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The Blow up Analysis of the General Curve Shortening Flow

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Abstract It is shown that the curvature function satisfies a nonlinear evolution equation under the general curve shortening flow and a detailed asymptotic behavior of the closed curves is presented when they contract to a point in finite time.

Keywords Asymptotic behavior, curve shortening flow, support function, Hausdorff metric

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1 Introduction

The curve shortening flows have been studied by many authors and have many applications (cf. [1]). Gage and Hamilton [2] discussed the mean curvature flow in one dimensional case. The flow is given by the equation

$$v = k,$$

where v and k are, respectively, the normal velocity and inward curvature of the plane curve. They proved that a convex closed curve stays convex and smooth and shrinks to a point in finite time with the limiting shape of a circle. In this paper, we generalize some results on curve shortening flow (cf. [2]) to the generalized curve shortening flow (cf. [3])

$$v = |k|^{p-1}k, \tag{1.1}$$

where p is the positive number. When $p = \frac{1}{3}$, the flow (1.1) is the affine plane curve evolution (cf. [4, 5]). More generally, we will also study the non-homogeneous flow (cf. [3]) which is given by

$$v = G(k)k, \tag{1.2}$$

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where $G(\cdot)$ is a positive function on $(0,\infty)$. In the high-dimensional case, Guo [6] derived the evolution equation of the integral of the Gauss curvature on an evolving hypersurface. Nien and Tsai [7] proved that the self-similar solutions under the contraction flow could happen only as (1.1). Andrews [3] studied some nonlinear expansion of contraction flow and obtained the limiting self-similar solutions. But the asymptotic behavior of solutions of the flow (1.1), (1.2) is little known. The main objective of this paper is to analyze the asymptotic behavior of the curvature under the generalized curve shortening flow.

Let $\mathbb{S}^1$ be a unit circle in the plane, and

$$\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$$

be a closed convex curve in the plane. We look for a family of closed curves

$$\gamma(u,t) : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2,$$

which satisfies

$$\begin{cases} \dfrac{\partial\gamma}{\partial t}(u,t) = |k|^{p-1}kN, & u\in\mathbb{S}^1,\ t\in[0,T), \\ \gamma(u,0)=\gamma_0(u), & u\in\mathbb{S}^1,\ t=0, \end{cases}$$

where p is a positive number, $k(\cdot,t)$ is the inward curvature of the plane curve $\gamma(\cdot,t)$ and $N(\cdot,t)$ is the unit inward normal vector. More generally, we consider $\gamma(\cdot,t)$ satisfying

$$\begin{cases} \dfrac{\partial\gamma}{\partial t}(u,t) = G(k)kN, & u\in\mathbb{S}^1,\ t\in[0,T), \\ \gamma(u,0)=\gamma_0(u), & u\in\mathbb{S}^1,\ t=0, \end{cases} \tag{1.3}$$

where G is a positive, non-decreasing smooth function on $(0,\infty)$.

In the following sections we assume that $A(t)$ is the area of a bounded domain enclosed by the curve $\gamma(\cdot,t)$, $L(t)$ is the length of $\gamma(\cdot,t)$, $r_{\text{out}}(t)$ and $r_{\text{in}}(t)$ are respectively the radii of the largest circumscribed circle and the smallest circumscribed circle of $\gamma(\cdot,t)$. Define

$$k_{\max}(t) = \max\{k(u,t)\mid u\in\mathbb{S}^1\},$$
$$k_{\min}(t) = \min\{k(u,t)\mid u\in\mathbb{S}^1\}.$$

Firstly we introduce the existence theorem, which belongs to Andrews [3, Theorem Π 4.1, Proposition Π 4.4].

Proposition 1.1 *Let γ_0 be a closed strictly convex curve. Then the unique classical solution $\gamma(\cdot,t)$ of* (1.3) *exists only at finite time interval $[0,\omega)$, and the solution $\gamma(\cdot,t)$ converges to a point ϑ as $t\to\omega$ and $A(t)$, $k_{\max}(t)$ satisfy the following properties:*

$$\forall\, t\in[0,\omega),\quad A(t)>0,\quad k_{\max}(t)<+\infty,$$
$$\lim_{t\to\omega} A(t)=0,\quad \lim_{t\to\omega} k_{\max}(t)=+\infty.$$

As $t\to\omega$, the normalized curves

$$\eta(\cdot,t)=\sqrt{\frac{\pi}{A(t)}}\gamma(\cdot,t)$$

converge to the unit circle centered at the point ϑ.

In this paper we always assume that the initial curve satisfies the conditions of Proposition 1.1, and that $G(x)$ is a function on $(0, \infty)$ satisfying

(H1) $G(x) \in C^3(0, \infty)$, $G'(x) \geq 0$ and $G(x) > 0$ for $x \in (0, \infty)$,

(H2) $G(x)x^2$ is convex in $(0, \infty)$ and there is a positive constant C_0 such that

$$G'(x)x \leq C_0 G(x), \text{ for sufficiently large } x.$$

We now state the main theorem of this paper.

Theorem 1.2 *Suppose $G(x)$ satisfies* (H1) *and* (H2). *Let $\gamma(\cdot, t)$ be the solution for Proposition* 1.1. *Then the following hold*:

i) $\lim_{t\to\omega} \frac{r_{\text{in}}(t)}{r_{\text{out}}(t)} = 1$,

ii) $\lim_{t\to\omega} \frac{k_{\min}(t)}{k_{\max}(t)} = 1$,

iii) $\lim_{t\to\omega} \frac{1}{\omega - t} \int_{k(\theta,t)}^{+\infty} \frac{dx}{G(x)x^3} = 1$ *is uniformly convergent on* $\mathbb{S}^1$.

Remark 1.3 Let $G(x) = |x|^{p-1}$ with $p \geq 1$ in Proposition 1.1. Then

$$k(\theta, t)[(p+1)(\omega - t)]^{\frac{1}{p+1}} \text{ converges uniformly to 1 as } t \to \omega, \tag{1.4}$$

for uniformly θ in $\mathbb{S}^1$.

Remark 1.4 When $p = 1$, by (1.4) it shows the asymptotic formula about curvature function of the curve shortening flow (1.1) which was discovered firstly by Gage and Hamilton [2, Corollary 5.6].

This paper is organized as follows: In the next section we transfer the flow (1.3) into an initial PDEs problem and establish some monotone geometric inequality. In Section 3, we obtain the global Harnack inequality of the curvature function according to the flow (1.3), see Lemma 3.6. And then we complete the proof of Theorem 1.2 by making use of Gage–Hamilton's methods (cf. [2]).

2 Evolutions

Using the idea in [2], we can drive the evolution equations under the flow (1.3) for the length and the curvature of the curves, and the area enclosed by the curves.

Let the curve be $\gamma(u) = (x(u), y(u))$ with parameter u (modulo 2π) and s be an arc-length parameter along the curve $\gamma(u)$ which is unique up to a constant. Then

$$ds = v du, \quad \frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

where

$$v = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}.$$

Suppose ξ and N are the unit tangent vector and the unit inward normal vector of the curve. Then the Frenet equations (cf. [8]) are

$$\frac{\partial \xi}{\partial u} = vkN, \quad \frac{\partial N}{\partial u} = -vk\xi,$$

or

$$\frac{\partial \xi}{\partial s} = kN, \quad \frac{\partial N}{\partial s} = -k\xi,$$

where $k(u)$ is the inward curvature of $\gamma(u)$.

Let θ be the tangent angle of the curve $\gamma(u)$ to the x-axis. We drive the following useful formula.

Lemma 2.1 *The geometric quantities v, L, ξ, N, θ and k of the flow* (1.3) *evolve according to*

i) $\dfrac{\partial v}{\partial t} = -G(k)k^2 v,$

ii) $\dfrac{dL}{dt} = -\displaystyle\int_0^L G(k)k^2 ds,$

iii) $\dfrac{\partial}{\partial t}\dfrac{\partial}{\partial s} = \dfrac{\partial}{\partial t}\dfrac{\partial}{\partial s} + G(k)k^2\dfrac{\partial}{\partial s},$

iv) $\dfrac{\partial \xi}{\partial t} = (G(k)k)'\dfrac{\partial k}{\partial s}N,\ \ \dfrac{\partial N}{\partial t} = -(G(k)k)'\dfrac{\partial k}{\partial s}\xi,$

v) $\dfrac{\partial \theta}{\partial t} = (G(k)k)'\dfrac{\partial k}{\partial s},\ \ \dfrac{\partial \theta}{\partial s} = k,$ *where* $\xi = (\cos\theta, \sin\theta),$

vi) $\dfrac{dA}{dt} = -\displaystyle\int_0^L G(k)kds,$

vii) $\dfrac{\partial k}{\partial t} = \dfrac{\partial}{\partial s}\left((G(k)k)'\dfrac{\partial k}{\partial s}\right) + G(k)k^3.$

Proof Let $\langle \cdot, \cdot \rangle$ be the inner product in $\mathbb{R}^2$.

