Discontinuous Galerkin method for hyperbolic equations with singularities

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Joint work with Qiang Zhang, Yang Yang and Dongming Wei
Outline

• Introduction to discontinuous Galerkin (DG) method for hyperbolic equations

• DG method for discontinuous solutions of linear hyperbolic equations: an error estimate

• DG method for hyperbolic equations with $\delta$-singularities: error estimates and applications
We are interested in solving a hyperbolic conservation law

\[ u_t + f(u)_x = 0 \]

In 2D it is

\[ u_t + f(u)_x + g(u)_y = 0 \]

and in system cases \( u \) is a vector, and the Jacobian \( f'(u) \) is diagonalizable with real eigenvalues.
Several properties of the solutions to hyperbolic conservation laws.

- The solution $u$ may become discontinuous regardless of the smoothness of the initial conditions.

- Weak solutions are not unique. The unique, physically relevant entropy solution satisfies additional entropy inequalities

$$U(u)_t + F(u)_x \leq 0$$

in the distribution sense, where $U(u)$ is a convex scalar function of $u$ and the entropy flux $F(u)$ satisfies $F'(u) = U'(u)f'(u)$. 
To solve the hyperbolic conservation law:

$$u_t + f(u)_x = 0,$$  \hspace{1cm} (1)

we multiply the equation with a test function $v$, integrate over a cell $I_j = [x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]$, and integrate by parts:

$$\int_{I_j} u_t v \, dx - \int_{I_j} f(u)_v x \, dx + f(u_{j+\frac{1}{2}})v_{j+\frac{1}{2}} - f(u_{j-\frac{1}{2}})v_{j-\frac{1}{2}} = 0$$
Now assume both the solution $u$ and the test function $v$ come from a finite dimensional approximation space $V_h$, which is usually taken as the space of piecewise polynomials of degree up to $k$:

$$V_h = \{ v : v|_{I_j} \in P^k(I_j), \ j = 1, \cdots, N \}$$

However, the boundary terms $f(u_{j+\frac{1}{2}}), v_{j+\frac{1}{2}}$ etc. are not well defined when $u$ and $v$ are in this space, as they are discontinuous at the cell interfaces.
From the conservation and stability (upwinding) considerations, we take

- A single valued monotone numerical flux to replace $f(u_{j+\frac{1}{2}})$:

$$\hat{f}_{j+\frac{1}{2}} = \hat{f}(u_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}^-)$$

where $\hat{f}(u, u) = f(u)$ (consistency); $\hat{f}(\uparrow, \downarrow)$ (monotonicity) and $\hat{f}$ is Lipschitz continuous with respect to both arguments.

- Values from inside $I_j$ for the test function $v$

$$v_{j+\frac{1}{2}}^-, v_{j-\frac{1}{2}}^+$$
Hence the DG scheme is: find \( u \in V_h \) such that

\[
\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0
\]

for all \( v \in V_h \).

Notice that, for the piecewise constant \( k = 0 \) case, we recover the well known first order monotone finite volume scheme:

\[
(u_j)_t + \frac{1}{h} \left( \hat{f}(u_j, u_{j+1}) - \hat{f}(u_{j-1}, u_j) \right) = 0.
\]
DG method for hyperbolic equations with singularities

Time discretization could be by the TVD Runge-Kutta method (Shu and Osher, JCP 88). For the semi-discrete scheme:

\[ \frac{du}{dt} = L(u) \]

where \( L(u) \) is a discretization of the spatial operator, the third order TVD Runge-Kutta is simply:

\[
\begin{align*}
  u^{(1)} &= u^n + \Delta t L(u^n) \\
  u^{(2)} &= \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t L(u^{(1)}) \\
  u^{n+1} &= \frac{1}{3} u^n + \frac{2}{3} u^{(2)} + \frac{2}{3} \Delta t L(u^{(2)})
\end{align*}
\]
Properties and advantages of the DG method:

- Easy handling of complicated geometry and boundary conditions (common to all finite element methods). Allowing hanging nodes in the mesh (more convenient for DG);

- Compact. Communication only with immediate neighbors, regardless of the order of the scheme;
DG method for hyperbolic equations with singularities

- Explicit. Because of the discontinuous basis, the mass matrix is local to the cell, resulting in explicit time stepping (no systems to solve);
- Parallel efficiency. Achieves 99% parallel efficiency for static mesh and over 80% parallel efficiency for dynamic load balancing with adaptive meshes (Biswas, Devine and Flaherty, APNUM 94; Remacle, Flaherty and Shephard, SIAM Rev 03); Also friendly to GPU environment (Klockner, Warburton, Bridge and Hesthaven, JCP10).
DG method for hyperbolic equations with singularities

