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The exterior Dirichlet problem for fully nonlinear elliptic equations related to the eigenvalues of the Hessian

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Abstract

In this paper, we establish the existence theorem for the exterior Dirichlet problems for a class of fully nonlinear elliptic equations, which are related to the eigenvalues of the Hessian matrix, with prescribed asymptotic behavior at infinity. This extends the previous results on Monge–Ampère equation and k-Hessian equation to more general cases, in particular, including the special Lagrangian equation.

1. Introduction

In this paper, we study the existence of viscosity solutions to exterior Dirichlet problem for the following fully nonlinear, second order partial differential equation of the form

$$\begin{cases} f\big(\lambda\big(D^2u\big)\big) = 1, & \text{in } \mathbb{R}^n \setminus \overline{D}, \quad \text{(a)} \\ u = \varphi, & \text{on } \partial D, \quad \text{(b)} \end{cases} \tag{1.1}$$

where D is a bounded open set in $\mathbb{R}^n$ ($n \geqslant 3$), $f(\lambda)$ is a given smooth symmetric function of the eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ of the Hessian matrix D^2u. The typical cases of f include the elementary symmetric functions

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$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n, \tag{1.2}$$

the quotients of elementary symmetric functions,

$$\sigma_{k,l}(\lambda) = \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}, \quad 1 \leqslant l < k \leqslant n, \tag{1.3}$$

and the special Lagrangian operator

$$\sum_{i=1}^{n} \arctan \lambda_i .$$

The elementary symmetric functions (1.2) are embraced by [6] and treated as well by Ivochkina [20]. Note that the case $k = 1$ corresponds to Laplace operator, while for $k = n$, we have the classical Monge–Ampère operator.

In bounded domains $\Omega \subset \mathbb{R}^n$, Caffarelli, Nirenberg and Spruck treated the traditional (or interior) Dirichlet problem in [6],

$$\begin{cases} f\big(\lambda\big(D^2 u\big)\big) = \psi(x), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where they demonstrated the existence of classical solutions, under various hypothesis on the function f and the domain Ω. The results in [6] extended their previous work [5], and that of Krylov [25], Ivochkina [19] and others on equations of Monge–Ampère type,

$$\det D^2 u = \psi(x), \tag{1.5}$$

where ψ is a given function in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Trudinger [28] provided a new method to obtain the double normal second derivative estimation and extended the result in [6] to the important examples of quotients of elementary symmetric functions (1.3), which do not satisfy the structure hypothesis on f in [6]. More results in a bounded domain on these types of equations can be referred to Trudinger [29], Urbas [31] and the references therein.

In contrast to numerous results on the traditional Dirichlet problems (1.4) in bounded domains, less is known about the exterior Dirichlet problems (1.1a)–(1.1b) where the domain is unbounded. Especially, in the whole space $\mathbb{R}^n$, a classical theorem of Jörgens [23], Calabi [7], and Pogorelov [27] states that any classical convex solution of

$$\det\big(D^2 u\big) = 1, \quad \text{in } \mathbb{R}^n \tag{1.6}$$

must be a quadratic polynomial. More extensive and outstanding results on (1.6) are given by Cheng and Yau [9], Caffarelli [3], Jost and Xin [24], Trudinger and Wang [30] and many others. Caffarelli and Li [4] extended the Jörgens–Calabi–Pogorelov theorem to exterior domains, namely that if u is a locally convex viscosity solution of

$$\det\big(D^2 u\big) = 1, \quad \text{in } \mathbb{R}^n \setminus \overline{D} \tag{1.7}$$

where D is a bounded open convex set in $\mathbb{R}^n$, $n \geqslant 3$, then there exist an $n \times n$ real symmetric positive definite matrix A with $\det(A) = 1$, a vector $b \in \mathbb{R}^n$, and a constant $c \in \mathbb{R}$ such that

$$\limsup_{|x|\to\infty} |x|^{n-2} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \right| < \infty. \tag{1.8}$$

The above mentioned asymptotic results motivate one to study the existence of solutions of the exterior Dirichlet problem with such prescribed asymptotic behavior at infinity for Monge–Ampère equation, even for general nonlinear equation (1.1a). Caffarelli and Li [4] proved the existence of solutions of (1.7) with prescribed asymptotic behavior (1.8) in dimension $n \geqslant 3$. Similar problems in dimension two were also studied by Delanoë [13], Ferrer, Martínez and Milán [14,15].

For Hessian equations

$$\sigma_k\big(\lambda\big(D^2u\big)\big) = 1, \quad \text{in } \mathbb{R}^n \setminus \overline{D},\ 1 \leqslant k \leqslant n,\ n \geqslant 3, \tag{1.9}$$

Dai and Bao [12] established the existence theorem under the asymptotic assumption

$$\limsup_{|x|\to\infty} |x|^{n-2} \left| u(x) - \left(\frac{c^*}{2} |x|^2 + b \cdot x + c \right) \right| < \infty, \tag{1.10}$$

where $c^* = (C_n^k)^{-1/k}$, $C_n^k = \frac{n!}{(n-k)!k!}$. For Hessian quotient equations

$$\sigma_{k,l}\big(\lambda\big(D^2u\big)\big) = 1, \quad \text{in } \mathbb{R}^n \setminus \overline{D},\ n \geqslant 3, \tag{1.11}$$

Dai [11] proved the existence theorem under suitable asymptotic assumption for the cases $k - l \geqslant 3$. However, this restriction on k and l rules out an important example $\frac{\det(D^2u)}{\Delta u} = 1$ in dimension three, which originates from the study on the Lagrangian submanifolds.

Generally, the (Lagrangian) graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ $(n \geqslant 2)$ is called special when the argument of the complex number $(1 + \sqrt{-1}\lambda_1) \cdots (1 + \sqrt{-1}\lambda_n)$ is a constant Θ or u satisfies

$$\sum_{i=1}^n \arctan \lambda_i\big(D^2u\big) = \Theta, \tag{1.12}$$

and it is special if and only if $(x, Du(x))$ is a volume-minimizing minimal submanifold in $\mathbb{R}^n \times \mathbb{R}^n$ (see Theorem 2.3, Proposition 2.17 in [26]). For $\Theta = k\pi$, the special Lagrangian equation (1.12) in $\mathbb{R}^3$ also takes the form

$$\det\big(D^2u\big) = \Delta u. \tag{1.13}$$

The regularity of convex strong solutions of (1.13) in $\mathbb{R}^3$ was studied by Bao and Chen [1]. In dimensions $n \geqslant 3$, the regularity of convex solutions to special Lagrangian equations was established by Chen, Warren and Yuan [8] and Wang and Yuan [33]. The Bernstein type results for global solutions of special Lagrangian equation were obtained by Fu [16] for $n = 2$ and Yuan [34] for higher dimensions.