i) By (1.3) and the Frenet equations, we have

$$\begin{aligned}\frac{\partial}{\partial t}(v^2) &= \frac{\partial}{\partial t}\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle \\ &= 2\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial^2 \gamma}{\partial u \partial t}\right\rangle \\ &= 2\left\langle v\xi, \frac{\partial}{\partial u}(G(k)kN)\right\rangle \\ &= 2\left\langle v\xi, \frac{\partial}{\partial u}(G(k)k)N + G(k)k\frac{\partial N}{\partial u}\right\rangle \\ &= 2\left\langle v\xi, \frac{\partial}{\partial u}(G(k)k)N - G(k)k^2 v\xi\right\rangle \\ &= -2G(k)k^2v^2.\end{aligned}$$

This implies that the identity (i) holds.

ii) Since $L = \int_0^{2\pi} vdu$, then from (i), there holds

$$\frac{\partial L}{\partial t} = \int_0^{2\pi}\frac{\partial v}{\partial t}du = -\int_0^{2\pi} G(k)k^2 vdu = -\int_0^L G(k)k^2 ds.$$

iii) By (i), we get

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial u}\right) &= -\frac{1}{v^2}\frac{\partial v}{\partial t}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial t}\frac{\partial}{\partial u} \\ &= -\frac{1}{v^2}(-G(k)k^2v)\frac{\partial}{\partial u} + \frac{\partial}{\partial s}\frac{\partial}{\partial t} \\ &= G(k)k^2\frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}.\end{aligned}$$

iv) It follows from (1.3) and (iii) that

$$\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\gamma = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\gamma + G(k)k^2\frac{\partial}{\partial s}\gamma = \frac{\partial}{\partial s}(G(k)kN) + G(k)k^2\xi.$$

Then by Frenet equations, we have

$$\begin{aligned}\frac{\partial \xi}{\partial t} &= \frac{\partial}{\partial s}(G(k)k)N + G(k)k\frac{\partial N}{\partial s} + G(k)k^2\xi \\ &= (G(k)k)'\frac{\partial k}{\partial s}N - G(k)k^2\xi + G(k)k^2\xi \\ &= (G(k)k)'\frac{\partial k}{\partial s}N. \end{aligned} \tag{2.1}$$

In terms of $\langle \xi, N\rangle \equiv 0$, we obtain

$$0 \equiv \frac{\partial}{\partial t}\langle \xi, N\rangle = \left\langle \frac{\partial \xi}{\partial t}, N\right\rangle + \left\langle \xi, \frac{\partial N}{\partial t}\right\rangle.$$

Thus

$$0 = \left\langle (G(k)k)'\frac{\partial k}{\partial s}N, N\right\rangle + \left\langle \xi, \frac{\partial N}{\partial t}\right\rangle = (G(k)k)'\frac{\partial k}{\partial s} + \left\langle \xi, \frac{\partial N}{\partial t}\right\rangle.$$

From $0 \equiv \langle \frac{\partial N}{\partial t}, N\rangle$, it follows that there exists λ such that

$$\frac{\partial N}{\partial t} = \lambda \xi,$$

and then combining with the above equality, we have

$$\lambda = -(G(k)k)'\frac{\partial k}{\partial s},$$

and

$$\frac{\partial N}{\partial t} = -(G(k)k)'\frac{\partial k}{\partial s}\xi.$$

v) Since $\xi = (\cos\theta, \sin\theta)$, then $N = (-\sin\theta, \cos\theta)$, we obtain

$$\frac{\partial \xi}{\partial t} = (-\sin\theta, \cos\theta)\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial t}N.$$

Comparing it with (iv), we conclude that

$$\frac{\partial \theta}{\partial t} = (G(k)k)'\frac{\partial k}{\partial s}.$$

In other cases,

$$\frac{\partial \xi}{\partial s} = (-\sin\theta, \cos\theta)\frac{\partial \theta}{\partial s} = \frac{\partial \theta}{\partial s}N.$$

It follows from $\frac{\partial \xi}{\partial s} = kN$ that

$$\frac{\partial \theta}{\partial s} = k.$$

vi) Consider the closed curve $\gamma = \{(x(u), y(u)) | u \in \mathbb{S}^1\}$ in $\mathbb{R}^2$. Then it is well known that the area of the domain by the curve γ can be expressed by the formula

$$A = \frac{1}{2}\int_0^{2\pi}\left(x\frac{\partial y}{\partial u} - y\frac{\partial x}{\partial u}\right)du = -\frac{1}{2}\int_0^{2\pi}\langle \gamma(u), vN\rangle du.$$

Then

$$\frac{dA}{dt} = -\frac{1}{2}\int_0^{2\pi}\left\langle \frac{\partial \gamma}{\partial t}, vN\right\rangle du - \frac{1}{2}\int_0^{2\pi}\left\langle \gamma, \frac{\partial v}{\partial t}N\right\rangle du - \frac{1}{2}\int_0^{2\pi}\left\langle \gamma, \frac{\partial N}{\partial t}v\right\rangle du.$$

By (1.3) and (i), (iv), we obtain

$$
\begin{aligned}
\frac{dA}{dt} &= -\frac{1}{2}\int_0^{2\pi} \langle G(k)kN, vN\rangle du + \frac{1}{2}\int_0^{2\pi} \langle \gamma, G(k)k^2 vN\rangle du + \frac{1}{2}\int_0^{2\pi} \left\langle \gamma, \frac{\partial}{\partial s}(G(k)k)v\xi \right\rangle du \\
&= -\frac{1}{2}\int_0^{2\pi} G(k)kvdu + \frac{1}{2}\int_0^{2\pi} \langle \gamma, G(k)k^2 N\rangle vdu + \frac{1}{2}\int_0^{2\pi} \left\langle \gamma, \frac{\partial}{\partial u}(G(k)k)\xi \right\rangle du \\
&= -\frac{1}{2}\int_0^{2\pi} G(k)kvdu + \frac{1}{2}\int_0^{2\pi} \langle \gamma, G(k)k^2 N\rangle vdu \\
&\quad + \frac{1}{2}\int_0^{2\pi} \left\langle \gamma, \frac{\partial}{\partial u}(G(k)k\xi) \right\rangle du - \frac{1}{2}\int_0^{2\pi} \left\langle \gamma, G(k)k\frac{\partial \xi}{\partial u} \right\rangle du.
\end{aligned}
$$

Thus from $\frac{\partial \xi}{\partial u} = vkN$, we arrive at

$$
\begin{aligned}
\frac{dA}{dt} &= -\frac{1}{2}\int_0^{2\pi} G(k)kvdu + \frac{1}{2}\int_0^{2\pi} \langle \gamma, G(k)k^2 N\rangle vdu \\
&\quad - \frac{1}{2}\int_0^{2\pi} \left\langle \frac{\partial \gamma}{\partial u}, G(k)k\xi \right\rangle du - \frac{1}{2}\int_0^{2\pi} \langle \gamma, G(k)k^2 N\rangle vdu \\
&= -\frac{1}{2}\int_0^{2\pi} G(k)kvdu - \frac{1}{2}\int_0^{2\pi} \langle \xi, G(k)k\xi\rangle vdu \\
&= -\frac{1}{2}\int_0^{2\pi} G(k)kvdu - \frac{1}{2}\int_0^{2\pi} G(k)kvdu \\
&= -\int_0^{2\pi} G(k)kvdu \\
&= -\int_0^{L} G(k)kds.
\end{aligned}
$$

vii) By (iii) and (v), we drive the following equation

$$
\frac{\partial k}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \theta}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \theta}{\partial t} + G(k)k^2\frac{\partial \theta}{\partial s} = \frac{\partial}{\partial s}\left((G(k)k)'\frac{\partial k}{\partial s}\right) + G(k)k^3.
$$

Thus, the proof of Lemma 2.1 is completed. □

Similarly to [2], we can use the angle θ of the tangent line as the parameter of the curve and then write the curvature $k = k(\theta)$ in terms of this parameter which is 2π periodic curvature function of convex curve. The following results give the necessary and sufficient condition for some one-parameter function as the curvature function of a simple closed curve (cf. Lemma 4.1.1 in [2]).

Lemma 2.2 *A positive 2π periodic function represents the curvature function of a closed and strictly convex C^2 curves in the plane if and only if*

$$
\int_0^{2\pi} \frac{\cos\theta}{k(\theta)} d\theta = \int_0^{2\pi} \frac{\sin\theta}{k(\theta)} d\theta = 0.
$$

According to the flow (1.3), we take $\tau = t$ as the time parameter and use θ as other coordinate and hence change variables from (u, t) to (θ, τ).

