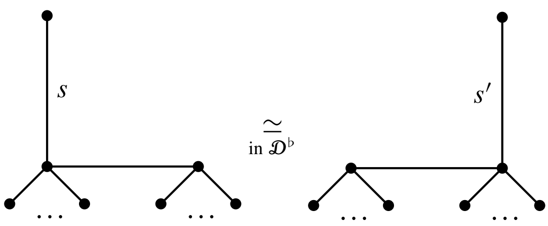
Then the complex $T = \left(\bigoplus_{S\in I} T_S\right) \oplus \left(\bigoplus_{S'\in I'} P(S')\right)$ is a tilting complex for A.

4.2 *Kauer Moves on the Brauer Graph*

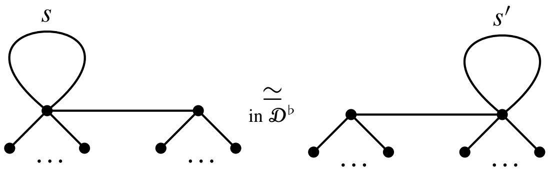
An example of an Okuyama–Rickard tilting complex is given by the following complex constructed by Kauer in [55]. Let A be a Brauer graph algebra with Brauer graph G. For $I = \{S_0\}$ where S_0 is a simple A-module such that the corresponding edge in the Brauer graph is not a loop where the corresponding half-edges are such that one is a direct successor of the other, set $T = T_{S_0} \oplus \bigoplus_{S\neq S_0, S \text{ simple}} P(S)$. Then in [55] this is shown to be a tilting complex for A by verifying its properties one by one. However, that T is a tilting complex also follows directly from the fact that T is an Okuyama tilting complex for A. Kauer further shows that the Brauer graph algebra $B = \mathrm{End}_{\mathcal{K}^\flat(P_A)}(T)$ has Brauer graph $G' = (G \setminus s) \cup s'$ where s is the edge in G corresponding to S_0 and where s' is obtained by one of the following local moves on G:



(1)



(2)



(3)

Definition 4.2. We call a local move as in (1)–(3) above a *Kauer move at s* and we adopt the following notation $\mu_s^+(G) = (G \setminus s) \cup s'$.

Remark 4.3. (1) In [2] Aihara refers to the above moves (1)–(3) as flips of Brauer graphs and he describes them in terms of quiver combinatorics.

(2) In [3] Aihara and Iyama introduce the notion of a (silting) mutation of an algebra in terms of left and right approximations and in [36] the Okuyama–Rickard tilting complexes giving rise to the Kauer moves are expressed in the more general case of weakly symmetric special biserial algebras in terms of silting mutations, that is in terms of left and right approximations.

Example 4.4. Let A be the Brauer graph algebra with Brauer graph as given by the left-hand graph in Fig. 10. The Kauer move at $s = 0$ gives rise to the Brauer graph on the right-hand graph in Fig. 10.

The Okuyama–Rickard complex T giving rise to the equivalence of $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$, where $B = \mathrm{End}_{\mathcal{K}^b(A)}(T)$ is given by $T = T_0 \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5$ where, for $i = 1, 2, 3, 4, 5$, P_i is the stalk complex concentrated in degree zero of the indecomposable projective associated to the edge i and $T_0 = P_4 \oplus P_2 \xrightarrow{\pi} P_0$, that is, T_0 is given by the following complex:

$$T_0 = \begin{smallmatrix} & 4 & \\ 5 & & \\ 1 & & 3 \\ 0 & & \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 3 & & 1 \\ 0 & & \\ & 2 & \end{smallmatrix} \xrightarrow{\pi_0} \begin{smallmatrix} & 0 & \\ 4 & & 2 \\ 5 & & 3 \\ 1 & & \\ & 0 & \end{smallmatrix}$$

concentrated in degrees zero and one and where $\mathrm{Im}(\pi_0) = \mathrm{rad}(P_0) = \begin{smallmatrix} 4 & & 2 \\ 5 & & 3 \\ 1 & & \\ & 0 & \end{smallmatrix}$.

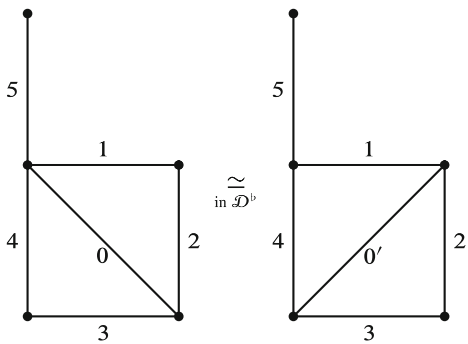


Fig. 10 Example of a Kauer move of type (1). The Brauer graph algebras associated to the two graphs are derived equivalent.

4.3 *Brauer Graph Algebras Associated to Triangulations of Marked Oriented Surfaces*

Let S be an oriented surface with boundary and let M be a set of marked points on S. Note here we don't necessarily require that M lies in the boundary of S, however, we do require that on each boundary component of S there is at least one marked point of M. We first recall that any two triangulations of (S, M) are flip connected, that is, one can be obtained from the other by a series of flips of diagonals, see for example [17, 40].

Theorem 4.5. *Let T and T' be triangulations of a marked oriented surface (S, M). Then the triangulations T and T' are flip connected.*

Recall that every triangulation of (S, M) is a ribbon graph where the cyclic ordering is induced by the orientation of S. In [66] we consider triangulations T of (S, M) (including the boundary arcs) as Brauer graphs. Note that the Brauer graph algebras we consider in [66] are different from the Brauer graph algebras associated to triangulations in Sect. 3. Namely, in Sect. 3, the Brauer graphs consist of the internal arcs of the triangulation whereas in [66] the boundary edges of the triangulation are part of the Brauer graphs, see Example 4.8. It is observed in [66] that when regarding T (including the boundary edges) as a Brauer graph, the flip of T at an internal edge s coincides with applying the Kauer move to T at s. Combining this with Theorem 4.5 we get

Theorem 4.6 ([66])**.** *Let T and T' be two triangulations of a marked oriented surface (S, M). Then the associated Brauer graph algebras Λ_T and $\Lambda_{T'}$ are derived equivalent, that is the bounded derived categories of finitely generated modules $\mathcal{D}^b(\Lambda_T)$ and $\mathcal{D}^b(\Lambda_{T'})$ are equivalent as triangulated categories.*

Furthermore, we have

Proposition 4.7 ([66])**.** *Let Q_T be the quiver of the Brauer graph algebra associated to a triangulation T of (S, M) and let $T' = (T \setminus s) \cup s'$. Then Q'_T is obtained from Q_T by Fomin–Zelevinsky quiver mutation, that is $Q_{T'} = \mu_s(Q_T)$.*

Note that in Proposition 4.7 we do not have any restrictions on the triangulation T. In particular, it can contain self-folded triangles or punctures with exactly two incident arcs.

