On the Riemann problem for a system modeling blood flow

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Outline

1. Background and model

2. Mathematically related models

3. Riemann problem and mathematical structure

4. L–M and R–M curves
   - L–M curve in Case $II_l$
   - Examples
   - L–M curve in Case $IV_l$
   - Examples

5. Conclusions and outlook
Figure: http://www.hopkinsmedicine.org
Blood in compliant vessels of medium and large diameter
Blood in compliant vessels of medium and large diameter

- continuum and incompressible liquid in thin collapsible tubes
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- continuum and incompressible liquid in thin collapsible tubes

**Figure:** Assume blood vessel in 3D is axially symmetric

- axisymmetric incompressible Navier–Stokes equations + boundary conditions
• Blood in compliant vessels of medium and large diameter
  • continuum and incompressible liquid in thin collapsible tubes

Figure: Assume blood vessel in 3D is axially symmetric

• axisymmetric incompressible Navier–Stokes equations + boundary conditions

• Apply averaging procedure

• The radius of blood vessel is much smaller than length of blood vessel
Governing equations

\[ \begin{align*} 
\partial_t A + \partial_x (Au) &= 0, \\
\partial_t (uA) + \partial_x (Au^2 + \frac{A\Psi}{\rho}) - \frac{\Psi A_x}{\rho} &= -Ru, \\
\partial_t K &= 0, 
\end{align*} \]

where

- \( A(x, t) \) - cross-sectional area of vessel
- \( u(x, t) \) - average velocity in cross sectional area

---

Governing equations

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where

- \( A(x,t) \) - cross-sectional area of vessel
- \( u(x,t) \) - average velocity in cross sectional area
- \( \rho \) - constant density of blood

Governing equations

\[ \frac{\partial}{\partial t} A + \frac{\partial}{\partial x} (Au) = 0, \]
\[ \frac{\partial}{\partial t} (uA) + \frac{\partial}{\partial x} (Au^2 + \frac{A \Psi}{\rho}) - \frac{\Psi A_x}{\rho} = -R u, \]
\[ \frac{\partial}{\partial t} K = 0, \]

where

- \( A(x, t) \) - cross-sectional area of vessel
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**Tube law for transmural pressure\(^1\)**

\[ \Psi(x, t) = K(x) \left[ \left( \frac{A}{A_0} \right)^m - 1 \right] \]

- \( 0 < m < 1 \) constant

---

Material property variable

\[ K(x) = \frac{\sqrt{\pi}}{(1 - \nu^2)R_0} \frac{E(x)h(x)}{\sqrt{A_0}} \]

- \( A_0, R_0 \) - equilibrium cross sectional area and radius
- \( h \) - thickness of vessel walls
- \( E \) - Young’s modulus of elasticity
- \( \nu \) - Poisson ratio
Material property variable

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- \( \mathcal{R} = \frac{2\pi \nu R}{\delta} \) due to viscous resistance of blood
Material property variable

\[ K(x) = \frac{\sqrt{\pi}}{(1 - \nu^2)R_0} \frac{E(x)h(x)}{\sqrt{A_0}} \]

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- \( h \) - thickness of vessel walls
- \( E \) - Young’s modulus of elasticity
- \( \nu \) - Poisson ratio

\[ R = \frac{2\pi\nu R}{\delta} \] due to viscous resistance of blood

Homogeneous simplified blood flow model

\[ \partial_t A + \partial_x (Au) = 0, \]
\[ \partial_t (uA) + \partial_x (Au^2 + \frac{A\Psi}{\rho}) - \frac{\Psi A_x}{\rho} = 0, \]
\[ \partial_t K = 0, \]

Wave speed

\[ c(A, K) := \sqrt{\frac{A}{\rho} \Psi A} = \sqrt{\frac{mK}{\rho} \left( \frac{A}{A_0} \right)^m}. \] (1)
Mathematically related models