Lemma 2.3

$$\begin{aligned}\frac{\partial k}{\partial \tau} &= k^2\left(\frac{\partial^2}{\partial \theta^2}(G(k)k) + G(k)k\right)\\ &= k^2(G(k)k)'\frac{\partial^2 k}{\partial \theta^2} + k^2(G(k)k)''\left(\frac{\partial k}{\partial \theta}\right)^2 + G(k)k^3.\end{aligned}$$

Proof By the chain rule and Lemma 2.1 (iii), (v), we get

$$\begin{aligned}\frac{\partial k}{\partial t} &= \frac{\partial k}{\partial \tau} + \frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial t}\\ &= \frac{\partial k}{\partial \tau} + \frac{\partial k}{\partial \theta}(G(k)k)'\frac{\partial k}{\partial s}\\ &= \frac{\partial k}{\partial \tau} + \frac{\partial k}{\partial \theta}(G(k)k)'\frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial s}\\ &= \frac{\partial k}{\partial \tau} + (G(k)k)'k\left(\frac{\partial k}{\partial \theta}\right)^2.\end{aligned}$$

On the other hand, from Lemma 2.1 (vii), we obtain

$$\begin{aligned}\frac{\partial k}{\partial t} &= \frac{\partial}{\partial s}\left((G(k)k)'\frac{\partial k}{\partial s}\right) + G(k)k^3\\ &= \frac{\partial \theta}{\partial s}\frac{\partial}{\partial \theta}\left((G(k)k)'\frac{\partial k}{\partial \theta}\frac{\partial \theta}{\partial s}\right) + G(k)k^3\\ &= k\frac{\partial}{\partial \theta}\left((G(k)k)'\frac{\partial k}{\partial \theta}k\right) + G(k)k^3\\ &= k\frac{\partial}{\partial \theta}\left(k\frac{\partial}{\partial \theta}(G(k)k)\right) + G(k)k^3\\ &= k^2\frac{\partial^2}{\partial \theta^2}(G(k)k) + (G(k)k)'k\left(\frac{\partial k}{\partial \theta}\right)^2 + G(k)k^3.\end{aligned}$$

By comparing the above two equalities, we have the desired results. □

Throughout this paper, we will deal with this equation and replace τ by t.

Lemma 2.4 *Suppose* (H1) *hold. Then the general curve shortening problem* (1.3) *for convex curves is equivalent to the Cauchy problem*

$$\begin{cases}\dfrac{\partial k}{\partial t} = k^2\left(\dfrac{\partial^2}{\partial \theta^2}(G(k)k) + G(k)k\right), & \theta \in \mathbb{S}^1,\ t \in [0,T),\\ k(\theta,0) = k_0(\theta), & \theta \in \mathbb{S}^1,\ t = 0,\end{cases} \tag{2.2}$$

where $0 < \alpha < 1$*,* $k \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{S}^1 \times (0,T))$*,* $k_0(\theta)$ *is the curvature function of the initial curve* $\gamma_0(\theta)$*.*

Proof If $\gamma(\cdot,t)$ is the classical solution of problem (1.3), then by Lemma 2.3 the curvature function, expressed in θ coordinates, satisfies (2.2).

If $k_0(\theta)$ is the curvature function for the curve $\gamma_0(\theta)$ and $k(\theta,t)$ satisfies (2.2), then for each $t \geq 0$, we can define the curves by the formula

$$x(\theta,t) = \int_0^\theta \frac{\cos u}{k(u,t)}du, \quad y(\theta,t) = \int_0^\theta \frac{\sin u}{k(u,t)}du. \tag{2.3}$$

Let $\gamma(\theta,t)=(x(\theta,t),y(\theta,t))$. Then $\zeta=(\cos\theta,\sin\theta)$ and $N=(-\sin\theta,\cos\theta)$ are respectively the tangent vector and the inward normal vector of the curve $\gamma(\cdot,t)$. Combining (2.2) with (2.3), we have

$$\begin{aligned}\frac{\partial x}{\partial t}&=-\int_0^\theta \frac{\cos u}{k^2}\frac{\partial k}{\partial t}du\\&=-\int_0^\theta \cos u\ \left(\frac{\partial^2}{\partial u^2}(G(k)k)+G(k)k\right)du\\&=-\int_0^\theta \cos u\ \frac{\partial^2}{\partial u^2}(G(k)k)du-\int_0^\theta \cos uG(k)kdu,\end{aligned}$$

so that

$$\begin{aligned}\frac{\partial x}{\partial t}=&-\int_0^\theta \sin u\frac{\partial}{\partial u}(G(k)k)du-\int_0^\theta \cos uG(k)kdu-\cos\theta\frac{\partial}{\partial\theta}(G(k)k)+\frac{\partial}{\partial\theta}(G(k)k)\Big|_{\theta=0}\\=&\int_0^\theta \cos uG(k)kdu-G(k)k\sin\theta-\int_0^\theta \cos uG(k)kdu-\cos\theta\frac{\partial}{\partial\theta}(G(k)k)+\frac{\partial}{\partial\theta}(G(k)k)\Big|_{\theta=0}\\=&-G(k)k\sin\theta-\cos\theta\frac{\partial}{\partial\theta}(G(k)k)+\frac{\partial}{\partial\theta}(G(k)k)\Big|_{\theta=0}.\end{aligned}$$

For the same reason the following equality holds:

$$\frac{\partial y}{\partial t}=G(k)k\cos\theta-\sin\theta\frac{\partial}{\partial\theta}(G(k)k)-G(k)k|_{\theta=0}.$$

By setting $\theta=2\pi$ to the above two equalities, we obtain $\frac{\partial x}{\partial t}=0$, $\frac{\partial y}{\partial t}=0$. On the other hand, $x(2\pi,0)=0$, $y(2\pi,0)=0$. Because γ_0 is the closed curve in the plane, then applying Lemma 2.2, we obtain $x(2\pi,0)=0$, $y(2\pi,0)=0$, and

$$\int_0^{2\pi}\frac{\cos\theta}{k(\theta,t)}d\theta=\int_0^{2\pi}\frac{\sin\theta}{k(\theta,t)}d\theta=0.$$

By applying Lemma 2.2 again, the curve $\gamma(\theta,t)$ which is defined by (2.3) is closed and then we have

$$\frac{\partial\gamma}{\partial t}(\theta,t)=G(k)kN-\frac{\partial}{\partial\theta}(G(k)k)\xi-(a(0,t),b(0,t)),\tag{2.4}$$

where

$$(a(0,t),b(0,t))=\left(-\frac{\partial}{\partial\theta}(G(k)k)\Big|_{\theta=0},G(k)k|_{\theta=0}\right).$$

Set

$$\begin{aligned}&\theta=\theta(u,\tau),\quad t=\tau,\\&\hat\gamma(u,\tau)=\gamma(\theta(u,\tau),\tau)+\left(\int_0^\tau a(0,t)dt,\int_0^\tau b(0,t)dt\right),\end{aligned}\tag{2.5}$$

where $\theta=\theta(u,\tau)$ is the unique solution of the following ordinary equation

$$\begin{cases}\dfrac{\partial\theta}{\partial\tau}=k\dfrac{\partial}{\partial\theta}(G(k)k),\quad \tau\in[0,T),\\ \theta(u,0)=u.\end{cases}$$

Combining (2.4) with(2.5), we know that $\hat\gamma(u,\tau)$ satisfies

$$\frac{\partial\hat\gamma}{\partial\tau}=G(k)kN$$

and hence we obtain the general curve shortening flow (1.3). □

Using the standard results on parabolic equations (cf. [9]), we obtain the existence results of the problem (2.2).

Lemma 2.5 *Suppose* $k_0(\theta) \in C(\mathbb{S}^1)$ *and* (H1) *holds. Then there exists*

$$T > 0, \quad k \in C^{2,1}(\mathbb{S}^1 \times (0,T)) \cap C(\mathbb{S}^1 \times [0,T)),$$

satisfying (2.2).

By the maximum principle it shows that if the initial curves are strictly convex, then the curves remain so under the flow (1.3).

Lemma 2.6 *Suppose* (H1) *holds and* $k(\theta,t)$ *is the classical solution of* (2.2). *If* $k_0(\theta)$ *is positive on* $\mathbb{S}^1$, *then* $k_{\min}(t) = \inf\{k(\theta,t) \mid \theta \in \mathbb{S}^1\}$ *is a nondecreasing function.*

Proof By contradiction, there exist $0 \le t_1' \le t_1 < T$ such that $k_{\min}(t)$ is nondecreasing in $[0,t_1']$ and $k_{\min}(t_1') > k_{\min}(t_1) > 0$. We suppose $t_1' = 0$ without loss of generality. Set

$$t_0 = \inf\{t \in [0,t_1] | k_{\min}(t) \le k_{\min}(t_1)\}.$$

By the regularity of $k(\theta,t)$, we know that there is $\theta_0 \in \mathbb{S}^1$ such that $k_{\min}(t_0) = k(\theta_0,t_0)$. It is easy to see that $k_{\min}(t_0) > 0$ and then we have

$$\frac{\partial^2 k}{\partial\theta^2}(\theta_0,t_0) \ge 0, \quad \frac{\partial k}{\partial\theta}(\theta_0,t_0) = 0, \quad k(\theta_0,t_0) > 0.$$

Hence from (2.2) and (H1) this yields

$$\frac{\partial k}{\partial t}(\theta_0,t)|_{t=t_0} > 0,$$

and it contradicts the hypothesis of t_0. So we obtain the desired results. □

Some further consequences of Lemmas 2.4–2.6 are part of Proposition 1.1.