In this paper, we present a new technique for the construction of subsolutions, and extend previous existence theorems for Monge–Ampère equation and Hessian (quotient) equation to more general fully nonlinear equations, including not only those cases considered in [6] and [28] but also the special Langragian equation case. Although we cannot present an explicit formula for the radial function to be a subsolution of (1.1a), just like dealing with Eqs. (1.7) and (1.9), we find that the solutions to the corresponding ordinary differential equations have a uniformly asymptotic behavior at infinity (see Proposition 2.1 below) whenever f satisfies our hypotheses. This fact is the key ingredient of this paper, which allows us to establish the existence of viscosity solution of (1.1a)–(1.1b).

Let $\Gamma \subsetneqq \mathbb{R}^n$ be an open convex cone, with vertex at the origin, containing the positive cone $\{\lambda \in \mathbb{R}^n \mid \lambda_j > 0,\ j = 1, \ldots, n\} := \Gamma_n$, symmetric in the λ_i, $i = 1, \ldots, n$. We assume that the symmetric function f is defined in Γ, $f \in C^2(\Gamma)$, and satisfies

$$f_{\lambda_i} > 0, \quad \text{on } \Gamma,\ i = 1, \ldots, n, \tag{1.14}$$

$$\limsup_{\lambda \to \lambda_0} f(\lambda) < 1, \quad \text{for every } \lambda_0 \in \partial\Gamma, \tag{1.15}$$

and

$$\text{there exists a constant } c^* \text{ such that } f\big(c^*(1, 1, \ldots, 1)\big) = 1. \tag{1.16}$$

Then our main result is

Theorem 1.1. *Let D be a smooth, bounded, strictly convex open set in $\mathbb{R}^n$, $n \geqslant 3$, and $\varphi \in C^2(\partial D)$. Assume f satisfies (1.14)–(1.16). Then there exists some constant $\tilde{c}$, depending only on n, D, f, and φ, such that for every $c > \tilde{c}$ there exists a function $u \in C^0(\mathbb{R}^n \setminus D)$ that satisfies (1.1a)–(1.1b) in the viscosity sense and satisfies (1.10), where c^* is given in (1.16).*

Remark 1.1. Condition (1.14) implies that $f(\lambda(D^2u)) = 1$ is elliptic at u if $\lambda(D^2u) \in \Gamma$. By Proposition A.1 proved in [31], if a viscosity solution u is of class C^2, then u is an admissible classical solution, so it is unique. Here we also would like to mention that the hypotheses on the function f in the paper [28], where Trudinger treated the interior Dirichlet problem (1.4) with $\psi(x) > 0$, include the requirement that

$$\limsup_{\lambda \to \lambda_0} f(\lambda) \leqslant 0, \quad \text{for every } \lambda_0 \in \partial\Gamma, \tag{1.17}$$

and

$$f(R\lambda) \to \infty, \quad \text{as } R \to \infty,\ \text{for all } \lambda \in \Gamma. \tag{1.18}$$

That the assumption (1.16) is weaker than (1.18) allows f cover more examples of fully nonlinear equation. On the other hand, we do not assume the concavity of f, like in [6] and [28].

Remark 1.2. We gives two examples of f for which our result is true.

Example 1. The first example is the Hessian quotient equations (1.11) with

$$\Gamma = \Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ j = 1, 2, \dots, k\},$$

where $c^* = (C_n^l / C_n^k)^{\frac{1}{k-l}}$. The main contribution of [28] is that it covers this case.

Example 2. The second example is the special Lagrangian equations (1.12). We write

$$f\big(\lambda\big(D^2 u\big)\big) = \frac{1}{\Theta} \sum_{i=1}^{n} \arctan \lambda_i\big(D^2 u\big) = 1,$$

then the assertion is still true for $\frac{(n-1)\pi}{2} < |\Theta| < \frac{n\pi}{2}$, where $c^* = \tan \frac{\Theta}{n}$. By Lemma C in [6], in case n is odd the corresponding cone Γ is the positive cone Γ_n; in case n is even the cone Γ is the cone Γ_{n-1}.

Remark 1.3. Finally, it is necessary to point out that in the special case of radial solutions of σ_k-Hessian equation on $\mathbb{R}^n \setminus B_1$, Wang and Bao show that there is no solution if c is small enough, see Theorem 2 in [32]. Recently, Bao, Li and Li [2] extended the result in [12] to allow more general behavior at infinity, where the construction of subsolutions essentially depends on the homogeneity of σ_k-Hessian operators. But for these general f considered in this paper, it still requires new idea involved even for the Hessian quotient equation (1.11) if the assumption at infinity is more general.

The paper is organized as follows. In Section 2, we derive the uniformly asymptotic property at infinity of the radial symmetric solutions of $f(\lambda(D^2 u)) = 1$. In Section 3 we prove Theorem 1.1 by Perron method. An adapted version for Dirichlet problems on unbounded domains is included in Appendix A.

2. Radial symmetric solutions

For $r > 0$, let $B_r := \{x \in \mathbb{R}^n \mid |x| < r\}$ be a ball in $\mathbb{R}^n$ with center 0 and radius r. In order to prove the existence of solution of (1.1a)–(1.1b) by using Perron method, we need to construct a sequence of subsolutions of (1.1a)–(1.1b). To this end, we consider the radial symmetric solutions of

$$f\big(\lambda\big(D^2 u\big)\big) = 1. \tag{2.1}$$

Let $u = u(r)$ be a smooth admissible (see Definition A.1) radial symmetric function, then

$$\lambda\big(D^2 u\big) = \left(u''(r), \frac{u'(r)}{r}, \dots, \frac{u'(r)}{r}\right) \in \Gamma.$$

Thus in order to find the subsolutions of (1.1a)–(1.1b), we solve the following ordinary differential equation

$$f\left(u''(r), \frac{u'(r)}{r}, \dots, \frac{u'(r)}{r}\right) = 1, \quad \text{for } r > 1, \tag{2.2}$$

with initial data

$$u(1) = a, \qquad u'(1) = b, \tag{2.3}$$

where a, b are constants to be determined later.