Example 4.8. The Brauer graph algebras associated to the two Brauer graphs in Fig. 11 given by the triangulations T and T' of a hexagon (together with the boundary arcs) are derived equivalent. Note that here, as opposed to the examples in Sect. 3.3, T and T' do not have the same number of boundary triangles, yet the corresponding Brauer graph algebras are derived equivalent.

Remark 4.9. By Theorem 4.6 up to derived equivalence there is a unique Brauer graph algebra associated to any marked surface (S, M). This is similar to the corresponding situation for generalised cluster categories as defined by Amiot, see for example [4]. Namely, if $\mathcal{C}$ and $\mathcal{C}'$ are two cluster categories associated to two quivers with potential (Q, W) and (Q', W') associated to two triangulations T and T' of (S, M), then $\mathcal{C}$ and $\mathcal{C}'$ are triangle equivalent [56].

4.4 *Brauer Graph Algebras and Frozen Jacobian Algebras*

In this section we show that the frozen Jacobian algebra associated to a triangulation of a marked disc is closely related to the Brauer graph algebra whose Brauer graph corresponds to the triangulation (including the boundary arcs).

Definition 4.10 ([15]). An *ice quiver with potential* (Q, W, F) is a quiver with potential (Q, W) and a subset F of vertices of Q, called the *frozen vertices*.

The complete path algebra $\widehat{KQ}$ is the completion of the path algebra KQ with respect to the ideal $\mathcal{R}$ generated by the arrows of Q. A potential on Q is an element of the closure $Pot(KQ)$ of the space generated by all non-trivial cyclic paths of Q. We say two potentials are cyclically equivalent if their difference is in the closure of the space generated by all differences $a_1 a_2 \cdots a_s - a_2 \cdots a_s a_1$, where $a_1 a_2 \cdots a_s$ is a cycle.

For a path p in Q, let $\partial_p \colon \mathrm{Pot}(KQ) \to \widehat{KQ}$ be the unique continuous linear map which for a cycle c is defined by $\partial_p(c) = \sum_{upv=c} vu$ where u and v might be the trivial path at $s(c)$ and $t(c)$, respectively. If $p = a$ for some arrow a in Q then ∂_a is called the *cyclic derivative with respect to* a.

The *frozen Jacobian algebra of* (Q, W, F) is given by $\mathcal{J}(Q, W, F) = \widehat{KQ}/I_F$ where $\widehat{KQ}$ is the complete path algebra, I_F is the closure of the ideal $\langle \partial_a w \mid w \in W, a \in Q_1, s(a) \notin F \text{ or } t(a) \notin F \rangle$.

In the case of a triangulation of a marked disc (S, M, T), consider the ice quiver (Q, W, F) given by the quiver $Q = (Q_0, Q_1)$ where Q_0 corresponds to the edges of T and where the set F of frozen vertices is given by the boundary arcs and Q_1 is induced by the orientation. So in particular Q_1 includes a boundary arrow for all marked points incident to more than 2 arcs. The potential is given by

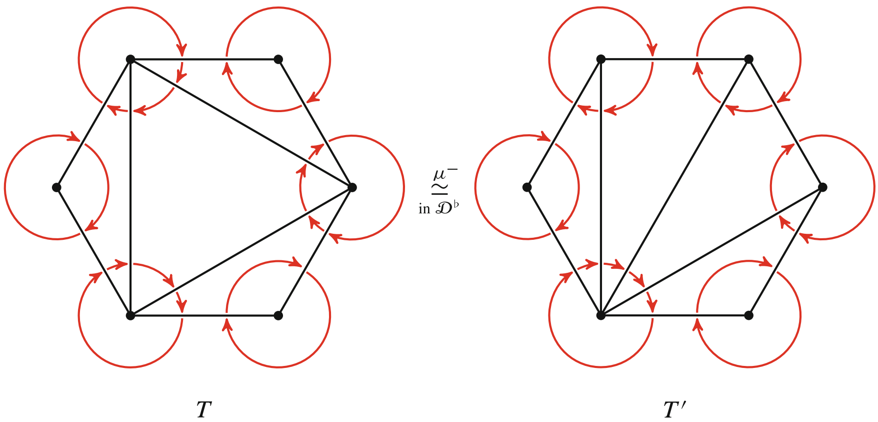


Fig. 11 Example of two triangulations where the Brauer graph algebras associated to the Brauer graphs given by the triangulations including the boundary arcs are derived equivalent

$$W = \sum_i C_i - \sum_j D_j$$

where the cycles C_i are given by (all) triangles in T and the cycles D_j are given by the cycles containing a boundary arrow. Then $\mathcal{J}_T = KQ/\langle \partial_a w | w \in W, a \in Q_1, s(a) \notin F \text{ or } t(a) \notin F \rangle$ is the frozen Jacobian algebra of (Q, W, F).

Remark 4.11. All relations in I_F are commutativity relations and the algebra $\mathcal{J}(Q, W, F)$ is infinite-dimensional.

Example 4.12. Given the following triangulation of a hexagon, consider the ice quiver with arrows as given in Fig. 12 where the frozen vertices correspond to the boundary arcs. We note that the quiver is almost identical to the quiver of the Brauer graph algebra in Example 4.8 associated to the same triangulation of the hexagon, the difference being the absence in the ice quiver of boundary arrows around vertices of valency two.

