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The exterior Dirichlet problem for special Lagrangian equations in dimensions $n \leq 4$

Jiguang Bao, Haigang Li*

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing, 100875, China

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ABSTRACT

In this paper, we establish the existence theorem for the exterior Dirichlet problem for special Lagrangian equations with prescribed asymptotic behavior at infinity. This extends the previous results on Monge–Ampère equations and Hessian equations to special Lagrangian equations in dimensions $n \leq 4$, which is from calibrated geometry. More generally, we prove that the result is also true for Hessian quotient equations with $0 \leq l < k \leq n$ in dimensions $n \geq 3$.

1. Introduction

In this paper, we consider the exterior Dirichlet problem for the following Hessian quotient equations

$$\begin{cases} F(D^2u) = S_{3,1}(\lambda(D^2u)) = 1, & \text{in } \mathbb{R}^n \setminus \overline{D}, \\ u = \varphi, & \text{on } \partial D \end{cases} \tag{1.1}$$

where D is a bounded open set in $\mathbb{R}^n$, $n \geq 3$, $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix D^2u,

$$S_{k,l}(\lambda) = \frac{S_k(\lambda)}{S_l(\lambda)}, \quad 0 \leq l < k \leq n,$$

and

$$S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k-th elementary symmetric function. We define $S_0(\lambda) = 1$.

For a bounded open set Ω in $\mathbb{R}^n$ $(n \geq 2)$, if u is a smooth function on Ω, then the graph of ∇u is automatically a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$ with the standard complex structure. Harvey and Lawson [1] showed that the partial differential

* Corresponding author. Tel.: +86 58807735.
E-mail addresses: jgbao@bnu.edu.cn (J. Bao), hgli@bnu.edu.cn (H. Li).

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equation

$$\mathrm{Im}\det(Id+\sqrt{-1}D^2u)\equiv\sum_{k=0}^{[(n-1)/2]}(-1)^k S_{2k+1}(\lambda(D^2u))=0 \tag{1.2}$$

is elliptic at every solution u and (see Theorem 2.7 in [1]) if u is a solution, then the graph of ∇u is an absolutely volume-minimizing submanifold of $\mathbb{R}^n\times\mathbb{R}^n$, which are called special Lagrangian submanifolds. For $n=2$, (1.2) is $\Delta u=0$; for $n=3$ and $n=4$, (1.2) corresponds to

$$S_3(\lambda(D^2u))=S_1(\lambda(D^2u)).$$

The special Lagrangian equation (1.2) in $\mathbb{R}^3$ also takes the form

$$\det(D^2u)=\Delta u. \tag{1.3}$$

The regularity of convex strong solutions of (1.3) in $\mathbb{R}^3$ was studied by Bao and Chen [2]. In dimensions $n\geq 3$, the regularity of viscosity solutions to special Lagrangian equations was established by Chen, Warren and Yuan [3] and Wang and Yuan [4]. When the domain is the entire space, the Bernstein type results for global solutions were obtained by Fu [5] for $n=2$ and Yuan [6] for higher dimensions. We mention that the Liouville property of global solutions to Hessian and Hessian quotient equations can be referred to [7] and the references therein.

For the Monge–Ampère equations

$$\det(D^2u)=1 \tag{1.4}$$

in $\mathbb{R}^n$, a classical theorem of Jörgens [8], Calabi [9], and Pogorelov [10] states that any classical convex solution of (1.4) must be a quadratic polynomial. Extensive studies and outstanding results on such an equation are given by Cheng and Yau [11], Caffarelli [12], Jost and Xin [13], Trudinger and Wang [14] and many others.

Caffarelli and Li [15] extended the Jörgens–Calabi–Pogorelov theorem to exterior domains, namely that if u is a locally convex viscosity solution of (1.4) in $\mathbb{R}^n\setminus\overline{D}$, where D is a bounded open set in $\mathbb{R}^n$, $n\geq 3$, then there exist an $n\times n$ real symmetric positive definite matrix A with $\det(A)=1$, a vector $b\in\mathbb{R}^n$, and a constant $c\in\mathbb{R}$ such that

$$\limsup_{|x|\to\infty}|x|^{n-2}\left|u(x)-\left(\frac{1}{2}x^TAx+b\cdot x+c\right)\right|<\infty. \tag{1.5}$$

We remark that the theorem of Jörgens–Calabi–Pogorelov is an easy consequence of the above results (see [15]). Motivated by the above mentioned asymptotic results, Caffarelli and Li [15] established the existence of solutions of the exterior Dirichlet problem for the Monge–Ampère equation $\det(D^2u)=1$ in $\mathbb{R}^n\setminus D$, $n\geq 3$ with prescribed asymptotic behavior (1.5). In $\mathbb{R}^2$, similar problems can be referred to [16–18].

