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J. Math. Anal. Appl. 303 (2005) 462–476
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The local regularity for strong solutions of the Hessian quotient equation

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Received 1 April 2004
Available online 18 September 2004
Submitted by N.S. Trudinger

Abstract

By means of the Reilly formula and the Alexandrov maximum principle, we obtain the local $C^{1,1}$ estimates of the $W^{2,p}$ strong solutions to the Hessian quotient equations for p sufficiently large, and then prove that these solutions are smooth. There are counterexamples to show that the integral exponent p is optimal in some cases. We modify partially the known result in the Hessian case, and extend the regularity result in the special Lagrangian case to the Hessian quotient case.

Keywords: Hessian equation; Strong solutions; Local regularity

1. Introduction

In this paper we consider the local $C^{1,1}$ estimate and the regularity of the strong solutions for the Hessian quotient equation

$$\frac{S_k(D^2u)}{S_l(D^2u)} = c, \quad \text{a.e. } x \in \Omega,\ 0 \leqslant l < k \leqslant n, \tag{1.1}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.08.046

where Ω is a domain in $\mathbb{R}^n$, c is a positive constant, D^2u denotes the Hessian of a function u on Ω, and $S_j(D^2u)$ is defined to be the jth elementary symmetric function of the eigenvalues $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of D^2u,

$$S_j(D^2u) = \sigma_j\big(\lambda(D^2u)\big) = \sum_{1\leqslant i_1<\cdots<i_j\leqslant n} \lambda_{i_1}\ldots\lambda_{i_j}, \quad j = 1, 2, \ldots, n.$$

By a classical theorem of Calderón and Zygmund [8], the functions in $W^{2,p}_{\mathrm{loc}}(\Omega)$, $p > \frac{n}{2}$, are pointwise twice differentiable almost everywhere in Ω. We say $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is k-admissible, denoted by $u \in \Phi^k(\Omega)$, if at almost every point of Ω the vector $\lambda = \lambda(D^2u)$ of eigenvalues of D^2u belongs to the cone

$$\Gamma_k = \big\{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ j = 1, \ldots, k\big\}.$$

We call a function $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for $p > \frac{n}{2}$ an admissible strong solution to (1.1) if $u \in \Phi^k(\Omega)$ and satisfies (1.1) almost everywhere in Ω (cf. [6]). From [13] and [7], we know that Eq. (1.1) is elliptic and

$$\left(\frac{S_k(D^2u)}{S_l(D^2u)}\right)^{\frac{1}{k-l}}$$

is a concave function of the second derivatives of u if $u \in \Phi^k(\Omega)$.

The Hessian quotient equation (1.1) is an important class of fully nonlinear elliptic equation which is closely related to geometry problems. Some well-known equations can be regarded as its special cases. When $l = 0$, it is a k-Hessian equation. In particular, it is a Poisson equation if $k = 1$, while it is a Monge–Ampère equation if $k = n$. When $k = n = 3$, $l = 1$ and $c = 1$, Eq. (1.1) arises from special Lagrangian geometry (cf. [10]): if u is a solution of (1.1), the graph of Du over $\mathbb{R}^3$ in $\mathbb{C}^3$ is a special Lagrangian submanifold in $\mathbb{C}^3$, i.e., its mean curvature vanishes everywhere and the complex structure on $\mathbb{C}^3$ sends the tangent space of the graph to the normal space at every point. Therefore the Hessian quotient equation (1.1) has drawn much attention.

The regularity of the strong solutions for above equations has been studied by many authors. It is a classical result (cf. [9]) that the $W^{2,p}_{\mathrm{loc}}(\Omega)$ strong solution of Poisson equation $\Delta u = \varphi(x)$ is smooth when $p > 1$ and $\varphi \in C^\infty(\Omega)$. Recently, Urbas (cf. [15] and [16]) has proved that when $p > \frac{k(n-1)}{2}$ such regularity result holds for the k-Hessian equations. For the special Lagrangian equation in dimension 3, $\det(D^2u) = \Delta u$, the same problem is resolved in [1]. It is verified that the $W^{2,p}_{\mathrm{loc}}(\Omega)$ solutions of the equations are smooth if $p > 3$. A counterexample is given to show that this regularity fails if $p < 3$.

The interior regularity for the Hessian quotient equations of the form

$$\frac{S_n(D^2u)}{S_l(D^2u)} = \varphi(x), \quad 1 \leqslant l < n, \tag{1.2}$$

has been discussed in [2], and the local $C^{1,1}$ estimate is established when $p > (n-1) \times \max\{n-l, 2\}$. Throughout this paper, setting

$$\gamma_1 = \max\left\{\frac{k(n-1)}{2}, \frac{n}{2}\right\}, \qquad \gamma_2 = \max\left\{(k-1)\max\{k-l, 2\}, \frac{n}{2}\right\},$$

and

$$\gamma = \begin{cases} \min\{\gamma_1, \gamma_2\}, & n \geqslant 3, \\ \gamma_2, & n = 2. \end{cases}$$

Our main results are as follows.

Theorem 1.1. *Let Ω be a domain in $\mathbb{R}^n$, and c be a positive constant. If $p > \gamma$, then every admissible strong solution of* (1.1) *in $W^{2,p}_{\mathrm{loc}}(\Omega)$ is smooth.*

Remark 1.2. There is a counterexample to show that Theorem 1.1 is false if $p < k$. Therefore our result is optimal for the following cases:

$$k = 1, \quad n = 2, \quad l = 0;$$
$$k = 2, \quad n = 2, 3, 4, \quad l = 0, 1;$$
$$k = 3, \quad n = 3, \quad l = 0, 1, 2.$$

It would be interesting to determine sharp lower bounds for p in Theorem 1.1. An example of Pogorelov [12] tells us that Theorem 1.1 fails in the Hessian case (i.e., $l = 0$) if $p < \frac{k(k-1)}{2}$. Moreover in both the Monge–Ampère case (i.e., $k = n > 2$ and $l = 0$) and the special Lagrangian case (i.e., $k = n = 3$ and $l = 1$) the optimal regularity results are $p > \frac{n(n-1)}{2}$ (cf. [3], [4] and [1], respectively), which can be obtained again by Theorem 1.1.

