Multi-domain Hybrid RKDG and WENO-FD Method for Hyperbolic Conservation Laws

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Joint work with Jian Cheng, Kun Wang
Outline

- Introduction
- RKDG and WENO-FD methods
- Hybrid RKDG+WENO-FD method
- Numerical results
- Conclusions and future work
Introduction—3rd order or higher is necessary

2nd order methods do not meet the requirements

Higher order efficient methods are demanding for 3D complex flow Simulations

Complex flow requires higher order methods

Higher order enables lower mesh numbers

Spatial accuracy $\epsilon = 10^{-6}$

<table>
<thead>
<tr>
<th>Order of method</th>
<th>Mesh number per unit volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd</td>
<td>$10 \times 10 \times 10 \approx 0(10^3)$</td>
</tr>
<tr>
<td>3rd</td>
<td>$4.6 \times 4.6 \times 4.6 \approx 0(10^2)$</td>
</tr>
</tbody>
</table>
Introduction—Cost & Efficiency

Comparison of computational costs for popular higher order methods

| High order FD methods | Good: Cheaper (Comparable to 2\textsuperscript{nd} order FV method) | Bad: Uniform mesh; not for complex geometry |

<table>
<thead>
<tr>
<th>3D Quadrilateral</th>
<th>Surface GPs</th>
<th>Volume GPs</th>
<th>Reconstr</th>
<th>Surface Flux</th>
<th>Surface Integral</th>
<th>Volume Integral</th>
<th>Sum</th>
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<tr>
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<td>0</td>
<td>24</td>
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<tr>
<td>3\textsuperscript{rd}-WENO-FV</td>
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<td>0</td>
<td>24\textsuperscript{*}8</td>
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<td>9\textsuperscript{*}6</td>
<td>9\textsuperscript{*}6</td>
<td>10</td>
<td>199</td>
</tr>
</tbody>
</table>

| High order FV/DG methods | Good: unstructured mesh; complex geometry | Bad: expensive (1 order higher than 2\textsuperscript{nd} methods) |

A single higher order method does not work out well for 3D complex flow over complex geometry
Introduction—Hybrid technique

• Multi-domain methods

  - Patched grids
  - Overlapping grids

• Hybrid methods
  - Hybrid finite compact-WENO scheme
  - Multi-domain hybrid spectral-WENO methods
  - Etc.
• Hybrid methods based on reconstruction
  – Balsara et al. [JCP, 2007], hybrid RKDG + HWENO schemes
  – Luo et al. [AIAA, 2010], Reconstructed DG
  – Dumbser et al. [JCP, 2008] one step finite volume +DG, PnPm
  – Zhang et al. [JCP, 2012], FV+DG

• Hybrid methods based on domain decomposition
  – Costa et al. [JCP, 2007], hybrid spectral-WENO methods
  – Shahbazi et al. [JCP, 2007], Fourier-continuation/WENO
  – Zhu et al. [CiCP, 2011], hybrid finite difference and finite element time domain (FDTD/FETD) method (Maxwell equations)
  – Utzmann et al. [AIAA, 2006], L’eger et al. [AIAA, 2012], DG+FD (Acoustic)

Hybrid FD + DG: Higher order hybrid WENO-FD+RKDG
Outline

- Introduction
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- Hybrid RKDG+WENO-FD method
- Numerical results
- Conclusions and future work
RKDG and WENO are the representatives of truly high order methods
RKDG methods

Two dimensional hyperbolic conservation laws:

\[
\begin{aligned}
&\quad u_t + f(u)_x + g(u)_y = 0 \quad \text{in } \Omega \times (0, T) \\
&\quad u(x, y, 0) = u_0(x, y)
\end{aligned}
\]

Spatial discretization:

The solution and test function space:

\[
V^K_h = \{v(x, y) : v(x, y)|_{\Delta_j} \in P^k(\Delta_j)\}
\]

DG adopts a series of local basis over target cell:

\[
\{v^{(l)}(x, y), l = 0, 1, \ldots, K; K = (k + 1)(k + 2) / 2 - 1\}
\]

The numerical solution can be written as:

\[
u^h(x, y, t) = \sum_l u_l(t)v^{(l)}(x, y)\]
RKDG methods

Multiply test functions and integrate over target cell:

\[
\frac{d}{dt} \int_{\Delta_j} u^h v^{(l)}(x, y) dx dy + \int_{\partial\Delta_j} (f(u^h), g(u^h))^T \cdot n v^{(l)}(x, y) ds
\]

\[
- \int_{\Delta_j} (f(u^h) \frac{\partial}{\partial x} v^{(l)}(x, y) + g(u^h) \frac{\partial}{\partial y} v^{(l)}(x, y)) dx dy = 0
\]

where \( n = (n_x, n_y) \)

