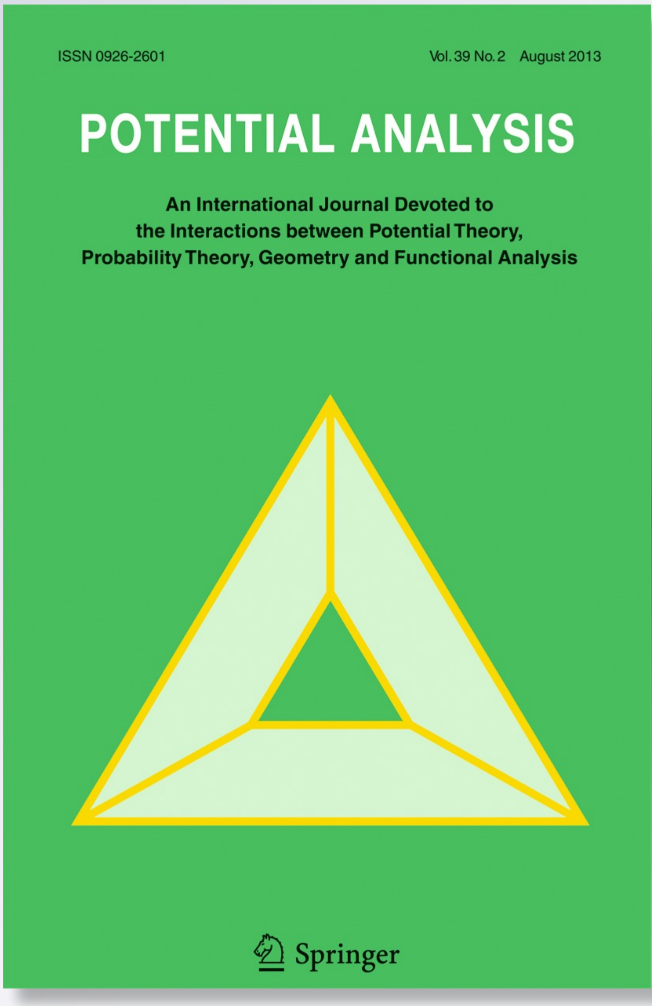
Sharp Regularity for Elliptic Systems Associated with Transmission Problems

Jingang Xiong & Jiguang Bao

Potential Analysis
An International Journal Devoted to the Interactions between Potential Theory, Probability Theory, Geometry and Functional Analysis

ISSN 0926-2601
Volume 39
Number 2

Potential Anal (2013) 39:169-194
DOI 10.1007/s11118-012-9325-6



Springer

Potential Anal (2013) 39:169–194
DOI 10.1007/s11118-012-9325-6

Sharp Regularity for Elliptic Systems Associated with Transmission Problems

Jingang Xiong · Jiguang Bao

Received: 23 January 2010 / Accepted: 21 November 2012 / Published online: 19 December 2012

Abstract The paper concerns regularity theory for linear elliptic systems with divergence form arising from transmission problems. Estimates in BMO, Dini and Hölder spaces are derived simultaneously and the gaps among of them are filled by using Campanato–John–Nirenberg spaces. Results obtained in the paper are parallel to the classical regularity theory for elliptic systems.

Keywords Elliptic systems · Transmission problem · Sharp regularity

Mathematics Subject Classifications (2010) Primary 35J55; Secondary 35D10

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing L disjoint subdomains $\Omega_1, \cdots, \Omega_L$, i.e., $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$ and $\Omega_m \subset\subset \Omega$ for $m = 1, \cdots, L$. Let $\Omega_{L+1} = \Omega \setminus \bigcup_{m=1}^{L} \overline{\Omega}_m$. We consider the elliptic systems of the form

$$-\partial_\alpha (a_{ij}^{\alpha\beta}(x)\partial_\beta u^j) = g_i(x) - \partial_\beta f_i^\beta(x), \text{ in } \Omega, \ i = 1, \cdots, N, \tag{1.1}$$

Supported by the National Natural Science Foundation of China (11071020) and Doctoral Programme Foundation of Institute of Higher Education of China (20100003110003).