There have been some partial results on the existence of solutions to the following exterior Dirichlet problem for Hessian quotient equations:

$$S_{k,l}(\lambda(D^2u))=1 \tag{1.6}$$

in $\mathbb{R}^n\setminus\overline{D}$. For the cases $l=0$, (1.6) is Hessian equation

$$S_k(\lambda(D^2u))=1.$$

Dai and Bao [19] established its existence theorem, under the asymptotic assumption

$$\limsup_{|x|\to\infty}|x|^{n-2}\left|u(x)-\left(\frac{c^*}{2}|x|^2+c\right)\right|<\infty, \tag{1.7}$$

where $c^*=(C_n^k)^{-1/k}$, $C_n^k=\frac{n!}{(n-k)!k!}$. A more general case for Hessian equations was considered in [20]. Dai [21] extended the result in [19] to Hessian quotient cases with $k-l\geq 3$ and proved the existence of the exterior Dirichlet problem of (1.6) with the following asymptotic behavior at infinity:

$$\limsup_{|x|\to\infty}\left(|x|^{k-l-2}\left|u(x)-\left(\frac{c^*}{2}|x|^2+c\right)\right|\right)<\infty, \tag{1.8}$$

with $c^*=\left(C_n^l/C_n^k\right)^{\frac{1}{k-l}}$.

If $n=3$ or 4, (1.6) with $k=3$ and $l=1$ corresponds to the special Lagrangian equation (1.2). However, the constraint that $k-l\geq 3$ in [21] precludes these special cases which are useful in geometry. Therefore, in this paper we will prove the existence theorems for the exterior Dirichlet problem for the special Lagrangian equation (1.2) in dimensions $n=3$ and 4, with prescribed asymptotic behavior at infinity (1.7).

First by solving the radial symmetric solution of the corresponding ordinary differential equation of

$$S_{3,1}(\lambda(D^2u))=1 \tag{1.9}$$

to be a subsolution of (1.1), and then using Perron's method, we obtain the first main theorem.

Theorem 1.1. *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^n, n \geq 3$, $\varphi \in C^2(\partial D)$. Then there exists some constant c_*, depending only on n, D, and φ, such that for every $c > c_*$ there exists a unique function $u \in C^0(\mathbb{R}^n \setminus D)$ that satisfies (1.1) in the viscosity sense and the asymptotic assumption (1.7) with $c^* = \left(C_n^1/C_n^3\right)^{1/2}$.*

More generally, for Hessian quotient equations (1.6) with $0 \leq l < k \leq n$, although we cannot obtain an explicit formula for its radial solution, we still can detect some properties from an observation on the corresponding ordinary differential equation, which guarantee the following theorem still true for the exterior Dirichlet problem of

$$\begin{cases} F(D^2u) = S_{k,l}(\lambda(D^2u)) = 1, \quad 0 \leq l < k \leq n, & \text{in } \mathbb{R}^n \setminus \overline{D}, \\ u = \varphi, & \text{on } \partial D. \end{cases} \tag{1.10}$$

Theorem 1.2. *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^n, n \geq 3$, $\varphi \in C^2(\partial D)$. Then there exists some constant c_*, depending only on n, D, and φ, such that for every $c > c_*$ there exists a unique function $u \in C^0(\mathbb{R}^n \setminus D)$ that satisfies (1.10) in the viscosity sense and the asymptotic assumption (1.7) with $c^* = \left(C_n^l/C_n^k\right)^{\frac{1}{k-l}}$.*

The plan of this paper is as follows. In the next section we give some elementary results on viscosity solutions and Perron's method. In Section 3, we study the properties of the radial solutions of Hessian quotient equations. In Section 4 we give the Proof of Theorems 1.1 and 1.2.

2. Preliminaries

For the reader's convenience, we recall the definition of viscosity solutions for fully nonlinear elliptic equations (1.6) (see, e.g., [22]).

Definition 2.1. A function $u \in C^2(\mathbb{R}^n \setminus \overline{D})$ is called admissible (or k-convex) if at every $x \in \mathbb{R}^n \setminus \overline{D}$, $\lambda(D^2u(x)) \in \overline{\Gamma}_k$, where $\Gamma_k = \{\lambda \in \mathbb{R}^n \mid S_j(\lambda) > 0,\ j = 1, 2, \dots, k\}$.

Definition 2.2. A function $u \in \mathrm{USC}(\mathbb{R}^n \setminus \overline{D})$ ($u \in \mathrm{LSC}(\mathbb{R}^n \setminus \overline{D})$) is said to be a viscosity subsolution (supersolution) of (1.6) in $\mathbb{R}^n \setminus \overline{D}$, (or say that u satisfies $F(D^2u) \geq 1 (\leq 1)$ in the viscosity sense), if for any open set N in $\mathbb{R}^n \setminus \overline{D}$, any admissible function $\psi \in C^2(\mathbb{R}^n \setminus \overline{D})$ and local maximum (minimum) $\bar{x} \in N$ of $u - \psi$ we have

$$F(D^2\psi(\bar{x})) \geq 1 (\leq 1). \tag{2.1}$$

A function $u \in C^0(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity solution of (1.6), if it is both a viscosity subsolution and supersolution of (1.6).

