The regularity of the surface of the Gauss curvature $0 \leqslant K \in C_0^\infty$ *

BAO Jiguang (保继光)

(School of Mathematical Sciences, Peking University, Beijing 100871, China;

Department of Mathematics, Beijing Normal University, Beijing 100875, China)

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Abstract Under the mild conditions, it is proved that the convex surface is global $C^{1,1}$, with the given Gaussian curvature $0 \leqslant K \in C_0^\infty$ and the given boundary curve. Examples are given to show that the regularity is optimal.

Keywords: **nonnegative curvature, degenerate elliptic equation, any boundary data, global $C^{1,1}$ regularity.**

In this paper we consider in a plane domain Ω the existence of the convex surface $z = u(x, y) \in C^{1,1}(\bar{\Omega})$, whose Gauss curvature is $0 \leqslant K \in C_0^\infty(\Omega)$ and whose boundary is a space curve $z = \varphi(x, y)((x, y) \in \partial\Omega)$. As is well known, the existence of the surface is equivalent to the solvability in $C^{1,1}(\bar{\Omega})$ of the Dirichlet problem for the degenerate elliptic Monge-Ampere equations

$$u_{xx}u_{yy} - u_{xy}^2 = K(x, y)(1 + u_x^2 + u_y^2)^2, \ (x, y) \in \Omega, \tag{1}$$

$$u = \varphi(x, y), \ (x, y) \in \partial\Omega. \tag{2}$$

Since refs. [1—3] solved successfully the famous Minkowski problem, the case of the surface with boundary has attracted wide interest. Trudinger and Urbas pointed out in ref. [4] that when $K > 0$ in Ω, (1) and (2) are uniquely solvable for arbitrary $C^{1,1}$ boundary data φ, with convex solution $u \in C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ if and only if

$$\iint_\Omega K(x, y)\mathrm{d}x\mathrm{d}y < \pi, \quad K\mid_{\partial\Omega} \equiv 0.$$

Therefore, the elliptic Monge-Ampere equations degenerate on the boundary become the focal point. By modifying the examples in ref. [5], we know that there exists $0 \leqslant K \in C_0^\infty(\Omega)$ such that (1) and (2) have a solution $u \in C^{1,1}(\bar{\Omega}) \setminus C^2(\bar{\Omega})$ if $\partial\Omega \in C^\infty$ and $\varphi \equiv 0$. This shows that the degenerate ellipticity of (1) near $\partial\Omega$ restricts rigorously the global regularity of its solution.

In some special cases, e.g. $K \equiv 0$ (see ref. [6]) and $\varphi \equiv 0$ (see ref. [7]) for $\partial\Omega$ and φ under some relations (see refs. [8, 9]), the $C^{1,1}(\bar{\Omega})$ regularity of the solution is obtained for (1) and (2). Using the ideas in ref. [6] and a preprint by Guan1), we prove

Theorem 1. *The problem* (1)—(2) *admits a unique convex solution in* $C^{1,1}(\bar{\Omega})$, *provided* Ω *is a smooth uniformly convex domain in the plane*, $\varphi \in C^\infty(\partial\Omega)$, $0 \leqslant K \in C_0^\infty(\Omega)$ *and*

$$\iint_\Omega K(x, y)\mathrm{d}x\mathrm{d}y < \pi.$$

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1) Guan, B., The Dirichlet problem for monge-Ampere equations in non-convex domains and spacelike hypersurface of constant Gauss curvature, Preprint.

The counterexamples in ref. [5] show that in general u is not in $C^2(\bar{\Omega})$. Therefore Theorem 1 improves the known results. Here the smoothness conditions on K, φ, Ω can be weakened.

(1) and (2) have a unique convex solution $u \in C^{1,1/4}(\bar{\Omega}) \cap C^{1,1}(\Omega)$[1)]. It will be proved that u is in $C^{1,1}(\bar{\Omega})$. From now on we denote by C the positive constants depending only on K, φ and Ω.

Let

$$\varepsilon > 0,\ N_i = \{(x,y) \in \Omega \mid \mathrm{dist}((x,y),\partial\Omega) < i\varepsilon\},\ i = 1,2.$$

Then $N_2 \subset \Omega \setminus \mathrm{supp}K$ for some small constant ε. We will prove that there is a constant C such that for every $P_0(x_0, y_0) \in N_1$, there exists $\delta = \delta(P_0)$ so that for any $P(x,y) \in N_1$, with $|PP_0| < \delta$, we have

$$u(x,y) - u(x_0,y_0) - u_x(x_0,y_0)(x - x_0) - u_y(x_0,y_0)(y - y_0) \leqslant C|PP_0|^2.$$

By the convexity of u, $u \in C^{1,1}(\overline{N_1})$ (see ref. [10]). The desired result $u \in C^{1,1}(\bar{\Omega})$ is obtained via combining the interior regularity. Without loss of generality, we may suppose $u = u_x = u_y = 0$ at P_0. Therefore $u \geqslant 0$ in $\bar{\Omega}$. Now we only need to prove

$$u(x,y) \leqslant C\,|PP_0|^2,\quad P(x,y) \in N_1,\ |PP_0| < \delta. \tag{3}$$

Let $P_0(x_0,y_0) \in N_1$, $S = \{(x,y) \in \bar{N}_2 \mid u(x,y) = 0\}$.