On cell boundaries, the numerical solution is discontinuous, a numerical flux based on Riemann solution is used to replace the original flux:

\[
(f(u^h), g(u^h))^T \cdot n \rightarrow h_{c\Delta_j, n}(u^+, u^-)
\]

Time discretization:

third-order Runge-Kutta method
WENO methods

- **WENO-FD**
  (finite difference based WENO)
  - Efficient for structured mesh
  - Not applicable for unstructured mesh
  - Difficult in treatment of complex boundaries

- **WENO-FV**
  (finite volume based WENO)
  - Easy in treatment of complex boundaries
  - Costly and troublesome for maintaining higher order for unstructured mesh

- WENO-FV has computational cost 4 times (2D)/9 times (3D) larger than WENO-FD for 3rd order accuracy!
WENO-FD schemes

Two dimensional hyperbolic conservation laws:

\[
\begin{cases}
  u_t + f(u)_x + g(u)_y = 0 & \text{in } \Omega \times (0, T) \\
  u(x, y, 0) = u_0(x, y)
\end{cases}
\]

Spatial discretization:

For a WENO-FD scheme, uniform grid is required and solve directly using a conservative approximation to the space derivative:

\[
\frac{d u_{i,j}}{dt} + \frac{1}{\Delta x} \left( \hat{f}_{i+\frac{1}{2},j} - \hat{f}_{i-\frac{1}{2},j} \right) + \frac{1}{\Delta y} \left( \hat{g}_{i+\frac{1}{2},j} - \hat{g}_{i-\frac{1}{2},j} \right) = 0
\]

\[
f(x) = \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi) d\xi
\]

The numerical fluxes $\hat{f}_{i+\frac{1}{2},j}, \hat{g}_{i+\frac{1}{2},j}$ are obtained by one dimensional WENO-FD approximation procedure.
WENO-FD schemes

One dimensional WENO-FD procedure: (5th-order WENO-FD)

WENO construct polynomial $q(x)$ on each candidate stencil $S_0, S_1, S_2$ and use the convex combination of all candidate stencils to achieve high order accurate.

$$q^0_{j+\frac{1}{2}} = a_1^0 f_{j-2} + a_2^0 f_{j-1} + a_3^0 f_j + O(\Delta x^3)$$

$$q^1_{j+\frac{1}{2}} = a_1^1 f_{j-2} + a_2^1 f_{j-1} + a_3^1 f_j + O(\Delta x^3)$$

$$q^2_{j+\frac{1}{2}} = a_1^2 f_{j-2} + a_2^2 f_{j-1} + a_3^2 f_j + O(\Delta x^3)$$

The numerical flux for 5th order WENO-FD:

$$\hat{f}_{j+\frac{1}{2}} = d_0 q^0_{j+\frac{1}{2}} + d_1 q^1_{j+\frac{1}{2}} + d_2 q^2_{j+\frac{1}{2}}$$
WENO-FD schemes

One dimensional WENO-FD procedure: (5th-order WENO-FD)

Classical WENO schemes use the smooth indicator (Jiang and Shu JCP, 1996) of each stencil as follows:

\[
\beta_k = \sum_{l=1}^{r-1} \int_{x_{j-l/2}}^{x_{j+(l-1)/2}} \Delta x^{2l-1} \left( \frac{\partial^l q^k(x)}{\partial^l x} \right)^2 dx
\]

The nonlinear weights are given by:

\[
w_k = \frac{\alpha_k}{\sum_{s=0}^{r-1} \alpha_s}, \quad \alpha_k = \frac{d_k}{(\epsilon + \beta_k)^2}, \quad k = 0, 1, ..., r-1
\]

The numerical flux for 5th order WENO-FD:

\[
\hat{f}_{j+1/2} = w_0 q^0_{j+1/2} + w_1 q^1_{j+1/2} + w_2 q^2_{j+1/2}
\]
## Summary

<table>
<thead>
<tr>
<th>Advantage</th>
<th>RKDG</th>
<th>WENO-FD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Well in handling complex geometries</td>
<td>Highly efficient in structured grid</td>
</tr>
<tr>
<td>Weakness</td>
<td>Expensive in computational costs and storage requirements</td>
<td>Only in uniform mesh and hard in handling complex geometry</td>
</tr>
</tbody>
</table>
Outline