Remark 1.3. Theorem 1.1 still holds for the following equations of more general form:

$$\frac{S_k(D^2u)}{S_l(D^2u)} = \varphi(x), \quad \text{a.e. } x \in \Omega,\ 0 \leqslant l < k \leqslant n, \tag{1.3}$$

if $n \geqslant 2$, $p > \gamma_2$ and $0 < \varphi \in C^\infty(\Omega)$ (cf. [2]).

The result in Remark 1.3 coincides with [2] in the case of $k = n$. Noting that if $l = 0$ and $\gamma_2 < \gamma_1$, we modify partially the corresponding result of the k-Hessian equations in [15] and [16].

It is natural for the solutions of Eq. (1.1) to be considered in $\Phi^k(\Omega)$. We overcome more difficulties in this paper than in [2], since the eigenvalues of D^2u are no longer all positive for $u \in \Phi^k(\Omega)$ if $k < n$.

The rest of the paper is organized as follows. In the next section, we present some preliminary inequalities of the elementary symmetric functions $\sigma_k(\lambda)$ and their quotients $\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}$, which will be used later. Section 3 is devoted to the locally second order derivative bound in the case of $p > \gamma_1$ where we obtain a weighted iterative inequality by means of the Reilly formula to improve the regularity step by step. In Section 4 we consider the $C^{1,1}$ estimates in another case of $p > \gamma_2$, by using the Alexandrov maximum principle. Combining above results achieved, we can prove in the last section that the admissible $W^{2,p}_{\mathrm{loc}}(\Omega)$ strong solutions to Eq. (1.1) are smooth if $p > \gamma$ with the standard regularity theory on the elliptic equations. At the end of this paper, we give the counterexample in Remark 1.2, and then get some optimal cases.

2. Some properties of $\sigma_k(\lambda)$ and $\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}$

We will give some properties of the elementary symmetric functions $\sigma_k(\lambda)$ and their quotients $\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}$ in this section. Following the notations in [11], we set

$$\sigma_{k;i}(\lambda) = \sigma_k|_{\lambda_i=0}, \qquad \sigma_{k;i_1 i_2,\dots,i_s}(\lambda) = \sigma_k|_{\lambda_{i_1}=\lambda_{i_2}=\cdots=\lambda_{i_s}=0}.$$

For any $i \in \{1, 2, \dots, n\}$, we have $\frac{\partial\sigma_k}{\partial\lambda_i}(\lambda) = \sigma_{k-1;i}(\lambda)$. For convenience we set $\sigma_0(\lambda) = 1$ and $\sigma_k(\lambda) = 0$ for $k > n$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma_k$,

$$\text{there are at least } k \text{ positive components in } \lambda, \tag{2.1}$$

and

$$\sigma_{l;i_1 i_2 \dots i_s}(\lambda) > 0 \quad \text{for any } \{i_1, i_2, \dots, i_s\} \subset \{1, 2, \dots, n\},\ l + s = k. \tag{2.2}$$

For $k \geqslant 2$, we have Newton inequality (cf. [11])

$$\sigma_k(\lambda)\sigma_{k-2}(\lambda) \leqslant \frac{(k-1)(n-k+1)}{k(n-k+2)}\big(\sigma_{k-1}(\lambda)\big)^2, \quad \lambda \in \mathbb{R}^n.$$

Denote $\sigma_{k,l}(\lambda) = \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}$. It follows that

$$\begin{aligned}\frac{\partial\sigma_{k,l}}{\partial\lambda_i}(\lambda) &= \frac{\sigma_{k-1;i}(\lambda)\sigma_l(\lambda) - \sigma_{l-1;i}(\lambda)\sigma_k(\lambda)}{(\sigma_l(\lambda))^2}\\ &= \frac{\sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda) - \sigma_{l-1;i}(\lambda)\sigma_{k;i}(\lambda)}{(\sigma_l(\lambda))^2}\\ &\geqslant \frac{\sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda)}{C(\sigma_l(\lambda))^2}, \quad \lambda \in \Gamma_k.\end{aligned} \tag{2.3}$$

From now on suppose that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma_k$, then

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0, \tag{2.4}$$

$$0 < \frac{\partial\sigma_k}{\partial\lambda_1}(\lambda) \leqslant \frac{\partial\sigma_k}{\partial\lambda_2}(\lambda) \leqslant \cdots \leqslant \frac{\partial\sigma_k}{\partial\lambda_n}(\lambda). \tag{2.5}$$

From [17] we have Maclaurin inequality

$$\big(\big(C_n^k\big)^{-1}\sigma_k(\lambda)\big)^{\frac{1}{k}} \leqslant \big(\big(C_n^l\big)^{-1}\sigma_l(\lambda)\big)^{\frac{1}{l}}, \quad \lambda \in \Gamma_k,\ k \geqslant l \geqslant 1. \tag{2.6}$$

If $k \geqslant r$, $l \geqslant s$, $k - l \geqslant r - s$, there are Newton–Maclaurin inequalities [13]

$$\left(\frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}\right)^{\frac{1}{k-l}} \leqslant C\left(\frac{\sigma_r(\lambda)}{\sigma_s(\lambda)}\right)^{\frac{1}{r-s}}, \quad \lambda \in \Gamma_k,\ C = C(n, k, l, r, s). \tag{2.7}$$

For $l = k - 1$ in view of (2.3) and (2.5), we see

$$\frac{\partial}{\partial\lambda_j}\left(\frac{\sigma_k(\lambda)}{\sigma_{k-1}(\lambda)}\right) = \frac{\sigma_{k-1;j}(\lambda)}{\sigma_{k-1}(\lambda)} - \frac{\sigma_{k-1}(\lambda)\sigma_{k-2;j}(\lambda)}{(\sigma_{k-1}(\lambda))^2} \geqslant 0$$

for $\lambda \in \Gamma_k$, $j = 1, 2, \dots, n$. Therefore

$$\frac{\sigma_{k-1;j}(\lambda)}{\sigma_k(\lambda)} \geqslant \frac{\sigma_{k-2;j}(\lambda)}{\sigma_{k-1}(\lambda)}.$$