J. Xiong · J. Bao (✉)
School of Mathematical Sciences and LMCS, Beijing Normal University,
Beijing 100875, China
e-mail: jgbao@bnu.edu.cn

Present Address:
J. Xiong
BICMR, Peking University, Beijing 100871, China
e-mail: jxiong@mail.bnu.edu.cn, jxiong@math.pku.edu.cn

with indices $i, j = 1, \cdots, N$; $\alpha, \beta = 1, \cdots, n$ and where we use the summation convention that repeated indices are summed. We assume that the system satisfies the following strong ellipticity condition

$$\Lambda|\xi|^2 \le a_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \le \Lambda^{-1}|\xi|^2, \tag{1.2}$$

for all $\xi \in \mathbb{R}^{nN}$, $x \in \Omega$, where $\Lambda > 0$ is a constant.

When $a_{ij}^{\alpha\beta}$ and $f = \{f_i^\beta\}$ satisfy continuity conditions uniformly in Ω, there are many papers devoted to system 1.1 and elliptic equation with divergence structure, see e.g., Huang [11], Byun and Wang [2] and references therein.

The aim of the paper is to establish estimates and regularity for weak solutions $u = (u^1, \cdots, u^N) \in H^1(\Omega; \mathbb{R}^N)$ to system 1.1 under the assumptions that $a_{ij}^{\alpha\beta}$ and $f = \{f_i^\beta\}$ are piecewise defined in Ω, more precisely, satisfying some kinds of continuity condition in each subdomain Ω_m but probably not cross some interfaces $\partial\Omega_m$, while $g = (g_1, \cdots, g_N)$ meets integrability conditions. The background of our investigation comes from transmission problems or interface problems which appear in many practical applications, in particular they are likely to appear in the situation that more than one type of material (or medium) are used.

When coefficients and right hand sides are smooth on each $\overline{\Omega}_m$, which is smooth also, weak solutions of Eq. 1.1 are smooth on each $\overline{\Omega}_m$, see Ladyzhenskaya et al. [15], Chipot et al. [6] or Li and Nirenberg [16]. When $a_{ij}^{\alpha\beta}$ are piecewise constants and $\partial\Omega_m$ merely Lipschitz continuous, Escauriaza, Fabes and Verchota [7] showed that weak solutions to homogenous Eq. 1.1 with $N = 1$ are $W^{3/2,2}$ in each subdomain. Along the same line those results were extended to $N \ge 1$ and parabolic type by Escauriaza and Seo [9]. Their approaches were based on layer potential techniques. Later, global $W^{1,p}$ estimates for Eq. 1.1 were established by Caffarelli and Peral [5] with piecewise continuous coefficients and C^1 interfaces. When $N = 1$, $a_{ij}^{\alpha\beta}$, f is C^μ $(0 < \mu < 1)$ on each $\overline{\Omega}_m$, $g \in L^\infty(\Omega)$ and $\partial\Omega_m \in C^{1,\nu}$ $(0 < \nu < 1)$, then Li and Vogelius [17] proved $u \in C^{1,\gamma}$ in each Ω_m with $0 < \gamma \le \min\{\mu, \frac{\nu}{2(\nu+1)}\}$ by the approach in Caffarelli [3] for fully nonlinear elliptic equations. Similar results were established for elliptic system by Li and Nirenberg [16]. The primary goal of [17] and [16] is to show that the L^∞ bound of gradient is independent of the distances between $\partial\Omega_m$, and hence verifies previous numerical observation in composite materials. From regularity point of view, the requirement $g \in L^\infty(\Omega)$ and $\gamma \le \frac{\nu}{2(\nu+1)}$, however, is a little stronger than those in the standard Schauder theory for elliptic systems (see, e.g., [10]). High order derivatives estimates were considered in [17] and [16] as well, but more demands were imposed on the coefficients because of the essential use of a special version of Sobolev embedding theorem.

Inspired by [11], we consider estimates and regularity in generalized Morrey space $L_\varphi^{2,\mu}$ and Campanato–John–Nirenberg space BMO_ψ via the procedure of Campanato for establishing Schauder estimates. An advantage of using the BMO_ψ space is that regularity for solutions in BMO, Dini and Hölder spaces can be derived simultaneously, and the gaps among them can be filled.