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Multidomain hybrid RKDG+WENO-FD method

RKDG+WENO-FD method

- Couple RKDG and WENO-FD based on domain decomposition
- Combine advantages of both RKDG and WENO-FD, 90-99% domain in WENO-FD and 10-1% domain in RKDG

Hybrid mesh approach

Cut-cell approach
RKDG+WENO-FD method: structured meshes

RKDG+WENO-FD method for one dimensional conservation laws

\[
\begin{aligned}
    &u_t + f(u)_x = 0, \\
    &u(x, 0) = u_0(x), \quad \text{in } \Omega \times (0, T).
\end{aligned}
\]

\[
\begin{align*}
    \frac{d}{dt} u_j + \frac{1}{\Delta x_j} \left( \widehat{f}^{(rkdg)}_{j+\frac{1}{2}} - \widehat{f}^{(rkdg)}_{j-\frac{1}{2}} \right) &= 0, \quad j = 1, \ldots, J - 1, \\
    \frac{d}{dt} u_J + \frac{1}{\Delta x_J} \left( \widehat{f}^{(weno)}_{J+\frac{1}{2}} - \widehat{f}^{(rkdg)}_{J-\frac{1}{2}} \right) &= 0, \quad j = J, \\
    \frac{d}{dt} u_{J+1} + \frac{1}{\Delta x_{J+1}} \left( \widehat{f}^{(weno)}_{J+\frac{1}{2}} - \widehat{f}^{(rkdg)}_{J-\frac{1}{2}} \right) &= 0, \quad j = J + 1, \\
    \frac{d}{dt} u_{J+2} + \frac{1}{\Delta x_{J+2}} \left( \widehat{f}^{(weno)}_{J+\frac{1}{2}} - \widehat{f}^{(weno)}_{J-\frac{1}{2}} \right) &= 0, \quad j = J + 2, \ldots, N.
\end{align*}
\]

Conservative coupling method: \( \widehat{f}^{(r)}_{J+\frac{1}{2}} = \widehat{f}^{(l)}_{J+\frac{1}{2}} \)

Non-conservative coupling method: \( \widehat{f}^{(r)}_{J+\frac{1}{2}} \neq \widehat{f}^{(l)}_{J+\frac{1}{2}} \)
RKDG+WENO-FD method: Construction of interface flux

Construction of WENO flux at the interface:
1. Deploy ghost nodes at the DG domain
2. Compute the point value at each ghost points via DG solution
3. Obtain the WENO flux at the cell interface J+1/2
RKDG+WENO-FD method: Construction of interface flux

Construction of DG flux at the interface:
1. Construct a WENO polynomial in cell J+1
2. Project the WENO polynomial to the DG space in cell J+1
3. Obtain the DG flux at cell boundary J+1/2

Figure: Construct DG flux with point values of WENO subdomain.
RKDG+WENO-FD method: Construction of interface flux

Non-conservative Coupling (method):

\[
\hat{f}^{(\downarrow)}_{J+\frac{1}{2}} = \hat{f}^{(rkdg)}_{J+\frac{1}{2}}, \quad \hat{f}^{(\uparrow)}_{J+\frac{1}{2}} = \hat{f}^{(weno)}_{J+\frac{1}{2}},
\]

\[
\begin{align*}
\frac{d}{dt}u_j + \frac{1}{\Delta x_j} \left( \hat{f}^{(rkdg)}_{j+\frac{1}{2}} - \hat{f}^{(rkdg)}_{j-\frac{1}{2}} \right) &= 0, \quad j = 1, \ldots, J, \\
\frac{d}{dt}u_j + \frac{1}{\Delta x_j} \left( \hat{f}^{(weno)}_{j+\frac{1}{2}} - \hat{f}^{(weno)}_{j-\frac{1}{2}} \right) &= 0, \quad j = J + 1, \ldots, N.
\end{align*}
\]