Corollary 2.7 *Suppose* (H1) *holds. Then there exist* $T > 0$ *and the unique* $\gamma(u,t) \in C^{2,1}(\mathbb{S}^1 \times (0,T)) \cap C(\mathbb{S}^1 \times [0,T))$ *satisfying the generalized curve shortening flow* (1.3).

According to the flow (1.3), we consider the support function (cf. [10, 11]) of $\gamma(u,t)$ by defining

$$h(\theta,t) = \langle \gamma(u(\theta,t),t), (\sin\theta, -\cos\theta)\rangle, \quad \theta \in [0,2\pi],$$

where θ is the tangent angle of $\gamma(\cdot,t)$ and the unit normal vector $N = -(\sin\theta, -\cos\theta)$.

Applying the equation (1.3), we see

$$\frac{\partial h}{\partial t} = \left\langle \frac{\partial \gamma}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial \gamma}{\partial u}, -N \right\rangle = \left\langle G(k)kN + \frac{\partial u}{\partial t}\left|\frac{\partial \gamma}{\partial u}\right|\xi, -N \right\rangle = -G(k)k.$$

Using the methods in [11], we can compute the inward curvature of $\gamma(\cdot,t)$ by the formula

$$k = \left(\frac{\partial^2 h}{\partial\theta^2} + h\right)^{-1}.$$

Then $h(\theta,t)$ satisfies the following equation

$$\frac{\partial h}{\partial t}\left(\frac{\partial^2 h}{\partial\theta^2} + h\right) G^{-1}\left(\left(\frac{\partial^2 h}{\partial\theta^2} + h\right)^{-1}\right) = -1. \tag{2.6}$$

By making use of the maximum principle, we can obtain the containment principle of the flow (1.3) (cf. [12]).

Lemma 2.8 *Let γ_1 and $\gamma_2 : \mathbb{S}^1 \times [0,T)$ be two classical solutions of the flow* (1.3)*. If $\gamma_2(\cdot,0)$ is in the domain enclosed by $\gamma_1(\cdot,0)$, then $\gamma_2(\cdot,t)$ is contained in the domain enclosed by $\gamma_1(\cdot,t)$ for all $t \in [0,T)$.*

Proof Set $h_1(\theta,t)$ and $h_2(\theta,t)$ to be the support functions of $\gamma_1(\cdot,t)$ and $\gamma_2(\cdot,t)$. Then $h_1(\theta,t)$ and $h_2(\theta,t)$ satisfy the equation (2.6). Because $\gamma_2(\cdot,0)$ is in the domain enclosed by $\gamma_1(\cdot,0)$ we can select $h_1(\theta,0)$ and $h_2(\theta,0)$ such that $h_1(\theta,0) \geq h_2(\theta,0)$ for $\theta \in \mathbb{S}^1$. Thus by applying the maximum principle of parabolic equations, we deduce that $h_1(\theta,t) \geq h_2(\theta,t)$ for all $t \in [0,T)$ and then we obtain the desired results. □

In order to prove some isometric inequalities, we need the following lemma which belongs to Andrews [3, Lemma Ⅱ 3.3].

Lemma 2.9 *Let M be a compact manifold with a volume form $d\mu$, and let ξ be a continuous function on M. Then for any non-decreasing function F, there holds*

$$\frac{\int_M \xi F(\xi) d\mu}{\int_M F(\xi) d\mu} \geq \frac{\int_M \xi d\mu}{\int_M d\mu}.$$

The next two lemmas roughly characterize the behavior of the geometric quantity when $\gamma(\cdot,t)$ is contracting to a point under the flow (1.3).

Lemma 2.10 *Suppose* (H1) *hold. Then under the flow* (1.3)*, we have*

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) \leq 0.$$

Proof By Lemma 2.1 (ii) and (vi), i.e.,

$$\frac{dL}{dt} = -\int_0^L G(k)k^2 ds = -\int_0^{2\pi} G(k)k d\theta,$$
$$\frac{dA}{dt} = -\int_0^L G(k)k ds = -\int_0^{2\pi} G(k) d\theta,$$

we obtain

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) = -\frac{2L}{A}\left(\int_0^{2\pi} G(k)k d\theta - \frac{L}{2A}\int_0^{2\pi} G(k) d\theta\right). \tag{2.7}$$

From the isometric inequality in [15], the following inequality holds for convex curves,

$$\frac{\pi L}{A} \leq \int_0^L k^2 ds = \int_0^{2\pi} k d\theta, \tag{2.8}$$

Substituting (2.8) into (2.7), we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{L^2}{A}\right) &\leq -\frac{2L}{A}\left(\int_0^{2\pi} G(k)k d\theta - \frac{1}{2\pi}\int_0^{2\pi} G(k) d\theta \int_0^{2\pi} k d\theta\right) \\ &= -\frac{2L}{A}\int_0^{2\pi} G(k) d\theta \left(\frac{\int_0^{2\pi} G(k)k d\theta}{\int_0^{2\pi} G(k) d\theta} - \frac{\int_0^{2\pi} k d\theta}{\int_0^{2\pi} 1 d\theta}\right).\end{aligned} \tag{2.9}$$

Setting $M = \mathbb{S}^1$, $\psi = k, d\mu = d\theta$ in Lemma 2.9, one can show that

$$\frac{\int_0^{2\pi} G(k)k d\theta}{\int_0^{2\pi} G(k) d\theta} \geq \frac{\int_0^{2\pi} k d\theta}{\int_0^{2\pi} d\theta}, \tag{2.10}$$

so the proof is completed by means of (2.9) and (2.10). □

Lemma 2.11 *Suppose* (H1) *holds. Under the flow* (1.3)*, if*

$$\lim_{t\to\omega} A(t) = 0,$$

we have

$$\liminf_{t\to\omega} L\biggl(\int_0^L k^2 ds - \frac{\pi L}{A}\biggr) \le 0. \tag{2.11}$$

Proof From (2.7), we see

$$\begin{aligned}\frac{d}{dt}\left(\frac{L^2}{A}\right) &= -\frac{2L}{A}\left(\int_0^{2\pi} G(k)kd\theta - \frac{L}{2A}\int_0^{2\pi} G(k)d\theta\right)\\ &= -\frac{\int_0^{2\pi} G(k)d\theta}{\pi A}\cdot L\biggl(\frac{2\pi\int_0^{2\pi} G(k)kd\theta}{\int_0^{2\pi} G(k)d\theta} - \frac{\pi L}{A}\biggr).\end{aligned} \tag{2.12}$$

Applying

$$\frac{dA}{dt} = -\int_0^{2\pi} G(k)d\theta$$

to (2.12), we conclude that

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) = \frac{1}{\pi}\frac{d}{dt}(\ln A)\cdot L\biggl(\frac{2\pi\int_0^{2\pi} G(k)kd\theta}{\int_0^{2\pi} G(k)d\theta} - \frac{\pi L}{A}\biggr).$$

Using (2.10), we have

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) \le \frac{1}{\pi}\frac{d}{dt}(\ln A)\cdot L\left(\int_0^{2\pi} kd\theta - \frac{\pi L}{A}\right). \tag{2.13}$$

Now we prove (2.11) by contradiction. If not, there exists $\delta > 0$ such that if $\lim_{t\to\omega} A(t) = 0$ then

$$\liminf_{t\to\omega} L\biggl(\int_0^L k^2 ds - \frac{\pi L}{A}\biggr) \ge 2\delta.$$

Hence there exists $\beta = \beta(\delta) \in (0,\omega)$ such that if $t \in (\beta,\omega)$, then the following inequality holds

$$L\biggl(\int_0^L k^2 ds - \frac{\pi L}{A}\biggr) \ge \delta. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) \le \frac{\delta}{\pi}\frac{d}{dt}(\ln A), \quad t \in (\beta,\omega).$$

Integrating from β to t, we have

$$\begin{aligned}\frac{L^2}{A}(t) - \frac{L^2}{A}(\beta) &\le \frac{\delta}{\pi}\left(\ln A(t) - \ln A(\beta)\right),\\ -\frac{L^2}{A}(\beta) &\le \frac{\delta}{\pi}\left(\ln A(t) - \ln A(\beta)\right).\end{aligned}$$

Using $\lim_{t\to\omega} A(t) = 0$, one easily verifies that

$$-\frac{L^2(\beta)}{A(\beta)} + \frac{\delta}{\pi}\ln A(\beta) = -\infty,$$

and this contradicts Proposition 1.1 and then the proof of Lemma 2.11 is completed. □

3 Asymptotic Behavior

In this section we will study the asymptotic behavior of the curvature under the flow (1.3) and prove the main theorem of this paper.

We recall the following two auxiliary results, which belong to Gage [15, 16] and Osserman [17].