The arrows $\alpha_0, \beta_0, \gamma_0, \delta, \varepsilon, \eta$ are frozen arrows and the boundary arrows are $\alpha_0, \beta_0, \gamma_0$. The potential is given by

$$W = \alpha_1\beta_3\delta + \gamma_3\varepsilon\beta_1 + \alpha_2\gamma_2\beta_2 + \eta\gamma_1\alpha_3 - \alpha_0\alpha_1\alpha_2\alpha_3 - \beta_0\beta_1\beta_2\beta_3 + \gamma_0\gamma_1\gamma_2\gamma_3.$$

Then

$$I_F = \langle \beta_3\delta - \alpha_2\alpha_3\alpha_0, \gamma_2\beta_2 - \alpha_1\alpha_0\alpha_1, \eta\gamma_1 - \alpha_0\alpha_1\alpha_2, \gamma_3\varepsilon - \beta_2\beta_3\beta_0, \alpha_2\gamma_2 - \beta_3\beta_0\beta_1,$$
$$\delta\alpha_1 - \beta_0\beta_1\beta_2, \alpha_3\eta - \gamma_2\gamma_3\gamma_0, \beta_2\alpha_2 - \gamma_3\gamma_0\gamma_1, \varepsilon\beta_1 - \gamma_0\gamma_1\gamma_2\rangle \, .$$

Let $\mathcal{F}$ be the category of maximal Cohen–Macaulay modules over the Gorenstein tiled $K[x]$-order defined in [26]. Then Demonet and Luo show the following:

Theorem 4.13 ([26])**.** *Let P_n be the disc with n marked points in the boundary and let $\mathcal{F}$ be the associated category of maximal Cohen–Macaulay modules over the Gorenstein tiled $K[x]$-order defined in* [26]*. Then*

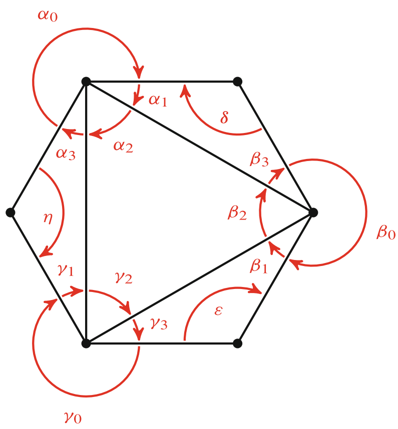


Fig. 12 Quiver of the frozen Jacobian algebra associated to this triangulation of the hexagon where the frozen vertices correspond to the boundary arcs of the triangulation

(1) *The stable category* $\underline{\mathcal{F}}$ *is triangle equivalent to the cluster category of type* A_{n-3}.
(2) *There are bijections between the indecomposable objects in* $\mathcal{F}$ *and the arcs between marked points in* P_n*, including boundary arcs.*
(3) *The bijection in* (2) *induces a bijection between triangulations* T *in* P_n *and the cluster-tilting objects* M_T *in* $\mathcal{F}$.
(4) *The frozen Jacobian algebra* $\mathcal{J}(Q, W, F)$ *associated to a triangulation* T *of* P_n *is isomorphic to* $\mathrm{End}_{\mathcal{F}}(M_T)^{\mathrm{op}}$.

We note that $\underline{\mathcal{F}}$ is a Hom-finite 2-Calabi–Yau triangulated category. So the following theorem by Palu can be applied.

Theorem 4.14 ([69])**.** *Let* $\underline{\mathcal{C}}$ *be a Hom-finite* 2*-Calabi–Yau triangulated category which is the stable category of a Frobenius category* $\mathcal{C}$. *Let* $\overline{M}, \overline{M}'$ *be cluster-tilting objects in* $\underline{\mathcal{C}}$ *with pre-images* M, M' *in* $\mathcal{C}$. *Then there is a triangle equivalence*

$$\mathcal{D}(\mathrm{End}_{\mathcal{C}}(M)^{\mathrm{op}}\text{-Mod}) \simeq \mathcal{D}\big(\mathrm{End}_{\mathcal{C}}\big(M'\big)^{\mathrm{op}}\text{-Mod}\big)$$

where, for A *an algebra,* $\mathcal{D}(A\text{-Mod})$ *denotes the derived category of all* A*-modules.*

Combining Theorems 4.13 and 4.14, it was observed in [66] that we obtain the following:

Theorem 4.15. *Let* $\mathcal{J}(Q, W, F)$ *and* $\mathcal{J}(Q', W', F')$ *be two frozen Jacobian algebras associated to two triangulations of a polygon. Then there is a triangle equivalence*

$$\mathcal{D}(\mathcal{J}(Q, W, F)\text{-Mod}) \simeq \mathcal{D}(\mathcal{J}(Q', W', F')\text{-Mod}) .$$

Summary We will summarise the comparison between frozen Jacobian algebras and Brauer graph algebras associated to the same triangulation of a polygon. Given a triangulation T of a polygon, let $\mathcal{J}_T$ be the frozen Jacobian algebra associated to T and let A_T be the Brauer graph algebra where the Brauer graph is given by T including the boundary arcs. Then the quiver of $\mathcal{J}_T$ is almost identical to the quiver of A_T (compare, for example the quiver in Fig. 12 with the quiver in left-hand side of Fig. 11). The only difference is the absence in the quiver of the frozen Jacobian algebra of boundary arrows around vertices that are incident to only 2 arcs (i.e. around vertices that are not incident with any internal arcs).

Furthermore, given triangulations T, T' of the same polygon, by Theorem 4.15 we have a derived equivalence of the unbounded derived categories of the module category of all modules over the associated frozen Jacobian algebras $\mathcal{J}_T$ and $\mathcal{J}_{T'}$,

$$\mathcal{D}(\mathcal{J}_T\text{-Mod}) \simeq \mathcal{D}(\mathcal{J}_{T'}\text{-Mod})$$

and by Theorem 4.6 we have a derived equivalence of the bounded derived categories of the module categories of finitely generated modules over the associated Brauer graph algebras A_T and $A_{T'}$

$$\mathcal{D}^b(A_T) \simeq \mathcal{D}^b(A_{T'}) .$$

This might indicate that there is a structural connection between the frozen Jacobian algebra and the Brauer graph algebra associated to the same triangulation (of a polygon). That this should be the case has also been highlighted by the results in [25] and in [64].

5 Auslander–Reiten Components

Auslander–Reiten theory for self-injective special biserial algebras has been well studied. Erdmann and Skowroński classify in [38] the Auslander–Reiten components for self-injective special biserial algebras. As Brauer graph algebras are symmetric special biserial, this gives the general structure of the Auslander–Reiten quiver for Brauer graph algebras. The result for representation-infinite algebras depends on the notions of domestic and polynomial growth. For the convenience of the reader, we recall these notions.

5.1 *Finite, Tame and Wild Representation Type, Domestic Algebras and Algebras of Polynomial Growth*

Let A be a finite-dimensional K-algebra, then we say that A is of *finite representation type*, if up to isomorphism there are only finitely many distinct indecomposable A-modules.