Then the relation between viscosity and classical solutions is that if u is an admissible classical solution of (1.6), then u is a viscosity solution; conversely, if u is a viscosity solution of (1.6) and u is of class C^2, then u is an admissible classical solution.

Definition 2.3. Let $\varphi \in C^0(\partial D)$. A function $u \in C^0(\mathbb{R}^n \setminus D)$ is a viscosity subsolution (resp. supersolution, solution) of the Dirichlet problem (1.10), if u is a viscosity subsolution (resp. supersolution, solution) of (1.6) and $u \leq$ (resp. $\geq, =$) $\varphi(x)$ on ∂D.

With the above definition of the viscosity solution, the well-known theory on the viscosity solution developed in [23] can be adapted to the present case with slight modifications. Under the assumptions $u, v \in C^0(\overline{\Omega})$, the comparison principle was proved in [22], based on Jensen approximations (see [24]). The proof remains valid under the weaker regularity assumptions on u and v.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u \in \mathrm{LSC}(\overline{\Omega})$ and $v \in \mathrm{USC}(\overline{\Omega})$ are respectively viscosity supersolutions and subsolutions of (1.6) in Ω satisfying $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω.*

The following ingredients for the viscosity adaptation of Perron's method (see [25]) are available.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\mathcal{S}$ be a non-empty family of viscosity subsolutions of (1.6) in Ω. Set*

$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}\},$$

and

$$u^*(x) = \lim_{r\to 0} \sup_{B_r(x)} u(y)$$

be the upper semicontinuous envelope of u. Then, if $u^* < \infty$ in Ω, u^* is a viscosity subsolution of (1.6) in Ω.

Lemma 2.2 can be proved by standard arguments; see e.g. [23]. Let

$$u_*(x) = \lim_{r\to 0} \inf_{B_r(x)} u(y)$$

be the lower semicontinuous envelope of u. We also need the following construction to apply Perron's method.

Lemma 2.3 ([23]). *Let Ω be open and u be subsolution of $F(D^2u) = 1$ in Ω. If u_* fails to be a supersolution at some point $\hat{x}$, then for any small $\kappa > 0$ there is a subsolution U_κ of $F(D^2u) = 1$ in Ω satisfying*

$$\begin{cases} U_\kappa(x) \geq u(x) \quad \text{and} \quad \sup\limits_{\Omega}(U_\kappa - u) > 0, \\ U_\kappa(x) = u(x) \quad \text{for } x \in \Omega, \ |x - \hat{x}| \geq \kappa. \end{cases} \tag{2.2}$$

3. Radial symmetric solutions

For $r > 0$, let $B_r := \{x \in \mathbb{R}^n | \, |x| < r\}$. In order to find a subsolution of (1.9), we first consider the radial symmetric solutions of (1.9). We have the following proposition.

Proposition 3.1. *For $n \geq 3$, let $u_a = u_a(r)$ be a smooth radial symmetric solution of*

$$S_{3,1}(\lambda(D^2u)) = 1, \quad \text{in } \mathbb{R}^n \setminus \overline{B_1}, \tag{3.1}$$

then u_a is locally convex in $\mathbb{R}^n \setminus \overline{B_1}$ and

$$u_a(r) = \frac{c^*}{2}r^2 + \mu_0(a) + O\left(r^{2-n}\right), \quad \text{as } r \to +\infty,$$

where $\mu_0(a)$ is a strictly increasing function of a.

Proof. Let $u = u(r)$ be a smooth radial function, then

$$\lambda(D^2u) = \left(u''(r), \frac{u'(r)}{r}, \ldots, \frac{u'(r)}{r}\right).$$

Thus if $u = u(r)$ is a real solution of (3.1), then

$$u''(r) \cdot C_{n-1}^2 \left(\frac{u'(r)}{r}\right)^2 + C_{n-1}^3 \left(\frac{u'(r)}{r}\right)^3 = u''(r) + C_{n-1}^1 \frac{u'(r)}{r}.$$

That is,

$$(r^{n-3}(u'(r))^3)' = \frac{C_n^1}{C_n^k}(r^{n-1}u'(r))'.$$

Integrating it, we have

$$\left(\frac{u'(r)}{r}\right)^3 - \frac{C_n^1}{C_n^3}\frac{u'(r)}{r} - \frac{2a}{r^n} = 0,$$

where a is an arbitrary positive constant. (Actually, we need a to be a sufficiently large positive constant, determined by c_* later.)