Conservative Coupling (method):

\[
\begin{align*}
\frac{d}{dt}u_j + \frac{1}{\Delta x_j} \left( \hat{f}^{(rkdg)}_{j+\frac{1}{2}} - \hat{f}^{(rkdg)}_{j-\frac{1}{2}} \right) &= 0, \quad j = 1, \ldots, J - 1, \\
\frac{d}{dt}u_j + \frac{1}{\Delta x_j} \left( \hat{f}^{(rkdg)}_{j+\frac{1}{2}} - \hat{f}^{(rkdg)}_{j-\frac{1}{2}} \right) &= 0, \quad j = J, \\
\frac{d}{dt}u_j + \frac{1}{\Delta x_j} \left( \hat{f}^{(weno)}_{j+\frac{1}{2}} - \hat{f}^{(weno)}_{j-\frac{1}{2}} \right) &= 0, \quad j = J + 1, \\
\frac{d}{dt}u_j + \frac{1}{\Delta x_j} \left( \hat{f}^{(weno)}_{j+\frac{1}{2}} - \hat{f}^{(weno)}_{j-\frac{1}{2}} \right) &= 0, \quad j = J + 2, \ldots, N,
\end{align*}
\]

\[\hat{f}^{(\downarrow)}_{J+\frac{1}{2}} - \hat{f}^{(rkdg)}_{J+\frac{1}{2}} \text{ or } \hat{f}^{(\downarrow)}_{J+\frac{1}{2}} - \hat{f}^{(weno)}_{J+\frac{1}{2}} \text{ or } \ldots\]
RKDG+WENO-FD method: theoretical results

We consider a general form of the hybrid RKDG+WENO-FD method which a $p^{th}$-order DG method couples with a $q^{th}$-order WENO-FD scheme (SISC, 2013):

**Accuracy:**

- Conservative multi-domain hybrid method of $p^{th}$-order RKDG and $q^{th}$-order WENO-FD is of $1^{st}$-order accuracy in max-norm.

- Non-conservative multi-domain hybrid method of $p^{th}$-order RKDG and $q^{th}$-order WENO-FD can preserve $r^{th}$-order ($r=\min(p,q)$) accuracy in smooth region.

**Conservation error:**

$$CE = \left| \sum_j \Delta x_j u_j^{n+1} - \sum_j \Delta x_j u_j^n \right|$$

- The conservative error of non-conservative multi-domain hybrid method of $p^{th}$-order RKDG and $q^{th}$-order WENO-FD is of $3^{rd}$-order accuracy.
Proof of accuracy for the conservative hybrid method

Using DG flux:

\[
\frac{d}{dt} u_{J+1} + \frac{1}{\Delta x_{J+1}} (\hat{f}^{\text{weno}}_{J+\frac{3}{2}} - \hat{f}^{\text{rkdg}}_{J+\frac{1}{2}}) = 0.
\]

Using WENO flux:

\[
\frac{d}{dt} u_J + \frac{1}{\Delta x_J} (\hat{f}^{\text{weno}}_{J+\frac{1}{2}} - \hat{f}^{\text{rkdg}}_{J-\frac{1}{2}}) = 0.
\]
Proof of accuracy for the conservative hybrid method—continued

For WENO flux:

$$\tilde{f}_{J+\frac{3}{2}}^{\text{(weno)}} = h_{J+\frac{3}{2}} + O(\Delta x^q)$$

For DG flux:

$$\tilde{f}_{J+\frac{1}{2}}^{(rkgd)} = f_{J+\frac{1}{2}} + O(\Delta x^p)$$

$$\tilde{f}_{J+\frac{3}{2}}^{\text{(weno)}} - \tilde{f}_{J+\frac{1}{2}}^{(rkgd)} = h_{J+\frac{3}{2}} - f_{J+\frac{1}{2}} + O(\Delta x^{\min(p,q)})$$

$$h_{J+\frac{3}{2}} = f_{J+\frac{3}{2}} - \frac{1}{24} \frac{d^2 f}{dx^2}\bigg|_{x=x_{J+\frac{3}{2}}} \Delta x^2 + O(\Delta x^4).$$
RKDG+WENO-FD method: theoretical results

**Stability:**

- Non-conservative Coupling Approach: Numerically stable

- Conservative Coupling Approach:
  - Numerically stable with DG flux
  - Non-linearly stable with WENO flux
RKDG+WENO-FD method: theoretical results

Stability:

Using WENO-FD flux at coupling interface:

(a) case-1

(b) case-2

<table>
<thead>
<tr>
<th></th>
<th>case-1</th>
<th>case-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(u) &gt; 0$</td>
<td>$\times$</td>
<td>$\sqrt{\text{ } }$</td>
</tr>
<tr>
<td>$f'(u) &lt; 0$</td>
<td>$\sqrt{\text{ } }$</td>
<td>$\sqrt{\text{ } }$</td>
</tr>
</tbody>
</table>

linear

nonlinear

Table: Stability of conservative coupling approach with WENO-FD flux.
RKDG+WENO-FD method: theoretical results