- Provable cell entropy inequality and $L^2$ stability, for arbitrary nonlinear equations in any spatial dimension and any triangulation, for any polynomial degrees, without limiters or assumption on solution regularity (Jiang and Shu, Math. Comp. 94 (scalar case); Hou and Liu, JSC 07 (symmetric systems)). For $U(u) = \frac{u^2}{2}$:

$$\frac{d}{dt} \int_{I_j} U(u) dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} \leq 0$$

Summing over $j$:

$$\frac{d}{dt} \int_a^b u^2 dx \leq 0.$$  

This also holds for fully discrete RKDG methods with third order TVD Runge-Kutta time discretization, for linear equations (Zhang and Shu, SINUM 10).
• At least \((k + \frac{1}{2})\)-th order accurate, and often \((k + 1)\)-th order accurate for smooth solutions when piecewise polynomials of degree \(k\) are used, regardless of the structure of the meshes, for smooth solutions (Lesaint and Raviart 74; Johnson and Pitkäranta, Math. Comp. 86 (linear steady state); Zhang and Shu, SINUM 04 and 06 (RKDG for nonlinear equations)).

• \((2k + 1)\)-th order superconvergence in negative norm and in strong \(L^2\)-norm for post-processed solution for linear and nonlinear equations with smooth solutions (Cockburn, Luskin, Shu and Süli, Math. Comp. 03; Ryan, Shu and Atkins, SISC 05; Curtis, Kirby, Ryan and Shu, SISC 07; Ji, Xu and Ryan, JSC 13).
• $(k + 3/2)$-th or $(k + 2)$-th order superconvergence of the DG solution to a special projection of the exact solution, and non-growth of the error in time up to $t = O\left(\frac{1}{\sqrt{h}}\right)$ or $t = O\left(\frac{1}{h}\right)$, for linear and nonlinear hyperbolic and convection diffusion equations (Cheng and Shu, JCP 08; Computers & Structures 09; SINUM 10; Meng, Shu, Zhang and Wu, SINUM 12 (nonlinear); Yang and Shu, SINUM 12 ($(k + 2)$-th order)).

• Easy $h$-$p$ adaptivity.

• Stable and convergent DG methods are now available for many nonlinear PDEs containing higher derivatives: convection diffusion equations, KdV equations, ...
Collected works on the DG methods:


DG method for hyperbolic equations with singularities

DG method for hyperbolic equations with singularities


Even though the major motivation to design the DG method for solving hyperbolic equations is to resolve discontinuous solutions more effectively, there are not many convergence and error estimate results for DG method with discontinuous solutions.
Previous work on this issue:

- Johnson et al CMAME 1984; Johnson and Pitkäranta Math Comp 1986; Johnson et al Math Comp 1987: error estimates for piecewise linear streamline diffusion and DG methods for stationary (or space-time) hyperbolic equations. Pollution region around discontinuity: $O(h^{1/2} \log(1/h))$.

- Cockburn and Guzmán SINUM 2008: RKDG2 (second order in space and time), Pollution region around discontinuity: $O(h^{1/2} \log(1/h))$ on the downwind side and $O(h^{2/3} \log(1/h))$ on the upwind side. The result is for uniform meshes and does not hold when the CFL number goes to zero, or for the semi-discrete DG scheme. Also it does not generalize easily to higher order either in space or in time.
In Zhang and Shu, Num Math 2014, we investigate the RKDG3 scheme with arbitrary polynomial degree \( k \geq 1 \) in space and third order TVD Runge-Kutta in time, on arbitrary quasi-uniform mesh, for solving the linear model equation

\[
    u_t + \beta u_x = 0
\]

\[
    u(x, 0) = u^0(x)
\]

where \( \beta \) is a constant, \( u^0(x) \) has compact support, has a sole discontinuity at \( x = 0 \) and is sufficiently smooth everywhere else.
We prove the following error estimate:

**Theorem:** Assume the CFL number \( \lambda := |\beta| \Delta t / h_{\text{min}} \) is small enough, there holds

\[
\| u(t^N) - u_h^N \|_{L^2(\mathbb{R} \setminus \mathcal{R}_T)} \leq M \left( h^{k+1} + \Delta t^3 \right),
\]

where \( M > 0 \) is independent of \( h \) and \( \Delta t \), but may depend on the final time \( T \), the norm of the exact solution in smooth regions, and the jump at the discontinuity point. Here \( \mathcal{R}_T \) is the pollution region at the final time \( T \), given by

\[
\mathcal{R}_T = (\beta T - C \sqrt{T \beta \nu^{-1}} h^{1/2} \log(1/h), \beta T + C \sqrt{T \beta \nu^{-1}} h^{1/2} \log(1/h)),
\]

where \( C > 0 \) is independent of \( \nu = h_{\text{min}} / h_{\text{max}}, \lambda, \beta, h, \Delta t \) and \( T \).
Several major ingredients of the proof:

- Introduction of a weight function for a weighted $L^2$-norm argument, similar to that in Johnson et al CMAME 1984 and Cockburn and Guzmán SINUM 2008.