Solving the cubic equation

$$z^3 + pz + q = 0,$$

using the Cardano formula, we have

$$z_j = A\omega^j + B\omega^{3-j}, \quad j = 1, 2, 3,$$

where

$$A = \left[-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right]^{1/3}, \qquad B = \left[-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right]^{1/3}, \qquad \omega = \frac{-1 + \sqrt{-3}}{2}.$$

Letting $p = -\frac{C_n^1}{C_n^3} = -(c^*)^2$, $q = -\frac{2a}{r^n}$, recalling u is a real-valued function, we have

$$\frac{u'(r)}{r} = \left[\frac{a}{r^n} + \sqrt{\left(\frac{a}{r^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}\right]^{1/3} + \left[\frac{a}{r^n} - \sqrt{\left(\frac{a}{r^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}\right]^{1/3}. \tag{3.2}$$

We first show that u is a locally convex function on $\mathbb{R}^n \setminus B_1$ (cf. [2]). From (3.2) it follows that

$$u''(r) = \frac{\left(\frac{c^*}{\sqrt{3}}\right)^6}{\sqrt{\left(\frac{a}{r^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}} \left(\left[\frac{a}{r^n} + \sqrt{\left(\frac{a}{r^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6} \right]^{2/3} - \left[\frac{a}{r^n} - \sqrt{\left(\frac{a}{r^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6} \right]^{2/3} \right) \tag{3.3}$$

for $r \neq \left((\frac{\sqrt{3}}{c^*})^3 a \right)^{1/n}$, and

$$u''\left(\left(\left(\frac{\sqrt{3}}{c^*}\right)^3 a \right)^{1/n} \right) = \frac{4}{3} \left(\left(\frac{\sqrt{3}}{c^*}\right)^3 a \right)^{\frac{2}{3n}}.$$

It is clear that if $r \leq \left((\frac{\sqrt{3}}{c^*})^3 a \right)^{1/n}$, $u'(r)$ and $u''(r)$ are real. For $r > \left((\frac{\sqrt{3}}{c^*})^3 a \right)^{1/n}$, letting

$$\frac{a}{\left(\frac{c^*}{\sqrt{3}}\right)^3 r^n} = \cos\theta,$$

then $0 < \theta < \frac{\pi}{2}$. Thus, (3.2) and (3.3) can be rewritten as follows:

$$u'(r) = \frac{c^* r}{\sqrt{3}} \left(\left[\cos\theta + \sqrt{-1}\sin\theta\right]^{1/3} + \left[\cos\theta - \sqrt{-1}\sin\theta\right]^{1/3} \right) = \frac{2c^* r}{\sqrt{3}} \cos\frac{\theta}{3},$$

and

$$\begin{aligned} u''(r) &= \left(\frac{c^*}{\sqrt{3}}\right) \frac{\left[\cos\theta + \sqrt{-1}\sin\theta\right]^{2/3} - \left[\cos\theta - \sqrt{-1}\sin\theta\right]^{2/3}}{i\sin\theta} \\ &= 2\left(\frac{c^*}{\sqrt{3}}\right) \frac{\sin(2\theta/3)}{\sin\theta}. \end{aligned}$$

By a direct calculation, we have

$$\lim_{r\to\infty} \frac{u'(r)}{r} = \lim_{r\to\infty} u''(r) = c^*,$$

and

$$u'(r) > 0, \qquad u''(r) > 0 \quad \text{on } [1, +\infty).$$

This shows that $u = u(r)$ is a locally convex function in $\mathbb{R}^3 \setminus B_1$.

From (3.2), we have

$$\begin{aligned} u(r) &= \frac{c^*}{2} r^2 + \int_1^r s \left\{ \left[\frac{a}{s^n} + \sqrt{\left(\frac{a}{s^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6} \right]^{1/3} + \left[\frac{a}{s^n} - \sqrt{\left(\frac{a}{s^n}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6} \right]^{1/3} - c^* \right\} ds + C \\ &= \frac{c^*}{2} r^2 + \int_1^r \left(\frac{c^* s}{\sqrt{3}}\right) \left(2\cos\left(\frac{1}{3} \arccos\left(\frac{a}{\left(\frac{c^*}{\sqrt{3}}\right)^3 s^n} \right) \right) - \sqrt{3} \right) ds + C. \end{aligned} \tag{3.4}$$

Let

$$f(s) = \cos\left(\frac{1}{3} \arccos\left(\frac{a}{\left(\frac{c^*}{\sqrt{3}}\right)^3 s^n} \right) \right),$$

then as $s \to +\infty$, by Taylor's expansion, we have

$$f(s) = \frac{\sqrt{3}}{2} + \frac{1}{6} \frac{a}{\left(\frac{c^*}{\sqrt{3}}\right)^3 s^n} + o\left(\frac{1}{s^n}\right).$$

So that

$$g(r)=\int_r^{+\infty}\left(\frac{c^*s}{\sqrt{3}}\right)\left(2\cos\left(\frac{1}{3}\arccos\left(\frac{a}{\left(\frac{c^*}{\sqrt{3}}\right)^3 s^n}\right)\right)-\sqrt{3}\right)ds$$

is convergent, and

$$g(r)=\frac{1}{3}\frac{(n-2)a}{\left(\frac{c^*}{\sqrt{3}}\right)^2}\frac{1}{r^{n-2}}+o\left(\frac{1}{r^{n-2}}\right)\quad \text{as } r\to+\infty.$$