**Stability:**

Using DG flux at coupling interface:

(c) case-1

<table>
<thead>
<tr>
<th></th>
<th>case-1</th>
<th>case-2</th>
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<tbody>
<tr>
<td></td>
<td>$f'(u) &gt; 0$</td>
<td>$f'(u) &lt; 0$</td>
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<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>nonlinear</td>
<td>✓</td>
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</tbody>
</table>

(d) case-2

Table: Stability of conservative coupling approach with DG flux.
RKDG+WENO-FD method: hybrid mesh

Construct WENO-FD flux

When construct WENO-FD flux, RKDG can provide the central point values for WENO construction.

FIG. RKDG provides point values for WENO-FD construction
RKDG+WENO-FD method: hybrid meshes

Construct RKDG flux

When construct RKDG flux, we use WENO point values to construct RKDG flux, we follow these three steps:

1. First, we construct a high order polynomial \( p(x, y) \) at target cell \( I_{i,j} \).

2. Second, we construct degrees of freedom of RKDG at the target cell with a local orthogonal basis

\[
 u_{i,j}^{(l)} \approx \frac{1}{\int_{I_{i,j}} (v_{i,j}^{(l)}(x,y))^2 dx dy} \int_{I_{i,j}} p(x,y)v_{i,j}^{(l)}(x,y) dx dy
\]

3. At last, we get Gauss quadrature point values and form the interface flux for RKDG

\[
 \hat{f}^{(RKDG)}_I = \hat{f}^{LF}(u^+, u^-)
\]
RKDG+WENO-FD method: shock approaching

Indicator of polluted cell:

We define
\[ \tilde{u} = u_{i,j} - u_{g}^{(r)-} \]
and
\[ \Delta_{+} u = u_{i+1,j} - u_{i,j} \quad \Delta_{-} u = u_{i,j} - u_{e} \]

A TVD(TVB) smooth indicator is applied at the coupling interface to indicate possible discontinuities:
\[ \tilde{u}^{(\text{mod})} = m(\tilde{u}, \Delta_{+} u, \Delta_{-} u) \]

where \( m(a1,a2,a3) \) is TVD(TVB) minmod function.
RKDG+WENO-FD method: hybrid meshes

Hybrid RKDG+WENO-FD approach

- Initiate WENO and RKDG data;
- Construct Lagrange interpolation at the interface and use coupling interface indicator;
- Choose the coupling scheme at the interface:
  1. If the stencil is polluted, choose conservative coupling at the interface (WENO-FD flux is used);
  2. If solution in the stencil is smooth enough, choose non-conservative coupling.
- Space discretization RKDG and WENO-FD domain;
- Time discretization.
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Numerical results: Accuracy tests

Example 1-1D: 1D linear scalar conservation law

\[
\begin{align*}
    u_t - u_x &= 0, \quad x \in (0.0, 2.0), \\
    u(x, 0) &= \sin(\pi(x + 0.25)).
\end{align*}
\]

Accuracy of conservative hybrid

<table>
<thead>
<tr>
<th>N</th>
<th>(L_\infty) error</th>
<th>(L_\infty) order</th>
<th>(L_\infty) error</th>
<th>(L_\infty) order</th>
<th>(L_\infty) error</th>
<th>(L_\infty) order</th>
<th>(L_\infty) error</th>
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<td>1.45E-2</td>
<td>1.62E-2</td>
<td>1.66E-2</td>
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<td></td>
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</tr>
<tr>
<td>80</td>
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<td>7.71E-3</td>
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<tr>
<td>160</td>
<td>3.63E-3</td>
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<td>3.75E-5</td>
<td>1.04</td>
<td>3.77E-3</td>
<td>1.05</td>
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<td></td>
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<tr>
<td>320</td>
<td>1.82E-3</td>
<td>1.00</td>
<td>1.85E-5</td>
<td>1.02</td>
<td>1.85E-3</td>
<td>1.03</td>
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Accuracy of non-conservative hybrid