Repeating above discussion we obtain

$$\frac{\sigma_{k-1;j}(\lambda)}{\sigma_k(\lambda)} \geqslant \frac{\sigma_{k-2;j}(\lambda)}{\sigma_{k-1}(\lambda)} \geqslant \cdots \geqslant \frac{\sigma_{0;j}(\lambda)}{\sigma_1(\lambda)} = \frac{1}{\sigma_1(\lambda)}, \tag{2.8}$$

i.e.,

$$\sigma_{k-1;j}(\lambda) \geqslant \frac{\sigma_k(\lambda)}{\sigma_1(\lambda)}, \quad \lambda \in \Gamma_k,\ j = 1, 2, \ldots, n. \tag{2.9}$$

By Theorem 1 of [11], there exists a constant $\theta = \theta(n, k)$, such that

$$\frac{\sigma_{k-1;i}(\lambda)}{\sigma_{k-1}(\lambda)} \geqslant \theta, \quad i \geqslant k,\ \lambda \in \Gamma_k. \tag{2.10}$$

Write $\sigma_{k-1}(\lambda)$ as follows:

$$\sigma_{k-1}(\lambda) = \sigma_{k-1;1}(\lambda) + \lambda_1\sigma_{k-2;12}(\lambda) + \cdots + \lambda_1 \ldots \lambda_{k-2}\sigma_{1;12\ldots(k-1)}(\lambda) \\ + \lambda_1\lambda_2 \ldots \lambda_{k-1},$$

we have by (2.2),

$$\sigma_{k-1}(\lambda) \geqslant \lambda_1\lambda_2 \ldots \lambda_{k-1}, \quad \lambda \in \Gamma_k. \tag{2.11}$$

Using (2.4) and (2.5), we obtain

$$\sigma_{k;1}(\lambda) \leqslant C\big(\sigma_{k-1;1}(\lambda)\big)^{\frac{k}{k-1}} = C\sigma_{k-1;1}(\lambda)\big(\sigma_{k-1;1}(\lambda)\big)^{\frac{1}{k-1}} \leqslant C\lambda_1\sigma_{k-1;1}(\lambda).$$

In light of $\sigma_k(\lambda) = \lambda_1\sigma_{k-1;1}(\lambda) + \sigma_{k;1}(\lambda)$, finally we arrive at

$$\lambda_1\sigma_{k-1;1}(\lambda) \geqslant C\sigma_k(\lambda). \tag{2.12}$$

3. The $C^{1,1}$ estimates for $p > \gamma_1$

This section is devoted to the establishing of the local $C^{1,1}$ estimates of strong solutions to Eq. (1.1) for $p > \gamma_1$.

In this section from now on we always assume that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ with $p > \gamma_1$ and $n \geqslant 3$ is a strong solution of Eq. (1.1) in $\Phi^k(\Omega)$, and Ω' is a compact sub-domain of Ω in $\mathbb{R}^n$. Let ψ be a mollifier (cf. Chapter 7 in [9]). For $\varepsilon > 0$, the regularization of u is defined by the convolution

$$u_\varepsilon(x) = \varepsilon^{-n}\int\limits_{\Omega} \psi\left(\frac{x-y}{\varepsilon}\right)u(y)\,dy = \int\limits_{B_1(0)} \psi(y)u(x-\varepsilon y)\,dy.$$

Then $u_\varepsilon \in \Phi^k(\Omega')$ and u_ε belongs to $C^\infty(\Omega')$ provided $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, and

$$u_\varepsilon \to u \quad \text{in } W^{2,p}(\Omega') \tag{3.1}$$

as $\varepsilon \to 0$, by Lemma 2.3 in [14].

We write Eq. (1.1) in the form

$$F(D^2u) := \left(\frac{S_k(D^2u)}{S_l(D^2u)}\right)^{\frac{1}{k-l}} = c^{\frac{1}{k-l}}, \quad \text{a.e. in } \Omega. \tag{3.2}$$

Equation (3.2) is elliptic and the function F is concave with respect to any function in $\Phi^k(\Omega)$ (cf. [13]).

For later use, we denote the first derivatives of $F(M)$ and $S_k(M)$ with respect to m_{ij} by

$$F^{ij}(M) := \frac{\partial F(M)}{\partial m_{ij}}, \qquad S_k^{ij}(M) := \frac{\partial S_k(M)}{\partial m_{ij}}, \quad k = 1, 2, \ldots, n, \tag{3.3}$$

and

$$a_\varepsilon^{ij} = a^{ij}(D^2u_\varepsilon) := S_k^{ij}(D^2u_\varepsilon) - cS_l^{ij}(D^2u_\varepsilon), \tag{3.4}$$

$$f_\varepsilon := \frac{S_k(D^2u_\varepsilon)}{S_l(D^2u_\varepsilon)}, \quad x \in \Omega', \tag{3.5}$$

where $M = (m_{ij})$ is any $n \times n$ matrix. Let λ_ε and Λ_ε be the minimal and maximal eigenvalues and $\mathcal{T}_\varepsilon$ be the trace of the matrix (a_ε^{ij}), respectively. Also, let $\mathcal{T}$ be the trace of $(a^{ij}) = (a^{ij}(D^2u))$.

The estimates is almost identical to that of [1]. Therefore we need only reevaluate the lower and the upper bounds of (a_ε^{ij}) and $\mathcal{T}_\varepsilon$. Using the similar argument of [1], we have the following

Proposition 3.1. *Let $p > \gamma_1$, then in Ω',*

$$S_k(D^2u_\varepsilon) \geqslant cS_l(D^2u_\varepsilon), \quad f_\varepsilon \geqslant c, \quad \left(a_\varepsilon^{ij}\right) > 0, \tag{3.6}$$

if $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$.