Therefore, by (3.4), we have

$$u(r)=\frac{c^*}{2}r^2+\mu_0(a)+O\left(\frac{1}{r^{n-2}}\right)\quad \text{as } r\to+\infty, \tag{3.5}$$

where

$$\mu_0(a)=\int_1^{+\infty}s\left\{\left[\frac{a}{s^n}+\sqrt{\left(\frac{a}{s^n}\right)^2-\left(\frac{c^*}{\sqrt{3}}\right)^6}\right]^{1/3}+\left[\frac{a}{s^n}-\sqrt{\left(\frac{a}{s^n}\right)^2-\left(\frac{c^*}{\sqrt{3}}\right)^6}\right]^{1/3}-c^*\right\}ds+C.$$

By a simple computation similar as above, we have

$$\begin{aligned}\frac{\partial\mu_0(a)}{\partial a}&=\int_1^{+\infty}\frac{s}{3s^n\sqrt{\left(\frac{a}{s^n}\right)^2-\left(\frac{c^*}{\sqrt{3}}\right)^6}}\left\{\left[\frac{a}{s^n}+\sqrt{\left(\frac{a}{s^n}\right)^2-\left(\frac{c^*}{\sqrt{3}}\right)^6}\right]^{1/3}-\left[\frac{a}{s^n}-\sqrt{\left(\frac{a}{s^n}\right)^2-\left(\frac{c^*}{\sqrt{3}}\right)^6}\right]^{1/3}\right\}ds\\&>0.\end{aligned}$$

The proof is completed. □

For Hessian quotient equations

$$S_{k,l}(\lambda(D^2u))=1,\quad \text{in } \mathbb{R}^n\setminus\overline{B_1}, \tag{3.6}$$

assume that $u=u(r)$ is a smooth convex radial symmetric solution of (3.6). By the direct calculation, we know that $w(r)=\frac{u'(r)}{r}$ satisfies

$$F(w,b):=w^k(r)-\left(C_n^l/C_n^k\right)w^l(r)-\frac{b}{r^n}=0,\quad w(r)>0, \tag{3.7}$$

for an arbitrary constant $b>0$. Although there is no explicit formula for the solution of (3.7), similarly as (3.2), by a straightforward analysis on (3.7) itself, we can obtain the desired properties.

Rewrite it as

$$w^l(r)\left(w^{k-l}(r)-(c^*)^{k-l}\right)=\frac{b}{r^n},$$

where $c^*=(C_n^l/C_n^k)^{\frac{1}{k-l}}$. If $b>0$, then there is a unique real solution $w(r,b)$ such that $w(r,b)>c^*$ for $r>1$. Letting $r\to+\infty$, we have

$$\lim_{r\to+\infty}w(r,b)=c^*.$$

Furthermore,

$$\begin{aligned}\lim_{r\to+\infty}\frac{w(r,b)-c^*}{br^{-n}}&=\lim_{r\to+\infty}\frac{1}{w^l(r)(w^{k-l-1}(r)+\cdots+(c^*)^{k-l-1})}\\&=\frac{1}{(k-l)(c^*)^{k-1}}.\end{aligned}$$

This implies

$$w(r,b)-c^*=O(r^{-n}),\quad \text{as } r\to+\infty. \tag{3.8}$$

Proposition 3.2. *For $n \geq 3$, let $u_b = u_b(r)$ be a smooth radial symmetric solution of (3.6) then u_b is locally convex in $\mathbb{R}^n \setminus \overline{B_1}$ and*

$$u_b(r) = \frac{c^*}{2}r^2 + \mu_0(b) + O\left(r^{2-n}\right), \quad \text{as } r \to +\infty,$$

where $\mu_0(b)$ is a strictly increasing function of b.

Proof. Recalling $w(r, b) = \frac{u_b'(r)}{r}$, then for any $\beta_0 \in \mathbb{R}$, the solution of (3.6) can be written as

$$\begin{aligned} u_b(r) &= \int_1^r u_b'(s)ds + \beta_0 \\ &= \frac{c^*}{2}r^2 + \int_1^r s\left(w(s, b) - c^*\right) ds - \frac{c^*}{2} + \beta_0. \end{aligned}$$

By (3.8), we know for a fixed $b > 0$, the integral

$$\int_1^{+\infty} s\left(w(s, b) - c^*\right) ds$$

is convergent. Therefore, we have

$$\lim_{r\to+\infty}\left(u_b(r) - \frac{c^*}{2}r^2\right) = \beta_0 - \frac{c^*}{2} + \int_1^{+\infty} s\left(w(s, b) - c^*\right) ds := \mu_0(b),$$

and in view of (3.8),

$$\begin{aligned} u_b(r) - \frac{c^*}{2}r^2 - \mu_0(b) &= -\int_r^{+\infty} s\left(w(s, b) - c^*\right) ds \\ &= O\left(r^{2-n}\right), \quad \text{as } r \to +\infty. \end{aligned} \tag{3.9}$$