We say A is of *tame representation type*, if for any positive integer n, there exists a finite number of A-$K[x]$-bimodules M_i, for $1 \leq i \leq d_n$ such that M_i is finitely generated and is free as a left $K[x]$-module and such that all but a finite number of isomorphism classes of indecomposable n-dimensional A-modules are isomorphic to $M_i \otimes_{K[x]} K[x]/(x-a)$, for $a \in K$. For each n, let $\mu(n)$ be the least number of such A-$K[x]$-bimodules. Following [83], we say that A is of *polynomial growth* if there exists a positive integer m such that $\mu(n) \leq n^m$ for all $n \geq 2$. We say that A is *domestic* if $\mu(n) \leq m$ for all $n \geq 1$ [78] and we say that A is d*-domestic* if d is the least such integer m.

Note that every domestic algebra is of polynomial growth [84].

Let $K\langle x, y\rangle$ be the group algebra of the free group in two generators. We say that A is of *wild representation type* if there is a $K\langle x, y\rangle$ -A bimodule M such that M is free as a $K\langle x, y\rangle$-module, and if X is an indecomposable $K\langle x, y\rangle$-module then $M \otimes_K X$ is an indecomposable A-module, and if for some $K\langle x, y\rangle$-module Y we have $M \otimes_K X \simeq M \otimes_K Y$ then $X \simeq Y$.

By [31] an algebra of infinite representation type that is not of tame representation type is of wild representation type.

5.2 *Domestic Brauer Graph Algebras*

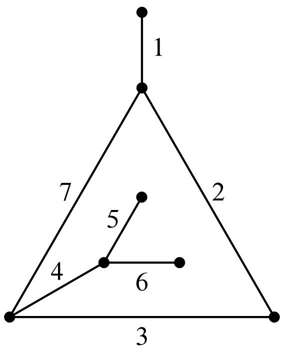
In the following let Λ be a representation-infinite Brauer graph algebra with Brauer graph $G = (G_0, G_1, m, \mathfrak{o})$. In [14] Bocian and Skowroński give a characterisation of the domestic Brauer graph algebras.

Theorem 5.1 ([14]). *Let Λ be a Brauer graph algebra with Brauer graph G. Then*

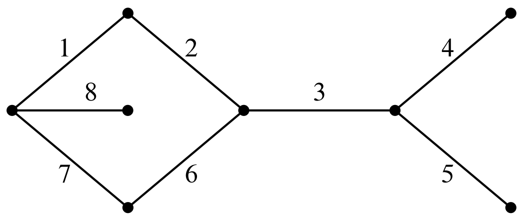
(1) *Λ is 1-domestic if and only if one of the following holds*
- *G is a tree with $m(i) = 2$ for exactly two vertices $i = i_0, i_1 \in G_0$ and $m(i) = 1$ for all $i \in G_0$, $i \neq i_0, i_1$.*
- *G is a graph with a unique cycle of odd length and $m \equiv 1$.*

(2) *Λ is 2-domestic if and only if G is a graph with a unique cycle of even length and $m \equiv 1$.*
(3) *There are no n-domestic Brauer graph algebras for $n \geq 3$.*

Example 5.2. (1) Example of a Brauer graph of a 1-domestic Brauer graph algebra.



(2) Example of Brauer graph of a 2-domestic Brauer graph algebra.



5.3 *The Stable Auslander–Reiten Quiver of a Self-Injective Special Biserial Algebra*

We refer the reader to the textbooks [8, 9, 80] for the definition and general set-up of Auslander–Reiten theory. Let Λ be a self-injective K-algebra (recall that a finite-dimensional K-algebra is self-injective if every projective module is injective). In the following denote by Γ_Λ the Auslander–Reiten quiver of Λ and by ${}_s\Gamma_\Lambda$ the stable Auslander–Reiten quiver of Λ. Denote by **mod**-Λ the module category of finitely generated Λ-modules and by $\underline{\textbf{mod}}$-Λ the stable category of Λ. Let $\nu\colon \Lambda\text{-}\mathbf{mod} \to \Lambda\text{-}\mathbf{mod}$ be the *Nakayama functor*, given by sending $N \in \Lambda\text{-}\mathbf{mod}$ to $D\big(\mathrm{Hom}_\Lambda(N, \Lambda)\big)$. Let Ω be *Heller's syzygy functor*, that is $\Omega\colon \underline{\mathbf{mod}}\text{-}\Lambda \to \underline{\mathbf{mod}}\text{-}\Lambda$ given by $M \in \underline{\mathbf{mod}}\Lambda$, $\Omega(M) = \ker \pi$ where $\pi\colon P_M \to M$ is the projective cover of M. Then the *Auslander–Reiten translate* τ is given by $\tau M = \nu\Omega^2 M$, for $M \in \mathbf{mod}\text{-}\Lambda$. Recall that if Λ is a symmetric K-algebra then $\nu = \mathrm{Id}$ and $\tau = \Omega^2$.

Following Riedtmann [76], any connected component of ${}_s\Gamma_\Lambda$ is of the form $\mathbb{Z}T/G$ where T is an oriented tree and G is an admissible automorphism group. We call T the tree class of ${}_s\Gamma_\Lambda$. The tree class of any component of the stable Auslander–Reiten quiver of an algebra containing a periodic module, that is a module M such that $\tau^n M \simeq M$, for some n, is equal to A_∞ [51]. The shapes of the translations quivers $\mathbb{Z}A_\infty$, $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}\tilde{A}_{p,q}$ are described in [51]. By $\tilde{A}_{p,q}$ we denote the following orientation of the quiver with underlying extended Dynkin diagram of type $\tilde{\mathrm{A}}_n$, see Fig. 13.

If a connected component of ${}_s\Gamma_\Lambda$ does not contain any projective or injective module, we call that component *regular*.

The following two theorems give a complete description of the possible components of the stable Auslander–Reiten quiver of a self-injective special biserial algebra of tame representation type.