Proposition 3.2. *Let $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, then λ_ε, Λ_ε and $\mathcal{T}_\varepsilon$ satisfy the following estimates in Ω':*

$$\begin{aligned} &\frac{1}{C\Delta u_\varepsilon} \leqslant a_\varepsilon^{ii} \leqslant C(\Delta u_\varepsilon)^{k-1}, \quad 1 \leqslant i \leqslant n; \\ &\frac{\Delta u_\varepsilon}{C} \leqslant a_\varepsilon^{ii} \leqslant C(\Delta u_\varepsilon)^{k-1}, \quad i \geqslant k; \\ &\frac{\Delta u_\varepsilon}{C} \leqslant \mathcal{T}_\varepsilon \leqslant C(\Delta u_\varepsilon)^{k-1}, \end{aligned} \tag{3.7}$$

where C is a positive constant depending only on n, k, l and c.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of D^2u_ε, here the ε-dependence of the λ_j's is omitted for simplicity. Without losing of generality we may assume $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$, and D^2u_ε is in diagonal form at the point under consideration. Therefore, $(S_k^{ij}(D^2u_\varepsilon))$ and $(S_l^{ij}(D^2u_\varepsilon))$ are in diagonal forms as well, and

$$\left(a_\varepsilon^{ij}\right) = \operatorname{diag}\left(\frac{\partial\sigma_k}{\partial\lambda_1} - c\frac{\partial\sigma_l}{\partial\lambda_1}, \frac{\partial\sigma_k}{\partial\lambda_2} - c\frac{\partial\sigma_l}{\partial\lambda_2}, \ldots, \frac{\partial\sigma_k}{\partial\lambda_n} - c\frac{\partial\sigma_l}{\partial\lambda_n}\right).$$

It follows from (3.4) and (3.6) that

$$\frac{\partial\sigma_{k,l}}{\partial\lambda_i}(\lambda) = \frac{\sigma_{k-1;i}(\lambda)\sigma_l(\lambda) - \sigma_{l-1;i}(\lambda)\sigma_k(\lambda)}{(\sigma_l(\lambda))^2} \leqslant \frac{\sigma_{k-1;i}(\lambda) - c\sigma_{l-1;i}(\lambda)}{\sigma_l(\lambda)} = \frac{a_\varepsilon^{ii}}{\sigma_l(\lambda)}.$$

In view of (2.3), we have

$$a_\varepsilon^{ii} \geqslant \frac{\sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda)}{C\sigma_l(\lambda)}. \tag{3.8}$$

Especially, by (2.10),

$$a_\varepsilon^{ii} \geqslant \frac{\sigma_{k-1;i}(\lambda)}{C} \quad \text{for } i \geqslant l+1. \tag{3.9}$$

From (3.6) and (2.11) we know

$$c \leqslant \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \leqslant \frac{C\lambda_1\lambda_2\ldots\lambda_k}{\lambda_1\lambda_2\ldots\lambda_l} = C\lambda_{l+1}\lambda_{l+2}\ldots\lambda_k \leqslant C(\lambda_{l+1})^{k-l}.$$

Therefore

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{l+1} \geqslant \frac{1}{C}. \tag{3.10}$$

By Maclaurin inequality we deduce

$$c \leqslant \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \leqslant \frac{C(\sigma_l(\lambda))^{\frac{k}{l}}}{\sigma_l(\lambda)} = C\sigma_l(\lambda)^{\frac{k-l}{l}} \leqslant C\sigma_1(\lambda)^{k-l} = C(\Delta u_\varepsilon)^{k-l}. \tag{3.11}$$

Again from (2.8), (2.11) and (3.10), when $l+1<k$, we obtain

$$\sigma_{l;i}(\lambda) \geqslant \frac{\sigma_{l+1}(\lambda)}{\sigma_1(\lambda)} \geqslant \frac{\lambda_1\ldots\lambda_{l+1}}{\sigma_1(\lambda)} \geqslant \frac{\lambda_2\ldots\lambda_{l+1}}{C} \geqslant \frac{1}{C}.$$

When $l+1=k$, it follows from (2.9), (3.6), (2.11) and (3.10),

$$\sigma_{l;i}(\lambda) \geqslant \frac{\sigma_{l+1}(\lambda)}{\sigma_1(\lambda)} = \frac{\sigma_k(\lambda)}{\sigma_1(\lambda)} \geqslant \frac{c\sigma_l(\lambda)}{\sigma_1(\lambda)} \geqslant \frac{c\lambda_1\ldots\lambda_l}{\sigma_1(\lambda)} \geqslant \frac{\lambda_2\ldots\lambda_l}{C} \geqslant \frac{1}{C}.$$

In light of (3.8), (2.5) and (2.12), we find

$$\lambda_1 a_\varepsilon^{ii} \geqslant \frac{\lambda_1\sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda)}{C\sigma_l(\lambda)} \geqslant \frac{\lambda_1\sigma_{k-1;1}(\lambda)}{C\sigma_l(\lambda)} \geqslant \frac{\sigma_k(\lambda)}{C\sigma_l(\lambda)} \geqslant \frac{1}{C}.$$

Using (2.11), we obtain

$$\frac{\Delta u_\varepsilon}{n} \leqslant \lambda_1 \leqslant \Delta u_\varepsilon,$$

$$a_\varepsilon^{ii} \geqslant \frac{1}{C\lambda_1} \geqslant \frac{1}{C\Delta u_\varepsilon}.$$

Hence it follows from (2.10), (2.11) and (3.10),

$$a_\varepsilon^{ii} \geqslant \frac{\sigma_{k-1;i}(\lambda)}{C} \geqslant \frac{\sigma_{k-1}(\lambda)}{C} \geqslant \frac{\lambda_1\lambda_2\ldots\lambda_{k-1}}{C} \geqslant \frac{\lambda_1}{C} \quad \text{for } i \geqslant k \geqslant l+1.$$

In view of

$$\sum_{i=1}^n \frac{\partial\sigma_k}{\partial\lambda_i}(\lambda) = (n-k+1)\sigma_{k-1}(\lambda), \tag{3.12}$$

we see

$$a_\varepsilon^{ii} = \frac{\partial\sigma_k}{\partial\lambda_i}(\lambda) - c\frac{\partial\sigma_l}{\partial\lambda_i}(\lambda) \leqslant \frac{\partial\sigma_k}{\partial\lambda_i}(\lambda) \leqslant C\sigma_{k-1}(\lambda) \leqslant C(\Delta u_\varepsilon)^{k-1}.$$

Finally we arrive at

$$\frac{\Delta u_\varepsilon}{C} \leqslant a_\varepsilon^{ii} \leqslant C(\Delta u_\varepsilon)^{k-1} \quad \text{for } i \geqslant k.$$