Proposition 2.1. *Assume f satisfies (1.14)–(1.16). Then (2.2) and (2.3) has a smooth admissible solution $u(r) = u_{a,b}(r)$ in $(1, \infty)$, such that $u'(r) \geqslant c^* r$, and for $b > c^*$,*

$$u_{a,b}(r) = \frac{c^*}{2} r^2 + \mu_1(b) + a - \frac{c^*}{2} + O\big(r^{2-n}\big), \quad \text{as } r \to +\infty,$$

where c^* is a constant given in (1.16), and $\mu_1(b)$ is a strictly increasing function of b and

$$\lim_{b \to +\infty} \mu_1(b) = +\infty.$$

In particular, if $b = c^*$, then $u_{a,c^*}(r) = \frac{c^*}{2} r^2 + a - \frac{c^*}{2}$.

To solve (2.2), we consider the following equation

$$f(p, q, \ldots, q) = 1, \quad \text{for } (p, q, \ldots, q) \in \Gamma. \tag{2.4}$$

First, the assumptions (1.14) and (1.16) imply that

$$f\big(c^*, q, \ldots, q\big) > 1, \quad \text{if } q > c^*.$$

Combining with (1.15) it follows from the intermediate value theorem and the smoothness and monotonicity of f that there exists a unique $g(q)$ for every $q > c^*$ such that

$$f\big(g(q), q, \ldots, q\big) = 1, \quad \big(g(q), q, \ldots, q\big) \in \Gamma, \ \text{for } q > c^*.$$

Thus we define a function g such that

$$p = g(q), \quad \text{for } q \geqslant c^*, \tag{2.5}$$

and g is continuous and differentiable on $[c^*, +\infty)$, in particular,

$$c^* = g\big(c^*\big). \tag{2.6}$$

Indeed, by differentiating the equation

$$f\big(g(q), q, \ldots, q\big) = 1$$

with respect to q, we have

$$f_{\lambda_1}\big(g(q), q, \ldots, q\big) g'(q) + \sum_{i=2}^{n} f_{\lambda_i}\big(g(q), q, \ldots, q\big) = 0, \quad \text{for } q \geqslant c^*.$$

From (1.14) it follows that

$$g'(q) = -\frac{\sum_{i=2}^{n} f_{\lambda_i}(g(q), q, \ldots, q)}{f_{\lambda_1}(g(q), q, \ldots, q)} < 0. \tag{2.7}$$

So that

$$g(q) < c^*, \quad \text{if } q > c^*. \tag{2.8}$$

Especially, using the symmetry of f, we obtain that

$$g'\big(c^*\big) = 1 - n. \tag{2.9}$$

Now, for $b > c^*$, we consider the following initial value problem

$$\begin{cases} \dfrac{dw}{dr} = \dfrac{g(w) - w}{r}, \\ w(1) = b, \end{cases} \tag{2.10}$$

where g is continuous and differentiable, and

$$\frac{g(q) - q}{r} \leqslant 0, \quad \text{if } r > 0 \text{ and } q \geqslant c^*. \tag{2.11}$$

It follows from the Picard–Lindelöf theorem and the extension theorem for initial value problem that (2.10) has a unique global solution and its maximal interval of existence is $[1, +\infty)$, due to the monotonicity (2.11) implies that the solution $w = w(r, b)$ always stays in the strip between $w = c^*$ and $w = b$ for $b > c^*$. In particular, if $b = c^*$, then

$$w\big(r, c^*\big) \equiv c^*, \quad \text{in } [1, +\infty). \tag{2.12}$$

Letting

$$w(r) = \frac{u'(r)}{r},$$

i.e., $u'(r) = r w(r)$, then

$$u''(r) = r w'(r) + w(r).$$

On the other hand, from (2.2) and (2.5), we have

$$u''(r) = g\big(w(r)\big), \quad \text{for } w(r) \geqslant c^*.$$

Thus

$$r w'(r) = g\big(w(r)\big) - w(r), \quad \text{for } w(r) \geqslant c^*.$$

That is, $w(r) = \frac{u'(r)}{r}$ satisfies the equation in (2.10).

By further analysis, we find that the solution of (2.10) has the following asymptotic behaviors. This is in fact the main point of Proposition 2.1.

Lemma 2.2. *If $w(r)$ is a solution of (2.10), then*

$$\lim_{r\to+\infty} w(r) = c^*, \tag{2.13}$$

and

$$w(r) - c^* = O\left(r^{-n}\right), \quad \text{as } r \to +\infty. \tag{2.14}$$

Proof. Since

$$\frac{d}{dr}\left(w(r) - c^*\right) = \frac{g(w(r)) - w(r)}{r} \leqslant 0,$$

it follows that there exists some constant c_0 such that

$$\lim_{r\to+\infty}\left(w(r) - c^*\right) = c_0 \geqslant 0.$$

That is,

$$\lim_{r\to+\infty} w(r) = c^* + c_0.$$

Next we prove $c_0 = 0$ by contradiction. Suppose that $c_0 > 0$. Then there exists $r_0 > 0$ sufficiently large such that

$$\left|w(r) - \left(c^* + c_0\right)\right| < c_0, \quad \text{for } r \geqslant r_0.$$

Hence for $r \geqslant r_0$, $w(r) > c^*$, and

$$g\left(w(r)\right) - g\left(c^*\right) = g\left(w(r)\right) - c^* < 0.$$

It follows that for $r \geqslant r_0$,

$$\frac{d}{dr}\left(w(r) - c^*\right) = \frac{g(w(r)) - g(c^*) - w(r) + c^*}{r} \leqslant -\frac{w(r) - c^*}{r}. \tag{2.15}$$

This implies that for $r \geqslant r_0$,

$$\frac{d(w(r) - c^*)}{w(r) - c^*} \leqslant \frac{-dr}{r}. \tag{2.16}$$

Integrating (2.16) from r_0 to r,

$$\ln\left(w(r) - c^*\right) + \ln r \leqslant \ln\left(w(r_0) - c^*\right) + \ln r_0 := C_0.$$

Thus,

$$w(r) - c^* \leqslant \frac{e^{C_0}}{r} \to 0, \quad \text{as } r \to +\infty,$$

which contradicts with the assumption that $c_0 > 0$. So (2.13) is proved.