Theorem 5.3 ([38, **Theorem 2.1**]). *Let Λ be a self-injective special biserial algebra. Then the following are equivalent*:

(1) *Λ is representation-infinite domestic.*
(2) *Λ is representation-infinite of polynomial growth.*
(3) *${}_s\Gamma_\Lambda$ has a component of the form $\mathbb{Z}\tilde{A}_{p,q}$.*
(4) *${}_s\Gamma_\Lambda$ is infinite but has no component of the form $\mathbb{Z}A_\infty^\infty$.*
(5) *All but a finite number of components of Γ_Λ are of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$.*

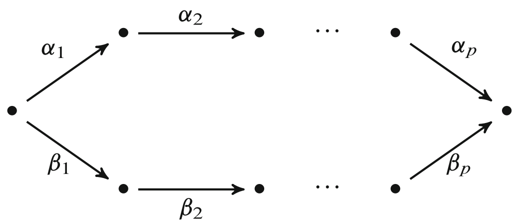


Fig. 13 $\tilde{A}_{p,q}$

(6) ${}_s\Gamma_\Lambda$ *is a disjoint union of m components of the form* $\mathbb{Z}\tilde{A}_{p,q}$*, m components of the form* $\mathbb{Z}A_\infty/\langle\tau^p\rangle$*, and m components of the form* $\mathbb{Z}A_\infty/\langle\tau^q\rangle$*.*

Theorem 5.4 ([38, **Theorem 2.2**]). *Let Λ be a self-injective special biserial algebra. Then the following are equivalent:*

(1) *Λ is not of polynomial growth.*
(2) ${}_s\Gamma_\Lambda$ *has a component of the form* $\mathbb{Z}A_\infty^\infty$*.*
(3) Γ_Λ *has infinitely many (regular) components of the form* $\mathbb{Z}A_\infty^\infty$*.*
(4) ${}_s\Gamma_\Lambda$ *is a disjoint union of a finite number of components of the form* $\mathbb{Z}A_\infty/\langle\tau^n\rangle$*, with* $n > 1$*, infinitely many components of the form* $\mathbb{Z}A_\infty/\langle\tau\rangle$ *and infinitely many components of the form* $\mathbb{Z}A_\infty^\infty$*.*

5.4 *Green Walks, Double-Stepped Green Walks and Exceptional Tubes*

In [49] Sandy Green defined a *walk around the Brauer tree* and showed that for certain modules it encodes their minimal projective resolution.

Let A be a Brauer graph algebra. We call a simple A-module with uniserial projective cover a *uniserial simple* A*-module*. The uniserial simple A-modules correspond exactly to the truncated edges in the Brauer graph. Recall that a truncated edge is a leaf in the Brauer graph where the leaf vertex has multiplicity one. Green showed that the successive terms in the minimal projective resolution of a uniserial simple A-module are given by 'walking' around the Brauer tree. Roggenkamp showed that the same is true for a Brauer graph algebra [79].

5.4.1 Definition of Green's walk around the Brauer graph

A Green walk on a Brauer graph is given by a graph theoretic path on the graph underlying the Brauer graph. It is defined in terms of successor relations. Let $A = KQ/I$ be a Brauer graph algebra with Brauer graph G.

Definition of Green Walks. Suppose first that G consists of a single loop i_0. Then there are two Green walks of period 1 on G both given by i_0, see Fig. 14.

Now suppose that G contains more than one edge and let i_0 be an edge in G with vertices a_0 and a_1. Suppose first that $a_0 \neq a_1$, that is suppose that i_0 is not a loop. Let i_1

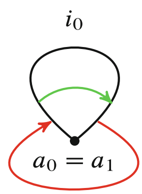


Fig. 14 Green walk of the Brauer graph consists of a single loop

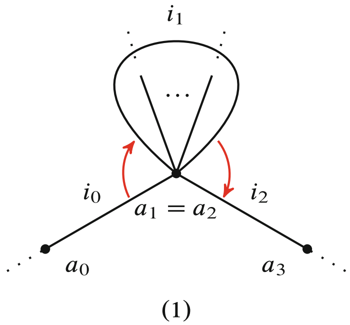


Fig. 15 Green walk at a loop

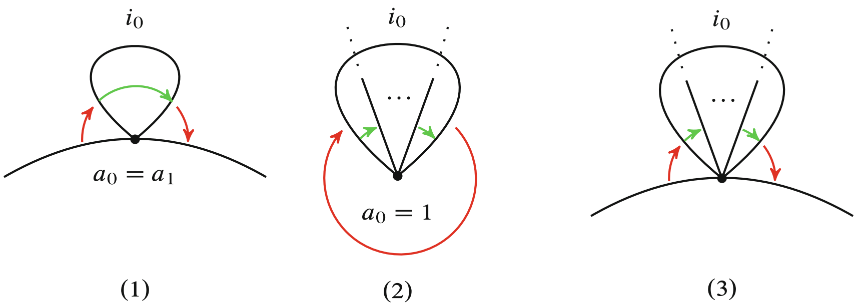


Fig. 16 Green walk starting on loops

be the successor of i_0 at the vertex a_1. Now let a_2 be the other vertex of i_1 and let i_2 be the successor of i_1 at vertex a_2. Note that if i_1 is a loop then $a_1 = a_2$ and the successor i_2 of i_1 appears as in Fig. 15.

Let a_3 be the other vertex of i_2 and let i_3 be the successor of i_2 at a_3 (treating the case that i_2 is a loop as above). In this way, we define an infinite sequence of edges $i_0, i_1, i_2, \ldots$, called a *Green walk starting at* i_0. Let k be the minimal number n such that $i_{n+k} = i_k$ for all $k \geq 0$. Then we say that $(i_0, i_1, \ldots, i_{k-1})$ is a *Green walk of period* k. Since G is a finite graph, every Green walk is periodic.

Note that if i_0 is a loop, that is if $a_0 = a_1$, then there are two distinct Green walks starting at i_0 depending on the successor i_1 of i_0 chosen. This is illustrated in Fig. 16.

Every edge appears in exactly two (not necessarily distinct) Green walks, corresponding to the two successors of a given edge. Any edge corresponding to a leaf (truncated or not) appears twice in the same Green walk.

In order to avoid the complication with loops in the definition of a Green walk, many authors consider half-edges instead of edges using, for example, the language of ribbon graphs, see [1, 30, 35, 79].

Remark 5.5. Green walks are independent of the multiplicities in the Brauer graph, they only depend on the graph and the orientation.