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<th>N</th>
<th>(L_\infty) error</th>
<th>(L_\infty) order</th>
<th>(L_1) error</th>
<th>(L_1) order</th>
<th>(L_\infty) error</th>
<th>(L_\infty) order</th>
<th>(L_1) error</th>
<th>(L_1) order</th>
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<tr>
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<td>2.29E-6</td>
<td>3.15E-7</td>
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</tr>
<tr>
<td>80</td>
<td>2.56E-5</td>
<td>2.53</td>
<td>7.66E-8</td>
<td>5.85E-9</td>
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<td></td>
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</tr>
<tr>
<td>160</td>
<td>3.61E-6</td>
<td>2.83</td>
<td>8.45E-10</td>
<td>1.42E-10</td>
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<tr>
<td>320</td>
<td>4.80E-7</td>
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<td>2.55E-11</td>
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</table>
### Numerical results: Accuracy tests

**Example 1-1D: 1D linear scalar conservation law**

Local conservation error of non-conservative hybrid

<table>
<thead>
<tr>
<th>N</th>
<th>$L_\infty$ error</th>
<th>$L_\infty$ order</th>
<th>$L_\infty$ error</th>
<th>$L_\infty$ order</th>
<th>$L_\infty$ error</th>
<th>$L_\infty$ order</th>
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</thead>
<tbody>
<tr>
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<td>8.40E-6</td>
<td>3.01</td>
<td>8.40E-6</td>
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<td>2.31E-6</td>
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<tr>
<td>80</td>
<td>1.04E-6</td>
<td>3.00</td>
<td>1.03E-6</td>
<td>3.02</td>
<td>2.89E-7</td>
<td>3.00</td>
</tr>
<tr>
<td>160</td>
<td>1.30E-7</td>
<td>3.00</td>
<td>1.30E-7</td>
<td>3.00</td>
<td>3.61E-8</td>
<td>3.00</td>
</tr>
<tr>
<td>320</td>
<td>1.63E-8</td>
<td>3.00</td>
<td>1.63E-8</td>
<td>3.00</td>
<td>4.52E-9</td>
<td>3.00</td>
</tr>
</tbody>
</table>
Numerical results: Accuracy tests

Example 1: 2D linear scalar conservation law

We test the accuracy of the hybrid RKDG+WENO-FD method when applied to solve a two dimensional linear scalar conservation law with exact boundary condition couple interface at \( x=0.5, \ 0<y<1 \).

\[
\begin{align*}
    u_t + u_x + u_y &= 0 \quad (x, y) \in (0,1) \times (0,1) \\
    u(x,0) &= \sin(2\pi x) \sin(2\pi y)
\end{align*}
\]

Table 1: Accuracy for hybrid RKDG+WENO-FD methods (Example 1)

<table>
<thead>
<tr>
<th>N</th>
<th>( L_\infty(DG) )</th>
<th>( L_\infty(WENO) )</th>
<th>Order</th>
<th>( L_1(DG) )</th>
<th>( L_1(WENO) )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/20</td>
<td>6.33E-04</td>
<td>6.17E-04</td>
<td></td>
<td>1.13E-04</td>
<td>1.99E-04</td>
<td></td>
</tr>
<tr>
<td>1/40</td>
<td>6.35E-05</td>
<td>4.08E-05</td>
<td>3.31</td>
<td>1.17E-05</td>
<td>9.16E-06</td>
<td>3.27</td>
</tr>
<tr>
<td>1/80</td>
<td>7.77E-06</td>
<td>5.07E-06</td>
<td>3.00</td>
<td>1.44E-06</td>
<td>7.24E-07</td>
<td>3.02</td>
</tr>
<tr>
<td>1/160</td>
<td>9.39E-07</td>
<td>7.51E-07</td>
<td>2.76</td>
<td>1.82E-07</td>
<td>7.41E-08</td>
<td>2.98</td>
</tr>
</tbody>
</table>
Numerical results: Accuracy tests

Example 2: 2D scalar Burgers’ equation

We test the accuracy of the hybrid RKDG+WENO-FD method when applied to solve a two dimensional Burgers’ equation with exact boundary condition couple interface at x=0, -1<y<1.

\[
\begin{align*}
    u_t + \left( \frac{1}{2} u^2 \right)_x + \left( \frac{1}{2} u^2 \right)_y &= 0 \quad (x, y) \in (-1,1) \times (-1,1) \\
    u(x, y, 0) &= 0.5 \sin(\pi(x + y)) + 0.25
\end{align*}
\]