The last inequality in (3.7) follows from the definition of $\mathcal{T}_\varepsilon$. □

In the proof of Proposition 3.5 of [1], taking

$$q = \frac{(p-k+1)^2(n-1)k}{p((n-3)k+2)}, \quad r = \frac{2(n-1)k}{(n-1)k+2}, \quad s = \frac{2pq}{(p-k+1)^2 r} > 1,$$

we have

$$r < 2, \quad \frac{2pq}{(p-k+1)^2} = \frac{(n-1)r}{n-1-r}, \quad \frac{r}{2-r} = \frac{(n-1)k}{2}, \quad p-k < q-1.$$

An argument similar to [1] gives the following

Proposition 3.3. *Let $n \geqslant 3$, $p > \gamma_1$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ be a k-admissible strong solution of* (1.1). *Then we have $u \in W^{2,\bar p}_{\mathrm{loc}}(\Omega)$ for any $\bar p < \infty$, and for any compact sub-domain Ω' of Ω there exists a positive constant C, depending only on n, k ,l, c, p, $\bar p$, Ω',* $\operatorname{dist}(\Omega', \partial\Omega)$ *and the local L^p norm of Δu in Ω, such that*

$$\|D^2u\|_{L^{\bar p}(\Omega')} \leqslant C.$$

4. The $C^{1,1}$ estimates for $p > \gamma_2$

In this section we establish local $C^{1,1}$ estimates of strong solutions to Eq. (1.1) for $p > \gamma_2$, by using Alexandrov maximum principle (cf. [9]).

We recall (see [9]) that the upper contact set of a function v, denoted $\Gamma_v^+(\Omega)$, is defined to be the subset of Ω where the graph of v lies below a support hyperplane in $\mathbb{R}^{n+1}$, that is,

$$\Gamma_v^+(\Omega) = \left\{x \in \Omega:\ v(z) \leqslant v(x) + \nu\cdot(z-x) \text{ for all } z \in \Omega,\ \text{for some } \nu \in \mathbb{R}^n\right\}.$$

We have the following form of the Alexandrov maximum principle.

Proposition 4.1. *Let (a^{ij}) be an $n \times n$ matrix which is positive definite a.e. in a bounded domain $\Omega \subset \mathbb{R}^n$, and $v \in W^{2,q}_{\mathrm{loc}}(\Omega) \cap C^0(\bar\Omega)$ with $v = 0$ on $\partial\Omega$, where $q > n$. Then*

$$\sup_\Omega v \leqslant Cd\left(\int_{\Gamma_v^+(\Omega)} \frac{(-a^{ij}D_{ij}v)^n}{\det(a^{ij})}\,dx\right)^{\frac{1}{n}},$$

provided that $(a^{ij}D_{ij}v)/\det(a^{ij})^{\frac{1}{n}} \in L^n(\Omega)$, where $d = \operatorname{diam}\Omega$ and C is a constant depending only on n.

Proposition 4.2. *Let Λ_F be the maximum eigenvalue of the matrix (F^{ij}). Then*

$$\frac{\Lambda_F^n}{\det(F^{ij})} \leqslant C(\Delta u)^{\gamma_2}, \quad a.e.\ x \in \Omega.$$

Proof. Applying

$$\frac{\partial \sigma_{k,l}}{\partial \lambda_i}(\lambda) = \frac{\sigma_{k-1;i}(\lambda)\sigma_l(\lambda) - \sigma_{l-1;i}(\lambda)\sigma_k(\lambda)}{(\sigma_l(\lambda))^2} \leqslant \frac{\sigma_{k-1;i}(\lambda)}{\sigma_l(\lambda)},$$

and (3.12), we know

$$\frac{\partial \sigma_{k,l}}{\partial \lambda_i}(\lambda) \leqslant \frac{C\sigma_{k-1}(\lambda)}{\sigma_l(\lambda)}.$$

Therefore it follows from (2.3), (2.9) and (2.10),

$$\begin{aligned}
\frac{\Lambda_F^n}{\det(F^{ij})} &\leqslant \frac{C(\frac{\sigma_{k-1}(\lambda)}{\sigma_l(\lambda)})^n}{\prod_{i=1}^n \frac{\sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda)}{\sigma_l^2(\lambda)}} = \frac{C(\sigma_{k-1}(\lambda))^n(\sigma_l(\lambda))^n}{\prod_{i=1}^n \sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda)} \\
&\leqslant \frac{C(\sigma_{k-1}(\lambda))^n(\sigma_l(\lambda))^n}{(\frac{\sigma_k(\lambda)}{\sigma_1(\lambda)})^{k-1}(\sigma_{k-1}(\lambda))^{n-k+1}\cdot(\frac{\sigma_{l+1}(\lambda)}{\sigma_1(\lambda)})^l(\sigma_l(\lambda))^{n-l}} \\
&= \frac{C(\sigma_1(\lambda))^{k+l-1}(\sigma_{k-1}(\lambda))^{k-1}(\sigma_l(\lambda))^l}{(\sigma_k(\lambda))^{k-1}(\sigma_{l+1}(\lambda))^l} \\
&= C\left(\frac{\sigma_l(\lambda)}{\sigma_k(\lambda)}\right)^{k-1}\frac{(\sigma_1(\lambda))^{k+l-1}(\sigma_{k-1}(\lambda))^{k-1}}{(\sigma_l(\lambda))^{k-l-1}(\sigma_{l+1}(\lambda))^l}.
\end{aligned}$$

If $l+1=k$, we have

$$\frac{\Lambda_F^n}{\det(F^{ij})} \leqslant C\left(\frac{\sigma_l(\lambda)}{\sigma_k(\lambda)}\right)^{2(k-1)}\big(\sigma_1(\lambda)\big)^{2(k-1)}.$$