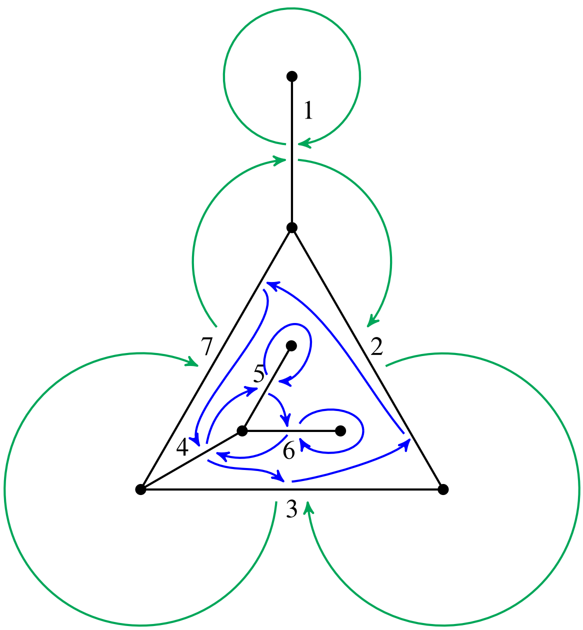
Given a Green walk $i_0, i_1, i_2, \ldots$ at some edge i_0 of a Brauer graph G, a *double-stepped Green walk at* i_0 is given by the sequence of edges $i_0, i_2, i_4, \ldots$ in G.

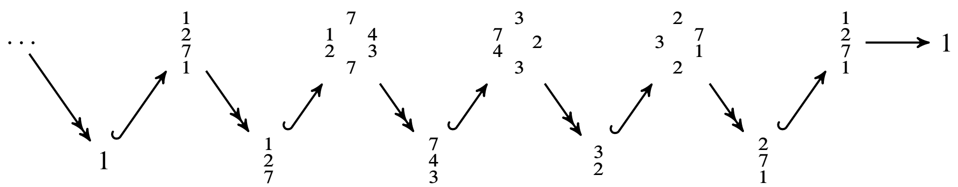
Theorem 5.6 ([49, 79])**.** *Let A be a Brauer graph algebra with Brauer graph G.*

(1) *Let S be a uniserial simple A-module and let i_0 be the edge in G corresponding to S and let $i_0, i_1, \ldots$ be the Green walk starting at i_0. Then the nth projective in a minimal projective resolution of S is given by P_{i_n}, the indecomposable projective associated to the edge i_n.*

(2) *Let j_0 be the edge in G corresponding to the simple top of a maximal uniserial submodule M of some projective indecomposable module and let $j_0, j_1, \ldots$ be the Green walk starting at j_0. Then the nth projective in a minimal projective resolution of M is given by P_{j_n}.*

Example 5.7. Example of the two Green walks on the 1-domestic Brauer graph given in Example 5.2 and the minimal projective resolution of the simple corresponding to the edge 1 given by the walk starting at 1.





It follows immediately from Theorem 5.6 that

Corollary 5.8. *Any uniserial simple A-module and any maximal uniserial submodule of an indecomposable projective A-module is Ω-periodic.*

As a consequence by [51], it follows that any uniserial simple A-module and any maximal uniserial submodule of an indecomposable projective A-module lies in a component of tree class A_∞.

Moreover, in [35] the precise location of these modules is given

Proposition 5.9 ([35])**.** *Let A be a Brauer graph algebra. An indecomposable (string) module M is at the mouth of an exceptional tube in the stable Auslander–Reiten quiver of A if and only if M is uniserial simple or a maximal uniserial submodule of an indecomposable projective A-module.*

5.5 *Exceptional Tubes*

Exceptional tubes are Auslander–Reiten components of the form $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, where the integer $n \geq 1$ denotes the rank of the tube and where for tubes of rank one, we distinguish between the exceptional tubes—that is, the tubes of rank one consisting of string modules, of which there are finitely many—and the homogeneous tubes of rank one, which consist only of band modules.

Theorem 5.10 ([35, **Theorem 4.3**])**.** *Let A be a representation-infinite Brauer graph algebra with Brauer graph G.*

(1) *The number of exceptional tubes in ${}_s\Gamma_A$ is given by the number of double-stepped Green walks on G.*
(2) *The length of the double-stepped Green walk gives the rank of the associated exceptional tube.*

Example 5.11. (1) Suppose that A is a Brauer graph algebra with Brauer graph as given in Fig. 17. Then there are three distinct Green walks on G of lengths 1, 5 and 6, and thus 4 double-stepped Green walks of lengths 1, 3, 3 and 5 giving rise to four exceptional tubes of respective ranks 1, 3, 3 and 5.
More precisely, the stable Auslander–Reiten quiver of A is a disjoint union of infinitely many components of the forms $\mathbb{Z}A_\infty/\langle\tau\rangle$ and $\mathbb{Z}A_\infty^\infty$ and two components of the form $\mathbb{Z}A_\infty/\langle\tau^3\rangle$, one component of the form $\mathbb{Z}A_\infty/\langle\tau^6\rangle$ and one component of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$ (given by the string corresponding to the arrow inside the unique loop and not a band module as the ones giving rise to the infinitely many other component of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$).

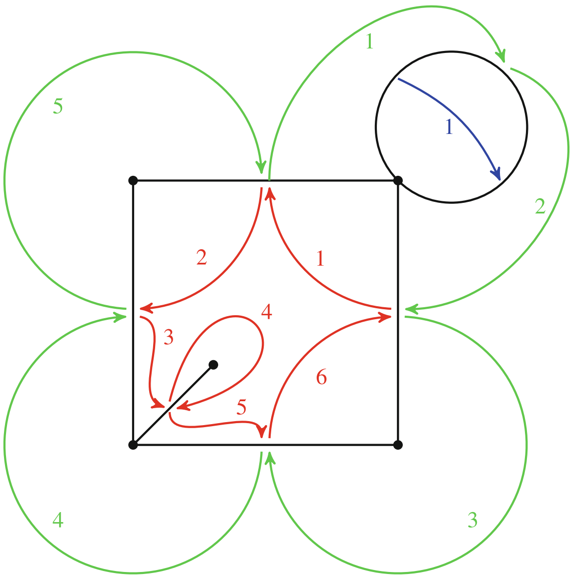


Fig. 17 Brauer graph with three distinct Green walks

(2) For completeness we include one example of a non-planar Brauer graph G' and the Green walks on G'. Suppose that A' is a Brauer graph algebra with Brauer graph G', see Fig. 18.