<table>
<thead>
<tr>
<th>N×N</th>
<th>(3^{rd})-DG/5(^{th})-WENO-FD</th>
<th>(5^{th})-DG/5(^{th})-WENO-FD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(L_\infty) error</td>
<td>(L_\infty) order</td>
</tr>
<tr>
<td>40×40</td>
<td>4.54e-4</td>
<td>2.96c 5</td>
</tr>
<tr>
<td>80×80</td>
<td>5.10e-5</td>
<td>3.15</td>
</tr>
<tr>
<td>160×160</td>
<td>3.44e-6</td>
<td>3.89</td>
</tr>
<tr>
<td>320×320</td>
<td>4.90e-7</td>
<td>2.81</td>
</tr>
</tbody>
</table>

\[ t = 0.5 / \pi \]
Example 2: 2D scalar Burgers’ equation—continued

(a) Hybrid 3\textsuperscript{rd}-DG/5\textsuperscript{th}-WENO-FD method, mesh size $h = 1/160$.

(b) Hybrid 5\textsuperscript{th}-DG/5\textsuperscript{th}-WENO-FD method, mesh size $h = 1/160$. 
Numerical results: 1D Euler systems

Example 3: Sod’s Shock Tube Problem

\[(\rho_L, q_L, P_L) = (1, 0, 1), (\rho_R, q_R, P_R) = (0.125, 0, 0.1)\]

(a) Density profile of Sod’s problem
(b) Velocity profile of Sod’s problem

Artificial boundary at \(x=-0.5 \& 0.5, t=0.4\)
Example 4: Two Interacting Blast Waves

\[ u(x,0) = \begin{cases} 
    u_L & \text{if } 0 < x < 0.1, \\
    u_M & \text{if } 0.1 < x < 0.9, \\
    u_R & \text{if } 0.9 < x < 1, 
\end{cases} \]

\[ \rho_L = \rho_M = \rho_R = 1, \ u_L = u_M = u_R = 0, \ P_L = 10^3, \ P_M = 10^{-2}, \ P_R = 10^2 \]

Artificial boundary at \( x = 0.25 \) & \( 0.75 \), \( t = 0.038 \)
Numerical results: 2D scalar conservation law

Example 5: 2D scalar Burgers’ equation

We test the accuracy of the hybrid RKDG+WENO-FD method when applied to solve a two dimensional Burgers’ equation with exact boundary condition couple interface at x=0, -1<y<1.

\[ t = \frac{1.5}{\pi} \]
Example 6: Interaction of isentropic vortex and weak shock wave

This problem describes the interaction between a moving vortex and a stationary shock wave.

(a) Interaction of isentropic vortex and weak shock wave, sample mesh, mesh size $h = 1/20$.

(b) $3rd$-hybrid RKDG+WENO-FD method, density 30 contours from 1.0 to 1.24, mesh size $h=1/100$, $t=0.4$, CPU time: 2148.7s.

(c) $3rd$-RKDG method, density 30 contours from 1.0 to 1.24, mesh size $h=1/100$, $t=0.4$, CPU time: 4105.9s.

(d) $5th$-WENO-FD scheme, density 30 contours from 1.0 to 1.24, mesh size $h=1/100$, $t=0.4$, CPU time: 37.88s.
Example 7: Flow through a channel with a smooth bump

The computational domain is bounded between $x = -1.5$ and $x = 1.5$, and between the bump and $y = 0.8$. The bump is defined as

$$y = 0.0625e^{-25x^2}$$

We test two cases which is a subsonic flow with inflow Mach number is 0.5 with 0 angle of attack and a supersonic flow with Mach 2.0 with 0 angle of attack.

FIG. Flow through a channel with a smooth bump, sample mesh, mesh size $h = 1/20$. 
Numerical results: 2D Euler systems

Example 8: Flow through a channel with a smooth bump

FIG. Subsonic flow, 3rd-hybrid RKDG+WENO-FD method, Mach number 15 contours from 0.44 to 0.74, mesh size h=1/20.

FIG. Supersonic flow, 3rd-hybrid RKDG+WENO-FD method, density 25 contours from 0.55 to 1.95, mesh size h=1/50.
Numerical results: 2D Euler systems

Example 9: Incident shock past a cylinder

The computational domain is a rectangle with length from $x = -1.5$ to $x = 1.5$ and height for $y = -1.0$ to $y = 1.0$ with a cylinder at the center. The diameter of the cylinder is 0.25 and its center is located at $(0, 0)$. The incident shock wave is at Mach number of 2.81 and the initial discontinuity is placed at $x = -1.0$.