Next, we will prove (2.14) as follows. We know that

$$c^* \leqslant w(r) \leqslant b, \quad \text{in } [1, +\infty).$$

If there exists $r_1 > 1$ such that $w(r_1) = c^*$, then by the monotonicity of w we have $w(r) \equiv c^*$ in $[r_1, +\infty)$. Thus (2.14) is proved. So in the following we suppose that

$$c^* < w(r) \leqslant b, \quad \text{in } [1, +\infty).$$

Notice that now (2.15) holds for $r > 1$. Integrating (2.16) from 1 to r, we have

$$0 < w(r) - c^* \leqslant \frac{b - c^*}{r}, \quad \text{for } r > 1.$$

Rewrite (2.15) as follows

$$\begin{aligned} \frac{d}{dr}\big(w(r) - c^*\big) &= \frac{g(w(r)) - g(c^*) - w(r) + c^*}{r} \\ &= \frac{w(r) - c^*}{r}\big(g'\big(\theta_0 w(r) + (1-\theta_0)c^*\big) - 1\big), \end{aligned}$$

for some $\theta_0 \in (0, 1)$. Since $c^* < \theta w(r) + (1-\theta)c^* < b$ for any $\theta \in (0, 1)$, then using (2.9), we have

$$\begin{aligned} g'\big(\theta_0 w(r) + (1-\theta_0)c^*\big) - 1 &= g'\big(\theta_0 w(r) + (1-\theta_0)c^*\big) - g'\big(c^*\big) - n \\ &\leqslant \omega_{g'}\big(w(r) - c^*\big) - n \\ &\leqslant \omega_{g'}\big(\big(b - c^*\big)r^{-1}\big) - n, \end{aligned}$$

where

$$\omega_{g'}(s) = \sup_{\substack{|q_1 - q_2| \leqslant s; \\ c^* < q_1 < q_2 < b}} \big|g'(q_1) - g'(q_2)\big|$$

denotes the modulus of continuity of the function g'. Therefore,

$$\frac{d}{dr}\big(w(r) - c^*\big) \leqslant \frac{-n + \omega_{g'}((b - c^*)r^{-1})}{r}\big(w(r) - c^*\big).$$

By integrating, we have

$$\ln\bigl(w(r)-c^*\bigr)-\ln\bigl(b-c^*\bigr)\leqslant\int_1^r\left(\frac{-n}{t}+\frac{\omega_{g'}((b-c^*)t^{-1})}{t}\right)dt$$
$$\leqslant -n\ln r+\int_1^{+\infty}\frac{\omega_{g'}((b-c^*)t^{-1})}{t}\,dt.$$

While the second term on the right hand side is bounded, because the integral is convergent,

$$\int_1^{+\infty}\frac{\omega_{g'}((b-c^*)t^{-1})}{t}\,dt=-\int_{b-c^*}^{0}\frac{\omega_{g'}(s)}{(\frac{s}{b-c^*})^{-1}}\cdot\frac{b-c^*}{s^2}\,ds=\int_0^{b-c^*}\frac{\omega_{g'}(s)}{s}\,ds<+\infty,$$

here we have used the fact that g' is Dini continuous, since $g\in C^2$. Therefore,

$$0<w(r)-c^*\leqslant C_1r^{-n},\quad \text{for } r\geqslant 1,$$

where C_1 depends only on the Dini modulus of continuity of g', b and c^*. The proof of Lemma 2.2 is completed. □

In order to study the dependence of solutions on the initial values, we rewrite the solution of (2.10) in the form $w(r)=w(r,b)$. By the smoothness of g, we know that the function on the right hand side of the equation in (2.10), $\frac{g(q)-q}{r}$, is differentiable with respect to q. Then by the differentiability of solution with respect to the initial value, $\frac{\partial w(r,b)}{\partial b}$ satisfies

$$\begin{cases}\dfrac{\partial z}{\partial r}=\left(\dfrac{g'(w(r,b))-1}{r}\right)z,\\ z(1)=1.\end{cases}$$

So that

$$\frac{\partial w(r,b)}{\partial b}=\exp\int_1^r\frac{g'(w(t,b))-1}{t}\,dt. \tag{2.17}$$

Since $g'(w(t,b))<0$, we have

$$0<\frac{\partial w(r,b)}{\partial b}\leqslant 1,\quad \text{for } r\geqslant 1. \tag{2.18}$$

Furthermore, differentiating equation (2.10) with respect to r, we have

$$\frac{d^2w(r)}{dr^2}=\frac{(g'(w(r))-2)(g(w(r))-w(r))}{r^2}>0,\quad r>1. \tag{2.19}$$

Remark 2.1. We remark that the above two facts stated in Lemma 2.2 are natural and intrinsic for Hessian equations (1.9) and Hessian quotient equations (1.11). Generally, assume that $u = u(r)$ is a smooth convex radial symmetric solution of (1.11), including the cases $l = 0$. By direct calculation, we know that $w(r) = \frac{u'(r)}{r}$ satisfies

$$w^k(r) - \left(C_n^l / C_n^k\right) w^l(r) - \frac{C}{r^n} = 0,$$

for an arbitrary constant C. Let $c^* = (C_n^l / C_n^k)^{\frac{1}{k-l}}$, the above equation can be rewritten in the form

$$w^l(r)\left[w^{k-l}(r) - \left(c^*\right)^{k-l}\right] = \frac{C}{r^n}. \tag{2.20}$$

Notice that c^* satisfies $f(c^*(1, 1, \dots, 1)) = 1$. Here we are interested only in the case that $C > 0$, corresponding the case $b > c^*$ in Lemma 2.2. If $C > 0$, then (2.20) always has a solution $w(r) > c^*$ for any $1 \leqslant l < k \leqslant n$. Letting $r \to +\infty$ on both side of (2.20), we have

$$\lim_{r \to +\infty} w(r) = c^*.$$

Furthermore, from (2.20),

$$\lim_{r \to +\infty} \frac{w(r) - c^*}{C r^{-n}} = \lim_{r \to +\infty} \frac{1}{w^l(r)(w^{k-l-1}(r) + \cdots + (c^*)^{k-l-1})} = \frac{1}{(k-l)(c^*)^{k-1}}.$$

This implies that (2.14) holds.

Proof of Proposition 2.1. If $u(r) = u_{a,b}(r)$ is a solution of (2.2) and (2.3), then

$$u_{a,b}(r) = \int_1^r u'_{a,b}(s)\, ds + a = \frac{c^*}{2} r^2 + \int_1^r s\left(w(s, b) - c^*\right) ds - \frac{c^*}{2} + a,$$

where $w(r, b) = \frac{u'_{a,b}(r)}{r}$ is the solution of (2.10). By (2.14), we know that for a fixed b, the limit

$$\lim_{r \to +\infty} \int_1^r s\left(w(s, b) - c^*\right) ds$$

is convergent. Denote

$$\mu_1(b) := \int_1^{+\infty} s\left(w(s, b) - c^*\right) ds.$$

Therefore, we have

$$\lim_{r\to+\infty}\left(u_{a,b}(r)-\frac{c^*}{2}r^2\right)=a-\frac{c^*}{2}+\mu_1(b),$$

and in view of (2.14),

$$\begin{aligned}u_{a,b}(r)-\left(\frac{c^*}{2}r^2+\mu_1(b)+a-\frac{c^*}{2}\right)&=-\int_r^{+\infty}s\big(w(s,b)-c^*\big)\,ds\\&=O\big(r^{2-n}\big),\quad \text{as } r\to+\infty.\end{aligned}\tag{2.21}$$

