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The Dirichlet Problem for the Degenerate Elliptic Monge–Ampère Equation*

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The existence and uniqueness of the global $C^{1,1/3}$ solution to the Dirichlet problem for the degenerate elliptic Monge–Ampère equation are proved, under mild conditions, and the application to the equation of the prescribed nonnegative Gauss curvature is also given. © 1999 Academic Press

1. INTRODUCTION

The present paper is devoted to the Dirichlet problems for the degenerate elliptic Monge–Ampère equations of the form

$$\det D^2 u = f(x, u, Du), \qquad \text{in } \Omega \tag{1.1}$$

$$u = \varphi(x), \qquad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a strictly convex domain in $\mathbf{R}^n$ with the boundary $\partial\Omega \in C^{2,1}$, $\varphi \in C^{2,1}(\partial\Omega)$, $0 \le f \in C^{1,1}(\Omega' \times \mathbf{R}^1 \times \mathbf{R}^n)$, Ω' is some neighborhood of Ω, and $Du = (D_i u)$ and $D^2 u = [D_{ij} u]$ are, respectively, the gradient vector and Hessian matrix of the function u.

The Dirichlet problem (1.1), (1.2) has received considerable study in both the nondegenerate case ($f > 0$) and the degenerate case ($f \ge 0$).

For the nondegenerate case, Caffarelli et al. [1] and Krylov [2] independently proved the well-known theorem: (1.1), (1.2) has a strictly convex solution in $C^\infty(\overline{\Omega})$ if $\partial\Omega$, φ, and f are smooth. In the case where

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$f \in C^{1,1}(\overline{\Omega})$ is positive on $\overline{\Omega}$ and $\partial\Omega$, $\varphi \in C^3$, Wang [3] claimed that the convex solution belongs to $C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

For the degenerate case, the counterexamples have been found by Bedford and Fornaess [4], who showed that in general the solution of (1.1), (1.2) is not of the class $C^2(\overline{\Omega})$. The question of whether the solution belongs to $C^{1,1}(\overline{\Omega})$ has attracted a lot of attention. The $C^{1,1}(\overline{\Omega})$ smoothness of the solution was proved in [5–9] but only for some special cases, e.g., for $f \equiv 0$, for $\varphi \equiv$ constant, and for general boundary data under some restrictions on φ and $\partial\Omega$. Recently we are informed that Guan et al. [10] got the global $C^{1,1}$ regularity of the solution for the problem (1.1), (1.2) in the cases where $f^{1/(n-1)} \in C^{1,1}(\Omega')$ and $\partial\Omega$, $\varphi \in C^{3,1}$.

If there is no additional assumption about f and φ, only a solution in $C^{0,1}(\overline{\Omega}) \cap C^{1,1}(\Omega)$ of the problem (1.1), (1.2) has been obtained so far (see [11–13]). Naturally, one can ask whether the solution has much better global regularity. This is the motivation of the present paper.

The purpose of this paper is to show the $C^{1,1/3}(\overline{\Omega}) \cap C^{1,1}(\Omega)$ regularity of the solution for the problem (1.1), (1.2) under the usual conditions. To the author's knowledge, the study for the global Hölder continuity of Du is much less in the degenerate case.

Throughout the paper we always assume that $f(x,z,p)$ satisfies the following structure conditions analogous to [8, 9, 12, and 13]:

$$f \geq 0, f_z \geq 0, \qquad (x,z,p) \in \Omega' \times \mathbf{R}^1 \times \mathbf{R}^n \tag{1.3}$$

$$f(x,-M,p) \leq \frac{g(x)}{h(p)}, \qquad (x,p) \in \overline{\Omega} \times \mathbf{R}^n \tag{1.4}$$

$$f(x,\varphi(x),p) \leq \mu d^{\beta}(x)\left(1+|p|^2\right)^{\alpha/2}, \qquad (x,p) \in N \times \mathbf{R}^n \tag{1.5}$$

$$f^{1/n} \in C^{1,1}(\Omega' \times \mathbf{R}^1 \times \mathbf{R}^n) \qquad \text{is convex with respect to } p, \tag{1.6}$$

where α, β, μ, and M are the nonnegative constants; $\beta > 0$; $\beta \geq \alpha - n - 1$; Ω', N are some neighborhoods of Ω; and $\partial\Omega$ and g, h are the positive functions, respectively, in $L^1(\Omega)$ and $L^1_{\text{loc}}(\mathbf{R}^n)$ such that

$$\int_\Omega g(x)\,dx < \int_{\mathbf{R}^n} h(p)\,dp. \tag{1.7}$$

Our main result is as follows.