When it comes to using Campanato method, difficulties occur near interfaces $\partial\Omega_m$ $(m \leq L)$. On the one hand, for piecewise continuous coefficients one cannot expect weak solutions of Eq. 1.1 to be in $C^{1,\gamma}(\Omega)$. For instance,

$$u(x) = \begin{cases} x, & x \in (0, 1] \\ 2x - 1, & x \in (1, 2] \\ 3x - 3, & x \in (2, 3) \end{cases}$$

is in $H^1((0, 3))$ and a weak solution to equation

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = 0 \; x \in (0, 3)$$

with

$$a(x) = \begin{cases} 1, & x \in (0, 1] \\ \frac{1}{2}, & x \in (1, 2] \\ \frac{1}{3}, & x \in (2, 3), \end{cases}$$

but u is only Lipschitz. Although the example is one dimensional, the same feature is valid in higher dimensions, see, e.g., [17]. Consequently, Campanato's method is not applied directly to the area cross interfaces $\partial\Omega_m$. On the other hand, if we restrict our attention to each $\overline{\Omega}_m$ and employ Campanato procedure on it, then troubles appear, too. That is because the lack of information of solutions on $\partial\Omega_m$ would lead to the failure of crucial Campanato's inequalities near $\partial\Omega_m$. To overcome them, we flatten the boundaries of Ω_m, then apply Companato procedure for partial variables and establish estimates for derivatives in those directions while for the other we take advantage of the system. Since weak solutions of Eq. 1.1 are defined in distribution sense, we estimate one partial derivative in terms of others by approximating.

The gradient estimates we obtain for elliptic systems with piecewise coefficients can compete with those in classical theory of elliptic systems and those derived in [11]. Under the assumption g is in some Morrey spaces, we shall show that, for $\Omega' \subset\subset \Omega$

1. If $a_{ij}^{\alpha\beta}\big|_{\Omega_m} \in BMO_\omega(\Omega_m)$, $\partial\Omega_m \in C^{1,BMO\omega}$ with $\omega(t) = o(-1/\log t)$ and $f\big|_{\Omega_m} \in BMO(\Omega_m)$, then $Du\big|_{\Omega_m} \in BMO(\Omega_m \cap \Omega')$.
2. If $a_{ij}^{\alpha\beta}\big|_{\Omega_m}$, $f\big|_{\Omega_m} \in BMO_\psi(\Omega_m)$ and $\partial\Omega_m \in C^{1,BMO\psi}$, where ψ satisfies a Dini condition, then $Du \in L^\infty(\Omega)$ and $Du\big|_{\Omega_m} \in BMO_\psi(\Omega_m)$.
   In particular, if $a_{ij}^{\alpha\beta}\big|_{\Omega_m}$, $f\big|_{\Omega_m} \in C^\gamma(\overline{\Omega}_m)$ and $\partial\Omega_m \in C^{1,\gamma}$, then $u \in C^{1,\gamma}(\overline{\Omega}_m \cap \Omega')$.

The organization of the paper is as following: In Section 2, several notations, definitions and the main results are given. In Section 3, gradient estimates is derived in generalized Morrey spaces by standard Campanato's method. Those estimates would take a crucial position in obtaining gradient estimates in generalized Campanato–John–Nirenberg space investigated in Section 4. In terms of gradient estimates, high order regularity is established in Section 5. Consequently, they depend on given data in a natural way.

2 Preliminaries and Main Results

First of all, Let us give some notations and definitions.