By (2.18), we have for $r\geqslant 1$,

$$w(r,b_1)>w(r,b_2),\quad \text{if } b_1>b_2>c^*.$$

This shows that $\mu_1(b)$ is strictly increasing in b, when $b>c^*$. Since

$$\frac{\partial w(r,b)}{\partial r}\bigg|_{r=1}=g(b)-b<0,\quad \text{for } b>c^*,$$

it follows from the monotonicity of $w(r,b)$ on r and (2.19) that

$$w(r,b)>\big(g(b)-b\big)(r-1)+b,\quad \text{for } r>1.$$

Then, by the convexity and symmetry of Γ, we have $-(n-1)q<g(q)<c^*$ for $q>c^*$. Hence,

$$\begin{aligned}\mu_1(b)&=\int_1^{+\infty}s\big(w(s,b)-c^*\big)\,ds\\&\geqslant\int_1^{\frac{b-c^*}{b-g(b)}+1}s\big(\big(g(b)-b\big)(s-1)+b-c^*\big)\,ds\\&\geqslant\frac{(b-c^*)^2}{2(b-g(b))}\\&\geqslant\frac{(b-c^*)^2}{2nb}.\end{aligned}$$

Therefore,

$$\lim_{b\to+\infty}\mu_1(b)=+\infty.$$

Proposition 2.1 is proved. □

Remark 2.2. Generally, for any fixed $r_0 \geqslant 1$, we define

$$\mu_{r_0}(b) := \int_{r_0}^{+\infty} s\big(w(s,b) - c^*\big)\,ds.$$

Since $w' \leqslant 0$ and (2.19), we claim that

$$\lim_{b\to+\infty} w(r_0, b) = +\infty, \quad \text{for fixed } r_0 \geqslant 1. \tag{2.22}$$

Indeed, suppose that there exists a constant $C_2 > 0$ such that

$$w(r_0, b) \leqslant C_2, \quad \text{for } b > c^*.$$

Then by (2.10), we have

$$\int_{w(r_0,b)}^{b} \frac{dw}{w - g(w)} = \ln r_0.$$

Using $-(n-1)q < g(q) < c^*$ for $q > c^*$ again, we have

$$\lim_{b\to+\infty} \int_{w(r_0,b)}^{b} \frac{dw}{w - g(w)} \geqslant \lim_{b\to+\infty} \int_{C_2}^{b} \frac{dw}{w - g(w)} \geqslant \lim_{b\to+\infty} \int_{C_2}^{b} \frac{dw}{nw} = +\infty.$$

This leads to a contradiction with the finiteness assumption, and hence (2.22) holds as claimed. Then, similarly as $\mu_1(b)$, we have

$$\begin{aligned}
\mu_{r_0}(b) &\geqslant \int_{r_0}^{\frac{r_0(w(r_0,b)-c^*)}{w(r_0)-g(w(r_0,b))}+r_0} s\Big(\big(g\big(w(r_0,b)\big) - w(r_0,b)\big)(s-1) + w(r_0,b) - c^*\Big)\,ds \\
&\geqslant \frac{r_0^2(w(r_0,b) - c^*)^2}{2(w(r_0,b) - g(w(r_0,b)))} \\
&\geqslant \frac{r_0^2(w(r_0,b) - c^*)^2}{2nw(r_0,b)}.
\end{aligned}$$

Therefore,

$$\lim_{b\to+\infty} \mu_{r_0}(b) = +\infty,$$

which will be used in the next section. This is the key point of this paper.

3. Proof of Theorem 1.1

Without loss of generality, we assume that $B_2 \subset D$. The following lemma was proved in [4].

Lemma 3.1. *Let D be a convex bounded domain in $\mathbb{R}^n$ with C^2 boundary, and $\varphi \in C^2(\partial D)$. There exists some constant C, depending only on n, the convexity of D, $\|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D, such that, for every $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$\left|\bar{x}(\xi)\right| \leqslant C \quad \text{and} \quad w_\xi < \varphi, \quad \text{on } \overline{D} \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\left(\left|x - \bar{x}(\xi)\right|^2 - \left|\xi - \bar{x}(\xi)\right|^2\right), \quad x \in \mathbb{R}^n. \tag{3.1}$$

Proof of Theorem 1.1. Fix an $\bar{r} > 2$, such that $\overline{D} \subset B_{\bar{r}}$. For $b > c^*$ and $\beta \in \mathbb{R}$, let

$$\omega_b(x) = \beta + u_{a,b}\left(|x|\right) - u_{a,b}(\bar{r}), \quad \text{for } |x| > 2,$$

where $u_{a,b}(|x|)$ is defined in Proposition 2.1. Clearly, $\omega_b(x)$ is a solution of (1.1a). Taking

$$\beta := \inf\left\{w_\xi(x) \;\middle|\; \xi \in \partial D,\ x \in \overline{B_{\bar{r}}} \setminus D\right\},$$

where $w_\xi(x)$ is given by Lemma 3.1, it is easy to see that

$$\omega_b \leqslant \beta, \quad \text{in } B_{\bar{r}} \setminus \overline{D}, \qquad \varphi \geqslant \beta, \quad \text{on } \partial D. \tag{3.2}$$

Then $\omega_b(x)$ is an admissible smooth subsolution of (1.1a)–(1.1b). On the other hand,

$$\begin{aligned}
\omega_b(x) &= \beta + \int_{\bar{r}}^{|x|} u'_{a,b}(s)\,ds = \beta + \int_{\bar{r}}^{|x|} w(s,b)\,ds \\
&= \beta + \frac{c^*}{2}\left(|x|^2 - \bar{r}^2\right) + \int_{\bar{r}}^{|x|} \left(w(s,b) - c^*\right) ds \\
&= \beta + \frac{c^*}{2}\left(|x|^2 - \bar{r}^2\right) + \mu_{\bar{r}}(b) - \int_{|x|}^{+\infty} \left(w(s,b) - c^*\right) ds \\
&= \frac{c^*}{2}|x|^2 + \mu(b) + O\left(|x|^{2-n}\right), \quad \text{as } |x| \to +\infty,
\end{aligned}$$

where

$$\mu(b) := \mu_{\bar{r}}(b) + \beta - \frac{c^*}{2}\bar{r}^2.$$

By the reason stated in Remark 2.2, we know that $\mu(b)$ is strictly increasing in $(c^*, +\infty)$, and

$$\lim_{b\to+\infty} \mu(b) = +\infty. \tag{3.3}$$

We will fix the value of $\tilde{c}$ as follows. First we require that $\tilde{c} > \beta$. If $b = c^*$, then in view of (2.12),

$$\mu\big(c^*\big) = \beta - \frac{c^*}{2}\bar{r}^2 < \beta < \tilde{c}.$$

Thus, in view of (3.3), for every $c > \tilde{c}$, there exists a unique $b(c)$ such that

$$\mu\big(b(c)\big) = c. \tag{3.4}$$

So $\omega_{b(c)}$ satisfies

$$\omega_{b(c)}(x) = \frac{c^*}{2}|x|^2 + c + O\big(|x|^{2-n}\big), \quad \text{as } |x| \to +\infty. \tag{3.5}$$