THEOREM 1.1. *Assume $\partial\Omega \in C^{2,1}$, $\varphi \in C^{2,1}(\partial\Omega)$, and $f \in C^{1,1}(\Omega' \times \mathbf{R}^1 \times \mathbf{R}^n)$ satisfy* (1.3)–(1.7). *Then there exists a unique convex solution of*

(1.1), (1.2) *in* $C^{1,1/3}(\overline{\Omega}) \cap C^{1,1}(\Omega) \cap C^{2,\gamma}(\Omega_f)$ *for all* $\gamma \in (0,1)$, *where*

$$\Omega_f = \{x \in \Omega \mid f(x, u(x), Du(x)) > 0\}.$$

Remark 1.2. The condition $\beta > 0$ implies that Eq. (1.1) must be degenerate on $\partial\Omega$, which is essential to our proof. But for $f = f(x,u)$ the result of Theorem 1.1 is also valid in the nondegenerate case (see [14]).

Monge–Ampère equations are closely related to the problems involving Gauss–Kronecker curvature in differential geometry. As a consequence, from Theorem 1.1, we provide the following result for the equation of the prescribed nonnegative Gauss curvature.

THEOREM 1.3. *Suppose that* $\partial\Omega \in C^{2,1}$, $\varphi \in C^{2,1}(\partial\Omega)$, *and*

$$K \geq 0 \quad \text{in } \Omega', \qquad K = 0 \quad \text{on } \partial\Omega \tag{1.8}$$

$$K, K^{1/n} \in C^{1,1}(\Omega') \tag{1.9}$$

$$\int_\Omega K(x)\, dx < \omega_n \tag{1.10}$$

hold, where ω_n *is the volume of the unit ball in* $\mathbf{R}^n$. *Then the problem*

$$\det D^2 u = K(x)\left(1 + |Du|^2\right)^{(n+2)/2}, \qquad \text{in } \Omega \tag{1.11}$$

$$u = \varphi(x), \qquad \text{on } \partial\Omega \tag{1.12}$$

has a unique convex solution $u \in C^{1,1/3}(\overline{\Omega}) \cap C^{1,1}(\Omega) \cap C^{2,\gamma}(\Omega_K)$ *for all* $\gamma \in (0,1)$, *where*

$$\Omega_K = \{x \in \Omega \mid K(x) > 0\}.$$

Theorem 1.3 tells us that one can find a $C^{1,1/3}(\overline{\Omega}) \cap C^{1,1}(\Omega)$ graph with any given nonnegative Gauss curvature, which meets any given curve over $\partial\Omega$.

2. REGULARIZED PROBLEM

This section is concerned with the construction of the regularized problem and the C^1 estimates of its solution.

First of all, we introduce f_m in a suitable way to approximate the problem (1.1), (1.2).

Denote $h_H(p) = \min\{h(p), H\}$, and choose $H > 1$ so large that by (1.7) and the Levi theorem,

$$\int_\Omega g(x)\,dx < \int_{\mathbf{R}^n} h_H(p)\,dp. \tag{2.1}$$

For $m = 1, 2, \dots,$ define

$$f_m(x, z, p) = \eta\left(\frac{p}{m}\right) f(x, z, p) + \frac{1}{mH},$$

where

$$\eta \in C_0^\infty(\mathbf{R}^n), \qquad 0 \le \eta \le 1$$

(p)

$$\eta = \begin{cases} 1, & |p| < 1 \\ 0, & |p| > 2. \end{cases}$$

Then by using (1.3), (1.4), and (2.1) f_m satisfies the conditions

$$0 < f_m(x, z, p) \le \mu_m(|z|) \equiv 1 + \sup\left\{ f(x, |z|, p) \mid x \in \overline{\Omega}, |p| \le 2m \right\} \tag{2.2}$$

$$-f_m(x, z, p)\text{sign } z \le f_m(x, -M, p) \le \frac{g(x) + 1/m}{h_H(p)}, \qquad |z| \ge M \tag{2.3}$$

$$\int_\Omega \left(g(x) + \frac{1}{m} \right) dx < \int_{\mathbf{R}^n} h_H(p)\,dp, \tag{2.4}$$

for sufficiently large m.