- Generalization of the $L^2$ energy estimate for RKDG3 schemes for smooth solutions in Zhang and Shu SINUM 2010, to weighted $L^2$ energy estimate for the current discontinuous case.
The generalized slope function as defined in Cheng and Shu SINUM 2010 and the highest frequency component in each cell, for a suitable projection of the numerical error, to cope with the troublesome terms resulting from the weight function.

A full utilization of the additional numerical stability in the time direction provided by the TVDRK3 time-marching. However, our result does hold when the CFL number goes to zero and for the semi-discrete case.
Numerical results:

We use uniform spatial meshes, together with the uniform time stepping with the CFL number $\lambda = 0.18$. We compute the errors and convergence orders at the final time $T = 0.25$, on the left and the right, respectively, of the singularity $x = 0.625$, namely,

$$\mathcal{R}_T^L = (-\infty, 0.625 - 0.5h^{1/2}), \quad \text{and} \quad \mathcal{R}_T^R = (0.625 + 0.8h^{1/2}, +\infty).$$

The errors and convergence orders for $k = 2$ are listed in Table 1. As we can see, the optimal orders of convergence are realized; this confirms the prediction of our theorem that the pollution region sizes on both sides of the discontinuity are no larger than about the order $O(h^{1/2})$. 

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Table 1: Errors and convergence orders in the $L^2$-norm and maximum norm, to the left and to the right of the singularity. Here $k = 2$ and $\lambda = 0.18$.

<table>
<thead>
<tr>
<th>$1/h_\ell$</th>
<th>Left-hand interval $\mathcal{R}^L_T$</th>
<th>Right-hand interval $\mathcal{R}^R_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$-err</td>
<td>order</td>
</tr>
<tr>
<td>1000</td>
<td>5.90e-8</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>7.90e-9</td>
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<tr>
<td>4000</td>
<td>1.03e-9</td>
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<td>8000</td>
<td>1.33e-10</td>
<td>2.96</td>
</tr>
<tr>
<td>16000</td>
<td>1.70e-11</td>
<td>2.97</td>
</tr>
<tr>
<td>32000</td>
<td>2.15e-12</td>
<td>2.98</td>
</tr>
</tbody>
</table>

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Now we use the piecewise cubic polynomials \((k = 3)\) on uniform meshes, as the finite element space in the RKDG3 method. We also compute the solution until the same final time \(T = 0.25\), To obtain the optimal fourth order accuracy, we take the time step \(\Delta t = 0.18h^{4/3}\), where \(h\) is the uniform mesh length. The errors and convergence orders on the two domains same as (5), are listed in Table 2. We can observe that, the optimal orders of convergence is also achieved.
Table 2: Errors and convergence orders in the $L^2$-norm and maximum norm, to the left and to the right of the singularity. Here $k = 3$.

<table>
<thead>
<tr>
<th>$1/h_\ell$</th>
<th>$L^2$-err</th>
<th>order</th>
<th>$L^\infty$-err</th>
<th>order</th>
<th>$L^2$-err</th>
<th>order</th>
<th>$L^\infty$-err</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
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<td>2.00e-9</td>
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<td>1.57e-10</td>
<td></td>
<td>2.00e-9</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>1.05e-10</td>
<td>3.91</td>
<td>1.25e-10</td>
<td>4.00</td>
<td>9.76e-12</td>
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<td>1.25e-10</td>
<td>4.00</td>
</tr>
<tr>
<td>4000</td>
<td>7.03e-13</td>
<td>3.90</td>
<td>9.42e-12</td>
<td>3.73</td>
<td>6.35e-13</td>
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<td>7.82e-12</td>
<td>4.00</td>
</tr>
<tr>
<td>8000</td>
<td>4.67e-14</td>
<td>3.91</td>
<td>6.52e-13</td>
<td>3.85</td>
<td>4.27e-14</td>
<td>3.89</td>
<td>5.48e-13</td>
<td>3.84</td>
</tr>
</tbody>
</table>
Several questions to be addressed:

- Is the bound on the pollution region $O(h^{1/2} \log(1/h))$ sharp, especially considering that Cockburn and Guzmán SINUM 2008 has a sharper bound $O(h^{2/3} \log(1/h))$ on the upwind side for RKDG2, and that it is expected in the numerical hyperbolic community (Harten, without formal proof) that higher order schemes should have a narrower pollution region.
Using a carefully designed numerical least square procedure over several mesh refinements, we can estimate the boundary of the pollution region (beyond which the normal $O(h^{k+1})$ convergence is achieved).