Set

$$\underline{w}(x) = \sup\big\{w_\xi(x) \bigm| \xi \in \partial D\big\}.$$

It is clear by the definition of $w_\xi(x)$, (3.1), that $\underline{w}$ is a locally Lipschitz function in $\mathbb{R}^n \setminus D$, and $\underline{w} = \varphi$ on ∂D. Since w_ξ is a smooth convex solution of (1.1a), $\underline{w}$ is a viscosity subsolution of (1.1a) in $\mathbb{R}^n \setminus \overline{D}$. We fix a number $\hat{r} > \bar{r}$, and then choose another number $\hat{b} > c^*$ such that

$$\min_{\partial B_{\hat{r}}} \omega_{\hat{b}} > \max_{\partial B_{\hat{r}}} \underline{w}.$$

We require that $\tilde{c}$ also satisfies $\tilde{c} \geqslant \mu(\hat{b})$. From now on, we fix once such $\tilde{c}$.

For $c \geqslant \tilde{c}$, we have $b(c) = \mu^{-1}(c) \geqslant \mu^{-1}(\tilde{c}) \geqslant \hat{b}$, and therefore

$$\omega_{b(c)} \geqslant \omega_{\hat{b}} > \underline{w}, \quad \text{on } \partial B_{\hat{r}}. \tag{3.6}$$

By (3.2), we have

$$\omega_{b(c)} \leqslant \beta \leqslant \underline{w}, \quad \text{in } B_{\bar{r}} \setminus \overline{D}. \tag{3.7}$$

Now we define, for $c > \tilde{c}$,

$$\underline{u}(x) = \begin{cases} \max\{\omega_{b(c)}(x), \underline{w}(x)\}, & x \in B_{\hat{r}} \setminus D, \\ \omega_{b(c)}(x), & x \in \mathbb{R}^n \setminus B_{\hat{r}}. \end{cases}$$

We know from (3.7) that

$$\underline{u} = \underline{w}, \quad \text{in } B_{\bar{r}} \setminus \overline{D}, \tag{3.8}$$

and in particular

$$\underline{u} = \underline{w} = \varphi, \quad \text{on } \partial D. \tag{3.9}$$

We know from (3.6) that $\underline{u} = \omega_{b(c)}$ in a neighborhood of $\partial B_{\hat{r}}$. Therefore $\underline{u}$ is locally Lipschitz in $\mathbb{R}^n \setminus D$. Since both $\omega_{b(c)}$ and $\underline{w}$ are viscosity subsolutions of (2.1) in $\mathbb{R}^n \setminus \overline{D}$, so is $\underline{u}$.

For $c > \tilde{c}$,

$$\bar{u}(x) := \frac{c^*}{2}|x|^2 + c$$

is an obvious smooth convex solution of (1.1a). By (3.7),

$$\omega_{b(c)} \leqslant \beta < \tilde{c} < \bar{u}, \quad \text{on } \partial D.$$

We also know by (3.5) that

$$\lim_{|x|\to\infty}\big(\omega_{b(c)}(x) - \bar{u}(x)\big) = 0.$$

Thus, in view of the comparison principle for smooth convex solutions of (1.1a) (see [6]), we have

$$\omega_{b(c)} \leqslant \bar{u}, \quad \text{on } \mathbb{R}^n \setminus D. \tag{3.10}$$

By (3.6) and the above, we have, for $c > \tilde{c}$,

$$w_\xi \leqslant \bar{u}, \quad \text{on } \partial(B_{\hat{r}} \setminus D),\ \forall \xi \in \partial D.$$

By the comparison principle again, we have

$$w_\xi \leqslant \bar{u}, \quad \text{in } B_{\hat{r}} \setminus \overline{D},\ \forall \xi \in \partial D.$$

Thus

$$\underline{w} \leqslant \bar{u}, \quad \text{in } B_{\hat{r}} \setminus \overline{D}.$$

This, combining with (3.10), implies that

$$\underline{u} \leqslant \bar{u}, \quad \text{in } \mathbb{R}^n \setminus D.$$

For any $c > \tilde{c}$, let $\mathcal{S}_c$ denote the set of $v \in C^0(\mathbb{R}^n \setminus D)$ which are viscosity subsolutions of (1.1a) in $\mathbb{R}^n \setminus \overline{D}$ satisfying

$$v = \varphi, \quad \text{on } \partial D, \tag{3.11}$$

and

$$\underline{u} \leqslant v \leqslant \bar{u}, \quad \text{in } \mathbb{R}^n \setminus D. \tag{3.12}$$

We know that $\underline{u} \in \mathcal{S}_c$. Let

$$u(x) := \sup\{v(x) \mid v \in \mathcal{S}_c\}, \quad x \in \mathbb{R}^n \setminus D.$$

By (3.5), and the definitions of $\underline{u}$ and $\bar{u}$,

$$u(x) \geqslant \underline{u}(x) = \omega_{b(c)}(x) = \frac{c^*}{2}|x|^2 + c + O\left(|x|^{2-n}\right), \quad \text{as } |x| \to \infty \tag{3.13}$$

and

$$u(x) \leqslant \bar{u}(x) = \frac{c^*}{2}|x|^2 + c.$$

The estimate (1.10) follows.