Now we assume without loss of generality that φ is extended to all of $\mathbf{R}^n$ and is contained in $C^{2,1}(\mathbf{R}^n)$. Consider the regularized problem

$$\det D^2 u = f_m(x, u, Du), \qquad \text{in } \Omega_m \tag{2.5}$$

$$u = \varphi_m(x), \qquad \text{on } \partial\Omega_m, \tag{2.6}$$

where $\{\Omega_m\}$ and $\{\varphi_m\}$ are, respectively, a sequence of strictly convex domains and C^4 functions in $\mathbf{R}^n$ such that

$$\Omega \subset \Omega_m, \quad \|\varphi_m\|_{C^{2,1}(\partial\Omega_m)} \le C, \qquad m = 1, 2, \dots,$$

$$\Omega_m \to \Omega, \quad \|\varphi_m - \varphi\|_{C^2(\partial\Omega)} \to 0, \qquad \text{as } m \to \infty.$$

From (2.2), (2.3), (2.4), and the existence theorem [15, Theorem 17.23], it follows that the problem (2.5), (2.6) always admits a unique convex solution $u_m \in C^3(\overline{\Omega}_m)$.

To prove that $\{u_m\}$ contains a converging subsequence, and the limit function u is the $C^{1,1/3}(\overline{\Omega}) \cap C^{1,1}(\Omega)$ solution of the problem (1.1), (1.2), we need to establish the $C^{1,1/3}(\overline{\Omega}_m)$ and $C^{1,1}(\Omega_m)$ estimates of the approximate solution u_m, which are independent of m.

For simplicity, in estimating u_m we shall omit the subscript m and write u, f, φ, Ω in place of u_m, f_m, φ_m, Ω_m; denote by C the positive constants, depending on only n, g, h, M, N, α, β, μ, Ω, and Ω'; and adopt the summation convention; i.e., the repeated indices indicate summation from 1 to n.

Our next step is to present the C^1-estimates. We give the outline of these estimates for sufficiently large m here for completeness.

THEOREM 2.1.

$$\sup_{\Omega} |u| \leq C. \tag{2.7}$$

Proof. The inequalities (2.3), (2.4) and the maximum principle [15, Theorem 17.4] enable us to get at once the bound for u.

THEOREM 2.2.

$$\sup_{\Omega} |Du| \leq C. \tag{2.8}$$

Proof. The convexity of u guarantees that $|Du|$ attains its maximum on $\partial\Omega$ and $\frac{\partial u}{\partial \nu} \leq C$ on $\partial\Omega$, where ν is the unit interior normal to $\partial\Omega$. The lower bound for $\frac{\partial u}{\partial \nu}$ follows from the inequality

$$f(x, \varphi(x), p) \leq \frac{1}{mH} + \mu d^{\beta}(x)\left(1 + |p|^2\right)^{\alpha/2}, \qquad (x, p) \in N \times \mathbf{R}^n.$$

For details, refer to [12]. Combining $D'u = D'\varphi$ on $\partial\Omega$ completes the proof of Theorem 2.2, where D' is the tangential boundary gradient operator.

3. $C^{1,1/3}$ ESTIMATES

In this section we derive the Hölder gradient estimates of the solution to the problem (2.5), (2.6) for sufficiently large m. We write Eq. (2.5) in the form

$$\log \det D^2 u = \log f(x, u, Du).$$

Its differentiation gives

$$u^{ij}D_{ijk}u = \frac{1}{f}\left(f_{x_k} + f_z D_k u + f_{p_i} D_{ik} u\right), \tag{3.1}$$

for $k = 1, 2, \dots, n$, where $[u^{ij}]$ denotes the inverse of D^2u. It follows from Eq. (2.5) that

$$f^{-1/n} = \left(\det[u^{ij}]\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n} u^{ii}. \tag{3.2}$$