Lemma 3.1 (Gage) (a) *There is a non-negative functional $F(\gamma)$ which is defined for all C^2 convex curves and satisfies*

$$(1-F(\gamma))\int_0^L k^2 ds - \frac{\pi L}{A} \geq 0. \tag{3.1}$$

(b) *Given a sequence of convex curves $\{\gamma_i\}$ such that $\lim_{i\to\infty} F(\gamma_i)=0$. If these normalized curves $\eta_i=\sqrt{\frac{\pi}{A}}\gamma_i$ lie in a fixed bounded region of the plane, then the domain H_i, enclosed by η_i, converges to the disk in the Hausdorff metric.*

(c) $F(\gamma)=0$ *if and only if γ is a circle.*

Lemma 3.2 (Bonneson inequality) *Let γ be a C^1 closed convex curve. Then*

$$\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{\text{out}} - r_{\text{in}})^2. \tag{3.2}$$

Definition 3.3 *Let A, B be two closed convex sets and $A_\epsilon = \{x \in \mathbb{R}^2 \mid \text{dist}(x, A) \leq \epsilon\}$. Then the Hausdorff distance between the sets A and B is given by*

$$d_H(A,B) = \inf\{\epsilon \mid A \subseteq B_\epsilon, B \subseteq A_\epsilon\}.$$

Proof of Theorem 1.2 (i) We use the idea in [16]. Consider the geometric quantities $L(t)$ and $A(t)$ according to the flow $\gamma(\cdot,t)$ satisfying (1.3). By (3.1), we have

$$\int_0^{L(t)} k^2(\theta,t)ds - \frac{\pi L(t)}{A(t)} \geq F(\gamma(t))\int_0^{L(t)} k^2(\theta,t)ds. \tag{3.3}$$

Using Schwarz inequality, we see that

$$(2\pi)^2 = \left(\int_0^{L(t)} k(\theta,t)ds\right)^2 \leq \int_0^{L(t)} k^2(\theta,t)ds \int_0^{L(t)} 1ds = L(t)\int_0^{L(t)} k^2(\theta,t)ds. \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$L(t)\left(\int_0^{L(t)} k^2(\theta,t)ds - \frac{\pi L(t)}{A(t)}\right) \geq F(\gamma)L(t)\int_0^{L(t)} k^2(\theta,t)ds \geq 4\pi^2 F(\gamma(t)). \tag{3.5}$$

In the following steps we will show that if $\forall\, t_i \in [0,\omega)$ it satisfies $\lim_{i\to+\infty} t_i = \omega$, then the limitations in Theorem 1.2 (i), (ii) hold.

By substituting $\gamma_i = \gamma(\cdot, t_i)$, $A(t_i)$, $L(t_i)$ into (3.5) and using (2.11), we have

$$\lim_{i\to+\infty} F(\gamma_i) = 0.$$

Next we show that the normalized curve $\eta_i = \sqrt{\frac{\pi}{A}}\gamma_i$ lies in a bound region. From Lemma 2.10, we observe that $\frac{L^2}{A}$ decreases under the flow (1.3). One easily verifies that

$$\frac{L^2(t_i)}{A(t_i)} = \frac{L_n^2(t_i)}{A_n(t_i)},$$

and

$$\frac{L^2(t_i)}{A(t_i)} - 4\pi = \frac{L_n^2(t_i)}{A_n(t_i)} - 4\pi \geq \frac{\pi^2}{A_n(t_i)}(r_{n,\text{out}}(t_i) - r_{n,\text{in}}(t_i))^2$$
$$= \pi(r_{n,\text{out}}(t_i) - r_{n,\text{in}}(t_i))^2, \tag{3.6}$$

and $r_{n,\text{in}}(t_i) \leq 1$, by using the Bonneson inequality, where $A_n(t_i)$ is the area about the bounded domain enclosed by the normalized curve η_i, and $L_n(t_i)$ is the perimeter of the curve η_i, $r_{n,\text{out}}(t_i)$ and $r_{n,\text{in}}(t_i)$ are respectively the radii of the largest and smallest circumscribed circles of the curve η_i. By (3.6) it shows that the outer radii of the normalized curve η_i are bounded for all $t_i \in [0, \omega)$. From Proposition 1.1 we know that γ_i shrinks to a point under the flow (1.3). Hence if we use ϑ as the origin in the homothetic expansion of $\mathbb{R}^2$, then all of the normalized curves η_i lie in a ball of radius 2C around this point.

Applying Lemma 3.1 (b), we see that the sequence of normalized domain $H(t_i)$ according to η_i converges to the unit disk in the Haudorff metric,

$$\lim_{i\to+\infty} H(t_i) = H_0, \tag{3.7}$$

where H_0 is the unit disk in the plane.

Since L and A are continuous functionals of convex domain then there holds

$$\lim_{i\to+\infty} \frac{L^2(t_i)}{A(t_i)} = \lim_{i\to+\infty} \frac{L_n^2(t_i)}{A_n(t_i)} = \lim_{i\to+\infty} \frac{L^2(H(t_i))}{A(H(t_i))} = \frac{L^2(H_0)}{A(H_0)} = 4\pi. \tag{3.8}$$

From (3.2), we have

$$\frac{L^2(t_i)}{A(t_i)} - 4\pi \geq \frac{\pi^2}{A(t_i)}(r_{\text{out}}(t_i) - r_{\text{in}}(t_i))^2 \geq \frac{\pi^2 r_{\text{out}}^2(t_i)}{A(t_i)}\left(1 - \frac{r_{\text{in}}(t_i)}{r_{\text{out}}(t_i)}\right)^2.$$

It is easy to see that $\pi r_{\text{out}}^2(t_i) \geq A(t_i)$, and then

$$\frac{L^2(t_i)}{A(t_i)} - 4\pi \geq \left(1 - \frac{r_{\text{in}}(t_i)}{r_{\text{out}}(t_i)}\right)^2,$$

such that combining this with (3.8), we have

$$\lim_{i\to+\infty} \frac{r_{\text{in}}(t_i)}{r_{\text{out}}(t_i)} = 1. \tag{3.9}$$

To prove Theorem 2.1 (ii), (iii), we need the following gradient estimates of the curvature. A similar proof can be found in [12].

Lemma 3.4 *Set $\Phi(k) = G(k)k$ and let $k = k(\theta, t)$ be the curvature function of the flow* (1.3), *where θ is the tangent angle of the curve $\gamma(\cdot, t)$. Suppose* (H1), (H2) *hold and $\varpi \in (0, \omega)$. Then the following inequality holds*:

$$\max_{0\leq t\leq\varpi, \theta\in\mathbb{S}^1} \left|\frac{\partial\Phi}{\partial\theta}\right|^2 \leq \max\left\{2\max_{0\leq t\leq\varpi, \theta\in\mathbb{S}^1} \Phi^2, \max_{t=0, \theta\in\mathbb{S}^1}\left(\left|\frac{\partial\Phi}{\partial\theta}\right|^2 + 2\Phi^2\right)\right\}. \tag{3.10}$$

Proof By Lemma 2.6, we know that $k_{\min}(t) > 0$ for $t \in [0, \omega)$. It follows from Lemma 2.3 that $\phi = \Phi(k)$ satisfies the following equation

$$\frac{\partial\Phi}{\partial t} = k^2\Phi'\frac{\partial^2\Phi}{\partial\theta^2} + k^2\Phi'\Phi. \tag{3.11}$$

Set

$$\Psi = \left(\frac{\partial \Phi}{\partial \theta}\right)^2 + \lambda \Phi^2,$$

where λ is a constant to be determined. Suppose $(\theta_0, t_0) \in \mathbb{S}^1 \times (0, \varpi]$ such that

$$\Psi(\theta_0, t_0) = \max_{\mathbb{S}^1 \times [0,\varpi]} \left(\left(\frac{\partial \Phi}{\partial \theta}\right)^2 + \lambda \Phi^2\right).$$

Then at (θ_0, t_0), Ψ satisfies the following properties

$$\frac{\partial \Psi}{\partial \theta} = 0, \quad \frac{\partial^2 \Psi}{\partial \theta^2} \leq 0, \quad \frac{\partial \Psi}{\partial t} \geq 0. \tag{3.12}$$