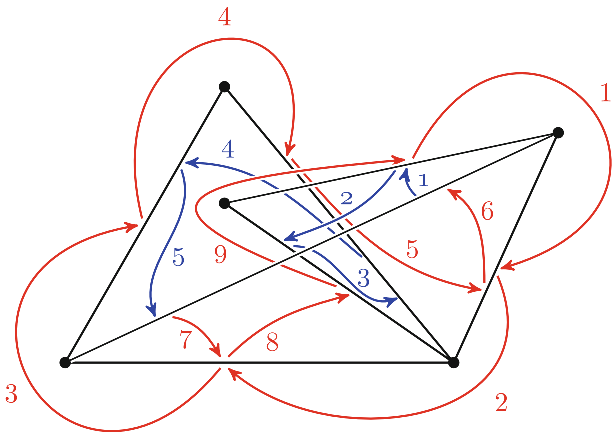


Fig. 18 Green walks on a non-planar Brauer graph

Then the stable Auslander–Reiten quiver of A' is a disjoint union of infinitely many components of the forms $\mathbb{Z}A_\infty/\langle\tau\rangle$ and $\mathbb{Z}A_\infty^\infty$ and one component of the form $\mathbb{Z}A_\infty/\langle\tau^5\rangle$ and one component of the form $\mathbb{Z}A_\infty/\langle\tau^9\rangle$.

5.6 *Auslander–Reiten Components of Non-domestic Brauer Graph Algebras*

Let A be a representation-infinite non-domestic Brauer graph algebra with Brauer graph G. It follows directly from Theorems 5.4 and 5.10 that ${}_s\Gamma_A$ is the disjoint union of

- k components of the form $\mathbb{Z}A_\infty/\langle\tau^{l_i}\rangle$, where k is the number of double-stepped Green walks and l_i is the length of the ith double-stepped Green walk, for $1 \le i \le k$,
- infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$
- infinitely many components of the form $\mathbb{Z}A_\infty^\infty$.

5.7 *Auslander–Reiten Components of Domestic Brauer Graph Algebras*

Let A be a representation-infinite m-domestic Brauer graph algebra with Brauer graph G. It follows from Theorems 5.3 and 5.10 that ${}_s\Gamma_A$ is the disjoint union of

- m components of the form $\mathbb{Z}\tilde{A}_{p,q}$
- m components of the form $\mathbb{Z}A_\infty/\langle\tau^p\rangle$
- m components of the form $\mathbb{Z}A_\infty/\langle\tau^q\rangle$
- infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$.

More precisely,

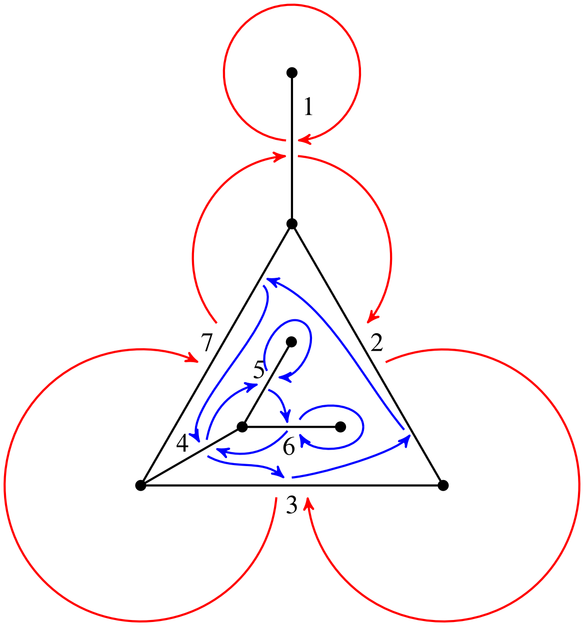
Theorem 5.12 ([35, **Theorem 4.4, Corollary 4.5**]). *Let A be a representation-infinite domestic Brauer graph algebra with Brauer graph G with n edges. If G has a cycle then it is unique and let n_1 be the number of* (*additional*) *edges on the inside of the cycle and n_2 the number of* (*additional*) *edges on the outside of the cycle. In the notation above:*

(1) *If A is 1-domestic, then $m = 1$ and $p + q = 2n$. Furthermore,*

- *if G is a tree, then $p = q = n$,*
- *if G has a unique cycle (of odd) length l, then $p = l + 2n_1$ and $q = l + 2n_2$.*

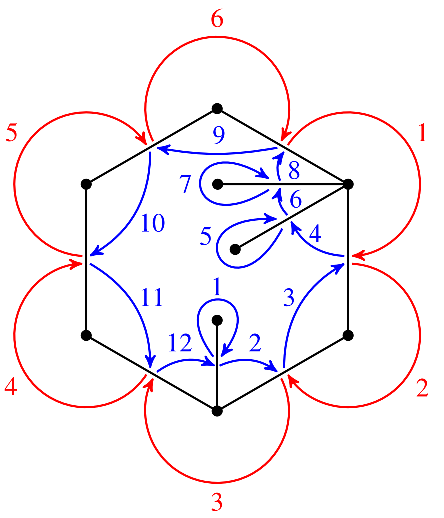
(2) *If A is 2-domestic with unique cycle (of even) length l, then $m = 2$ and $p = l/2 + n_1$ and $q = l/2 + n_2$ such that $p + q = n$.*

Example 5.13. Example of the stable Auslander–Reiten quiver of a 1-domestic Brauer graph algebra with Brauer graph (including the two distinct Green walks) given by



Here $l = 3$, $n_1 = 3$ and $n_2 = 1$ and therefore by Theorem 5.12, $p = 9$ and $q = 5$ and the stable Auslander–Reiten quiver consists of one component of the form $\mathbb{Z}\tilde{A}_{9,5}$, one of the form $\mathbb{Z}A_\infty/\langle\tau^9\rangle$ and one of the form $\mathbb{Z}A_\infty/\langle\tau^5\rangle$ as well as infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. Note that $p = 9$ is the length of the unique double-stepped Green walk on the inside of the 3-cycle and $q = 5$ is the length of the unique double stepped Green walk on the outside of the 3-cycle.