For $k = 2$ (third order RKDG in space and time), the result, showing in the next figure, clearly indicates that the least square process provides $s = 0.490$ for the left side and $s = 0.522$ for the right side of the pollution region $O(h^s)$. That is, the pollution region size is almost of the same order $O(h^{1/2})$ on both sides of the discontinuity, suggesting that our estimate about the pollution region size, $O(h^{1/2} \log h^{-1})$, is sharp.
Figure 1: Dependence of the pollution region size and the mesh length. RKDG3 with $k = 2$. 
If we repeat the process above for the $k = 3$ case (fourth order in space and third order in time, with $\Delta t = 0.18h^{4/3}$ so that the global error is fourth order), we obtain similar results. The least square process provides $s = 0.500000000000000097$ for the left boundary and $s = 0.49999999999999937$ for the right boundary of the pollution region $O(h^s)$. That is, the pollution region size is almost of the same order $O(h^{1/2})$ on both sides of the discontinuity, again suggesting that our estimate about the pollution region size, $O(h^{1/2} \log h^{-1})$, is sharp for higher order $k$. 
Figure 2: Dependence of the pollution region size and the mesh length. RKDG3 with $k = 3$. 
The result also holds for one-dimensional linear systems. The generalizations to two dimensions and especially to nonlinear hyperbolic equations are non-trivial and are left for future work.
We develop and analyze DG methods for solving hyperbolic conservation laws

\[ u_t + f(u)_x = g(x, t), \quad (x, t) \in \mathbb{R} \times (0, T], \]
\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \]

where the initial condition \( u_0 \), or the source term \( g(x, t) \), or the solution \( u(x, t) \) contains \( \delta \)-singularities.
Such problems appear often in applications and are difficult to approximate numerically, especially for finite difference schemes.

Many numerical techniques rely on modifications with smooth kernels (mollification) and hence may severely smear such singularities, leading to large errors in the approximation.

In Yang and Shu, Num Math 2013 and Yang, Wei and Shu, JCP 2013, we develop, analyze and apply DG methods for solve hyperbolic equations with $\delta$-singularities. The DG methods are based on weak formulations and can be designed directly to solve such problems without modifications, leading to very accurate results.
Linear equations with singular initial condition

We consider the linear model equation

\[ u_t + \beta u_x = 0 \]
\[ u(x, 0) = u^0(x) \]

where \( \beta \) is a constant, \( u^0(x) \) has compact support, has a sole \( \delta \)-singularity at \( x = 0 \) and is sufficiently smooth everywhere else.

Even though the initial condition \( u^0(x) \) is no longer in \( L^2 \), it does have an \( L^2 \)-projection to the DG space \( V_h \), which we use as the initial condition for the DG scheme. For problems involving \( \delta \)-singularities, negative-order norm estimates are more natural. We have the following theorem in Yang and Shu, Num Math 2013:
**Theorem:** By taking \( \Omega_0 + 2 \text{supp}(K_h^{2k+2,k+1}) \subset \subset \Omega_1 \subset \subset \Omega \setminus \mathcal{R}_T \), we have

\[
\| u(T) - u_h(T) \|_{-(k+1)} \leq Ch^k, \tag{7}
\]
\[
\| u(T) - u_h(T) \|_{-(k+2)} \leq Ch^{k+1/2}, \tag{8}
\]
\[
\| u(T) - u_h(T) \|_{-(k+1),\Omega_1} \leq Ch^{2k+1}, \tag{9}
\]
\[
\| u(T) - K_h^{2k+2,k+1} * u_h(T) \|_{\Omega_0} \leq Ch^{2k+1}, \tag{10}
\]

where the positive constant \( C \) does not depend on \( h \). Here the mesh is assumed to be uniform for (10) but can be regular and non-uniform for the other three inequalities.
Several comments:

- We use the results about the pollution region in Zhang and Shu, Num Math 2014, which is also valid in the current case with \( \delta \)-singularities.

- We follow the proof of negative-order error estimates and post-processing for DG methods solving linear hyperbolic equations with smooth solutions in Cockburn, Luskin, Shu and Süli Math Comp 2003 with suitable adjustments.
Numerical example: We solve the following problem

\[ u_t + u_x = 0, \quad (x, t) \in [0, \pi] \times (0, 1], \]
\[ u(x, 0) = \sin(2x) + \delta(x - 0.5), \quad x \in [0, \pi], \]

(11)

with periodic boundary condition \( u(0, t) = u(\pi, t) \). Clearly, the exact solution is

\[ u(x, t) = \sin(2x - 2t) + \delta(x - t - 0.5). \]
Table 3: $L^2$-norm of the error between the numerical solution and the exact solution for equation (11) after post-processing in the region away from the singularity.