By (3.7), we have, for $w > c^*$,

$$\frac{\partial w}{\partial b} = -\frac{\partial F}{\partial b}\Big/\frac{\partial F}{\partial w} = \frac{r^{-n}}{lw^{l-1}(w^{k-l} - (c^*)^{k-l}) + (k-l)w^{k-1}} > 0.$$

Hence

$$\frac{\partial \mu_0(b)}{\partial b} = \int_1^{+\infty} s\frac{\partial\left(w(s, b) - c^*\right)}{\partial b}ds > 0.$$

This shows that $\mu_0(b)$ is strictly increasing in b, if $b > 0$. Proposition 3.2 is proved. □

Remark 3.1. For any fixed $r_0 > 1$, we define

$$\mu_{r_0}(b) = \int_{r_0}^{+\infty} s(w(s, b) - c^*)ds.$$

Then

$$\mu_{r_0}(b) > \int_{r_0}^{r_0+1} s(w(s, b) - c^*)ds := h(b).$$

By (3.7), we have, for $b > 0$ and $w > c^*$,

$$\frac{\partial w}{\partial r} = -\frac{\partial F}{\partial r}\Big/\frac{\partial F}{\partial w} = \frac{-nbr^{-n-1}}{lw^{l-1}(w^{k-l} - (c^*)^{k-l}) + (k-l)w^{k-1}} < 0.$$

So

$$\begin{aligned} \frac{\partial h(b)}{\partial b} &= \int_{r_0}^{r_0+1} s\frac{\partial w(s, b)}{\partial b}ds \\ &= \int_{r_0}^{r_0+1} \frac{s^{1-n}}{lw^{l-1}(s, b)[w^{k-l}(s, b) - (c^*)^{k-l}] + (k-l)w^{k-1}(s, b)}ds \\ &> \frac{(r_0+1)^{1-n}}{lw^{l-1}(r_0, b)[w^{k-l}(r_0, b) - (c^*)^{k-l}] + (k-l)w^{k-1}(r_0, b)} \\ &> 0. \end{aligned}$$

This shows that $h(b)$ is strictly increasing in b and $\lim_{b\to+\infty} h(b) = +\infty$, so is $\mu_{r_0}(b)$. In the next section we will use

$$\lim_{b\to+\infty} \mu_{r_0}(b) = +\infty.$$

4. Proof of main results

Without loss of generality, we also assume that $B_2(0) \subset D$. This will be assumed below. The following lemma and its proof can be found in [15].

Lemma 4.1. *Let $\varphi \in C^2(\partial D)$. There exists some constant C, depending only on n, the convexity of D, $\|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D, such that, for every $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi)| \le C \quad \text{and} \quad w_\xi < \varphi \quad \text{on } \overline{D} \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\left(|x - \bar{x}(\xi)|^2 - |\xi - \bar{x}(\xi)|^2\right), \quad x \in \mathbb{R}^n. \tag{4.1}$$

Proof of Theorem 1.1. Fix $r_0 > 0$, such that $\overline{D} \subset B_{r_0}$. For $a > 0$ and $\beta \in \mathbb{R}$, let

$$\omega_a(x) = \beta + \int_{r_0}^{|x|} s\left\{\left[\frac{a^3}{s^3} + \sqrt{\left(\frac{a^3}{s^3}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}\,\right]^{1/3} + \left[\frac{a^3}{s^3} - \sqrt{\left(\frac{a^3}{s^3}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}\,\right]^{1/3}\right\} ds.$$

Then we claim that ω_a is a locally convex smooth subsolution of (1.1) with some large $a > r_0$ and some small β. From Proposition 3.1, we have

$$\omega_a(x) = \frac{c^*}{2}|x|^2 + \mu(a) + O(|x|^{2-n}), \quad \text{as } |x| \to +\infty, \tag{4.2}$$

where

$$\begin{aligned}\mu(a) &= \beta - \frac{c^*}{2}r_0^2 + \int_{r_0}^{+\infty} s\left\{\left[\frac{a^3}{s^3} + \sqrt{\left(\frac{a^3}{s^3}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}\,\right]^{1/3} + \left[\frac{a^3}{s^3} - \sqrt{\left(\frac{a^3}{s^3}\right)^2 - \left(\frac{c^*}{\sqrt{3}}\right)^6}\,\right]^{1/3} - c^*\right\} ds \\ &= \beta - \frac{c^*}{2}r_0^2 + \int_{r_0}^{+\infty}\left(\frac{c^* s}{\sqrt{3}}\right)\left(2\cos\left(\frac{1}{3}\arccos\left(\frac{a}{\left(\frac{c^*}{\sqrt{3}}\right)^3 s^n}\right)\right) - \sqrt{3}\right) ds.\end{aligned}$$

Moreover, $\mu(a)$ is strictly increasing in $(0, +\infty)$, and

$$\lim_{a\to\infty} \mu(a) = \infty. \tag{4.3}$$

On the other hand,

$$\omega_a \le \beta, \quad \text{in } B_{r_0} \setminus \overline{D},\ \forall a > r_0. \tag{4.4}$$

Let

$$\beta := \min\left\{w_\xi(x) \mid \xi \in \partial D,\ x \in \overline{B_{r_0}} \setminus D\right\},$$

where $w_\xi(x)$ is given by Lemma 4.1. This shows that ω_a is a locally convex smooth subsolution of (1.1).