If $l+1<k$, it follows from the Maclaurin inequality and (2.5),

$$\begin{aligned}
\frac{\Lambda_F^n}{\det(F^{ij})} &\leqslant C\left(\frac{\sigma_l(\lambda)}{\sigma_k(\lambda)}\right)^{k-1}\frac{(\sigma_1(\lambda))^{k+l-1}(\sigma_{k-1}(\lambda))^{k-1}}{(\sigma_{k-1}(\lambda))^{\frac{(k-l-1)l}{k-1}+\frac{(l+1)l}{k-1}}} \\
&= C\left(\frac{\sigma_l(\lambda)}{\sigma_k(\lambda)}\right)^{k-1}\big(\sigma_1(\lambda)\big)^{k+l-1}\big(\sigma_{k-1}(\lambda)\big)^{k-1-\frac{kl}{k-1}} \\
&\leqslant C\left(\frac{\sigma_l(\lambda)}{\sigma_k(\lambda)}\right)^{k-1}\big(\sigma_1(\lambda)\big)^{k+l-1}\big(\sigma_1(\lambda)\big)^{(k-1)^2-kl} \\
&= C\left(\frac{\sigma_l(\lambda)}{\sigma_k(\lambda)}\right)^{k-1}\big(\sigma_1(\lambda)\big)^{(k-1)(k-l)}.
\end{aligned}$$

Thus we get the conclusion by Eq. (1.1) and the definition of γ_2. □

Proposition 4.3. *Let Ω be a domain in $\mathbb{R}^n$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ an admissible strong solution of* (1.1) *where $p > \gamma_2$, $c > 0$ is a positive constant. Then $u \in C^{1,1}(\Omega)$ and, for any $y \in \Omega$ and $0 < R < 1$ with $\overline{B_{3R}(y)} \subset \Omega$,*

$$\sup_{B_R(y)} |D^2 u| \leqslant C\big(R^{-n}\|\Delta u\|^p_{L^p(B_{3R}(y))}\big)^{\frac{1}{q}},$$

where $q = \min\{n, p-\gamma_2\}$ and C is a positive constant depending on n, k, l, p and c.

Proof. We consider the second order difference quotient

$$\Delta^h_{\xi\xi} u(x) = \frac{u(x+h\xi) - 2u(x) + u(x-h\xi)}{h^2}, \quad \text{a.e. } x \in \Omega_h,$$

where $\Omega_h \equiv \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > h\}$. By the concavity of F we obtain from (3.2) and (3.3) that for a.e. $x \in \Omega_h$,

$$\begin{aligned} 0 &= F\big(D^2 u(x \pm h\xi)\big) - F\big(D^2 u(x)\big) \\ &\leqslant F^{ij}\big(D^2 u(x)\big)\big(D_{ij}u(x \pm h\xi) - D_{ij}u(x)\big). \end{aligned} \tag{4.1}$$

Let $y \in \Omega$ with $\overline{B_{3R}(y)} \subset \Omega$. Without loss of generality we may assume $c \geqslant 1$. For simplicity we will write $B_r = B_r(y)$ for $r > 0$ and $F^{ij} := F^{ij}(D^2 u(x))$ in the rest of this proof. It follows from (4.1) that

$$F^{ij} D_{ij}\big(\Delta^h_{\xi\xi} u(x)\big) \geqslant 0, \quad \text{a.e. } x \in B_{2R}, \tag{4.2}$$

when $h \leqslant R$. Consider the function

$$v = \eta \Delta^h_{\xi\xi} u,$$

where

$$\eta(x) = \left(1 - \frac{|x-y|^2}{4R^2}\right)^{\beta},$$

and $\beta > 2$ is a constant to be determined later. Direct calculation leads to

$$|D\eta| \leqslant \frac{\beta}{R}\eta^{1-\frac{1}{\beta}}, \qquad |D^2\eta| \leqslant \frac{C(n,\beta)}{R^2}\eta^{1-\frac{2}{\beta}}, \tag{4.3}$$

and

$$\begin{aligned} F^{ij} D_{ij} v &= F^{ij}\big(\eta D_{ij}\big(\Delta^h_{\xi\xi} u\big) + 2 D_i\eta D_j\big(\Delta^h_{\xi\xi} u\big) + \big(\Delta^h_{\xi\xi} u\big) D_{ij}\eta\big) \\ &\geqslant 2F^{ij} D_i\eta D_j\big(\Delta^h_{\xi\xi} u\big) + \big(\Delta^h_{\xi\xi} u\big) F^{ij} D_{ij}\eta \\ &\geqslant -\Lambda_F\big(2|D\eta|\big|D\big(\Delta^h_{\xi\xi} u\big)\big| + \big|\Delta^h_{\xi\xi} u\big||D^2\eta|\big) \\ &\geqslant -\frac{C(n,\beta)\Lambda_F}{R^2\eta^{\frac{2}{\beta}}}\big(v + R\eta^{1+\frac{1}{\beta}}\big|D\big(\Delta^h_{\xi\xi} u\big)\big|\big), \quad \text{a.e. } x \in B_{2R}, \end{aligned} \tag{4.4}$$

by (4.2) and (4.3).

For $x \in \Gamma^+_v(B_{2R})$ we take $z \in \partial B_{2R}$ with

$$\frac{z-x}{|z-x|} = -\frac{Dv(x)}{|Dv(x)|}.$$

Since $v = 0$ on ∂B_{2R}, it follows that

$$v(x) \geqslant v(z) - Dv(x)\cdot(z-x) = |z-x|\left|Dv(x)\right| \geqslant R\eta^{\frac{1}{\beta}}\left|Dv(x)\right|$$

as

$$|x-z| \geqslant 2R - |x-y| \geqslant R\eta^{\frac{1}{\beta}}.$$

Consequently on $\Gamma_v^+(B_{2R})$,

$$\eta\left|D\left(\Delta_{\xi\xi}^h u\right)\right| = \left|Dv - \left(\Delta_{\xi\xi}^h u\right)D\eta\right| \leqslant |Dv| + \left(\Delta_{\xi\xi}^h u\right)|D\eta| \leqslant \frac{(1+\beta)v}{R\eta^{\frac{1}{\beta}}}. \tag{4.5}$$

By (4.4), (4.5) and the concavity of v on $\Gamma_v^+(B_{2R})$, we have

$$0 \leqslant -F^{ij}D_{ij}v \leqslant \frac{C\Lambda_F v}{R^2\eta^{\frac{2}{\beta}}}, \quad \text{a.e. in } \Gamma_v^+(B_{2R}), \tag{4.6}$$

where $C > 0$ is a constant depending only on n, β and c.