(1) $\mathbb{R}^n$ is the n dimensional real Euclidean space.
(2) A typical point in $\mathbb{R}^n$ is $x = (x', x_n)$.
(3) $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and $\mathbb{R}^n_- = \{(x', x_n) \in \mathbb{R}^n : x_n < 0\}$.
(4) For $r > 0$, $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $B_r^+(x_0) = \{x \in B_r(x_0) : x_n > 0\}$ and $B_r^-(x_0) = \{x \in B_r(x_0) : x_n < 0\}$; $Q_r(x_0) = \{x \in \mathbb{R}^n : |x_i - x_{i0}| < r,\ i = 1, \cdots, n\}$, $Q_r^+(x_0) = \{x \in Q_r(x_0) : x_n > 0\}$ and $Q_r^-(x_0) = \{x \in Q_r(x_0) : x_n < 0\}$; $\Gamma_r(x_0) = \{x \in Q_r(x_0) : x_n = 0\}$.
(5) Let E be a measurable set in $\mathbb{R}^n$ and $f \in L^1(E)$. Denote

$$(f)_E = \fint_E f \, \mathrm{d}x,$$

the integral average over the set E. Denote $(f)_E^+ = (f)_{E \cap \mathbb{R}^n_+}$, $(f)_E^- = (f)_{E \cap \mathbb{R}^n_-}$ and

$$(f)_E^\pm = \begin{cases} (f)_E^+, & \text{in } E \cap \mathbb{R}^n_+, \\ (f)_E^-, & \text{in } E \cap \mathbb{R}^n_-. \end{cases}$$

When $E = Q_R(x_0)$, we write $(f)_{x_0,R}$ instead of $(f)_{Q_R(x_0)}$ for convenience.

Definition 2.1 We say the function $\varphi : [0, d] \to [0, \infty)$ is almost increasing if there exists a constant $K_\varphi \geq 1$ such that

$$\varphi(s) \leq K_\varphi \cdot \varphi(t) \quad \text{for } 0 \leq s \leq t \leq d.$$

Let D be a bounded domain in $\mathbb{R}^n$, $D(x_0, \rho) = D \cap Q_\rho(x_0)$ and d be the diameter of D. We recall definitions of some function spaces on D, which can be found in [1, 11, 19].

Definition 2.2 (Generalized Morrey Space) Let $1 \leq p < \infty$, $0 \leq \mu \leq n$, and φ be a nonnegative continuous function on $[0, d]$ satisfying $r^{n-\mu} \leq c\varphi^p(r)$ for some positive constant c. A function $f \in L_\varphi^{p,\mu}(D)$ if

$$\sup_{\substack{x_0 \in D, \\ 0 < \rho \leq d}} \frac{1}{\varphi(\rho)} \left(\frac{1}{\rho^\mu} \int_{D(x_0,\rho)} |f(x)|^p \, \mathrm{d}x \right)^{1/p} < \infty.$$

It is easy to verify that $L_\varphi^{p,\mu}(D)$ is a Banach space under the norm

$$\|f\|_{L_\varphi^{p,\mu}(D)} = \sup_{\substack{x_0 \in D, \\ 0 < \rho \leq d}} \frac{1}{\varphi(\rho)} \left(\frac{1}{\rho^\mu} \int_{D(x_0,\rho)} |f(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

In the case $\varphi = 1$, $L_\varphi^{p,\mu}(D)$ is the standard Morrey space $L^{p,\mu}(D)$.

Definition 2.3 (Campanato–John–Nirenberg Space) Let ψ be a nonnegative continuous function on $[0, d]$ satisfying $r \le c\psi(r)$ for some positive constant c. A function $f \in L^2(D)$ is said to be in $BMO_\psi(D)$, if

$$[f]_{BMO_\psi(D)} = \sup_{\substack{x_0\in D,\\ 0<\rho\le d}} \frac{1}{\psi(\rho)} \left(\fint_{D(x_0,\rho)} |f(x) - (f)_{D(x_0,\rho)}|^2 \, dx \right)^{1/2} < \infty.$$

A function f belongs to $VMO_\psi(D)$, if $f \in BMO_\psi(D)$ and satisfies

$$[f]_{BMO_\psi(D;r)} = \sup_{\substack{x_0\in D,\\ 0<\rho\le r}} \frac{1}{\psi(\rho)} \left(\fint_{D(x_0,\rho)} |f(x) - (f)_{D(x_0,\rho)}|^2 \, dx \right)^{1/2} \to 0,$$

as $r \to 0$.