Next, we prove that u satisfies the boundary condition on ∂D. It is obvious from (3.9) that

$$\liminf_{x\to\xi} u(x) \geqslant \lim_{x\to\xi} \underline{u}(x) = \varphi(\xi), \quad \forall \xi \in \partial D.$$

So we need only to prove that

$$\limsup_{x\to\xi} u(x) \leqslant \varphi(\xi), \quad \forall \xi \in \partial D.$$

Let $\omega^+ \in C^2(\overline{B_{\bar{r}} \setminus D})$ be defined by

$$\begin{cases} \Delta\omega^+ = 0, & \text{in } B_{\bar{r}} \setminus \overline{D}, \\ \omega^+ = \varphi, & \text{on } \partial D, \\ \omega^+ = \max\limits_{\partial B_{\bar{r}}} \bar{u}, & \text{on } \partial B_{\bar{r}}. \end{cases}$$

By Lemma A.4 below, we have u^*, the upper semicontinuous envelope of u, is a viscosity subsolution of (1.1a) in $B_{\bar{r}} \setminus \overline{D}$. Hence, by the definition of viscosity solution,

$$\begin{cases} \Delta u^* \geqslant 0, & \text{in } B_{\bar{r}} \setminus D, \\ u^* \leqslant \varphi, & \text{on } \partial D, \\ u^* \leqslant \max\limits_{\partial B_{\bar{r}}} \bar{u}, & \text{on } \partial B_{\bar{r}}. \end{cases}$$

By comparison principle, it follows that

$$u \leqslant u^* \leqslant \omega^+, \quad \text{in } B_{\bar{r}} \setminus \overline{D}.$$

Therefore,

$$\limsup_{x\to\xi} u(x) \leqslant \lim_{x\to\xi} \omega^+(x) = \varphi(\xi), \quad \forall \xi \in \partial D.$$

Finally, by Theorem A.2 below, an adapted version of Perron method for exterior domain, we have u is a viscosity solution of (1.1a)–(1.1b). Theorem 1.1 is established. □

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Appendix A. Perron method

For the reader's convenience, here we include an adapted version of Perron method for the following Dirichlet problem

$$\begin{cases} f\big(\lambda\big(D^2u\big)\big) = 1, & \text{in } \Omega, \quad \text{(a)} \\ u = \varphi, & \text{on } \partial\Omega, \quad \text{(b)} \end{cases} \tag{A.1}$$

where Ω is an open (bounded or unbounded) subset of $\mathbb{R}^n$. Especially, Ω can be the exterior domain $\mathbb{R}^n \setminus \overline{D}$, defined as in Section 1.

Recall that $\Gamma \subsetneqq \mathbb{R}^n$ be an open convex cone, with vertex at the origin, containing the positive cone Γ_n, and symmetric in the λ_i, $i = 1, \ldots, n$.

Definition A.1. A function $u \in C^2(\Omega)$ is called admissible, if at every $x \in \Omega$, $\lambda(D^2u(x)) \in \overline{\Gamma}$.

We use $\mathrm{USC}(\Omega)$ and $\mathrm{LSC}(\Omega)$ to denote respectively the set of upper and lower semicontinuous real valued functions on Ω. We recall the definition of viscosity solutions for fully nonlinear elliptic equations (see, e.g. [10]).

Definition A.2. A function $u \in \mathrm{USC}(\Omega)$ ($u \in \mathrm{LSC}(\Omega)$) is said to be a viscosity subsolution (supersolution) of Eq. (A.1a) (or say that u satisfies $f(\lambda(D^2u)) \geqslant (\leqslant) 1$ in the viscosity sense), if for any open set $\mathcal{N}$ in Ω, any admissible function $\psi \in C^2(\Omega)$ and local maximum (minimum) point $\bar{x} \in \mathcal{N}$ of $u - \psi$ we have

$$f\big(\lambda\big(D^2\psi(\bar{x})\big)\big) \geqslant (\leqslant) 1.$$

A function $u \in C^0(\Omega)$ is said to be a viscosity solution of (A.1a), if it is both a viscosity subsolution and supersolution of (A.1a).

The viscosity and classical solutions have the following relation.

Proposition A.1. *(See [31].) If u is an admissible classical solution of (A.1a), then u is a viscosity solution. Conversely, if u is a viscosity solution of (A.1a), and u is of class C^2, then u is an admissible classical solution.*

Remark A.1. The proof can be referred to [31]. From the proof, it is evident that if u is a viscosity subsolution of (A.1a), then u is admissible at each point at which u is twice differentiable.

Definition A.3. Let $\varphi \in C^0(\partial\Omega)$. A function $u \in \mathrm{USC}(\overline{\Omega})$ ($u \in \mathrm{LSC}(\overline{\Omega})$) is a viscosity subsolution (supersolution) of the Dirichlet problem (A.1a)–(A.1b), if u is a viscosity subsolution (supersolution) of (A.1a) and $u \leqslant (\geqslant)\ \varphi(x)$ on $\partial\Omega$. A function $u \in C^0(\overline{\Omega})$ is said to be a viscosity solution of (A.1a)–(A.1b) if it is both a subsolution and a supersolution.

With the above theory of viscosity solution can be applied. In particular, the hypotheses on the function f guarantee a comparison principle holds (see Proposition 3.3 in [10]). When a comparison principle for viscosity subsolution and supersolution holds, the argument made in [10] adapts and yields easily the following existence theorem in exterior domains.

Theorem A.2 *(Perron method). Assume that there exist $\underline{u}, \bar{u} \in C^0(\overline{\Omega})$ respectively to be viscosity subsolution and supersolution of (A.1a) such that $\underline{u} \leqslant \bar{u}$ in Ω, and $\underline{u} = \varphi$ on $\partial\Omega$. Then there exists a unique viscosity solution $u \in C^0(\overline{\Omega})$ of (A.1a)–(A.1b).*

To prove Theorem A.2, we need the following comparison principle.

Lemma A.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. If $u \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ are respectively viscosity subsolution and supersolution of (A.1a) in Ω and $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.*

Under the assumptions $u, v \in C^0(\overline{\Omega})$, the lemma was proved in [31] for $f(\lambda(D^2u)) = \sigma_k(\lambda(D^2u))$, based on Jensen approximations (see [21]). The proof remains valid under the weaker regularity assumptions on u and v.

Proof of Lemma A.3. As in [22], for $\epsilon > 0$ we define functions $u_\epsilon^+, v_\epsilon^-$ by

$$u_\epsilon^+(x) = \sup_{y\in\Omega}\left\{u(y) - \omega_0\frac{|x-y|^2}{\epsilon^2}\right\}, \qquad v_\epsilon^-(x) = \sup_{y\in\Omega}\left\{v(y) + \omega_0\frac{|x-y|^2}{\epsilon^2}\right\}, \tag{A.2}$$

where

$$\omega_0 = \max\{\operatorname{osc}_\Omega u, \operatorname{osc}_\Omega v\}.$$

Note that if $x \in \Omega_\epsilon = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \epsilon\}$, then the supremum and infimum in (A.2) will be achieved at points $x_\epsilon^\pm \in \Omega$. Moreover

$$\left|Du_\epsilon^+\right|, \left|Dv_\epsilon^-\right| \leqslant \frac{2\omega_0}{\epsilon}, \qquad D^2u_\epsilon^+, -D^2v_\epsilon^- \geqslant \frac{-2\omega_0}{\epsilon^2}$$

in the sense of distributions. Furthermore, u_ϵ^+ and v_ϵ^- are easily seen to be viscosity subsolution and supersolution, respectively, of (A.1a).