We will fix the value of c_* in the proof. First we require that c_* satisfies $c_* > \beta$. It follows that

$$\mu(0) \le \beta - \frac{c^*}{2}{r_0}^2 < \beta < c_*.$$

Thus, in view of (4.3), for every $c > c_*$, there exists a unique $a(c) \in [r_0, +\infty)$ such that

$$\mu(a(c)) = c. \tag{4.5}$$

So $\omega_{a(c)}$ satisfies

$$\omega_{a(c)}(x) = \frac{c^*}{2}|x|^2 + c + O\left(|x|^{2-n}\right), \quad \text{as } x \to \infty. \tag{4.6}$$

Set

$$\underline{w}(x) = \max\left\{w_\xi(x) \mid \xi \in \partial D\right\}.$$

It is clear by Lemma 4.1 that $\underline{w}$ is a locally Lipschitz function in $\mathbb{R}^n \setminus D$, and $\underline{w} = \varphi$ on ∂D. Since w_ξ is a smooth locally convex solution of (1.9), $\underline{w}$ is a viscosity subsolution of (1.9) in $\mathbb{R}^n \setminus \overline{D}$. We fix a number $r_1 > r_0$, and then choose another number $a_1 > 0$ such that

$$\min_{\partial B_{r_1}} \omega_{a_1} > \max_{\partial B_{r_1}} \underline{w}.$$

We require that c_* also satisfies $c_* \ge \mu(a_1)$. We now fix the value of c_*.

For $c \ge c_*$, we have $a(c) = \mu^{-1}(c) \ge \mu^{-1}(c_*) \ge a_1$, and therefore

$$\omega_{a(c)} \ge \omega_{a_1} > \underline{w}, \quad \text{on } \partial B_{r_1}. \tag{4.7}$$

By (4.4), we have

$$\omega_{a(c)} \le \beta \le \underline{w}, \quad \text{in } B_{r_0} \setminus \overline{D}. \tag{4.8}$$

Now we define, for $c > c_*$,

$$\underline{u}(x) = \begin{cases} \max\left\{\omega_{a(c)}(x), \underline{w}(x)\right\}, & x \in B_{r_1} \setminus D, \\ \omega_{a(c)}(x), & x \in \mathbb{R}^n \setminus B_{r_1}. \end{cases}$$

We know from (4.8) that

$$\underline{u} = \underline{w}, \quad \text{in } B_{r_0} \setminus \overline{D}, \tag{4.9}$$

and in particular

$$\underline{u} = \underline{w} = \varphi, \quad \text{on } \partial D. \tag{4.10}$$

We know from (4.7) that $\underline{u} = \omega_{a(c)}$ in a neighborhood of ∂B_{r_1}. Therefore $\underline{u}$ is locally Lipschitz in $\mathbb{R}^n \setminus D$. Since both $\omega_{a(c)}$ and $\underline{w}$ are viscosity subsolutions of (1.9) in $\mathbb{R}^n \setminus \overline{D}$, so is $\underline{u}$.

For $c > c_*$,

$$\overline{u}(x) := \frac{c^*}{2}|x|^2 + c$$

is a smooth convex solution of (1.9). By (4.8),

$$\omega_{a(c)} \le \beta < c_* < \overline{u}, \quad \text{on } \partial D.$$

We also know by (4.6) that

$$\lim_{|x|\to\infty} \left(\omega_{a(c)}(x) - \overline{u}(x)\right) = 0.$$

Thus, in view of Lemma 2.1, we have

$$\omega_{a(c)} \le \overline{u}, \quad \text{on } \mathbb{R}^n \setminus D. \tag{4.11}$$

By (4.7) and the above, we have, for $c > c_*$,

$$w_\xi \le \overline{u}, \quad \text{on } \partial(B_{r_1} \setminus D),\ \forall \xi \in \partial D.$$

By the comparison principle for smooth convex solutions of (1.9), we have

$$w_\xi \le \overline{u}, \quad \text{in } B_{r_1} \setminus \overline{D},\ \forall \xi \in \partial D.$$

Thus

$$\underline{w} \le \overline{u}, \quad \text{in } B_{r_1} \setminus \overline{D}.$$

This, combining with (4.11), implies that

$$\underline{u} \le \overline{u}, \quad \text{in } \mathbb{R}^n \setminus D.$$

For any $c > c_*$, let $\mathcal{S}_c$ denote the set of $v \in C^0(\mathbb{R}^n \setminus D)$ which are viscosity subsolutions of (1.9) in $\mathbb{R}^n \setminus \overline{D}$ satisfying

$$v = \varphi, \quad \text{on } \partial D, \tag{4.12}$$

and

$$\underline{u} \le v \le \overline{u}, \quad \text{in } \mathbb{R}^n \setminus D. \tag{4.13}$$