<table>
<thead>
<tr>
<th>$N$</th>
<th>d</th>
<th>$P^1$ polynomial</th>
<th>$P^2$ polynomial</th>
<th>$P^3$ polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>error</td>
<td>order</td>
<td>error</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P^1$ polynomial</td>
<td>$P^2$ polynomial</td>
<td>$P^3$ polynomial</td>
</tr>
<tr>
<td>200</td>
<td>0.2</td>
<td>6.88E-05</td>
<td>-</td>
<td>8.40e-07</td>
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<td>300</td>
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<td>3.92</td>
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<td></td>
<td></td>
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<td>3.98E-13</td>
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<tr>
<td>400</td>
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<td></td>
<td></td>
<td></td>
<td>4.42E-16</td>
</tr>
<tr>
<td>500</td>
<td>0.2</td>
<td>3.01E-06</td>
<td>3.01</td>
<td>6.13e-12</td>
</tr>
<tr>
<td></td>
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<tr>
<td>600</td>
<td>0.2</td>
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<td>3.00</td>
<td>2.37e-12</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.76E-17</td>
</tr>
</tbody>
</table>
We consider the following two dimensional problem

\[ u_t + u_x + u_y = 0, \quad (x, y, t) \in [0, 2\pi] \times [0, 2\pi] \]

\[ u(x, 0) = \sin(x + y) + \delta(x + y - 2\pi), \quad (x, y) \in [0, 2\pi] \times [0, 2\pi], \quad (12) \]

with periodic boundary condition. Clearly, the exact solution is

\[ u(x, t) = \sin(x + y - 2t) + \delta(x + y - 2t) + \delta(x + y - 2t - 2\pi). \]

We use \( Q^k \) polynomial approximation spaces with \( k = 1 \) and 2.
Table 4: $L^2$-norm of the error between the numerical solution and the exact solution for equation (12) after post-processing in the region away from the singularity.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$d$</th>
<th>$Q^1$ polynomial error</th>
<th>$Q^1$ polynomial order</th>
<th>$Q^2$ polynomial error</th>
<th>$Q^2$ polynomial order</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>0.4</td>
<td>2.60E-05</td>
<td>-</td>
<td>3.23E-08</td>
<td>-</td>
</tr>
<tr>
<td>500</td>
<td>0.4</td>
<td>1.24E-05</td>
<td>3.32</td>
<td>2.47E-10</td>
<td>20.0</td>
</tr>
<tr>
<td>600</td>
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<td>7.16E-06</td>
<td>3.01</td>
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</tr>
<tr>
<td>700</td>
<td>0.4</td>
<td>4.50E-06</td>
<td>3.01</td>
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<td>5.47</td>
</tr>
<tr>
<td>800</td>
<td>0.4</td>
<td>3.01E-06</td>
<td>3.02</td>
<td>2.53E-12</td>
<td>5.29</td>
</tr>
</tbody>
</table>
The theory generalizes to linear systems in a straightforward way. We solve the following linear system

\[ u_t - v_x = 0, \quad (x, t) \in [0, 2] \times (0, 0.4], \]
\[ v_t - u_x = 0, \quad (x, t) \in [0, 2] \times (0, 0.4], \]
\[ u(x, 0) = \delta(x - 1), \quad v(x, 0) = 0, \quad x \in [0, 2]. \]

(13)

Clearly, the exact solution (the Green’s function) is

\[ u(x, t) = \frac{1}{2} \delta(x - 1 - t) + \frac{1}{2} \delta(x - 1 + t), \]
\[ v(x, t) = \frac{1}{2} \delta(x - 1 + t) - \frac{1}{2} \delta(x - 1 - t). \]
Figure 3: Solutions of $u$ (left) and $v$ (right) for (13) at $t = 0.4$. 
**Linear equations with singular source terms**

We consider the linear model equation

\[
\begin{align*}
    u_t(x, t) + Lu(x, t) &= g(x, t), \\
    u(x, 0) &= 0,
\end{align*}
\]

\((x, t) \in \Omega \times (0, \infty)\),

with \(L\) being a linear differential operator that does not involve time derivatives and \(g(x, t)\) is a singular source term, for example \(g(x, t) = \delta(x)\). The singular source term can be implemented in the DG scheme in a straightforward way, since it involves only the integrals of the singular source term with test functions in \(V_h\).

By using Duhamel’s principle, we can prove the following theorem (Yang and Shu, Num Math 2013):
**Theorem:** Denote
\[ \mathcal{R}_T = I_i \cup (T - C \log(1/h) h^{1/2}, T + C \log(1/h) h^{1/2}), \]
where \( I_i \) is the cell which contains the concentration of the \( \delta \)-singularity on the source term. Then we have the following estimates

\[ \| u(T) - u_h(T) \|_{-k+1} \leq Ch^k, \]  
(14)

\[ \| u(T) - u_h(T) \|_{-k+2} \leq Ch^{k+1/2}, \]  
(15)

\[ \| u - u_h \|_{-(k+1),\Omega_1} \leq Ch^{2k+1}, \]  
(16)

\[ \| u(T) - K^{2k+2,k+1}_h * u_h(T) \|_{\Omega_0} \leq Ch^{2k+1}, \]  
(17)

where \( \Omega_0 + 2 \text{supp}(K^{2k+2,k+1}_h) \subset \subset \Omega_1 \subset \subset \mathbb{R} \setminus \mathcal{R}_T \). Here the mesh is assumed to be uniform for (17) but can be regular and non-uniform for the other three inequalities.
DG method for hyperbolic equations with singularities