Next we will prove that if we select some constant λ so large, then at (θ_0, t_0) we have

$$\frac{\partial \Phi}{\partial \theta} = 0. \tag{3.13}$$

Suppose not, then using

$$0 = \frac{\partial \Psi}{\partial \theta} = 2\frac{\partial \Phi}{\partial \theta}\left(\frac{\partial^2 \Phi}{\partial \theta^2} + \lambda \Phi\right)$$

we see

$$0 = \frac{\partial^2 \Phi}{\partial \theta^2} + \lambda \Phi. \tag{3.14}$$

From (3.11) and (3.12), we have

$$\begin{aligned}
0 &\leq \frac{1}{2}\frac{\partial \Psi}{\partial t} \\
&= \frac{\partial \Phi}{\partial \theta}\frac{\partial^2 \Phi}{\partial \theta \partial t} + \lambda \Phi \frac{\partial \Phi}{\partial t} \\
&= \frac{\partial}{\partial \theta}(\Phi' k^2)\frac{\partial \Phi}{\partial \theta}\frac{\partial^2 \Phi}{\partial \theta^2} + \Phi' k^2 \frac{\partial \Phi}{\partial \theta}\frac{\partial^3 \Phi}{\partial \theta^3} + \frac{\partial}{\partial \theta}(\Phi' k^2)\Phi\frac{\partial \Phi}{\partial \theta} \\
&\quad + k^2\Phi'(k)\left(\frac{\partial \Phi}{\partial \theta}\right)^2 + \lambda \Phi \Phi' k^2 \frac{\partial^2 \Phi}{\partial \theta^2} + \lambda k^2 \Phi' \Phi^2.
\end{aligned} \tag{3.15}$$

By $\frac{\partial^2 \Psi}{\partial \theta^2} \leq 0$, we see that

$$0 \geq \frac{\partial \Phi}{\partial \theta}\left(\frac{\partial^3 \Phi}{\partial \theta^3} + \lambda \frac{\partial \Phi}{\partial \theta}\right). \tag{3.16}$$

Substituting (3.14), (3.16) into (3.15), we obtain

$$\begin{aligned}
0 &\leq -\lambda \frac{\partial}{\partial \theta}(\Phi' k^2)\Phi\frac{\partial \Phi}{\partial \theta} - \lambda k^2 \Phi'\left(\frac{\partial \Phi}{\partial \theta}\right)^2 + \frac{\partial}{\partial \theta}(\Phi' k^2)\Phi \frac{\partial \Phi}{\partial \theta} \\
&\quad + k^2 \Phi'\left(\frac{\partial \Phi}{\partial \theta}\right)^2 - \lambda^2 k^2 \Phi^2 \Phi' + \lambda k^2 \Phi' \Phi^2 \\
&= (1-\lambda)\frac{\partial}{\partial \theta}(\Phi' k^2)\Phi\frac{\partial \Phi}{\partial \theta} + (1-\lambda)\Phi' k^2\left(\frac{\partial \Phi}{\partial \theta}\right)^2 + (\lambda - \lambda^2)k^2 \Phi' \Phi^2.
\end{aligned} \tag{3.17}$$

By the definition of Φ and (H1), (H2), we have

$$\begin{aligned}
&\Phi > 0, \qquad \Phi' > 0, \\
&(\Phi' k^2)' = G''(k)k^3 + 4k^2 G'(k) + 2G(k)k = k(G(k)k^2)'' \geq 0, \\
&\frac{\partial}{\partial \theta}(\Phi' k^2)\Phi\frac{\partial \Phi}{\partial \theta} = (\Phi' k^2)'\left(\frac{\partial k}{\partial \theta}\right)^2 \Phi' \Phi \geq 0,
\end{aligned}$$

$$\Phi' k^2 \Phi \left(\frac{\partial \Phi}{\partial \theta}\right)^2 > 0, \quad k^2 \Phi' \Phi^2 > 0. \tag{3.18}$$

By selecting $\lambda = 2$ and hence substituting it into (3.17) and using (3.18), we obtain the contradiction. So (3.13) holds.

By (3.13), we arrive at

$$\begin{aligned}\max_{0\le t\le\varpi,\theta\in\mathbb{S}^1}\left|\frac{\partial\Phi}{\partial\theta}\right|^2 &\le \max_{\mathbb{S}^1\times[0,\varpi]}\left(\left(\frac{\partial\Phi}{\partial\theta}\right)^2+\lambda\Phi^2\right)\\ &= \Psi(\theta_0,t_0)\\ &= 2\Phi^2|_{\theta=\theta_0,t=t_0}\\ &\le \max\left\{2\max_{0\le t\le\varpi,\theta\in\mathbb{S}^1}\Phi^2,\ \max_{t=0,\theta\in\mathbb{S}^1}\left(\left|\frac{\partial\Phi}{\partial\theta}\right|^2+2\Phi^2\right)\right\}.\end{aligned}$$

Thus the proof is completed. □

Lemma 3.5 *Let $q(t)$ be continuous function on $[0,\omega)$. Suppose for each $\varpi \in [0,\omega)$, one has*

$$\sup_{0\le t\le\varpi} q(t) < +\infty, \quad \lim_{t\to\omega} q(t) = +\infty.$$

Then there exists $\{t_i\} \subset [0,\omega)$ satisfying

$$\begin{aligned}&\forall\, i \in \{1,2,\dots\}, t_i < t_{i+1}, \lim_{i\to+\infty} t_i = \omega,\\ &q(t_i) = \sup_{0\le t\le t_i} q(t).\end{aligned} \tag{3.19}$$

Proof Consider the sequence $\{t_i'\} \triangleq \{T - \frac{T}{i+1}\}$. Firstly select $t_1 \in [0, t_1']$ satisfying

$$q(t_1) = \sup_{0\le t\le t_1'} q(t).$$

Then

$$q(t_1) = \sup_{0\le t\le t_1} q(t).$$

It follows from $\lim_{t\to\omega} q(t) = +\infty$ that we can choose $t''_{j_1} \in \{t_i'\}$ satisfying $q(t''_{j_1}) > q(t_1) + 2$, so that we can take $t_2 \in [0, t''_{j_1}]$ satisfying

$$q(t_2) = \sup_{0\le t\le t''_{j_1}} q(t).$$

Thus

$$q(t_2) = \sup_{0\le t\le t_2} q(t).$$

In general we can select $t''_{j_n} \in \{t_i'\}$ satisfying $q(t''_{j_n}) > q(t_n)+n+1$ and then choose $t_{n+1} \in [0, t''_{j_n}]$ satisfying

$$q(t_{n+1}) = \sup_{0\le t\le t''_{j_n}} q(t).$$

Then there holds

$$q(t_{n+1}) = \sup_{0\le t\le t_{n+1}} q(t).$$

The desired result follows by taking trace. □

Set $q(t) = k_{\max}(t)$. Then by Proposition 1.1 it is easy to verify that $k_{\max}(t)$ satisfies the conditions of Lemma 3.5.

Lemma 3.6 *Suppose* (H1) *and* (H2) *hold. If we take the sequence* $\{t_i\}$ *satisfying* (3.19) *such that the following holds*

$$\forall\, i \in \{1, 2, \dots\}, \quad k_{\max}(t_i) = \sup_{0 \le t \le t_i} k_{\max}(t)$$

and for each $i \in \{1, 2, \dots\}$ *there exists* $\theta_{i0} \in \mathrm{S}^1$ *such that* $k_{\max}(t_i) = k(\theta_{i0}, t_i)$, *then there exist constants* $C, C_1 > 0$ *depending only on* γ_0 *such that*

$$(1 - 2|\theta - \theta_{i0}|)\Phi(k_{\max}(t_i)) \le \Phi(k(\theta, t_i)) + C, \quad \forall\, \theta \in \mathbb{S}^1, \tag{3.20}$$

$$\Phi(k_{\max}(t_i)) \le C_1 \Phi(k(\theta, t_i)), \quad \forall\, \theta \in \mathbb{S}^1, \tag{3.21}$$

where $\Phi(x) = G(x)x$.

Proof **Step 1** Set $\Phi(\theta) = \Phi(k(\theta, t_i))$. For each $\theta \in \mathbb{S}^1$ by the medium theorem and (3.10), we have

$$\Phi(\theta_{i0}) - \Phi(\theta) = \frac{\partial \Phi}{\partial \theta}(\hat{\theta})(\theta_{i0} - \theta) \le (2\Phi(\theta_{i0}) + C)|\theta_{i0} - \theta| \le 2\Phi(\theta_{i0})|\theta_{i0} - \theta| + C. \tag{3.22}$$

This yields the inequality (3.20).