Since $f^{1/n} \in C^{1,1}(\Omega' \times \mathbf{R}^1 \times \mathbf{R}^n)$, we have by virtue of [16, Lemma 1.7.1]

$$\begin{aligned} |D(f^{1/n})| &\le C\left(f^{1/n}\right)^{1/2}, \\ \frac{1}{f}|f_x, f_z, f_p| &\le Cf^{-1/2n}. \end{aligned} \tag{3.3}$$

Using condition (1.5), we obtain

$$f(y, u(y), Du(x)) = 0$$

for $y \in \partial\Omega$, $|x - y| = d(x)$, and

$$\begin{aligned} f(x, u, Du) &\le C\left(|x - y| + |u(x) - u(y)| + \frac{1}{m}\right) \\ &\le C\left(d(x) + \frac{1}{m}\right), \qquad x \in N. \end{aligned} \tag{3.4}$$

Without loss of generality we may assume that $\varphi(x)$ and $\nu(x)$ have C^3 extensions on $\overline{\Omega}$, continuously denoted by $\varphi(x)$ and $\nu(x)$. Let $\tau(x)$ be the orthonormal vector field to $\nu(x)$. First we give the following result.

LEMMA 3.1.

$$\left|\frac{\partial u}{\partial \tau}(x) - \frac{\partial \varphi}{\partial \tau}(x)\right| \le Cd(x), \qquad x \in \overline{\Omega}. \tag{3.5}$$

Proof. Suppose the origin is on the boundary and the x_n axis is the interior normal at 0. It suffices to prove

$$\left|\frac{\partial u}{\partial \tau}(0', x_n) - \frac{\partial \varphi}{\partial \tau}(0', x_n)\right| \le Cx_n.$$

For $A > 1$ and $\varepsilon \in (0,1)$, set

$$w(x) = \frac{1}{\varepsilon}\left(Ax_n - |x|^2\right) - \frac{\partial}{\partial\tau}(u - \varphi)(x)$$

as the auxiliary function in $\Omega_\varepsilon = \Omega \cap \{x_n < \varepsilon\}$. Since Ω is strictly convex, there exists some positive constant c_0 such that

$$\varepsilon \geq x_n \geq c_0|x|^2, \qquad \text{on } \overline{\Omega_\varepsilon},$$

for sufficiently small ε. Hence we can choose A, independent of ε, so large that

$$w = \frac{1}{\varepsilon}\left(Ax_n - |x|^2\right) \geq \frac{1}{\varepsilon}(Ac_0 - 1)|x|^2 \geq 0, \qquad \text{on } \partial\Omega \cap \partial\Omega_\varepsilon,$$

$$w = \frac{1}{\varepsilon}\left(A\varepsilon - |x|^2\right) - \frac{\partial}{\partial\tau}(u - \varphi) \geq A - \frac{1}{c_0} - C \geq 0, \qquad \text{on } \Omega \cap \partial\Omega_\varepsilon.$$

From (3.1)–(3.4), we infer that

$$\begin{aligned}
u^{ij}D_{ij}w - \frac{f_{p_i}}{f}D_i w &= \frac{1}{\varepsilon}\left[-2\sum_{i=1}^n u^{ii} + \frac{1}{f}\left(Af_{p_n} - 2x_i f_{p_i}\right)\right] \\
&\quad - 2\operatorname{div}\tau - u^{ij}\left[D_{ij}\tau_k D_k u - D_{ij}\left(\frac{\partial\varphi}{\partial\tau}\right)\right] \\
&\quad - \frac{1}{f}\left[\tau\cdot f_x + f_z\frac{\partial u}{\partial\tau} + f_{p_i}\left(D_i\left(\frac{\partial\varphi}{\partial\tau}\right) - D_i\tau_k D_k u\right)\right] \\
&\leq \frac{1}{\varepsilon}\left[-2\sum_{i=1}^n u^{ii} + C\left(\sum_{i=1}^n u^{ii}\right)^{\frac{1}{2}}\right] + C\left(\sum_{i=1}^n u^{ii} + 1\right) \\
&\leq \left(C - \frac{1}{\varepsilon}\right)\sum_{i=1}^n u^{ii} + \frac{C}{\varepsilon} \\
&\leq n\left(C - \frac{1}{\varepsilon}\right)f^{-\frac{1}{n}} + \frac{C}{\varepsilon} \\
&\leq n\left(C - \frac{1}{\varepsilon}\right)C\left(\varepsilon + \frac{1}{m}\right)^{-\frac{1}{n}} + \frac{C}{\varepsilon} \\
&\leq 0,
\end{aligned}$$

if ε is small enough and m is sufficiently large.