Under the norm

$$\|f\|_{BMO_\psi(D)} = \|f\|_{L^2(D)} + [f]_{BMO_\psi(D)},$$

$BMO_\psi(D)$ and $VMO_\psi(D)$ are Banach spaces. When $\psi = 1$ and ρ^α, $0 < \alpha \le 1$, BMO_ψ is just the John-Nirenberg space BMO and Campannato space respectively. When $\psi = 1$, VMO_ψ is the Sarason class VMO space, see [18].

In the following we always assume that φ and ψ are almost increasing. The properties of classical Morrey spaces, BMO and VMO can be extended directly to generalized Morrey space, BMO_ψ and VMO_ψ, see [1, 4, 11, 19].

The following proposition is due to [19].

Proposition 2.1 *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Assume that ψ and $t/\psi(t)$ are almost increasing and satisfy*

$$\lim_{t\to 0} \psi(t) = 0 \textit{ and } \int_0^d \frac{\psi(t)}{t} \, dt < \infty. \tag{2.1}$$

Then $BMO_\psi(D) \subset C(\overline{D})$ and we have continuous modulus estimate

$$|f(x) - f(y)| \le C[f]_{BMO_\psi(D)} \int_0^{|x-y|} \frac{\psi(t)}{t} \, dt \;\; \textit{for any } x, y \in D. \tag{2.2}$$

The following interpolation inequality is proved in [4].

Proposition 2.2 *Let $f \in BMO_\psi(Q_R(0))$. Assume that ψ and $t/\psi(t)$ are almost increasing and satisfying Eq. 2.1. Then for $0 < \varepsilon \le R$ and $0 < \delta < \infty$, there exists a constant C depending only on n, δ such that*

$$\|f\|_{L^\infty(Q_R(0))} \le C \left\{ \int_0^\varepsilon \frac{\psi(t)}{t} \, dt \cdot [f]_{BMO_\psi(Q_R(0))} + \varepsilon^{-n/\delta} \left(\int_{Q_R(0)} |f|^\delta \, dx \right)^{1/\delta} \right\}. \tag{2.3}$$

In particular, for $\psi(t) = t^\gamma,\ 0 < \gamma \le 1$,

$$\|f\|_{L^\infty(Q_R(0))} \le C\left\{\varepsilon^\gamma [f]_{BMO_\psi(Q_R(0))} + \varepsilon^{-n/\delta}\left(\int_{Q_R(0)} |f|^\delta \,\mathrm{d}x\right)^{1/\delta}\right\}. \tag{2.4}$$

To estimate of solutions to Eq. 1.1 near the boundary $\partial\Omega_m$ $(m = 1, \cdots, L)$, we should pose some regularity requirements on the those subdomains.

Definition 2.4 We say $\partial D \in C^{k,BMO_\psi}$ (or C^{k,VMO_ψ}), if for any $x_0 \in \partial D$ there exists a $C^{k-1,1}$ transform $\mathcal{T}$ and neighborhood N_{x_0} such that $\mathcal{T} : N_{x_0} \cap D \to B_1^+(0)$ is one to one and onto with $\mathcal{T}(N_{x_0} \cap \partial\Omega) = \overline{B_1^+}(0) \cap \{x_n = 0\}$. Moreover, the norms of $\mathcal{T},\ \mathcal{T}^{-1}$ and their derivatives $D^\nu\mathcal{T},\ D^\nu(\mathcal{T}^{-1})$ $(|\nu| \le k)$ in L^∞ and BMO_ψ are uniformly bounded (or VMO_ψ modulus is uniform).

Now we are well-prepared to state our main results.