The Baer-Nuntiato model for two-phase mixtures\(^2\) \(a\) - solid, \(b\) - gas

\[
\begin{align*}
\frac{\partial c_a}{\partial t} + u_a \frac{\partial c_a}{\partial x} &= \Pi_c^a \\
\frac{\partial c_a \rho_a}{\partial t} + \frac{\partial (c_a \rho_a u_a^2 + c_a p_a)}{\partial x} &= \Pi_p^a \\
\frac{\partial c_a \rho_a E_a}{\partial t} + \frac{\partial c_a u_a (\rho_a E_a + p_a)}{\partial x} &= p_b \frac{\partial c_a}{\partial x} + \Pi_M^a \\
\frac{\partial c_b \rho_b}{\partial t} + \frac{\partial (c_b \rho_b u_b^2 + c_b p_b)}{\partial x} &= -\Pi_p^a \\
\frac{\partial c_b \rho_b u_b}{\partial t} + \frac{\partial (c_b \rho_b u_b^2 + c_b p_b)}{\partial x} &= -p_b \frac{\partial c_a}{\partial x} - \Pi_M^a \\
\frac{\partial c_b \rho_b E_b}{\partial t} + \frac{\partial c_b u_b (\rho_b E_b + p_b)}{\partial x} &= -p_b u_a \frac{\partial c_a}{\partial x} - \Pi_E^a
\end{align*}
\]

Mathematically related models

Duct flow, discontinuous duct area, extended model\(^3\)

\[
\begin{align*}
\frac{\partial a}{\partial t} &= 0, \\
\frac{\partial a\rho}{\partial t} + \frac{\partial a\rho v}{\partial x} &= 0, \\
\frac{\partial a\rho v}{\partial t} + \frac{\partial a(\rho v^2+p)}{\partial x} &= p\frac{\partial a}{\partial x}, \\
\frac{\partial a\rho E}{\partial t} + \frac{\partial a(vE+p)}{\partial x} &= 0,
\end{align*}
\]

\(a(x)\) duct area, \\
\(\rho(t, x), v(t, x), p(t, x)\) density, velocity and pressure respectively

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\(^3\)Ee Han, M. Hantke, W., J. Hyperbolic Diff. Eqns. 9(3) (2012) 403-449
Mathematically related models

Shallow water system

*discontinuous bottom topography*

extended model

\[
\begin{align*}
\frac{\partial z}{\partial t} &= 0, \\
\frac{\partial h}{\partial t} + \frac{\partial hv}{\partial x} &= 0, \\
\frac{\partial hv}{\partial t} + \frac{\partial (hv^2 + gh^2)}{\partial x} &= gh \frac{\partial z}{\partial x},
\end{align*}
\]

*\(z(x)\) bottom topography, \(h(t, x), v(t, x), g\) water depth, velocity and gravitational constant respectively*
Riemann problem and mathematical structure

Initial data \((A, u, K) = \begin{cases} (A_L, u_L, K_L), & x < 0, \\ (A_R, u_R, K_R), & x > 0. \end{cases}\) Serves as building block in theory and numerical methods.
Riemann problem

- Initial data \((A, u, K) = \begin{cases} (A_L, u_L, K_L), & x < 0, \\ (A_R, u_R, K_R), & x > 0. \end{cases}\)

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- Serves as building block in theory and numerical methods

- Applications

Figure: Abdominal aortic aneurysms (AAA)
Riemann problem

- Initial data \((A, u, K) = \begin{cases} (A_L, u_L, K_L), & x < 0, \\ (A_R, u_R, K_R), & x > 0. \end{cases}\)
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- Applications

**Figure:** Abdominal aortic aneurysms (AAA)

**Figure:** Surgical treatment
Eigenvalues: \( \lambda_0 = 0, \quad \lambda_1 = u - c, \quad \lambda_2 = u + c \)

Eigenvectors:

\[
\mathbf{R}_0 = \begin{pmatrix} 1 \\ 0 \\ \frac{(u^2 - c^2)}{c^2} \end{pmatrix}, \quad \mathbf{R}_1 = \begin{pmatrix} 1 \\ u - c \\ 0 \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} 1 \\ u + c \\ 0 \end{pmatrix}.
\]

- 0–wave (stationary wave), 1–wave, 2–wave
- 1– and 2–waves consist of shock and rarefaction waves
Mathematical structure