An application of the maximum principle gives $w \geq 0$ in Ω_ε. Therefore

$$\frac{\partial}{\partial\tau}(u - \varphi) \leq \frac{A}{\varepsilon}x_n, \qquad \text{in } \Omega_\varepsilon.$$

In a similar fashion we can obtain the lower estimate.

Now we show that $\frac{\partial u}{\partial \nu}$ is Lipschitz equicontinuous on $\partial\Omega$.

LEMMA 3.2.

$$\left|\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(y)\right| \le C|x-y|, \qquad x, y \in \partial\Omega. \tag{3.6}$$

Proof. From (3.5), we see

$$\left|\frac{\partial}{\partial \tau}(u-\varphi)(x) - \frac{\partial}{\partial \tau}(u-\varphi)(\bar{x})\right| \le C|x-\bar{x}|$$

for $x \in \overline{\Omega}$, $\bar{x} \in \partial\Omega$, and $|x-\bar{x}| = d(x)$. Thus

$$\left|\frac{\partial^2 u}{\partial \nu \partial \tau}\right| \le C, \qquad \text{on } \partial\Omega.$$

By virtue of the Mean Value Theorem, for $x, y \in \partial\Omega$, there is $\xi \in \partial\Omega$, such that

$$\left|\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(y)\right| = \left|D'\left(\frac{\partial u}{\partial \nu}(\xi)\right)\cdot(x-y)\right| \le C|x-y|.$$

This completes the proof of the lemma.

Next we present the boundary Hölder estimate for $\frac{\partial u}{\partial \nu}$.

LEMMA 3.3.

$$\left|\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(y)\right| \le C|x-y|^{1/2}, \qquad x \in \overline{\Omega}, y \in \partial\Omega. \tag{3.7}$$

Proof. There is no loss of generality in assuming that $y = 0$, the x_n axis is the interior normal at 0, and $D^2\varphi \ge c_0 I$ for some $c_0 > 0$. We consider the auxiliary function in $\Omega_\varepsilon = \Omega \cap \{x_n < \varepsilon\}$,

$$w(x) = A(u-\varphi)(x) + Bx_n^{1/2} - \varepsilon^{1/2}\left[\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(0)\right],$$

where $A, B > 1$ and $\varepsilon \in (0,1)$ are the positive constants to be determined.

On $\partial\Omega_\varepsilon \cap \partial\Omega$, by (3.6)

$$w = Bx_n^{1/2} - \varepsilon^{1/2}\left[\frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu}(0)\right] \ge Bx_n^{1/2} - C|x| \ge (B-C)x_n^{1/2} \ge 0,$$

if $B > C$. On $\partial\Omega_\varepsilon \cap \Omega$,

$$w = A(u - \varphi) + B\varepsilon^{1/2} - \varepsilon^{1/2}\left[\frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu}(0)\right]$$
$$\geq (B - CA\varepsilon^{1/2} - C)\varepsilon^{1/2} \geq 0,$$