Step 2 Take i such that $\Phi(\theta_{i0})$ is large sufficiently. Being likely with (3.22), $\forall\, \theta_1, \theta \in \mathbb{S}^1$, we obtain

$$\Phi(\theta_1) - \Phi(\theta) \le 3\Phi(\theta_{i0})|\theta_1 - \theta|. \tag{3.23}$$

It is clear that $\Phi(\theta) = \Phi(k(\theta, t_i)) \ge \Phi(k_{\min}(0)) > 0$. Let $\theta_1 = \theta_{i0}$. If $|\theta - \theta_{i0}| \le \frac{1}{6}$, then from (3.23), we have

$$\Phi(\theta_{i0}) - \Phi(\theta) \le \frac{1}{2}\Phi(\theta_{i0}).$$

So

$$\frac{1}{2}\Phi(\theta_{i0}) \le \Phi(\theta).$$

Assume a to be the maximal constant such that if $|\theta - \theta_{i0}| \le a$, then

$$C\Phi(\theta_{i0}) \le \Phi(\theta), \tag{3.24}$$

for a suitable positive constant C under control. We will prove that $a \ge \pi$. If not, we take $\theta_1 = \theta_{i0} + a$ or $\theta_1 = \theta_{i0} - a$. It follows from (3.23) and (3.24) that $\forall\, \theta \in \mathbb{S}^1$, we have

$$\Phi(\theta_1) - \Phi(\theta) \le \frac{3}{C}\Phi(\theta_1)|\theta_1 - \theta|.$$

If $|\theta - \theta_1| \le \frac{C}{6}$, then

$$\frac{1}{2}\Phi(\theta_1) \le \Phi(\theta).$$

Combining with (3.24), if $|\theta - \theta_{i0}| \le a + \frac{C}{6}$, then

$$\frac{C}{2}\Phi(\theta_{i0}) \le \Phi(\theta).$$

It is a contradiction to the assumption of the constant a. By $\Phi(\theta + 2\pi) = \Phi(\theta)$, we obtain the desired results. □

Set

$$\hat{k_\sigma}(t_i) = \sup\Big\{\inf_{[a,b]} k(\theta, t_i) | [a,b] \subset (-\infty, +\infty), b-a=\sigma\Big\}.$$

We introduce a lemma of Gage and Hamilton [2, Lemma 5.1], which is crucial for studying the asymptotic behavior of the curvature under the general curve shortening flow.

Lemma 3.7

$$\hat{k}_\sigma(t_i) r_{\rm in}(t_i) \le \frac{1}{1-\Lambda(\sigma)\big(\frac{r_{\rm out}(t_i)}{r_{\rm in}(t_i)}-1\big)},$$

where

$$\Lambda(\sigma) = \frac{2\cos\frac{\sigma}{2}}{1-\cos\frac{\sigma}{2}}.$$

Remark 3.8 The proof of Lemma 5.1 in [2] follows only from the convexity of the closed curve $\gamma(\cdot, t_i)$.

Corollary 3.9 *Suppose* (H1) *and* (H2) *hold. Consider the sequence* $\{t_i\}$ *satisfying the conditions of Lemma* 3.6*. Then for the positive* ϵ *being small sufficiently, we have*

$$k_{\max}(t_i) r_{\rm in}(t_i) \le \frac{2}{1-\epsilon}\cdot \frac{1}{1-\Lambda(\epsilon)\big(\frac{r_{\rm out}(t_i)}{r_{\rm in}(t_i)}-1\big)}. \tag{3.25}$$

Proof It follows from (3.20) that

$$\begin{aligned}(1-2|\theta-\theta_{i0}|)G(k_{\max}(t_i))k_{\max}(t_i) &\le G(k(\theta,t_i))k(\theta,t_i) + C \\ &\le G(k_{\max}(t_i))k(\theta,t_i)+C, \quad \forall\,\theta\in\mathbb{S}^1,\end{aligned}$$

so that

$$(1-2|\theta-\theta_{i0}|)k_{\max}(t_i) \le k(\theta,t_i)+C, \quad \forall\,\theta\in\mathbb{S}^1. \tag{3.26}$$

By Proposition 1.1, we have

$$\lim_{i\to+\infty} k_{\max}(t_i) = +\infty.$$

Hence from (3.21) and (H1), one can easily verify that

$$\lim_{i\to+\infty} k(\theta,t_i) = +\infty, \quad \forall\,\theta\in\mathbb{S}^1.$$

Combining this with (3.26), we obtain

$$(1-2|\theta-\theta_{i0}|)k_{\max}(t_i) \le 2k(\theta,t_i)$$

for i being large enough, and $\forall\,\theta\in\mathbb{S}^1$. Given any $\epsilon>0$, if $|\theta-\theta_{i0}|\le\frac{\epsilon}{2}$, then

$$2k(\theta,t_i) \ge k_{\max}(t_i)(1-\epsilon).$$

Take $\sigma=\epsilon$. It follows from the definition of $\hat{k_\sigma}(t_i)$ that we have

$$2\hat{k}_\sigma(t_i) \ge k_{\max}(t_i)(1-\epsilon).$$

Then using Lemma 3.7, we obtain

$$k_{\max}(t_i)r_{\rm in}(t_i)(1-\epsilon) \le 2\hat{k}_\sigma(t_i)r_{\rm in}(t_i) \le \frac{2}{1-\Lambda(\epsilon)\big(\frac{r_{\rm out}(t_i)}{r_{\rm in}(t_i)}-1\big)}.$$

This yields the desired results. □

Corollary 3.10 *Suppose* (H1) *and* (H2) *hold. Consider the sequence* $\{t_i\}$ *satisfying the conditions of Lemma* 3.6. *Then for the positive* ϵ *being small sufficiently, there exists* $i(\epsilon) \in \mathbb{N}$, *such that if* $i > i(\epsilon)$, *we have*

$$k_{\max}(t_i)r_{\text{in}}(t_i) \leq \frac{2}{(1-\epsilon)^2}.$$

Proof By Theorem 1.2 (ii),

$$\lim_{i\to+\infty} \frac{r_{\text{in}}(t_i)}{r_{\text{out}}(t_i)} = 1.$$

For i being so large, we have

$$1 - \Lambda(\epsilon)\left(\frac{r_{\text{out}}(t_i)}{r_{\text{in}}(t_i)} - 1\right) \geq 1 - \epsilon. \tag{3.27}$$

Then substituting (3.27) into (3.25), we obtain the desired results. □

Theorem 3.11 *Suppose* (H1) *and* (H2) *hold. Consider the sequence* $\{t_i\}$ *satisfying the conditions of Lemma* 3.6. *Then we have*

$$\lim_{i\to+\infty} k(\theta, t_i)r_{\text{in}}(t_i) = 1, \quad \forall\, \theta \in \mathbb{S}^1. \tag{3.28}$$

Proof Set $f_i(\theta) = k(\theta, t_i)r_{\text{in}}(t_i)$ and $\Phi(x) = G(x)x$.

Step 1 We will prove that $f_i(\theta)$ is equi-continuous and bounded uniformly.

Because $G(x)$ is non-decreasing function for $x \in (0, +\infty)$, then by Lemma 3.4, for $\theta \in \mathbb{S}^1$, we arrive at

$$\begin{aligned}\left|G(k(\theta,t_i))\frac{\partial k}{\partial\theta}(\theta,t_i)\right| &\leq \left|\frac{\partial\Phi}{\partial\theta}(\theta,t_i)\right| \\ &\leq \max_{0\leq t\leq t_i,\theta\in\mathbb{S}^1}\left|\frac{\partial\Phi}{\partial\theta}\right| \\ &\leq 2\max_{0\leq t\leq t_i,\theta\in\mathbb{S}^1}|\Phi| + C \\ &= 2\Phi(k_{\max}(t_i)) + C,\end{aligned}$$

so that

$$\Phi(k(\theta,t_i))\left|\frac{\partial k}{\partial\theta}(\theta,t_i)\right| = \left|G(k(\theta,t_i))k(\theta,t_i)\frac{\partial k}{\partial\theta}(\theta,t_i)\right| \leq 2\Phi(k_{\max}(t_i))k_{\max}(t_i) + Ck_{\max}(t_i),$$

$$\left|\frac{\partial k}{\partial\theta}(\theta,t_i)\right| \leq 2\frac{\Phi(k_{\max}(t_i))}{\Phi(k(\theta,t_i))}k_{\max}(t_i) + C\frac{k_{\max}(t_i)}{\Phi(k(\theta,t_i))} \leq 2\frac{\Phi(k_{\max}(t_i))}{\Phi(k(\theta,t_i))}k_{\max}(t_i) + C\frac{k_{\max}(t_i)}{\Phi(k_{\min}(0))}.$$

It follows from (3.21) that

$$\left|\frac{\partial k}{\partial\theta}(\theta,t_i)\right| \leq (2C + C)k_{\max}(t_i), \quad \forall\, \theta \in \mathbb{S}^1.$$

By Corollary 3.10, we obtain

$$\left|\frac{\partial k}{\partial\theta}(\theta,t_i)r_{\text{in}}(t_i)\right| \leq (2C + C)k_{\max}(t_i)r_{\text{in}}(t_i) \leq C$$

for i being so large. This yields

$$\left|\frac{df_i}{d\theta}(\theta)\right| \leq C.$$

On the other hand, by Corollary 3.10,

$$|f_i(\theta)| \le C.$$

The proof of Step 1 is completed.

Step 2 Because $f_i(\theta)$ is equi-continuous and bounded uniformly, then by Ascoli–Arzela theorem, there exists $f(\theta) \in C(\mathbb{S}^1)$ such that

$$\lim_{i\to+\infty} f_i(\theta) = f(\theta), \quad \forall\, \theta \in \mathbb{S}^1. \tag{3.29}$$

Step 3 We will prove that $f(\theta) \le 1$, $\forall\, \theta \in \mathbb{S}^1$.