Theorem 2.1 *Let* $u \in H^1(\Omega; \mathbb{R}^N)$ *be a weak solution to system* 1.1 *with Eq.* 1.2*. Suppose there exist* μ, λ *such that* $\mu < \lambda < n$ *and* $r^{\lambda-\mu}/\varphi^2(r)$ *is almost increasing. Assume that* $\partial\Omega_m \in C^{1,VMO}$,

$$a_{ij}^{\alpha\beta}\big|_{\Omega_m} \in VMO(\Omega_m),\ m = 1, \cdots, L+1 \tag{2.5}$$

and

$$g \in L_\varphi^{2n/(n+2),\mu n/(n+2)}(\Omega),\quad f \in L_\varphi^{2,\mu}(\Omega), \tag{2.6}$$

where $g = (g_1, \cdots, g_N)$ *and* $f = \{f_i^\beta\}$*. Then for any* $\Omega' \subset\subset \Omega$ *and* $0 < \sigma \le \frac{1}{\sqrt{n}} dist(\Omega', \Omega)$*, we have* $Du \in L_\varphi^{2,\mu}(\Omega')$*. Moreover,*

$$\|Du\|_{L_\varphi^{2,\mu}(\Omega')} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L_\varphi^{2,\mu}(\Omega)} + \|g\|_{L_\varphi^{2n/(n+2),\mu n/(n+2)}(\Omega)}\right), \tag{2.7}$$

where $C > 0$ *depends only on* $n,\ N,\ \Lambda,\ \mu,\ \lambda,\ K_{r^{\lambda-\mu}/\varphi^2(r)},\ [a_{ij}^{\alpha\beta}]_{BMO(\Omega_m;\sigma)},\ C^{1,BMO(\Omega_m;\sigma)}$ *modulus of* $\partial\Omega_m$ $(0 \le m \le L)$ *and* $dist(\Omega', \partial\Omega)$.

If the coefficients $a_{ij}^{\alpha\beta} \in C(\Omega)$, estimates Eq. 2.7 in the standard Morrey space $L^{2,\mu}$ were proved by Campanato, see [10]. If $a_{ij}^{\alpha\beta} \in VMO(\Omega)$, estimates Eq. 2.7 were obtained in [11]. Therefore, here we extend their results.

Theorem 2.2 *Let* $u \in H^1(\Omega; \mathbb{R}^N)$ *be a weak solution to system* 1.1 *with Eq.* 1.2*. Suppose* ψ *and* $r^\lambda/\psi(r)$ *are almost increasing for some* $\lambda \in (0, 1)$*. Assume that* $\partial\Omega_m \in C^{1,VMO_\omega}$,

$$a_{ij}^{\alpha\beta}\big|_{\Omega_m} \in VMO_\omega(\Omega_m),\ m = 1, \cdots, L+1 \tag{2.8}$$

and

$$g \in L_\psi^{2n/(n+2),n^2/(n+2)}(\Omega),\quad f\big|_{\Omega_m} \in BMO_\psi(\Omega_m), \tag{2.9}$$

where $g = (g_1, \cdots, g_N)$, $f = \{f_i^\beta\}$ *and*

$$\omega(r) = \psi(r)\left(\int_r^d \frac{\psi(t)}{t}\right)^{-1}$$

with $d = diam\Omega$. *Then for any* $\Omega' \subset\subset \Omega$, $Du \in BMO_\psi(\Omega' \cap \Omega_m)$ *and*

$$\|Du\|_{BMO_\psi(\Omega'\cap\Omega_m)} \le C\left(\|u\|_{L^2(\Omega)} + \sum_{m=1}^{L+1} \|f\|_{BMO_\psi(\Omega_m)} + \|g\|_{L_\psi^{2n/(n+2),n^2/(n+2)}(\Omega)}\right), \tag{2.10}$$

where $C > 0$ *depends only on* n, N, Λ, λ, $K_{r^\lambda/\psi(r)}$, $[a_{ij}^{\alpha\beta}]_{BMO_\omega(\Omega_m;\sigma)}$, $dist(\Omega', \partial\Omega)$ *and* $C^{1,BMO_\omega(\Omega_m;\sigma)}$ *modulus of* $\partial\Omega_m$ $(0 \le m \le L)$.