- **Eigenvalues:** \( \lambda_0 = 0, \quad \lambda_1 = u - c, \quad \lambda_2 = u + c \)
- **Eigenvectors:**
  
  \[
  R_0 = \begin{pmatrix}
  1 \\
  0 \\
  (u^2 - c^2)
  \end{pmatrix}, \quad
  R_1 = \begin{pmatrix}
  1 \\
  u - c \\
  0
  \end{pmatrix}, \quad
  R_2 = \begin{pmatrix}
  1 \\
  u + c \\
  0
  \end{pmatrix}.
  \]

- 0–wave (stationary wave), 1–wave, 2–wave
- 1– and 2–waves consist of shock and rarefaction waves
- **Features**
  
  - Nonstrictly hyperbolic: Mutual position of stationary wave with the remaining two waves cannot be determined a priori
  - **Resonance** occurs at the critical state \( u^2 = c^2 \)
    
    \( R_0 \to R_k \) as \( \lambda_k \to 0 \) when \( k = 1, 2 \)
  - T. Liu (1982): In the resonant state waves of different families are not well separated and coincide with each other
Elementary wave curves of Riemann problem

- The 1–wave curve $T_1(w_L)$ and 2–wave curve $T_2(w_R)$ are classical given by

$$T_k(w_q) = \{ w| u = u_q \pm f_q(A; w_q) \}$$

where

$$f_q(A; w_q) := \begin{cases} 
\frac{2}{m} \left( \sqrt{\frac{mK_q}{\rho}} \left( \frac{A}{A_0} \right)^m - c_q \right), & \text{if } A \leq A_q \\
\frac{c_q}{\sqrt{m+1}} \left[ \left( \left( \frac{A}{A_q} \right)^{m+1} - 1 \right) \left( 1 - \frac{A_q}{A} \right) \right]^{\frac{1}{2}}, & \text{if } A > A_q
\end{cases}$$

and $q = L$ when $k = 1$ and $q = R$ when $k = 2$

- $T_1(w_L)$ is strictly decreasing while $T_2(w_R)$ is strictly increasing in the $(u, \Psi)$ phase plane.
Stationary wave curve

- The additional *stationary wave curve* satisfies

\[
A_{out} u_{out} = A_{in} u_{in} \tag{2}
\]

\[
\frac{1}{2} \rho u_{out}^2 + K_{out} \left[ \left( \frac{A_{out}}{A_0} \right)^m - 1 \right] = \frac{1}{2} \rho u_{in}^2 + K_{in} \left[ \left( \frac{A_{in}}{A_0} \right)^m - 1 \right] \tag{3}
\]

- Denote \( w_{out} = J(K_{out}; w_{in}, K_{in}) \)
The additional *stationary wave curve* satisfies

\[
    \frac{1}{2} \rho u_{\text{out}}^2 + K_{\text{out}} \left[ \left( \frac{A_{\text{out}}}{A_0} \right)^m - 1 \right] = \frac{1}{2} \rho u_{\text{in}}^2 + K_{\text{in}} \left[ \left( \frac{A_{\text{in}}}{A_0} \right)^m - 1 \right]
\]

Denote \( w_{\text{out}} = J(K_{\text{out}}; w_{\text{in}}, K_{\text{in}}) \)

Define a *velocity* function \( \phi(u; w_{\text{in}}, K_{\text{in}}, K_{\text{out}}) \) by inserting (2) into (3)

\[
    \phi(u; w_{\text{in}}, K_{\text{in}}, K_{\text{out}}) = \frac{1}{2} \rho u^2 + K_{\text{out}} \left[ \left( \frac{A_{\text{in}} u_{\text{in}}}{A_0 u} \right)^m - 1 \right] - \frac{1}{2} \rho u_{\text{in}}^2 - K_{\text{in}} \left[ \left( \frac{A_{\text{in}}}{A_0} \right)^m - 1 \right]
\]
The additional stationary wave curve satisfies

\[
\frac{1}{2} \rho u_{out}^2 + K_{out} \left[ \left( \frac{A_{out}}{A_0} \right)^m - 1 \right] = \frac{1}{2} \rho u_{in}^2 + K_{in} \left[ \left( \frac{A_{in}}{A_0} \right)^m - 1 \right]
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\]

Then the properties of the velocity function are the following:

1. \( \phi(u; w_{in}, K_{in}, K_{out}) \) decreases if \( u < u^* \);
2. \( \phi(u; w_{in}, K_{in}, K_{out}) \) increases if \( u > u^* \);
3. \( \phi(u; w_{in}, K_{in}, K_{out}) \) has the minimum value at \( u = u^* \) and \( |u^*| = c^* \), where \( c^* = c(K_{out}, A^*) \) is defined in (1) and \( A^* = \frac{A_{in}u_{in}}{u^*} \).
L–M and R–M curves

- Closely resemble those of the Euler equations in discontinuous ducts and the shallow water equations with discontinuous bottom topography.