Lemma 1. *Each component Γ of $\partial S \cap N_2$ is a segment, which stretches to ∂N_2.*

Proof. If Γ is not a segment which stretches to ∂N_2, there is a point on Γ such that the tangent line and Γ only intersect at one point. After a translation of coordinates we may assume that the point is the origin, $B_\delta(0) \subset N_2$ and $S \cap B_\delta(0) \cap \{y \geqslant 0\} = \{0\}$ for some $\delta > 0$. Therefore, there is a constant $\delta_0 > 0$ such that u has positive lower bound on $\partial B_\delta(0) \cap \{y > -\delta_0\}$.

Let

$$U = \{(x,y) \in B_\delta(0) \mid y > -\delta_0\},$$
$$v(x,y) = \delta_1[\delta_0 + 2y + \delta_2(x^2 + y^2)],\quad \delta_1, \delta_2 > 0.$$

Then in U

$$v_{xx}v_{yy} - v_{xy}^2 = (2\delta_1\delta_2)^2 > 0 = u_{xx}u_{yy} - u_{xy}^2,$$

and for δ_1, δ_2 sufficiently small, $v \leqslant u$ on ∂U. Consequently, $u \geqslant v$ in $\bar{U}$ by the Aleksandrov maximum principle. In particular, $u(0,0) \geqslant v(0,0) = \delta_0\delta_1 > 0$. This contradicts the fact that $(0,0) \in S$ and $u(0,0) = 0$.

If $P_0 \in \mathrm{int}\,S$, then (3) holds obviously. If $P_0 \in \partial S$, then by Lemma 1, P_0 lies on a segment with end points $P_1(x_1,y_1)$, $P_2(x_2,y_2) \in \partial N_2$. By the convexity of $\partial N_2 \cap \Omega$ it is impossible for P_1 and P_2 to lie on Ω in the meantime, and one of them is at least on $\partial\Omega$. We may assume $P_2 = 0 \in \partial\Omega$, the positive y axis is the interior normal to $\partial\Omega$ at 0, $\partial\Omega$ is represented near 0, $y = \rho(x)$, $|x| < r$, and $x_0 \geqslant 0$, $y_0 = x_0 \tan\theta$, and $0 < \theta \leqslant \frac{\pi}{2}$.

Let $P(x,y) \in N_1$, $|PP_0| < \delta$, and L, L_0 be straight lines passing through P, P_1 and P_0, P_1, respectively. L intersects $\partial\Omega$ at a point $\bar{P}(\bar{x},\bar{y})$, and the angle between L and L_0 is α. Then

$$L_0: y = x\tan\theta,\quad L: y - x_1\tan\theta = (x - x_1)\tan(\theta \pm \alpha).$$

Denote by d the distance from P_0 to L and regard $\bar{x}$, $\bar{y}$ and d as the functions of the angle α. We have

1) Bao Jiguang, The global $C^{1,\frac{1}{4}}$ hypersurface of the nonnegative Gauss curvature, *Chin. Ann. of Math.*, to appear.

Lemma 2.

$$\lim_{\alpha\to 0}\frac{|\bar{x}(\alpha)|}{d(\alpha)} = \frac{|x_1|}{|x_1 - x_0|\sin\theta}.$$

Proof. It follows from $\bar{P} \in \partial\Omega \cap L$ that $\bar{x}(\alpha)$ satisfies the implicit equation

$$\rho(\bar{x}(\alpha)) - x_1\tan\theta = (\bar{x}(\alpha) - x_1)\tan(\theta \pm \alpha)$$

for α sufficiently small. Thus

$$\frac{d\bar{x}}{d\alpha} = \frac{\pm(x_1 - \bar{x})\sec^2(\theta \pm \alpha)}{\tan(\theta \pm \alpha) - \rho'(\bar{x}(\alpha))}.$$

With the distance formula between point and line we have

$$d(\alpha) = \frac{|x_1 - x_0|\sin\alpha}{\cos\theta}.$$

By viture of L' Hosptial rule and noting x, d, $\rho'(\bar{x}) \to 0$, if $\alpha \to 0$, we get

$$\lim_{\alpha\to 0}\frac{\bar{x}(\alpha)}{d(\alpha)} = \frac{\cos\theta}{|x_1 - x_0|}\lim_{\alpha\to 0}\frac{\bar{x}(\alpha)}{\sin\alpha} = \frac{\pm x_1}{|x_1 - x_0|\sin\theta}.$$

This completes the proof of Lemma 2.