if

$$\varepsilon < \left(\frac{B}{2CA}\right)^2, \qquad B > 2C.$$

By (3.1)–(3.4), a direct calculation yields

$$\begin{aligned}
&u^{ij}D_{ij}w - \frac{f_{p_i}}{f}D_i w \\
&\quad = A\left(n - u^{ij}D_{ij}\varphi\right) - \frac{B}{4}x_n^{-3/2}u^{nn} \\
&\qquad - \frac{1}{f}\left[Af_{p_i}D_i(u - \varphi) + \frac{B}{2}x_n^{-1/2}f_{p_n}\right] \\
&\qquad - \varepsilon^{1/2}\left[2\operatorname{div}\nu + u^{ij}D_{ij}\nu_k D_k u + \frac{1}{f}\left(\nu\cdot f_x + f_z\frac{\partial u}{\partial \nu} - f_{p_i}D_i\nu_k D_k u\right)\right] \\
&\quad \leq A\left(-c_0\sum_{i=1}^n u^{ii} + Cf^{-\frac{1}{2n}}\right) + BCx_n^{-1/2}f^{-1/2n} \\
&\qquad + C\varepsilon^{1/2}\left(\sum_{i=1}^n u^{ii} + f^{-1/2n}\right) \\
&\quad \leq \left(-c_0A + C\varepsilon^{1/2}\right)\sum_{i=1}^n u^{ii} + Cf^{-1/n}\left[Af^{1/2n} + B\left(\frac{f^{1/n}}{x_n}\right)^{1/2}\right] \\
&\quad \leq -\frac{c_0A}{2}\sum_{i=1}^n u^{ii} + Cf^{-1/n}\left[Af^{1/2n} + B\left(\frac{f^{1/n}}{x_n}\right)^{1/2}\right] \\
&\quad \leq f^{-1/n}\left[A\left(-\frac{nc_0}{2} + Cf^{1/2n}\right) + BC\right] \\
&\quad \leq f^{-1/n}\left[A\left(-c_0 + C\left(\varepsilon + \frac{1}{m}\right)^{1/2n}\right) + BC\right] \\
&\quad \leq 0,
\end{aligned}$$

if

$$\varepsilon < \left(\frac{c_0 A}{2C}\right)^2, \quad \varepsilon + \frac{1}{m} < \left(\frac{c_0}{2C}\right)^{1/2n}, \quad A > \frac{2BC}{c_0}.$$

Here we have used the fact that f in the term $f^{1/n}/x_n$, which is not f_m, satisfies

$$f(x, u(x), p) = 0, \qquad (x, p) \in \partial\Omega \times \mathbf{R}^n,$$

and $f^{1/n} \in C^{0,1}(\overline{\Omega} \times \mathbf{R}^1 \times \mathbf{R}^n)$.

By the maximum principle we have that $w \geq 0$ on $\overline{\Omega_\varepsilon}$. This implies that

$$\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(0) \leq \varepsilon^{-1/2}\left[A(u - \varphi)(x) + Bx_n^{1/2}\right] \leq C|x|^{1/2}.$$

The lower estimate follows from a similar argument.

After having proved the above three lemmas, we can deduce the global Hölder estimates for Du.

THEOREM 3.4.

$$|Du(x) - Du(y)| \leq C|x - y|^{1/3}, \qquad x, y \in \overline{\Omega}. \tag{3.8}$$

Proof. Let $x, y \in \overline{\Omega}$, $0 < |x - y| < \frac{1}{4}$, and $d(x) \leq d(y)$. The proof of the theorem is divided into two cases.

Case 1. $|x - y|^{2/3} > d(x)$. There exists a point $\bar{x} \in \partial\Omega$ with $d(x) = |x - \bar{x}|$. From (3.7) and

$$d(y) \leq |y - \bar{x}| \leq |y - x| + |x - \bar{x}| \leq 2|x - y|^{2/3},$$

it is easy to see that

$$\begin{aligned}\left|\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(y)\right| &\leq \left|\frac{\partial u}{\partial \nu}(x) - \frac{\partial u}{\partial \nu}(\bar{x})\right| + \left|\frac{\partial u}{\partial \nu}(y) - \frac{\partial u}{\partial \nu}(\bar{x})\right| \\ &\leq C\left(|x - \bar{x}|^{1/2} + |y - \bar{x}|^{1/2}\right) \\ &\leq C|x - y|^{1/3}.\end{aligned}$$

By (3.5) we have

$$\begin{aligned}\left|\frac{\partial u}{\partial \tau}(x) - \frac{\partial u}{\partial \tau}(y)\right| &\leq \left|\frac{\partial u}{\partial \tau}(x) - \frac{\partial \varphi}{\partial \tau}(x)\right| + \left|\frac{\partial \varphi}{\partial \tau}(x) - \frac{\partial \varphi}{\partial \tau}(y)\right| \\ &\quad + \left|\frac{\partial \varphi}{\partial \tau}(y) - \frac{\partial u}{\partial \tau}(y)\right| \\ &\leq C\left[d(x) + |x - y| + d(y)\right] \\ &\leq C|x - y|^{1/3}.\end{aligned}$$

Hence (3.8) holds in this case.