Han et al. (2011, 2012, 2013)
L–M and R–M curves

- Closely resemble those of the Euler equations in discontinuous ducts and the shallow water equations with discontinuous bottom topography.
- The discontinuous material property variable can be viewed as the limiting case of piecewise linear material property variables.
L–M and R–M curves

- Closely resemble those of the Euler equations in discontinuous ducts and the shallow water equations with discontinuous bottom topography \(^5\)
- The discontinuous material property variable can be viewed as the limiting case of piecewise linear material property variables
- Stationary wave to the Riemann problem is a \(^0\) width transition layer located at \(x = 0\)

\(^5\) Han et al. (2011, 2012, 2013)
Closely resemble those of the Euler equations in discontinuous ducts and the shallow water equations with discontinuous bottom topography. The discontinuous material property variable can be viewed as the limiting case of piecewise linear material property variables. Stationary wave to the Riemann problem is a width transition layer located at $x = 0$. We merge the stationary wave curve into the 1– or 2–wave curves and name these L–M and R–M curves.

5 Han et al. (2011, 2012, 2013)
Closely resemble those of the Euler equations in discontinuous ducts and the shallow water equations with discontinuous bottom topography \(^5\)

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Stationary wave to the Riemann problem is a width transition layer located at \(x = 0\)

We merge the stationary wave curve into the 1– or 2–wave curves and name these L–M and R–M curves

The L–M curve starting from \((A_L, u_L)\) to \((A_M, u_M)\), consists of a 1–shock curve \(S_1\), a 1–rarefaction curve \(R_1\), and/or the stationary wave curve (not necessarily present)

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- The algorithm for finding the exact Riemann solutions:
  - calculate \((A_M, u_M)\) for given Riemann initial data
  - construct the L–M and R–M curves

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\(^5\) Han et al. (2011,2012,2013)
# Cases of L–M curves with $K_L > K_R$

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_l$</td>
<td>$A_{max}^P \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$; $u_L \leq c_L$; $u_c^l \leq u_{sc}^l$</td>
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<td>$II_l$</td>
<td>$A_{max}^P \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$; $u_L \leq c_L$; $u_c^l &gt; u_{sc}^l$</td>
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<td>$III_l$</td>
<td>$A_{max}^P \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$; $u_L &gt; c_L$; $\varphi \left( \frac{K_R}{K_L}; w_L \right) \leq 0$</td>
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<tr>
<td>$IV_l$</td>
<td>$A_{max}^P \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$; $u_L &gt; c_L$; $\varphi \left( \frac{K_R}{K_L}; w_L \right) &gt; 0$; $\tau \left( \frac{K_R}{K_L}; w_L \right) \leq 0$</td>
</tr>
<tr>
<td>$V_l$</td>
<td>$A_{max}^P \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$; $u_L &gt; c_L$; $\tau \left( \frac{K_R}{K_L}; w_L \right) &gt; 0$</td>
</tr>
<tr>
<td>$VI_l$</td>
<td>$A_{max}^P &lt; A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$</td>
</tr>
</tbody>
</table>

Where $A_{max}^P = \begin{cases} A_0 \left( \frac{\rho}{mK_L} \right)^{\frac{1}{m}} \left( \frac{m}{2} u_L + c_L \right)^{\frac{2}{m}}, & \text{if } u_L \leq 0 \\ A_L x_{u_1^0}, & \text{if } u_L > 0 \end{cases}$ and $x_{u_1^0}$ is the solution to

$$x^{m+2} - x^{m+1} - \left[ 1 + (m+1) \left( \frac{u_L}{c_L} \right)^2 \right] x + 1 = 0$$

