Interface Treating Method for Multi-Medium Flow on Adaptive Triangular Meshes

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Outline

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- Equations
- The RGFM on triangular meshes
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1. Introduction

- The RKDG method is used to discretize of the Euler equation.
  - Advantages: Easy to handle the complicated fluid boundary, compactness,......
  - Disadvantages: More degrees of freedom, not cost efficient,......

- Triangular mesh: Easy to decompose the fluid field with complicated boundary.

Motivation

- Present an interface treating method for multi-medium flow on triangular meshes, easy to handle complicated geometry and fluid boundary.

- Couple the adaptive moving mesh method to the interface treating method to reduce the numerical and conservative errors and improve the solution resolution.
The ghost fluid method (GFM)

1D Original GFM (R. Fedkiw et al, JCP, 1999).

1D New GFM (R. Fedkiw et al, JCP, 2002).

The 1D and 2D Real GFM (C.W. Wang et al, SISC, 2006).
2. Equations

**Euler equation**
(Chi-Wang Shu et al, JCP, 1998)

The two-dimensional system for a compressible fluid can be written as follows:

\[
\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}}{\partial x} + \frac{\partial \vec{G}}{\partial y} = 0
\]

where

\[
\vec{U} = [\rho, \rho u, \rho v, E], \\
\vec{F}(\vec{U}) = [\rho u, \rho u^2 + p, \rho uv, (E + p)u]^T, \\
\vec{G}(\vec{U}) = [\rho v, \rho uv, \rho v^2 + p, (E + p)v]^T.
\]

\[
E = \rho e + \frac{1}{2}\rho (u^2 + v^2)
\]

For closure of the system, the equation of state (EOS) is required. In the present work, our interest centered on the compressible gas and water media. The EOS for gases or water medium can be written uniformly as

\[
p = (\gamma - 1)\rho e - \gamma B
\]

where \(\gamma\) and \(B\) are treated as fluid constants.
**Level set equation**

(Chi-Wang Shu et al, JCP, 2007)

To track the moving fluid interface, we employ the level set technique. The level set equation can be written as

$$\frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi = 0.$$
3. The RGFM on triangular meshes

**Case 1** There is one neighbor of triangle 1 in another medium.

**Case 2** There are two neighbors of triangle 1 in another medium.
**The unit outward normal**

The normal is needed in the construction of the Riemann problem. In this work, the normal direction of the level set function at the barycenter of triangle $\Delta_l$ can be approximated by using the Green formula as

$$\int_{\triangle_l} \frac{\partial P(x,y)}{\partial x} - \frac{\partial Q(x,y)}{\partial y} dxdy = \int_{\partial \triangle_l} Q(x,y) dx + P(x,y) dy$$

If $P(x,y) = \phi(x,y), Q(x,y) = 0$, we have

$$\int_{\triangle_l} \frac{\partial \phi(x,y)}{\partial x} dxdy = \int_{\partial \triangle_l} \phi(x,y) dy$$

So $\phi_x(x,y)$ can be approximated at the barycenter of triangle $\triangle_l$ as

$$\phi_x(x_b,y_b) = \frac{\int_{\partial \triangle_l} \phi(x,y) dy}{\int_{\triangle_l} dxdy} = \frac{1}{|\triangle_l|} \int_{\partial \triangle_l} \phi(x,y) dy$$

Similar procedures can be carried out to obtain $\phi_y(x_b,y_b)$. So the unit normal can be written as

$$N = \left( \frac{\phi_x(x_b,y_b)}{\sqrt{\phi_x^2(x_b,y_b) + \phi_y^2(x_b,y_b)}}, \frac{\phi_y(x_b,y_b)}{\sqrt{\phi_x^2(x_b,y_b) + \phi_y^2(x_b,y_b)}} \right)$$
Redefinition of the fluid velocity

When large gradient appears in the velocity field, the level set contours may be severely distorted or even the code may collapse. To overcome this difficulty, the extension velocity is applied by solving a convection equation

$$\frac{\partial I}{\partial t} + N \cdot \nabla I = 0.$$ 

to extrapolate the interface velocity to the whole computational domain. By solving the level set function with the extension velocity, the uniformly distributed level set contour can be obtained.
The moving triangular meshes
(H. Z. Tang et al, JSC, 2008)

A one-to-one coordinate transformation from the logical or computational domain to the physical domain is denoted by

\[ x = x(\xi), \xi \in \Omega_l \]

The map is to find the minimizer of the following functional

\[ \tilde{E}(x) = \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega_l} (\tilde{\nabla}x_i)^T G_i(\tilde{\nabla}x_i) d\xi \]