Case 2. $|x-y|^{2/3} \le d(x)$. By means of the interior estimates for the second derivatives [13],

$$|D^2u(x)| \le \frac{C}{d(x)}, \qquad x \in \Omega, \tag{3.9}$$

and

$$\begin{aligned} d(tx+(1-t)y) &\ge d(x) - |(tx+(1-t)y) - x| \\ &\ge |x-y|^{2/3} - |x-y| \\ &\ge \tfrac{1}{2}|x-y|^{2/3}, \qquad t \in [0,1], \end{aligned}$$

it follows that

$$\begin{aligned} |Du(x) - Du(y)| &= \left| \int_0^1 D^2u(tx+(1-t)y)\cdot(x-y)\,dt \right| \\ &\le \int_0^1 \frac{C|x-y|}{d(tx+(1-t)y)}\,dt \\ &\le C|x-y|^{1/3}. \end{aligned}$$

So far we have proved Theorem 3.4.

4. THE PROOF OF THEOREM 1.1

The estimates (2.7), (2.8), (3.8), and (3.9) tell us that the solution u_m of the regularized problem (2.5), (2.6) is uniformly bounded in $C^{1,1/3}(\overline{\Omega}_m) \cap C^{1,1}(\Omega_m)$. Consequently there is a subsequence of $\{u_m\}$ such that it converges in $C^1(\overline{\Omega})$ to $u \in C^{1,1/3}(\overline{\Omega}) \cap C^{1,1}(\Omega)$. It is not difficult to get that u is the unique convex solution of the problem (1.1), (1.2). $u \in C^{2,\gamma}(\Omega_f)$ follows from the regularity theorem in [17].

To conclude the paper we point out two counterexamples to explain the conditions in Theorem 1.1.

The maximum global smoothness possible for the solution u in Theorem 1.1 is $C^{1,1/2}$ because of the following.

EXAMPLE 4.1. In the C^∞ domain $\Omega = \{x \in \mathbf{R}^n \mid |x| < 1\}$, the convex function $u(x) = (1-x_n)^{3/2} \in C^{1,1/2}(\overline{\Omega})$ satisfies the equation $\det D^2u = 0$ and the boundary condition $u = \varphi(x) \in C^{2,1}(\partial\Omega)$. But u is not in $C^{1,1/2+\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$.

On the other hand, the smoothness of the boundary data $\varphi \in C^{2,1}(\partial\Omega)$ is the minimum smoothness possible for the global $C^{1,1/2}$ solution of the problem (1.1), (1.2), as shown in the following example.

EXAMPLE 4.2. Let $\Omega \subset \mathbf{R}^2$ be the unit ball centered at $(0,1)$, $\delta \in (0, \frac{1}{2})$, $\varphi(x, y) = |x|^{3-2\delta} + y^2$. Then $\partial\Omega \in C^\infty$, $\partial\Omega \subset \{2y > x^2\}$, and $\varphi \in C^{2,1-2\delta}(\partial\Omega)$.

Suppose u is the viscosity solution to the problem

$$\det D^2 u = 0, \quad \text{in } \Omega, \qquad u = \varphi(x, y), \quad \text{on } \partial\Omega.$$

Let

$$v = (2y)^{3/2-\delta} + y^2,$$
$$w = \varepsilon\left(x^2 y^{1/2-\delta} + 2y^{3/2-\delta}\right).$$

Then

$$\det D^2 w = 2\varepsilon^2\left(\tfrac{1}{2} - \delta\right)\left(\tfrac{3}{2} - \delta\right) y^{-1-2\delta}(2y - x^2) \geq 0$$
$$= \det D^2 u = \det D^2 v, \qquad \text{in } \Omega.$$

Fix $\varepsilon > 0$ small such that

$$w \leq 4\varepsilon y^{3/2-\delta} \leq 4\varepsilon C\left((x^2)^{3/2-\delta} + y^2\right)$$
$$\leq \varphi \leq (2y)^{3/2-\delta} + y^2 = v, \qquad \text{on } \partial\Omega.$$

By the definition of viscosity solution [17],

$$2\varepsilon y^{3/2-\delta} \leq w \leq u \leq v \leq C y^{3/2-\delta}, \qquad \text{on } \overline{\Omega}.$$

This shows that u does not belong to $C^{1,1/2}(\overline{\Omega})$.

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