The critical velocity $u_c^l = \frac{m}{m+2} u_L + \frac{2}{m+2} c_L$ and $u_{sc}^l = \sqrt{\frac{2m}{\rho(m+2)} (K_R - K_L) \left( \frac{K_R}{K_L} \right)^{\frac{2}{m+2}} - 1}$
Cases of R–M curves with $K_L > K_R$

<table>
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<tr>
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<th>Equation</th>
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</thead>
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<tr>
<td>$I_r$</td>
<td>$u_R + c_R \geq 0$; $u_c^r &lt; -u_{sc}^r$</td>
<td></td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>$III_r$</td>
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</tr>
<tr>
<td>$IV_r$</td>
<td>$u_R + c_R &lt; 0$; $\varphi \left( \frac{K_L}{K_R}; w_R \right) &gt; 0$; $\tau \left( \frac{K_L}{K_R}; w_R \right) \leq 0$</td>
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</tr>
<tr>
<td>$V_r$</td>
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<td></td>
</tr>
</tbody>
</table>

where

$$\varphi(\kappa; w_q) = \frac{m+2}{2} \left( \frac{u_q}{c_q} \right)^{\frac{2m}{m+2}} \kappa^{\frac{2}{m+2}} - \left( \frac{A_0}{A_q} \right)^m \kappa - \frac{m}{2} \left( \frac{u_q}{c_q} \right)^2 \left( \frac{\hat{A}_1}{A_q} \right)^{-2}$$

(4)

and

$$\tau(\kappa; w_q) = \frac{m + 2}{2} F_q^{\frac{2m}{m+2}} \kappa^{\frac{2}{m+2}} - \left( \frac{A_0}{A_q} \right)^m \kappa - \frac{m}{2} F_q^2 - 1 + \left( \frac{A_0}{A_q} \right)^m.$$  

(5)
Case $II_l$: $A_{max}^{P_2^l} \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right] \frac{m}{\bar{m}}$; $u_L \leq c_L$; $u_c^l \leq u_{sc}^l$

Possible wave configurations with $u_M > 0$

**Figure: A**

**Figure: B**
Case II: \( A_{\text{max}}^{P_2^l} \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right] \frac{1}{m} \); \( u_L \leq c_L \); \( u_c^l \leq u_{sc}^l \)

The L–M curve \( C_L(w_L) = \bigcup_{k=1}^{3} P_k^l(w_L) \):

\[
P_1^l(w_L) = \{ w | w \in T_1(w_L) \text{ with } u \leq 0 \}
\]
\[
P_2^l(w_L) = \{ w | w = J(K_R; w^-, K_L) \ w^- \in T_1(w_L) \text{ and } 0 < u < u_c \}
\]
\[
P_3^l(w_L) = \{ w | w \in T_1(w_c) \text{ with } u > u_c \}
\]

where \( w_c = J(a_R; \tilde{w}_c, a_L) \) and \( \tilde{w}_c \in T_1(w_L) \)
Case $II_l$: $A_{max}^{P_2^l} \geq A_0 \left(1 - \frac{K_R}{K_L}\right)^{\frac{1}{m}}$; $u_L \leq c_L$; $u_c^l \leq u_{sc}^l$

The L–M curve $C_L(w_L) = \bigcup_{k=1}^{3} P_k^l(w_L)$:

- $P_1^l(w_L) = \{w \mid w \in T_1(w_L) \text{ with } u \leq 0\}$
- $P_2^l(w_L) = \{w \mid w = J(K_R; w_, K_L) w_ - \in T_1(w_L) \text{ and } 0 < u < u_c\}$
- $P_3^l(w_L) = \{w \mid w \in T_1(w_c) \text{ with } u > u_c\}$

where $w_c = J(a_R; \tilde{w}_c, a_L)$ and $\tilde{w}_c \in T_1(w_L)$

L–M curve is continuous and decreasing in the $(u, \psi)$ phase plane.
Case $II_l$: $A_{max}^{P_2^l} \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^{\frac{1}{m}}$; $u_L \leq c_L$; $u_{c} \leq u_{sc}^l$