By using the choice of Winslow, we deduce the Euler-Lagrange equations of the functional

\[ \tilde{\nabla}x \cdot (\omega \tilde{\nabla}x) = 0 \]
Discretize the equation to solve it with a relaxed iteration method as

\[ \hat{x}_i = \sum_{j=1}^{\mathbb{N}(i)} W_{ij} \frac{x_{ij}^y}{\sum_{j=1}^{\mathbb{N}(i)} W_{ij}} \]

\[ x_{i}^{y+1} = \mu_i \hat{x}_i + (1 - \mu_i) x_{i}^{y} \]

where

\[ W_{ij} = \frac{\Delta \tau}{|V_i|} \omega_{ij} \frac{|\hat{l}_{ij}|}{|\xi_{ij}\xi_i|} \]

\[ \mu_i = \max \left( \sum_{j=1}^{\mathbb{N}(i)} W_{ij}, \sigma \right) \]

The monitor function is taken as

\[ \omega_{E_i} = \sqrt{1 + \alpha_p \bar{\omega}^2_{E_i}(\beta_p, \rho) + \alpha_p \bar{\omega}^2_{E_i}(\beta_p, \rho) + \alpha_q \bar{\omega}^2_{E_i}(\beta_q, q)} \]

where

\[ \bar{\omega}_{E_i}(\beta_p, \rho) =: \min \left( 1, \frac{|\nabla_{\xi} \rho|}{\beta_p \max_{E_i} \{|\nabla_{\xi} \rho|\}} \right) \]

\[ q = \sqrt{u^2 + v^2} \]
Remarks

- It is necessary to avoid the mesh moving across the interface which may lead to the mixed fluid around the interface.
- It is worth noting that the fluid interface does not move physically in the iterative mesh redistribution.
**Update the solution**

After each mesh iterative step, we need to update the approximate solutions $U$ and $\phi$ on the new mesh from the old mesh.

When the high-resolution conservative interpolation formulas for single-medium flow are directly applied to updating the solutions $U$ of multi-medium or multi-phase compressible flows, numerical inaccuracies will occur at the material interfaces due to the mesh-redistribution.
For the $P_1$ case in the RKDG method, the approximate solution $u_h(x, y, t)$ inside the triangle $K$ can be written as

$$u_h(x, y, t) = \sum_{i=1}^{3} u_i(t) \phi_i(x, y)$$

where the degrees of freedom $u_i(t)$ are values of the numerical solution at the midpoints of edges. The basis function

$$\phi_i(x, y) = a_i + b_i x + c_i y, i = 1, 2, 3$$

and

$$\phi_i(x_j, y_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

The $L_2$ projection method is used to remap the numerical solution to the new mesh from the old mesh

$$\int_K u_h^N(x, y) \phi_i(x, y) \, dx \, dy = \sum_i \int_{K_i} u_h^O(x, y) \phi_i(x, y) \, dx \, dy$$
4. Some numerical results

Problem 1. Sod problem

\[(\rho, u, v, p, \gamma, B) = \begin{cases} 
(1, 0, 0, 1, 1.4, 0), & x < 0.5 \\
(0.125, 0, 0, 0.1, 1.2, 0), & x \geq 0.5 
\end{cases} \]

The positive parameters in the monitor function are respectively taken as

\[\alpha_{\rho} = 20, \alpha_{p} = 20, \alpha_{q} = 0,\]
\[\beta_{\rho} = 0.1, \beta_{p} = 0.1, \beta_{q} = 0.1,\]
Problem 2. gas bubble explosion problem
The initial interface lies at the middle of the computational domain. The initial states on the left and right sides of the interface are taken as

\[
(\rho, u, v, p, \gamma, B) = \begin{cases} 
(1.5, 0, 0, 100, 1.4, 0), & \text{inside the bubble,} \\
(5, 0, 0, 1, 1.67, 0), & \text{outside of the bubble.}
\end{cases}
\]

\[
\alpha_\rho = 50, \alpha_p = 0, \alpha_q = 0,
\]
\[
\beta_\rho = 0.1, \beta_p = 0.1, \beta_q = 0.1,
\]
**Problem 4. Shock bubble interaction problem**

The gas shock with strength 2.35 impact on the cylindrical water bubble. The pre- and post-shock states of the incident shock wave are

\[
(\rho, u, v, p, \gamma, B) = \begin{cases} 
(1, 0, 0, 1, 2.8, 3306), & \text{inside the water cylinder}, \\
(0.001, 0, 0, 1, 1.4, 0), & \text{outside of the water cylinder},
\end{cases}
\]

Schematics of the computational domain

\[
\alpha_\rho = 10, \alpha_p = 0, \alpha_q = 30, \\
\beta_\rho = 0.1, \beta_p = 0.1, \beta_q = 0.1,
\]
Pressure contours at time $t = 0.02, t = 0.04$, respectively.
Pressure contours at time $t = 0.06, t = 0.075$, respectively.
5. Conclusions

- The RGFM is extended to simulate the multi-medium flow on triangular meshes, the construction of Riemann problem is presented to provide the fluid states at the interface.
- In the adaptive moving mesh method, we present the remapping method of the degree of freedom in RKDG and the Level-Set function from the old mesh to the new mesh.
- Several examples are given to test the robustness and the efficiency of the algorithm; the numerical results show that this method can improve the resolution of the discontinuities and capture sharp interface and shock wave accurately.
Thank you for your attention!