Numerical example: We solve the following problem

\[ u_t + u_x = \delta(x - \pi), \quad (x, t) \in [0, 2\pi] \times (0, 1], \]
\[ u(x, 0) = \sin(x), \quad x \in [0, 2\pi], \]
\[ u(0, t) = 0, \quad t \in (0, 1]. \]  

(18)

Clearly, the exact solution is

\[ u(x, t) = \sin(x - t) + \chi[\pi, \pi + t], \]

where \( \chi[a,b] \) denotes the indicator function of the interval \([a, b]\).
Table 5: $L^2$-norm of the error between the numerical solution and the exact solution for equation (18) after post-processing in the region away from the singularity.

<table>
<thead>
<tr>
<th>$N$</th>
<th>d</th>
<th>$P^1$ polynomial error</th>
<th>order</th>
<th>$P^2$ polynomial error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>401</td>
<td>0.2</td>
<td>1.74E-06</td>
<td>-</td>
<td>4.29E-08</td>
<td>-</td>
</tr>
<tr>
<td>801</td>
<td>0.2</td>
<td>5.92E-09</td>
<td>8.22</td>
<td>6.80E-13</td>
<td>15.9</td>
</tr>
<tr>
<td>1601</td>
<td>0.2</td>
<td>7.36E-10</td>
<td>3.03</td>
<td>1.34E-17</td>
<td>12.3</td>
</tr>
<tr>
<td>3201</td>
<td>0.2</td>
<td>9.19E-11</td>
<td>3.01</td>
<td>3.86E-18</td>
<td>5.13</td>
</tr>
<tr>
<td>6401</td>
<td>0.2</td>
<td>1.15E-11</td>
<td>3.01</td>
<td>1.16E-19</td>
<td>5.07</td>
</tr>
</tbody>
</table>
Even though our theory is established only for linear equations, the DG algorithm can be easily implemented for nonlinear hyperbolic equations involving $\delta$-singularities.

In Canuto, Fagnani and Tilli, SIAM J Control and Optimization 2012, the following problem

$$\rho_t + F_x = 0, \quad x \in [0, 1], \ t > 0,$$

$$\rho(0, t) = u_0(x), \quad t > 0,$$

(19)

is studied. Here $\rho$ is the density function, which is always positive.
The flux $F$ is given by

$$F(t, x) = v(t, x)\rho(t, x),$$

and the velocity $v$ is defined by

$$v(t, x) = \int_{\mathbb{R}^n} (y - x)\xi(y - x)\rho(t, y) dy,$$

where $\xi(x)$ is a positive function and supported on a ball centered at zero with radius $R$. Canuto et al. proved that when $t$ tends to infinity, the density function $\rho$ will converge to some $\delta$-singularities, and the distances between any of them cannot be less than $R$. Some computational results are shown in Canuto et al. based on a first order finite volume method.
We use the DG algorithm with the positivity-preserving limiter in Zhang and Shu JCP 2010, which can maintain positivity without affecting the high order accuracy, to both the one and two dimensional Rendez-vous algorithms, in Yang and Shu, Num Math 2013 and Yang, Wei and Shu, JCP 2013.
Figure 4: Numerical density at $t = 1000$ with $N = 400$ when using $\mathcal{P}^0$ (left) and $\mathcal{P}^1$ (right) polynomials.
In 2D, the model is

\[
\rho_t + \text{div}(\mathbf{v}\rho) = 0, \quad \mathbf{x} \in [-1, 1]^2, \quad t > 0, \\
\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad t > 0, 
\]

where the velocity \( \mathbf{v} \) is defined by

\[
\mathbf{v}(\mathbf{x}, t) = \int_{B_R(\mathbf{x})} (\mathbf{y} - \mathbf{x})\rho(\mathbf{y}, t) d\mathbf{y}.
\]

In this example, we take \( R = 0.1 \) and

\[
\rho_0(\mathbf{x}) = \begin{cases} 1 & r < 0.5, \\ 0 & r > 0.5, \end{cases}
\]

where \( r = \|\mathbf{x}\| \) is the Euclidean norm of \( \mathbf{x} \).
In Canuto et al., the authors demonstrated that the exact solution should be a single delta placed at the origin.