The L–M curve $C_L(w_L) = \bigcup_{k=1}^{3} P_k^l(w_L)$:

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- $P_2^l(w_L) = \{ w \mid w = J(K_R; w_-, K_L) \ w_- \in T_1(w_L) \text{ and } 0 < u < u_c \}$
- $P_3^l(w_L) = \{ w \mid w \in T_1(w_c) \text{ with } u > u_c \}$

where $w_c = J(a_R; \tilde{w}_c, a_L)$ and $\tilde{w}_c \in T_1(w_L)$

**Theorem of monotonicity.**

$L–M$ curve is continuous and decreasing in the $(u, \psi)$ phase plane

- The Riemann solution uniquely exists
Example: the solution with wave configuration A in Case $II_l$ by Toro and Siviglia

### Table: Initial and intermediate states

<table>
<thead>
<tr>
<th></th>
<th>$K$</th>
<th>$A (m^2)$</th>
<th>$u (m/s)$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_L$</td>
<td>2000003.266554</td>
<td>$3.0 \times 10^{-4}$</td>
<td>$-2.6575 \times 10^{-5}$</td>
<td>$-7.88826 \times 10^{-7}$</td>
</tr>
<tr>
<td>$w_-$</td>
<td>2000003.26654</td>
<td>$2.01185 \times 10^{-4}$</td>
<td>$12.81037$</td>
<td>$0.420195$</td>
</tr>
<tr>
<td>$w_M$</td>
<td>40000.06533</td>
<td>$6.34034 \times 10^{-4}$</td>
<td>$4.0648486$</td>
<td>$0.107114$</td>
</tr>
<tr>
<td>$w_R$</td>
<td>40000.06533</td>
<td>$3 \times 10^{-4}$</td>
<td>$6.123 \times 10^{-6}$</td>
<td>$1.28516 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

**Figure:** The exact vessel area and velocity at $t = 0.012$ s.
Example: the solution with wave configuration B in Case $II_l$

**Table: Initial and intermediate states**

<table>
<thead>
<tr>
<th></th>
<th>$K$</th>
<th>$A \ (m^2)$</th>
<th>$u \ (m/s)$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_L$</td>
<td>2000003.266554</td>
<td>$3.0 \times 10^{-4}$</td>
<td>$-2.6575 \times 10^{-5}$</td>
<td>$-7.88826 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\tilde{w}_c$</td>
<td>2000003.26654</td>
<td>$2.0034 \times 10^{-4}$</td>
<td>12.938557</td>
<td>0.424847</td>
</tr>
<tr>
<td>$w_c$</td>
<td>40000.06533</td>
<td>$4.82994 \times 10^{-4}$</td>
<td>5.36675</td>
<td>1</td>
</tr>
<tr>
<td>$w_M$</td>
<td>40000.06533</td>
<td>$4.13503 \times 10^{-4}$</td>
<td>6.18445</td>
<td>1.197995</td>
</tr>
<tr>
<td>$w_R$</td>
<td>40000.06533</td>
<td>$1.17 \times 10^{-4}$</td>
<td>$1.5 \times 10^{-7}$</td>
<td>$3.984 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

**Figure:** The exact vessel area and velocity at $t = 0.012 \ s$. 
Case $IV_1$: $A_{max}^{P_2} \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right]^\frac{1}{m}; \ u_L > c_L; \ \varphi \left( \frac{K_R}{K_L}; w_L \right) \leq 0$

Possible wave configurations with $u_M > 0$

(a) A

(b) E

(c) F
Case $IV_l$: $A_{max}^{P_2^l} \geq A_0 \left[1 - \frac{K_R}{K_L}\right]^\frac{1}{m}; u_L > c_L; \varphi \left(\frac{K_R}{K_L}; w_L\right) \leq 0$