We know that $\underline{u} \in \mathcal{S}_c$. Let

$$u_c(x) := \sup\{v(x) \mid v \in \mathcal{S}_c\}, \quad x \in \mathbb{R}^n \setminus D.$$

By (4.6), and the definitions of $\underline{u}$ and $\overline{u}$,

$$u_c(x) \geq \underline{u}(x) = \omega_{a(c)}(x) = \frac{c^*}{2}|x|^2 + c + O(|x|^{2-n}), \quad \text{as } |x| \to +\infty \tag{4.14}$$

and

$$u_c(x) \leq \overline{u}(x) = \frac{c^*}{2}|x|^2 + c.$$

The estimate (1.7) follows.

Next, we prove that u_c satisfies the boundary condition. It is obvious from (4.10) that

$$\liminf_{x\to\xi} u_c(x) \geq \lim_{x\to\xi} \underline{u}(x) = \varphi(\xi), \quad \forall \xi \in \partial D.$$

So we only need to prove that

$$\limsup_{x\to\xi} u_c(x) \leq \varphi(\xi), \quad \forall \xi \in \partial D.$$

Let $\omega_c^+ \in C^2(\overline{B_{r_0} \setminus D})$ be defined by

$$\begin{cases} \Delta\omega_c^+ = 0, & \text{in } B_{r_0} \setminus \overline{D}, \\ \omega_c^+ = \varphi, & \text{on } \partial D, \\ \omega_c^+ = \max\limits_{\partial B_{r_0}} \overline{u}, & \text{on } \partial B_{r_0}. \end{cases}$$

By Lemma 2.2, we have u_c^* is a viscosity subsolution of (1.9). Hence, by the definition of viscosity solution,

$$\begin{cases} \Delta u_c^* \geq 0, & \text{in } B_{r_0} \setminus D, \\ u_c^* \leq \varphi & \text{on } \partial D, \\ u_c^* \leq \max\limits_{\partial B_{r_0}} \overline{u} & \text{on } \partial B_{r_0}. \end{cases}$$

By the comparison principle, it follows that

$$u_c \leq u_c^* \leq \omega_c^+ \quad \text{in } B_{r_0} \setminus \overline{D},$$

and then

$$\limsup_{x\to\xi} u_c(x) \leq \lim_{x\to\xi} \omega_c^+(x) = \varphi(\xi), \quad \forall \xi \in \partial D.$$

Finally, we prove that $u_c \in C^0(\mathbb{R}^n \setminus D)$ is a viscosity solution of (1.9). We observe that

$$\underline{u}_* \leq (u_c)_* \leq u_c \leq u_c^* \leq \overline{u}^* = \overline{u}.$$

In particular,

$$(u_c)_* = u_c = u_c^* = \varphi \quad \text{on } \partial D,$$

and $(u_c)_*, u_c, u_c^*$ all satisfy the asymptotic assumption (1.7). By Lemma 2.2 u_c^* is a subsolution of (1.9) and hence, by comparison, $u_c^* \leq \overline{u}$. It then follows from the definition of u_c that $u_c = u_c^*$, so u_c is a subsolution.

If $(u_c)_*$ fails to be a supersolution at some point $\hat{x} \in \mathbb{R}^n \setminus \overline{D}$, let U_κ be provided by Lemma 2.3. Clearly $u_c \leq U_\kappa$ and U_κ satisfies the boundary conditions for sufficiently small κ. By comparison, $U_\kappa \leq \overline{u}$ and since u is the maximal subsolution between $\underline{u}$ and $\overline{u}$, we arrive at the contradiction $U_\kappa \leq u_c$. Hence $(u_c)_*$ is a supersolution of (1.9) and then, by comparison for (1.9), $u_c^* = u_c \leq (u_c)_*$, showing that u_c is continuous and is a solution.

Theorem 1.1 is established. □

Proof of Theorem 1.2. Fix an $\bar{r} > 2$, such that $\overline{D} \subset B_{\bar{r}}$. For $b > c^*$ and $\beta \in \mathbb{R}$, let

$$\omega_b(x) = \beta + u_b(|x|) - u_b(\bar{r}), \quad \text{for } |x| > 2,$$

where $u_b(|x|)$ is defined in Proposition 3.2. Then we claim that $\omega_b(x)$ is an admissible smooth subsolution of (1.10) with some small β. It is easy to see that

$$\omega_b(x) = \frac{c^*}{2}|x|^2 + \mu(b) + O\left(|x|^{2-n}\right), \quad \text{as } |x| \to +\infty,$$

where

$$\mu(b) := \mu_{\bar{r}}(b) + \beta - \frac{c^*}{2}\bar{r}^2.$$

By Remark 3.1, we have that $\mu(b)$ is strictly increasing in $(c^*, +\infty)$, and

$$\lim_{b\to+\infty} \mu(b) = +\infty.$$

Similar arguments as in the Proof of Theorem 1.1 can complete the rest of the proof. □

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