However, when we use rectangle meshes, we observe more than one delta singularity for $R$ sufficiently small. This is because the meshes are not invariant under rotation.
Figure 5: Numerical density $\rho$ with a rectangular $100 \times 100$ mesh using $P^0$ elements. $R = 0.08$ (left) and $R = 0.1$ (right).
To tackle this problem, we follow the same ideas in Cheng and Shu, JCP 2010; CiCP 2012, and construct a special equal-angle-zoned mesh. The structure of the mesh is given in figure 6. By using such a special mesh, the limit density is a single delta placed at the origin.
Figure 6: Left: Equal-angle-zoned mesh. Right: Numerical density $\rho$ for (20) at $t = 2000$ with $N = 200$ using $P^0$ elements.
Another important system admitting $\delta$-singularities in its solutions is the pressureless Euler equation

$$w_t + f(w)_x = 0, \quad t > 0, \quad x \in \mathbb{R},$$

(21)

with $w = \begin{pmatrix} \rho \\ m \end{pmatrix}$, $f(w) = \begin{pmatrix} m \\ \rho u^2 \end{pmatrix}$, where $m = \rho u$, $\rho$ is the density function and $u$ is the velocity.

It is quite difficult to obtain stable schemes for solve this system, especially for high order schemes.
A good property of this system is that the density is always positive and the velocity satisfies a maximum-principle. Thus, in 1D, the convex set

\[ G = \left\{ \mathbf{w} = \begin{pmatrix} \rho \\ m \end{pmatrix} : \rho > 0, a\rho \leq m \leq b\rho \right\}, \]

where

\[ a = \min u_0(x), \quad b = \max u_0(x), \]

with \( u_0 \) being the initial velocity, is invariant. In Yang, Wei and Shu, JCP 2013, we adapt the techniques in Zhang and Shu, JCP 2010 to design a limiter to guarantee that our DG solution stays in set \( G \) without affecting high order accuracy. This is also generalized to 2D. Our scheme is thus very robust, stable and high order accurate for this pressureless Euler system.
We consider the following initial data

\[ \rho_0(x) = \sin(x) + 2, \quad u_0(x) = \sin(x) + 2, \]  

(23)

with periodic boundary condition. Clearly, the exact solution is

\[ u(x, t) = u_0(x_0), \quad \rho(x, t) = \frac{\rho_0(x_0)}{1 + u_0'(x_0)}, \]

where \( x_0 \) is given implicitly by

\[ x_0 + tu_0(x_0) = x. \]
Table 6: $L^2$-norm of the error between the numerical density and the exact density for initial condition (23).

<table>
<thead>
<tr>
<th>$N$</th>
<th>k=1</th>
<th>k=2</th>
<th>k=3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error</td>
<td>order</td>
<td>error</td>
</tr>
<tr>
<td>20</td>
<td>1.41E-02</td>
<td>-</td>
<td>6.84E-04</td>
</tr>
<tr>
<td>40</td>
<td>4.18E-03</td>
<td>1.76</td>
<td>1.04E-04</td>
</tr>
<tr>
<td>80</td>
<td>1.30E-03</td>
<td>1.68</td>
<td>1.55E-05</td>
</tr>
<tr>
<td>160</td>
<td>4.24E-04</td>
<td>1.62</td>
<td>2.41E-06</td>
</tr>
<tr>
<td>320</td>
<td>1.51E-04</td>
<td>1.49</td>
<td>3.80E-07</td>
</tr>
</tbody>
</table>
We consider the following initial condition

\[
\rho_0(x) = \begin{cases} 
1 & x < 0, \\
0.25 & x > 0,
\end{cases} \quad u_0(x) = \begin{cases} 
1 & x < 0, \\
0 & x > 0.
\end{cases}
\]  

Clearly, the exact solution is

\[
(\rho(x, t), u(x, t)) = \begin{cases} 
(1, 1) & x < 2t/3, \\
(0.25, 0) & x > 2t/3,
\end{cases}
\]

and at \( x = \frac{2t}{3} \), the density should be a \( \delta \)-function.
Figure 7: Numerical density (left) and velocity (right) at $t = 0.5$ with $P^1$ polynomials for initial condition (24).
We consider the following initial condition

\[
\begin{align*}
\rho(x, y, 0) &= \rho_0(x + y) = \exp(\sin(x + y)), \\
u(x, y, 0) &= u_0(x + y) = \frac{1}{3}(\cos(x + y) + 2), \\
v(x, y, 0) &= v_0(x + y) = \frac{1}{3}(\sin(x + y) + 2).
\end{align*}
\] (25)

The exact solution is

\[
\begin{align*}
u(x, y, t) &= u_0(z_0), \quad v(x, y, t) = v_0(z_0), \quad \rho(x, y, t) = \frac{\rho_0(z_0)}{1 + u'_0(z_0) + v'_0(z_0)},
\end{align*}
\]

where \(z_0\) is given implicitly by

\[
z_0 + t(u_0(z_0) + v_0(z_0)) = x + y.
\]
Table 7: $L^2$-norm of the error between the numerical density and the exact density for initial condition (25).