Now let $\omega_\epsilon = u_\epsilon^+ - v_\epsilon^-$ and let Γ^ϵ denote the upper contact set of ω_ϵ on Ω_ϵ, i.e.,

$$\Gamma^\epsilon = \{ y \in \Omega_\epsilon \mid \exists p \in \mathbb{R}^n, \text{ such that } \omega_\epsilon(x) \leqslant \omega_\epsilon(y) + p \cdot (x-y), \ \forall x \in \Omega_\epsilon \}.$$

Since u_ϵ^+ and v_ϵ^- are semi-convex and semi-concave, respectively, it follows that $D^2 u_\epsilon^+$ and $D^2 v_\epsilon^+$ exist almost everywhere, and $D^2\omega_\epsilon = D^2 u_\epsilon^+ - D^2 v_\epsilon^- \leqslant 0$ almost everywhere on Γ^ϵ. Thus,

$$\frac{-2\omega_0}{\epsilon^2} \leqslant D^2 u_\epsilon^+ \leqslant D^2 v_\epsilon^- \leqslant \frac{2\omega_0}{\epsilon^2}, \quad \text{almost everywhere on } \Gamma^\epsilon.$$

By Remark A.1, we have $\lambda(D^2 u_\epsilon^+) \in \Gamma$, and hence also $\lambda(D^2 v_\epsilon^-) \in \Gamma$, at almost all points of Γ^ϵ. We see that $\omega_\epsilon \in C^{1,1}(\Gamma^\epsilon)$ and almost everywhere on Γ^ϵ, ω_ϵ satisfies the elliptic differential inequality

$$a^{ij} D_{ij}\omega_\epsilon \geqslant 0,$$

where

$$a^{ij}(x) = \int_0^1 F_{ij}\big(D^2\big(\theta u_\epsilon^+ + (1-\theta)v_\epsilon^+\big)\big)\,d\theta, \qquad F_{ij}\big(D^2u\big) = \frac{\partial f(\lambda(D^2u))}{\partial u_{ij}}.$$

Consequently, by the Aleksandrov maximum principle (see the proof of Theorem 9.1 in [17]), we obtain, for any $\Omega' \subset\subset \Omega$ and any $\epsilon > 0$ so small that $\Omega' \subset\subset \Omega_\epsilon$,

$$\sup_{\Omega'}(u-v) \leqslant \sup_{\Omega'}\omega_\epsilon \leqslant \limsup_{x\to\partial\Omega'}\omega_\epsilon^+ \to \sup_{\partial\Omega'}(u-v)^+, \quad \text{as } \epsilon \to 0.$$

Letting $\Omega' \to \Omega$ we conclude that $u - v \leqslant 0$ in Ω, so the proposition is proved. □

Let

$$u^*(u_*)(x) = \lim_{r\to 0} \sup_{B_r(x)} \Big(\inf_{B_r(x)} \Big) u(y)$$

be the upper (lower) semicontinuous envelope of u. Theorem A.2 is an immediate application of Perron's method as in [10,18]: indeed, one considers the function u defined as the supremum of, say, all USC subsolution v of (A.1a) equal to φ on $\partial\Omega$ and satisfying $\underline{u} \leqslant v \leqslant \bar{u}$. That is,

$$u(x) = \sup\{ v(x) \mid \underline{u} \leqslant v \leqslant \bar{u}, \text{ in } \Omega \text{ and } v \text{ is a subsolution of (A.1a)}, \ v|_{\partial\Omega} = \varphi \}.$$

Observe that $\underline{u} \leqslant u \leqslant \bar{u}$ in Ω. The main idea of Perron's construction (maximality) translates into the statement that u_*, which is LSC in Ω (and continuous at $\partial\Omega$), is a viscosity supersolution of (A.1a), while the stability of viscosity subsolutions through sup operations yields the fact that u^*, which is USC on Ω (and $u = \varphi$ on $\partial\Omega$), is a viscosity subsolution of (A.1a). Therefore, by comparison principle, $u^* \leqslant u_*$ in Ω while obviously by definition $u^* \geqslant u_*$ in Ω. Thus, u is continuous on Ω and is the unique viscosity solution of (A.1a)–(A.1b).

Lemma A.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\mathcal{S}$ be a non-empty family of viscosity subsolutions (supersolutions) of (A.1a) in Ω. Set*

$$u(x) = \sup(\inf)\{v(x) \mid v \in \mathcal{S}\}.$$

Then, if $u^* < \infty$ ($u_* > -\infty$) in Ω, u^* (u_*) is a viscosity subsolution (supersolution) of (A.1a) in Ω.

Lemma A.4 can be proved by standard arguments, see e.g. [10]. Finally, we need also the following construction argument (see e.g. Lemma 4.4 in [10]).

Lemma A.5. *Let Ω be open and u be a subsolution of (A.1a) in Ω. If u_* fails to be a supersolution at some point $\hat{x}$, then for any small $\kappa > 0$ there is a subsolution U_κ of (A.1a) in Ω satisfying*

$$\begin{cases} U_\kappa(x) \geqslant u(x) & \text{and} \quad \sup\limits_{\Omega}(U_\kappa - u) > 0, \\ U_\kappa(x) = u(x) & \text{for } x \in \Omega, \quad |x - \hat{x}| \geqslant \kappa. \end{cases} \tag{A.3}$$

Proof of Theorem A.2. With the notation of the theorem, we observe that $(\underline{u})_* \leqslant u_* \leqslant u \leqslant u^* \leqslant (\bar{u})^* = \bar{u}$ and, in particular,

$$u_* = u = u^* = \varphi, \quad \text{on } \partial\Omega.$$

By Lemma A.4 u^* is a subsolution of (A.1a). It then follows from the definition of u that $u = u^*$, so u is a subsolution.

If u_* fails to be a supersolution at some point $\hat{x} \in \mathbb{R}^n \setminus \overline{D}$, let U_κ be provided by Lemma A.5. Clearly $\underline{u} \leqslant u \leqslant U_\kappa$ and U_κ satisfies the boundary conditions for sufficiently small κ. By Lemma A.3, $U_\kappa \leqslant \bar{u}$ and since u is the maximal subsolution between $\underline{u}$ and $\bar{u}$, we arrive at the contradiction $U_\kappa \leqslant u$. Hence u_* is a supersolution of (A.1a) and then, $u^* = u \leqslant u_*$, showing that u is continuous and $u = u^* = u_*$ is a solution. □

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