Suppose the assertion is false. Then there exists $\theta_0 \in \mathbb{S}^1, \beta > 0$, such that $f(\theta_0) \ge 1 + 3\beta$. Hence there exists also $\delta > 0$, such that if $\theta \in [\theta_0 - \delta, \theta_0 - \delta]$, we have

$$f(\theta) \ge 1 + 2\beta.$$

By (3.29) for i being so large, we have

$$f_i(\theta) \ge 1 + \beta, \quad \forall\, \theta \in [\theta_0 - \delta, \theta_0 - \delta],$$

i.e.,

$$k(\theta, t_i) r_{\rm in}(t_i) \ge 1 + \beta, \quad \forall\, \theta \in [\theta_0 - \delta, \theta_0 - \delta].$$

Take $\sigma = 2\delta$. Then according to the definition of $\hat{k}_\sigma(t_i)$, we obtain

$$1 + \beta \le \hat{k}_{2\delta}(t_i) r_{\rm in}(t_i) \le \frac{1}{1 - \Lambda(2\delta)\big(\frac{r_{\rm out}(t_i)}{r_{\rm in}(t_i)} - 1\big)}.$$

Then using

$$\lim_{i\to+\infty} \frac{r_{\rm in}(t_i)}{r_{\rm out}(t_i)} = 1.$$

we have

$$1 + \beta \le 1,$$

and it is impossible. So $f(\theta) \le 1$, $\forall\, \theta \in \mathbb{S}^1$.

Step 4 We will prove that $f(\theta) \equiv 1$.

By Fatou lemma, we have

$$\int_0^{2\pi} \frac{d\theta}{f(\theta)} \le \liminf_{i\to+\infty} \int_0^{2\pi} \frac{d\theta}{f_i(\theta)} = \liminf_{i\to+\infty} \int_0^{2\pi} \frac{d\theta}{k(\theta, t_i) r_{\rm in}(t_i)}. \tag{3.30}$$

By the convexity of $\gamma(\cdot, t_i)$, it is easy to verify that

$$L(t_i) = \int_0^{2\pi} \frac{d\theta}{k(\theta, t_i)},$$

and substitute it into (3.30), we obtain

$$\int_0^{2\pi} \frac{d\theta}{f(\theta)} \le \liminf_{i\to+\infty} \frac{L(t_i)}{r_{\rm in}(t_i)} = \liminf_{i\to+\infty} \frac{L(t_i)}{r_{\rm out}(t_i)} \cdot \frac{r_{\rm out}(t_i)}{r_{\rm in}(t_i)}. \tag{3.31}$$

By the geometric property of $r_{\rm out}$ one can easily verify that $2\pi r_{\rm out} \ge L$. Then combining this with (3.31), we have

$$\int_0^{2\pi} \frac{d\theta}{f(\theta)} \le \liminf_{i\to+\infty} \frac{L(t_i)}{r_{\rm in}(t_i)} \le 2\pi \cdot \liminf_{i\to+\infty} \frac{r_{\rm out}(t_i)}{r_{\rm in}(t_i)}.$$

By making use of Theorem 1.2 (i) again, we have

$$\int_0^{2\pi} \frac{d\theta}{f(\theta)} \le 2\pi.$$

On the other hand, by $f(\theta) \le 1$ in Step 3, we obtain

$$\int_0^{2\pi} \frac{d\theta}{f(\theta)} \ge 2\pi.$$

This yields

$$\int_0^{2\pi} \frac{d\theta}{f(\theta)} = 2\pi.$$

Using $f(\theta) \le 1$ again, we have $f(\theta) \equiv 1$.

Combining (3.29) with Step 4 we complete the proof of the theorem. □

Remark 3.12 It follows from Cauchy criterion that the following limitation holds:

$$\lim_{t\to\omega} k(\theta,t) r_{\rm in}(t) = 1, \quad \forall\, \theta \in \mathbb{S}^1. \tag{3.32}$$

Proof of Theorem 1.2 (ii), (iii) **Step 1** According to (3.31) we conclude that

$$\lim_{t\to\omega} k_{\max}(t) r_{\rm in}(t) = 1, \quad \lim_{t\to\omega} k_{\min}(t) r_{\rm in}(t) = 1.$$

Combining this with Theorem 1.2 (ii) it shows that

$$\lim_{t\to\omega} \frac{k_{\max}(t)}{k_{\min}(t)} = 1. \tag{3.33}$$

Step 2 Given $t \in (0,\omega)$, consider $k_{\max}(t)$. By the property of continuous function, there exists $\theta = \theta(t) \in \mathbb{S}^1$ such that $k_{\max}(t) = k(\theta(t),t)$. Then at $(\theta(t),t)$ by the regularity of $k(\theta,t)$, we have

$$\frac{\partial k}{\partial \theta} = 0, \quad \frac{\partial^2 k}{\partial \theta^2} \le 0, \quad \frac{dk}{dt} = \frac{\partial k}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial k}{\partial t} = \frac{\partial k}{\partial t}. \tag{3.34}$$

By Lemma 2.3,

$$\frac{\partial k}{\partial t} = k^2 \left(\frac{\partial^2}{\partial \theta^2}(G(k)k) + G(k)k \right). \tag{3.35}$$

Combining (3.34) with (3.35), we see

$$\frac{dk_{\max}(t)}{dt} \le G(k_{\max}(t)) k_{\max}^3(t).$$

By the differential inequality and using $k_{\max}(\omega) = +\infty$, we get

$$\frac{1}{\omega - t} \int_{k_{\max}(t)}^{+\infty} \frac{dx}{G(x)x^3} \le 1. \tag{3.36}$$

From (3.21) it is easy to see that $k_{\min}(\omega) = +\infty$. Similarly, we have

$$\frac{1}{\omega - t} \int_{k_{\min}(t)}^{+\infty} \frac{dx}{G(x)x^3} \ge 1. \tag{3.37}$$

Since

$$\begin{aligned} \frac{\int_{k_{\min}(t)}^{k_{\max}(t)} \frac{dx}{G(x)x^3}}{\int_{k_{\min}(t)}^{+\infty} \frac{dx}{G(x)x^3}} &\le \frac{\frac{k_{\max}(t)-k_{\min}(t)}{G(k_{\min}(t))k_{\min}^3(t)}}{\frac{1}{2G(k_{\min}(t))k_{\min}^2(t)} - \frac{1}{2}\int_{k_{\min}(t)}^{+\infty} \frac{G'(x)}{G^2(x)x^2}dx} \\ &= 2\left(\frac{k_{\max}(t)}{k_{\min}(t)} - 1\right) \frac{1}{1 - G(k_{\min}(t))k_{\min}^2(t)\int_{k_{\min}(t)}^{+\infty} \frac{G'(x)}{G^2(x)x^2}dx}, \end{aligned} \tag{3.38}$$

we now claim that for the positive z being so large, there holds

$$1 - G(z)z^2 \int_z^{+\infty} \frac{G'(x)}{G^2(x)x^2} dx \geq \frac{2}{2 + C_0}. \tag{3.39}$$

Indeed, by (H2), we obtain

$$\int_z^{+\infty} \frac{G'(x)}{G^2(x)x^2} dx \leq C_0 \int_z^{+\infty} \frac{dx}{G(x)x^3}.$$

Then

$$\int_z^{+\infty} \frac{G'(x)}{G^2(x)x^2} dx \leq \frac{C_0}{C_0 + 2} \int_z^{+\infty} \frac{G'(x)}{G(x)x^2} dx + \frac{2C_0}{C_0 + 2} \int_z^{+\infty} \frac{dx}{G(x)x^3},$$

$$\int_z^{+\infty} \frac{G'(x)}{G^2(x)x^2} dx \leq \frac{-C_0}{C_0 + 2} \int_z^{+\infty} \left(\frac{1}{G(x)x^2} \right)' dx = \frac{C_0}{C_0 + 2} \cdot \frac{1}{G(z)z^2},$$

and this yields (3.39).

By (3.33), (3.38), (3.39) and applying $k_{\min}(\omega) = +\infty$, we have

$$\lim_{t \to \omega} \frac{\int_{k_{\min}(t)}^{k_{\max}(t)} \frac{dx}{G(x)x^3}}{\int_{k_{\min}(t)}^{+\infty} \frac{dx}{G(x)x^3}} = 0.$$

Then we obtain

$$\lim_{t \to \omega} \frac{\int_{k_{\max}(t)}^{+\infty} \frac{dx}{G(x)x^3}}{\int_{k_{\min}(t)}^{+\infty} \frac{dx}{G(x)x^3}} = \lim_{t \to \omega} \frac{\int_{k_{\min}(t)}^{+\infty} \frac{dx}{G(x)x^3} - \int_{k_{\min}(t)}^{k_{\max}(t)} \frac{dx}{G(x)x^3}}{\int_{k_{\min}(t)}^{+\infty} \frac{dx}{G(x)x^3}} = 1. \tag{3.40}$$

Combining (3.36), (3.37) with (3.40) we arrive at

$$\lim_{t \to \omega} \frac{1}{\omega - t} \int_{k(\theta,t)}^{+\infty} \frac{dx}{G(x)x^3} = 1, \quad \forall\, \theta \in \mathbb{S}^1. \qquad \square$$

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