The L–M curve $C_L(w_L) = \bigcup_{k=1}^{4} P_k^l(w_L)$:

\[
P_1^l(w_L) = \{w | w \in T_1(w_L) \text{ with } u < 0\}, \]
\[
P_2^l(w_L) = \{w | w = J(K_R; w, K_L) \text{ and } w \in T_1(w_L) 0 < u < \tilde{u}_1L\} \]
\[
P_3^l(w_L) = \{w | w = J(K_R; w_+, K); w_+ = S^0_1(w_-); w_- = J(K; w_L, K_L), K \in [K_R, K_L]\}, \]
\[
P_4^l(w_L) = \{w | w \in T_1(\tilde{w}_1L) u > \tilde{u}_L\}, \]
Case IV₁: \( A_{max}^{P_2^L} \geq A_0 \left[ 1 - \frac{K_R}{K_L} \right] \frac{1}{m} ; u_L > c_L ; \varphi \left( \frac{K_R}{K_L}; w_L \right) \leq 0 \)

The L–M curve \( C_L(w_L) = \bigcup_{k=1}^{4} P_k^l(w_L) \):

\[
\begin{align*}
P^l_1(w_L) &= \{ w | w \in T_1(w_L) \text{ with } u < 0 \}, \\
P^l_2(w_L) &= \{ w | w = J(K_R; w_-, K_L) \text{ and } w_- \in T_1(w_L) \text{ with } 0 < u < \tilde{u}_{1L} \} \\
P^l_3(w_L) &= \{ w | w = J(K_R; w_+, K); w_+ = S^0_1(w_-); w_- = J(K; w_L, K_L), K \in ]K_R, K_L[ \} \\
P^l_4(w_L) &= \{ w | w \in T_1(\tilde{w}_{1L}) \text{ with } u > \hat{u}_L \},
\end{align*}
\]

Theorem of Uniqueness.

- If \( \frac{\rho \hat{u}^2_L}{m K_R} - \frac{\hat{A}_L}{A_L} > 0 \), the L–M curve on \((u, \psi)\) phase plane is monotonically decreasing.
- The Riemann solution uniquely exists.
The L–M curve $C_L(w_L) = \bigcup_{k=1}^{4} P_k^l(w_L)$:

\[
\begin{align*}
P_1^l(w_L) &= \{ w | w \in T_1(w_L) \text{ with } u < 0 \}, \\
P_2^l(w_L) &= \{ w | w = J(K_R; w_-, K_L) \text{ and } w_- \in T_1(w_L) \ 0 < u < \tilde{u}_1 L \}, \\
P_3^l(w_L) &= \{ w | w = J(K_R; w_+, K); \ w_+ = S_1^0(w_-); \ w_- = J(K; w_L, K_L), \ K \in ]K_R, K_L[ \}, \\
P_4^l(w_L) &= \{ w | w \in T_1(\tilde{w}_1 L) \ u > \hat{u}_L \},
\end{align*}
\]

Theorem of Nonuniqueness.

- If $\frac{\rho \tilde{u}_L^2}{mK_R} - \hat{A}_L < 0$, the L–M curve on $(u, \psi)$ phase plane contains a bifurcation.

- For the given initial data we have three possible solutions, with wave configurations A, E, and F respectively.
Example: the unique solution with wave configuration E in Case IV$_1$

<table>
<thead>
<tr>
<th></th>
<th>K</th>
<th>$A , (m^2)$</th>
<th>$u , (m/s)$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_L$</td>
<td>40000.065331</td>
<td>$4.2248 \times 10^{-4}$</td>
<td>10.380259</td>
<td>2.0</td>
</tr>
<tr>
<td>$w_-$</td>
<td>29017.111011</td>
<td>$4.03365 \times 10^{-4}$</td>
<td>10.872157</td>
<td>2.488099</td>
</tr>
<tr>
<td>$w_+$</td>
<td>29017.111011</td>
<td>$1.607526 \times 10^{-3}$</td>
<td>2.728076</td>
<td>0.441869</td>
</tr>
<tr>
<td>$w_M$</td>
<td>28000.045732</td>
<td>$1.700599 \times 10^{-3}$</td>
<td>2.578769</td>
<td>0.419263</td>
</tr>
<tr>
<td>$w_R$</td>
<td>28000.045732</td>
<td>$1.098391 \times 10^{-3}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Figure:** The exact vessel area and velocity at $t = 0.02 \, s$, where $\varphi \left( \frac{K_R}{K_L} ; w_L \right) = -0.368912$, $\tau \left( \frac{K_R}{K_L} ; w_L \right) = -0.547927$ and $\frac{\rho \hat{u}_L^2}{mK_R} - \frac{\hat{A}_L}{A_L} = 4.814028$
**Example: the nonunique solutions with wave configurations A, E and F**