<table>
<thead>
<tr>
<th>$N$</th>
<th>$k=1$ error</th>
<th>$k=1$ order</th>
<th>$k=2$ error</th>
<th>$k=2$ order</th>
<th>$k=3$ error</th>
<th>$k=3$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.512</td>
<td>-</td>
<td>0.107</td>
<td>-</td>
<td>3.42E-02</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>0.176</td>
<td>1.54</td>
<td>3.12E-02</td>
<td>1.78</td>
<td>3.57E-03</td>
<td>3.26</td>
</tr>
<tr>
<td>40</td>
<td>6.48E-02</td>
<td>1.44</td>
<td>8.52E-03</td>
<td>1.87</td>
<td>4.86E-04</td>
<td>2.88</td>
</tr>
<tr>
<td>80</td>
<td>2.32E-02</td>
<td>1.48</td>
<td>1.39E-03</td>
<td>2.62</td>
<td>3.97E-05</td>
<td>3.61</td>
</tr>
<tr>
<td>160</td>
<td>9.08E-03</td>
<td>1.35</td>
<td>1.92E-04</td>
<td>2.86</td>
<td>3.65E-06</td>
<td>3.45</td>
</tr>
</tbody>
</table>
We consider the following initial condition

$$\rho(x, y, 0) = \frac{1}{100}, \quad (u, v)(x, y, 0) = \left(-\frac{1}{10} \cos \theta, -\frac{1}{10} \sin \theta\right), \quad (26)$$

where $\theta$ is the polar angle.

Since all the particles are moving towards the origin, the density function at $t > 0$ should be a single delta at the origin.
Figure 8: Numerical density (left) and velocity field (right) at $t = 0.5$ for the initial condition (26).
We consider the following initial condition

$$\rho(x, y, 0) = \frac{1}{10}, \quad (u, v)(x, y, 0) = \begin{cases} 
(-0.25, -0.25) & x > 0, y > 0, \\
(0.25, -0.25) & x < 0, y > 0, \\
(0.25, 0.25) & x < 0, y < 0, \\
(-0.25, 0.25) & x > 0, y < 0.
\end{cases}$$

(27)

Figure 9 shows the numerical density and velocity field at $t = 0.5$. From the figure, we can observe $\delta$-singularities located at the origin and the two axes.
Figure 9: Numerical density (left) and velocity field (right) at $t = 0.5$ for initial condition (27).
We consider the following initial condition

\[ \rho(x, y, 0) = \frac{1}{100}, \quad (u, v)(x, y, 0) = \begin{cases} 
(\cos \theta, \sin \theta) & r < 0.3, \\
(-\frac{1}{2} \cos \theta, -\frac{1}{2} \sin \theta) & r > 0.3,
\end{cases} \]

(28)

where \( r = \sqrt{x^2 + y^2} \) and \( \theta \) is the polar angle.

Figure 10 shows the numerical density (contour plot) and velocity field at \( t = 0.5 \). From the figure, we can observe \( \delta \)-shocks located on a circle and vacuum inside.
Figure 10: Numerical density (left) and velocity field (right) at $t = 0.5$ for initial condition (28).
We consider the following initial condition

\[
\rho(x, y, 0) = 0.5, \quad (u, v)(x, y, 0) = \begin{cases} 
(0.3, 0.4) & x > 0, y > 0, \\
(-0.4, 0.3) & x < 0, y > 0, \\
(-0.3, -0.4) & x < 0, y < 0, \\
(0.4, -0.3) & x > 0, y < 0.
\end{cases}
\] 

(29)

Figure 11 shows the numerical density (contour plot) and velocity field with \( N = 50 \) at \( t = 0.4 \). From the figure, we can observe that the numerical solution approximates the vacuum quite well.
Figure 11: Numerical density (left) and velocity field (right) at $t = 0.4$ with $N = 50$ for initial condition (29).
Concluding remarks

- DG methods are suitable for computing solutions with discontinuities or $\delta$-singularities, because it satisfies a cell entropy inequality and is based on a weak formulation.

- For linear model equations, the DG methods can be shown to converge in optimal $L^2$ error $O(h^{1/2} \log(1/h))$ away from the singularity for discontinuous but piecewise smooth solutions.
• For linear model equations, the DG methods can be shown to converge in negative norms when either the initial condition or the source term contains $\delta$-singularities. This convergence is of $O(h^{k+1/2})$ order in the whole domain and of $O(h^{2k+1})$ order $O(h^{1/2} \log(1/h))$ away from the singularities. Post-processing then produces $O(h^{2k+1})$ order superconvergence in the strong $L^2$ norm $O(h^{1/2} \log(1/h))$ away from the singularities.

• DG methods work well for nonlinear problems containing $\delta$-singularities. It is usually important to design and apply a bound-preserving limiter which keeps high order accuracy and can effectively prevent nonlinear instability caused by overshoots of the numerical solution into the ill-posed regime of the nonlinear PDEs.
DG METHOD FOR HYPERBOLIC EQUATIONS WITH SINGULARITIES

The End

THANK YOU!