By Proposition 4.2 we obtain

$$0 \leqslant \frac{-F^{ij}D_{ij}v}{\det(F^{ij})^{\frac{1}{n}}} \leqslant \frac{Cv(\Delta u)^{\frac{\gamma_2}{n}}}{R^2\eta^{\frac{2}{\beta}}}, \quad \text{a.e. } x \in \Gamma_v^+(B_{2R}), \tag{4.7}$$

where C depends on n, β and c. Choosing $\beta = \frac{2n}{p-\gamma_2}$, we see from (4.7) and Proposition 4.1 that

$$\begin{aligned}
\sup_{B_{2R}} v &\leqslant CR\left(\int_{\Gamma_v^+(B_{2R})} \frac{(-F^{ij}D_{ij}v)^n}{\det(F^{ij})}\,dx\right)^{\frac{1}{n}} \\
&\leqslant CR^{-1}\left(\int_{\Gamma_v^+(B_{2R})} (\Delta u)^{\gamma_2}\left(\eta^{\frac{-2}{\beta}}v\right)^n\,dx\right)^{\frac{1}{n}} \\
&\leqslant CR^{-1}\left(\sup_{B_{2R}} v\right)^{1-\frac{2}{\beta}}\left(\int_{B_{2R}} (\Delta u)^{\gamma_2}\left(\Delta_{\xi\xi}^h u\right)^{p-\gamma_2}\,dx\right)^{\frac{1}{n}}.
\end{aligned}$$

It follows from Hölder inequality and (cf. [9])

$$\left\|\Delta_{\xi\xi}^h u\right\|_{L^p(B_{2R})} \leqslant \|\Delta u\|_{L^p(B_{3R})}.$$

Therefore

$$\int_{B_{2R}} (\Delta u)^{\gamma_2}\left(\Delta_{\xi\xi}^h u\right)^{p-\gamma_2}\,dx \leqslant \|\Delta u\|_{L^p(B_{2R})}^{\gamma_2}\left\|\Delta_{\xi\xi}^h u\right\|_{L^p(B_{2R})}^{p-\gamma_2} \leqslant \|\Delta u\|_{L^p(B_{3R})}^p, \tag{4.8}$$

where the constant C depends on n, k, l, p and c. If $\sup_{B_{2R}} v < 1$, Proposition 4.3 holds trivially. So one may assume $\sup_{B_{2R}} v \geqslant 1$. If $p \leqslant n + \gamma_2$ then $1 - \frac{2}{\beta} \geqslant 0$ and

$$\sup_{B_{2R}} v \leqslant CR^{-1}\left(\sup_{B_{2R}} v\right)^{1-\frac{2}{\beta}}\|\Delta u\|_{L^p(B_{3R})}^{\frac{p}{n}}.$$

It follows that

$$\sup_{B_{2R}} v \leqslant C R^{-1\frac{n}{p-\gamma_2}} \|\Delta u\|_{L^p(B_{3R})}^{\frac{p}{p-\gamma_2}}.$$

In view of (4.8) we know that $p \geqslant n + \gamma_2$ as $1 - \frac{2}{\beta} \leqslant 0$, and

$$\sup_{B_{2R}} v \leqslant C R^{-1} \|\Delta u\|_{L^p(B_{3R})}^{\frac{p}{n}}.$$

Finally we conclude that for $h < R$,

$$\sup_{B_R} \Delta_{\xi\xi}^h u \leqslant \left(\frac{4}{3}\right)^{\frac{2n}{p-\gamma_2}} \sup_{B_{2R}} v \leqslant C\left(R^{-q} \|\Delta u\|_{L^p(B_{3R})}^p\right)^{\frac{1}{q}}.$$

As ξ is an arbitrary unit vector in $\mathbb{R}^n$, this completes the proof of Proposition 4.3. □

5. The proofs of Theorem 1.1 and Remark 1.2

In the last section, we give the proof of Theorem 1.1. On the other hand, we illustrate a counterexample to show that Theorem 1.1 fails if $p < k$, and then obtain several optimal cases.

Proof of Theorem 1.1. Let Ω' be a bounded subdomain of Ω, $\bar{\Omega}' \subset \Omega$. If $n = 2$ or $n \geqslant 3$, $\gamma_1 \geqslant \gamma_2$. Then $\gamma = \gamma_2$ and $u \in C^{1,1}(\bar{\Omega}')$ by Proposition 4.3. As (3.2) is concave and uniformly elliptic in a strictly admissible solution with bounded second derivatives, the Evans–Krylov regularity theorem [5] then implies that $u \in C^{2,\alpha}(\bar{\Omega}')$ for some $0 < \alpha < 1$. The smoothness of u now follows from the standard elliptic regularity theory (cf. [9]).

If $n \geqslant 3$ and $\gamma_2 > \gamma_1$. Then $\gamma = \gamma_1$ and $u \in W_{\mathrm{loc}}^{2,p}(\Omega)$ by Proposition 3.3, where $p > \gamma_2$. Therefore Theorem 1.1 holds by using above discussion. □

Proof of Remark 1.2. For $k \geqslant 4$, we have

$$\gamma_1 \geqslant \max\left\{2(n-1), \frac{n}{2}\right\} = 2(n-1),$$
$$\gamma_2 \geqslant \max\left\{2(k-1), \frac{k}{2}\right\} \geqslant 2(k-1),$$

and

$$\gamma \geqslant \min\left\{2(n-1), 2(k-1)\right\} = 2(k-1) > k+1.$$

Therefore it is impossible that the results in Theorem 1.1 is optimal for $k \geqslant 4$.

A direct calculation gives that

$$\gamma = \gamma_1 = \gamma_2 = \frac{n}{2}, \quad \text{if } k = 1;$$
$$\gamma_1 = n-1, \quad \gamma_2 = \max\left\{2, \frac{n}{2}\right\}, \quad \gamma = \min\left\{n-1, \frac{n}{2}\right\}, \quad \text{if } k = 2;$$

and

$$\gamma_1 = \frac{3}{2}(n-1), \quad \gamma_2 = \begin{cases} \max\{6, \frac{n}{2}\}, & l = 0, \\ \max\{4, \frac{n}{2}\}, & l = 1, 2, \end{cases}$$
$$\gamma \geqslant \min\left\{\frac{3}{2}(n-1), 4\right\}, \quad \text{if } k = 3.$$

So we derive the optimal cases that has been mentioned in Remark 1.2.