Case 1. $P_1 \in \Omega$.

In this case $|OP_0| \leqslant |P_0P_1|$ since $P_0 \in N_1$, and there is a uniformly positive constant θ^0 such that $\theta \geqslant \theta^0$ since $P_1 \in \partial N_2 \cap \Omega$. From Lemma 2 and $|x_1| \leqslant 2|x_1 - x_0|$, we obtain

$$\frac{|\bar{x}|}{|PP_0|} \leqslant \frac{|\bar{x}(\alpha)|}{d(\alpha)} \leqslant \frac{2|x_1|}{|x_1 - x_0|\sin\theta} \leqslant \frac{4}{\sin\theta^0},$$

if δ is sufficiently small. It follows from $u(x_1, y_1) = 0$ and the convexity and nonnegativity of u that

$$u(x,y) \leqslant u(\bar{x},\bar{y}) = \varphi(\bar{x},\rho(\bar{x})) \leqslant C\bar{x}^2 \leqslant C|PP_0|^2.$$

Case 2. $P_1 \in \partial\Omega$.

We may suppose $|P_0P_2| \leqslant |P_0P_1|$, or exchange the position of P_1 and P_2 in the following proof.

Lemma 3. $|OP_1| \leqslant C\tan\theta$.

Proof. Since Ω is smooth and convex, the boundary data φ admits a smooth strict convex extension in $\bar{\Omega}$. Therefore

$$\varphi(x_1, y_1) \geqslant \varphi(0,0) + \varphi_x(0,0)x_1 + \varphi_y(0,0)y_1 + \frac{1}{C}(x_1^2 + y_1^2).$$

Using $\varphi(x_1, y_1) = \varphi(0,0) = 0$ and $\varphi \geqslant 0$ on $\partial\Omega$, we have

$$\varphi_x(0,0) = 0,$$

$$|OP_1|^2 \leqslant -C\varphi_y(0,0)y_1 = -C\varphi_y(0,0)x_1\tan\theta \leqslant C|OP_1|\tan\theta.$$

Hence Lemma 3 is proved.

Now fix $\theta_0 \in (0, \frac{\pi}{2})$ such that $C\tan\theta_0 < r$. If $\theta \geqslant \theta_0$, the proof of (3) is analogous to that of case 1. If $\theta < \theta_0$, denote $f(x) = \varphi(x, \rho(x))$.

Lemma 4. $0 \leqslant f''(0) \leqslant Cx_1^2$.

Proof. By the nonnegativity of u, we have $f(x) \geqslant f(0) = 0, f'(0) = 0$; thus for some $\xi \in (-r, r)$,

$$f(x) = x^2\left[\frac{1}{2}f''(0) + \frac{1}{6}f'''(0)x + \frac{1}{24}f''''(\xi)x^2\right], \quad x \in (-r, r).$$

It follows that for sufficiently large C,

$$\frac{1}{2}f''(0)+\frac{1}{6}f'''(0)x+Cx^2\geqslant 0,\ x\in(-\infty,+\infty),$$

$$(f'''(0))^2\leqslant Cf''(0).$$

By $f(x_1)=0$, we obtain

$$\frac{1}{2}f''(0)+\frac{1}{6}f'''(0)x_1+\frac{1}{24}f''''(\xi)x_1^2=0,$$

$$f'''(0)\leqslant C(x_1\sqrt{f''(0)}+x_1^2)\leqslant \eta f''(0)+C_\eta x_1^2.$$

Taking $\eta=\frac{1}{2}$ completes the proof of Lemma 4.

It follows from Lemma 4 that

$$u(x,y)\leqslant \varphi(\bar{x},\rho(\bar{x}))=(\varphi(x,\rho(x)))''(0)\bar{x}^2+O(\bar{x}^3)\leqslant C(x_1^2\bar{x}^2+|\bar{x}|^3).$$

By Lemmas 2 and 3, when $\delta<\sin^3\theta$ is sufficiently small, we have

$$u(x,y)\leqslant C\left(\tan^2\theta\frac{|PP_0|^2}{\sin^2\theta}+\frac{|PP_0|^3}{\sin^3\theta}\right)\leqslant C|PP_0|^2.$$

The proof of Theorem 1 is completed.

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