FIG. Comparison of sample Mesh. Left for RKDG; Right for hybrid RKDG+WENO-FD, mesh size $h = 1/20$. 
Numerical results: 2D Euler systems

Example 9: Incident shock past a cylinder

(a) 3rd- hybrid RKDG+WENO-FD method, pressure 25 contours from 1.0 to 20.0, mesh size $h=1/100$, $t=0.5$, CPU time: 7100.6s.

(b) 3rd- RKDG method, pressure 25 contours from 1.0 to 20.0, mesh size $h=1/100$, $t=0.5$, CPU time: 41266.7s.
Numerical results: 2D Euler systems

Example 10: Supersonic flow past a tri-airfoil

This is a test of supersonic flow past three airfoils (NACA0012) with Mach number 1.2 and the attack angle $0.0^\circ$. In the sample hybrid mesh for this test case, unstructured meshes are applied in domain $[-1.0, 3.0] \times [-2.0, 2.0]$ around airfoils and structured meshes used other computational domains.

FIG. Left Sample hybrid mesh, mesh size $h=1/10$. Right $3^{rd}$-hybrid RKDG+WENO-FD method, 20 density contours from 0.7 to 1.8, mesh size $h=1/20$. 
Numerical results: 2D Euler systems

**Example 11: Double Mach Reflection**

This is a standard test case for high resolution schemes which a mach 10 shock initially makes a $60^\circ$ angle with a reflecting wall.

(a) Hybrid mesh $h=1/20$, interface $y=0.2$

(b) Hybrid RKDG+WENO-FD, $h=1/120$, CPU times: 39894.4s

(c) RKDG, $h=1/120$, CPU times: 92743.4s

(d) WENO-FD, $h=1/120$, CPU times: 607.8s

FIG. Double mach reflection problem
### Numerical results: Efficiency Comparison

#### Double mach reflection problem:

<table>
<thead>
<tr>
<th>Method</th>
<th>Grid size</th>
<th>CFL</th>
<th>RK steps</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5th-WENO-FD</td>
<td>480 × 120</td>
<td>0.6</td>
<td>1022</td>
<td>607.8</td>
</tr>
<tr>
<td>3rd-WENO-FV</td>
<td>480 × 120</td>
<td>0.6</td>
<td>997</td>
<td>1041.3</td>
</tr>
<tr>
<td>3rd-RKDG</td>
<td>480 × 120</td>
<td>0.18</td>
<td>3378</td>
<td>17059.6</td>
</tr>
<tr>
<td>3rd-DG/WENO-FD(10%)</td>
<td>480 × 120</td>
<td>0.18</td>
<td>3373</td>
<td>4385.9</td>
</tr>
<tr>
<td>3rd-DG/WENO-FD(5%)</td>
<td>480 × 120</td>
<td>0.18</td>
<td>3378</td>
<td>3206.2</td>
</tr>
</tbody>
</table>

Table: Efficiency comparison among WENO-FD scheme, WENO-FV scheme, RKDG method and hybrid DG/WENO-FD method.
Numerical results: Efficiency Comparison

In fact, the grid handled by DG method can be limited to only one layer near the boundary.

(a) Sample grid with only one layer employed by DG
(b) 3rd-DG/WENO-FD method (Grid: 480 × 120)

Figure: Double mach reflection problem.
Numerical results: 2D Euler systems

Example 12: Subsonic flow past NACA0012 airfoil

This is a subsonic flow around a NACA0012 airfoil at Mach number 0.8 and the attack angle 1.25°. Unstructured meshes are applied in domain $[-1.0, 2.0] \times [-1.0, 1.0]$ around airfoils.

FIG. 3rd-hybrid RKDG+WENO-FD method, 20 pressure contours from 0.6 to 0.8.
Outline

- Introduction
- RKDG and WENO-FD methods
- Hybrid RKDG+WENO-FD method
- Numerical results
- Conclusions and future work
Conclusions

- A relative simple approach is presented to combine a point-value based WENO-FD scheme with an averaged-value based RKDG method to higher order accuracy.

- Special strategy is applied at coupling interface to preserve high order accurate for smooth solution and avoid loss of conservation for discontinuities.

- Numerical results are demonstrated the flexibility of the hybrid RKDG+WENO-FD method in handling complex geometries and the capability of saving computational cost in comparison to the traditional RKDG method.
Future work

- Accelerate convergence for steady flow
- Adopt local mesh refinement and cut-cell approach
- Extend to two dimensional N-S equations
THANK YOU!

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