We conclude this paper with showing that a radially symmetric admissible solutions to Eq. (1.1) must be a quadratic polynomial or a strong solution in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p < k$ but not in $W^{2,k}_{\mathrm{loc}}(\mathbb{R}^n)$.

The counterexample of the case $k = n$ has been found in [2]. Hence we may assume $1 < k < n$. Let $u(x) = y(r)$, where $|x| = r$. Then for $i, j = 1, 2, \ldots, n$,

$$\frac{\partial u}{\partial x_i} = y' \frac{x_i}{r},$$
$$\frac{\partial^2 u}{\partial x_i \partial x_j} = y'' \frac{x_i x_j}{r^2} + y' \left(\frac{r^2 \delta_{ij} - x_i x_j}{r^3}\right).$$

At $x = (r, 0, \ldots, 0)$, $u_{11} = y''$, $u_{ii} = \frac{y'}{r}$ $(i \geqslant 2)$. Since the rotation invariance of $S_j(D^2u)$, the eigenvalues of D^2u are $\lambda_1 = y''$, $\lambda_i = \frac{y'}{r}$ $(i \geqslant 2)$. Therefore

$$S_j(D^2u) = C^{j-1}_{n-1} y'' \left(\frac{y'}{r}\right)^{j-1} + C^j_{n-1} \left(\frac{y'}{r}\right)^j$$
$$= C^{j-1}_{n-1} r^{-(n-1)} \left(\frac{r^{n-j}}{j} (y')^j\right)', \quad j = 1, 2, \ldots, n-1. \tag{5.1}$$

Let $u(x) = y(r)$ be such a solution of (1.1). Then u satisfies

$$C^{k-1}_{n-1} y'' \left(\frac{y'}{r}\right)^{k-1} + C^k_{n-1} \left(\frac{y'}{r}\right)^k = c C^{l-1}_{n-1} y'' \left(\frac{y'}{r}\right)^{l-1} + c C^l_{n-1} \left(\frac{y'}{r}\right)^l, \tag{5.2}$$

i.e.,

$$C^{k-1}_{n-1} r^{-(n-1)} \left(\frac{r^{n-k}}{k} (y')^k\right)' = c C^{l-1}_{n-1} r^{-(n-1)} \left(\frac{r^{n-l}}{l} (y')^l\right)'.$$

We integrate to obtain

$$r^n \left(\frac{y'}{r}\right)^k = C_0 r^n \left(\frac{y'}{r}\right)^l + C, \tag{5.3}$$

where C_0 is a positive constant depending only on n, k, l and c, and C is an arbitrary constant.

If $C = 0$, then either $y' = 0$ or $y' = (C_0)^{\frac{1}{k-l}} r$. Therefore y is a constant or quadratic polynomial in this case. If $C > 0$ for r sufficiently small, take positive $\frac{y'}{r}$ in (5.3). Then

$$r^n \left(\frac{y'}{r}\right)^k > C := \delta^k > 0.$$

Furthermore

$$r^n\left(\frac{y'}{\delta r}\right)^l = r^{n-\frac{nl}{k}}\cdot\left(\frac{r^{\frac{n}{k}}y'}{\delta r}\right)^l \leqslant r^{n-\frac{nl}{k}}\cdot\left(\frac{r^{\frac{n}{k}}y'}{\delta r}\right)^k = \frac{r^{n-\frac{nl}{k}}}{\delta^k}\cdot r^n\left(\frac{y'}{r}\right)^k.$$

In light of $k > l$, we see

$$\frac{C_0 r^{n-\frac{nl}{k}}}{\delta^{k-l}} \leqslant \frac{1}{2}$$

for r sufficiently small. Thus $r^n(\frac{y'}{r})^k \leqslant 2C$. We may deduce $r^n(\frac{y'}{r})^l \to 0$ as $r \to 0$. Let $r \to 0$ in (5.3), we have $r^n(\frac{y'}{r})^k \to C$, i.e.,

$$\lim_{r\to 0}\frac{y'}{r^{\frac{k-n}{k}}} = \delta. \tag{5.4}$$

Applying (5.2)–(5.4), we see

$$\lim_{r\to 0}\frac{y''}{r^{-\frac{n}{k}}} = -\frac{(n-k)}{k}\delta. \tag{5.5}$$

Therefore

$$\lim_{r\to 0}\frac{\Delta u}{r^{-\frac{n}{k}}} = \lim_{r\to 0}\frac{y'' + (n-1)\frac{y'}{r}}{r^{-\frac{n}{k}}} = n\left(1-\frac{1}{k}\right)\delta > 0.$$

So $\Delta u \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for $p < k$, but $\Delta u \notin L^k_{\mathrm{loc}}(\mathbb{R}^n)$. Hence if we can show $u \in \Phi^k(\Omega)$, Theorem 1.1 fails when $p < k$. Indeed, it follows from (5.1),

$$S_j(D^2u) = \frac{(n-1)!}{(n-j-1)!(j-1)!}\cdot\left(\frac{y'}{r}\right)^{j-1}\cdot\left(\frac{y''}{n-j}+\frac{y'}{jr}\right).$$

In view of (5.4) and (5.5), we see

$$\lim_{r\to 0}\frac{\frac{y''}{n-j}+\frac{y'}{jr}}{r^{-\frac{n}{k}}} = \frac{n(k-j)}{(n-j)kj}\delta.$$

Therefore

$$\lim_{r\to 0}\frac{S_j(D^2u)}{r^{\frac{nj}{k}}} = \frac{k-j}{k}C_n^j\delta^j > 0 \quad \text{for } j = 1, 2, \ldots, k-1.$$

It follows that $S_j(D^2u) > 0$ for $j = 1, 2, \ldots, k-1$ and r sufficiently small. In view of Eq. (1.1), we know that $S_k(D^2u) > 0$ and then $u \in \Phi^k(\Omega)$. □

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