Example of the stable Auslander–Reiten quiver of a 2-domestic Brauer graph algebra with Brauer graph (including the two distinct Green walks) given by



Here $l = 6$, $n_1 = 3$ and $n_2 = 0$ and therefore by Theorem 5.12, $p = 6$ and $q = 3$ and the stable Auslander–Reiten quiver consists of two components of the form $\mathbb{Z}\tilde{A}_{6,3}$, two of the form $\mathbb{Z}A_\infty/\langle\tau^6\rangle$ and one of the form $\mathbb{Z}A_\infty/\langle\tau^3\rangle$ as well as infinitely many components of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$. Note that $p = 6$ is the length of the two double-stepped Green walks on the inside of the 6-cycle and $q = 3$ is the length of the two double stepped Green walks on the outside of the 6-cycle.

5.8 *Position of Modules in the Auslander–Reiten Quiver*

In this section we determine, for any representation-infinite Brauer graph algebra (domestic or non-domestic) which simple modules and radicals of indecomposable projective modules (and hence which projective modules) lie in exceptional tubes and which do not. For that we define the notion of exceptional edges and we will see that the simple modules and the radicals of the projective indecomposable modules associated to the exceptional edges lie in the exceptional tubes of ${}_s\Gamma_A$.

Definition 5.14 ([35]).

(1) Let G be a Brauer graph such that G is not a Brauer tree (that is if G is a tree then there are at least two vertices of multiplicity greater than 2). An *exceptional subtree* of G is a subgraph T of G such that

(i) T is a tree,
(ii) there exists at unique vertex v in T such that $(G \setminus T) \cup v$ is connected,
(iii) every vertex of T has multiplicity 1 except perhaps v.

We call v the *connecting vertex* of T.

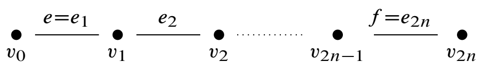
(2) Let G be a Brauer graph. An edge e in G is called an *exceptional edge* if e lies in an exceptional subtree of G and it is called a *non-exceptional edge* otherwise.

Note that the non-exceptional edges in a Brauer graph G are all connected to each other and that a non-exceptional edge is never truncated.

Theorem 5.15 ([35])**.** *Let A be a representation-infinite Brauer graph algebra with Brauer graph G and let e be an edge in G. Then the simple module S_e associated to e and the radical* rad P_e *of the indecomposable projective module P_e associated to e lie in an exceptional tube of ${}_s\Gamma_A$ if and only if e is an exceptional edge.*

Furthermore, S_e and rad P_e *lie in the same exceptional tube if and only if e occurs twice within the same double-stepped Green walk.*

Theorem 5.16 ([35])**.** *Let A be a representation-infinite Brauer graph algebra with Brauer graph G and let e and f be two not necessarily distinct non-exceptional edges in $G = (G_0, G_1, m, \mathfrak{o})$. Then S_e and* rad P_f *lie in the same component of ${}_s\Gamma_A$ if and only if A is 1-domestic or there exists an even length path*



of non-exceptional edges such that

(i) *none of the edges e_i, for $1 \le i \le 2n$, is a loop,*
(ii) *e_i and e_{i+1} are the only non-exceptional edges incident with v_i, for $1 \le i \le 2n-1$,*
(iii) *$m(v_i) = 1$, for $1 \le i \le 2n-1$, except if $e_i = e_{i+1}$ in which case $m(v_i) = 2$, for $1 \le i \le 2n-1$.*

Example 5.17. For the Brauer graph algebra with Brauer graph given by the graph in Figure 19, the simple module S_e associated to the edge e and the rad P_f where P_f is the indecomposable projective module associated to the edge f, are in the same component of the Auslander–Reiten quiver. Since the edge corresponding to e is not exceptional, this component is not an exceptional tube but of the form $\mathbb{Z}A_\infty^\infty$. On the other hand, the simple module S_g as well as rad P_g, the radical of the indecomposable projective associated to the edge g lie in the same exceptional tube (of rank 27) since the edge g is exceptional and occurs twice in the same double-stepped Green walk.

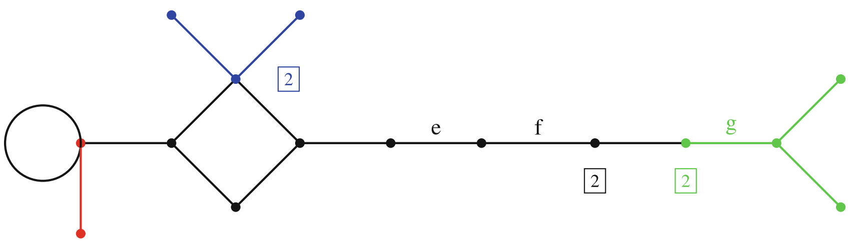


Fig. 19 Example of a Brauer graph with three exceptional subtrees

A Appendix

A.1 *Derived Equivalences: Rickard's Theorem*

Let A be a finite-dimensional K-algebra. We denote by $\mathcal{D}^b(A)$ the bounded derived category of finite-dimensional A-modules.

If M is a tilting A-module then the categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$, where $B = End_A(M)$, are equivalent [19, 50].

Rickard showed that all derived equivalences of finite-dimensional algebras are of a similar nature. Namely let **mod**-A be the category of all finitely presented A-modules and P_A the category of all finitely generated projective A-modules. We denote by $\mathcal{K}^b(P_A)$ the homotopy category of bounded complexes in P_A.

Theorem A.1 ([73, **1.1**]). *Let A and B be two finite-dimensional algebras. The following are equivalent*:

(a) $\mathcal{D}^b(A)$ *and* $\mathcal{D}^b(B)$ *are equivalent as triangulated categories.*
(b) $\mathcal{K}^b(P_A)$ *and* $\mathcal{K}^b(P_B)$ *are equivalent as triangulated categories.*
(c) B *is isomorphic to the endomorphism ring of an object* T *of* $\mathcal{K}^b(P_A)$*, that is* $B \simeq \mathrm{End}_{\mathcal{K}^b(P_A)}(T)$, *such that*

   (i) *For* $n \neq 0$, $\mathrm{Hom}(T, T[n]) = 0$.
   (ii) $\mathrm{add}(T)$, *the full subcategory of* $\mathcal{K}^b(P_A)$ *consisting of direct summands of direct sums of copies of* T, *generates* $\mathcal{K}^b(P_A)$ *as a triangulated category.*

Moreover, any equivalence as in (a) *restricts to an equivalence between the full subcategories consisting of objects isomorphic to bounded complexes of projectives* (*which are equivalent to* $\mathcal{K}^b(P_A)$ *and* $\mathcal{K}^b(P_B)$ *respectively*).

A complex T as in Theorem A.1(c) is called a *tilting complex over* A.

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