<table>
<thead>
<tr>
<th></th>
<th>$K$</th>
<th>$A$ (m²)</th>
<th>$u$ (m/s)</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_L$</td>
<td>58136.483963</td>
<td>$1.0 \times 10^{-6}$</td>
<td>6.655409</td>
<td>4</td>
</tr>
<tr>
<td>$w_R$</td>
<td>56973.754284</td>
<td>$5.038 \times 10^{-6}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

---

**Figure:** The Exact vessel areas and velocities at $t = 0.15$ s, where $\varphi \left( \frac{K_R}{K_L}; w_L \right) = -0.405688$, $\tau \left( \frac{K_R}{K_L}; w_L \right) = -1.19155$, and $\frac{\rho u_L^2}{mK_R} - \frac{\hat{A}_L}{A_L} = -5.156529$
The L–M curve is monotone decreasing in the \((u, \Psi)\) phase plane in Cases \(II_l, III_l,\) and \(VII_l\).
The L–M curve is monotone decreasing in the $(u, \Psi)$ phase plane in Cases $II_l$, $III_l$, and $VI_l$.

In Case $IV_l$ it is also monotone decreasing if \[ \frac{\rho \tilde{u}^2}{mK_R} - \frac{\hat{A}_L}{A_L} > 0 \]
The L–M curve is monotone decreasing in the \((u, \Psi)\) phase plane in Cases \(II_l, III_l,\) and \(VI_l\).

In Case \(IV_l\) it is also monotone decreasing if \(\frac{\rho \bar{u}_L^2}{mK_R} - \frac{\hat{A}_L}{A_L} > 0\).

The Riemann solution uniquely exists.
The L–M curve is monotone decreasing in the \((u, \Psi)\) phase plane in Cases \(II_l, III_l,\) and \(V I_l\).

In Case \(IV_l\) it is also monotone decreasing if \(\frac{\rho \bar{u}^2}{mK_R} - \frac{\hat{A}_L}{A_L} > 0\).

The Riemann solution uniquely exists.

The bifurcation occurs in Case \(IV_l\) if \(\frac{\rho \bar{u}^2}{mK_R} - \frac{\hat{A}_L}{A_L} < 0\) and always in Case \(V_l\).
The L–M curve is monotone decreasing in the \((u, \Psi)\) phase plane in Cases \(II_l, III_l,\) and \(V I_l\).

In Case \(IV_l\) it is also monotone decreasing if \(\frac{\rho \tilde{u}_L^2}{mK_R} - \frac{\hat{A}_L}{A_L} > 0\).

The Riemann solution uniquely exists.

The bifurcation occurs in Case \(IV_l\) if \(\frac{\rho \tilde{u}_L^2}{mK_R} - \frac{\hat{A}_L}{A_L} < 0\) and always in Case \(VI_l\).

The nonunique Riemann solutions occur due to the bifurcation.
The L–M curve is monotone decreasing in the \((u, \Psi)\) phase plane in Cases \(II_l, III_l,\) and \(VI_l\).

In Case \(IV_l\) it is also monotone decreasing if \(\frac{\rho \bar{u}_L^2}{m K_R} - \frac{\hat{A}_L}{A_L} > 0\).

The Riemann solution uniquely exists.

The bifurcation occurs in Case \(IV_l\) if \(\frac{\rho \bar{u}_L^2}{m K_R} - \frac{\hat{A}_L}{A_L} < 0\) and always in Case \(V_l\).

The nonunique Riemann solutions occur due to the bifurcation.

The cases for the R–M curves are analogous to the cases of the L–M curves.
Conclusions and outlook

- For any given Riemann initial data we obtained all possible exact solutions to the simplified blood flow model with discontinuous material properties.
- The behaviors of the L–M and R–M curves for all basic cases were completely analyzed.
- May help to assess numerical schemes for much more complicated and realistic models.
- Possibly a tool to evaluate the effect of the stent graft to the body.
- Planned: Numerical schemes based on these exact Riemann solutions.
Thank you very much for your attention