Furthermore, if $\psi = \psi_1 \cdot \psi_2$ *where* $\lim_{r\to 0} \psi_1(r) = 0$, ψ_2 *satisfies Eq.* 2.1 *and* ψ_2, $r^\lambda/\psi_2(r)$ *are almost increasing,* $\partial\Omega_m$ *and* $a_{ij}^{\alpha\beta}\big|_{\Omega_m}$ *can be reduced to be in* C^{1,BMO_ψ} *and* $BMO_\psi(\Omega_m)$, *respectively.*

For convenience, we say ψ satisfies *condition* (A) if $\psi = \psi_1 \cdot \psi_2$ where $\lim_{r\to 0} \psi_1(r) = 0$, ψ_2 satisfies Eq. 2.1 and $\psi_2, r^\lambda/\psi_2(r)$ are almost increasing. There are a large number of functions satisfying condition (A), see [11]. From Theorem 2.2 and Proposition 2.2, one can see that condition 2.1 imposed on ψ implies L^∞-estimates for gradient. In addition, since $L_\psi^{2n/(n+2),n^2/(n+2)} = L^\infty$ if $\psi(r) = r$, and thus for $C^{1,\gamma}$ $(\gamma \in (0,1))$ estimates, we do not need g to be L^∞. Finally, if $\psi(r) = r^\gamma$, i.e., $a_{ij}^{\alpha\beta}$, $f \in C^\gamma(\overline{\Omega}_m)$ and $\partial\Omega_m \in C^{1,\gamma}$, then $u \in C^{1,\gamma}(\Omega' \cap \overline{\Omega}_m)$. From the regularity perspective, above theorem improves previous work of Li et al., see [16, 17].

What is more, when $a_{ij}^{\alpha\beta} \in C^\gamma(\Omega)$, gradient estimates in the Campanato space and BMO was obtained by Campanato (Theorem 3,2 of [10]) using the celebrated Campanato method. An inspection of Campanato's proof gave a refinement under the assumption $a_{ij}^{\alpha\beta} \in VMO_\psi(\Omega)$ with

$$\psi(r) = O\left(\frac{1}{\log 1/r}\right),$$

see [1]. When $a_{ij}^{\alpha\beta} \in VMO_\omega(\Omega)$, estimates of Eq. 2.10 type were obtained in [11]. Therefore, we also extend their results to elliptic systems with piecewise coefficients.

For simplicity, we assume ψ satisfies condition (A) and $g = 0$ in the next theorem.

Theorem 2.3 *Assume* u *and* ψ *as above. Suppose that* $\partial\Omega_m \in C^{k+1,BMO_\psi}$ *with* $k \ge 1$,

$$a_{ij}^{\alpha\beta}\big|_{\Omega_m}, f\big|_{\Omega_m} \in C^{k-1,1}(\overline{\Omega}_m) \text{ and } D^k a_{ij}^{\alpha\beta}\big|_{\Omega_m}, D^k f\big|_{\Omega_m} \in BMO_\psi(\Omega_m) \tag{2.11}$$

Then for any $\Omega' \subset\subset \Omega$, $u\big|_{\Omega_m} \in C^k(\overline{\Omega}_m \cap \Omega')$ *and* $D^{k+1}u \in BMO_\psi(\Omega' \cap \Omega_m)$.

This theorem sharpens Proposition 2.1 and Proposition 1.6 in [17] and [16] respectively, since the regularity of solutions depends on given data naturally.

3 Estimates in Generalized Morrey Space

In this section we prove Theorem 2.1.

Consider system

$$-\partial_\alpha \left(A_{ij}^{\alpha\beta}(x)\partial_\beta v^j \right) = G_i(x) - \partial_\beta F_i^\beta(x), \ \ i = 1, \cdots, N, \tag{3.1}$$

in $Q_1(0)$, where $\{A_{ij}^{\alpha\beta}\}$ satisfies elliptic condition 1.2. Suppose $A_{ij}^{\alpha\beta}$, G_i, F_i^β are smooth in $Q_1^+(0)$ and $Q_1^-(0)$ respectively, but may be not continuous cross the hyperplane $\{x_n = 0\}$. Then we have following simpler version of Proposition 1.6 of [16].

Proposition 3.1 *Let $v \in H^1(Q_1(0); \mathbb{R}^N)$ be a weak solution to system 3.1 with Eq. 1.2. Then for all γ', $D_{x'}^{\gamma'} v \in C^0(Q_1(0))$, $v \in C^\infty(Q_1^\pm(0))$, and when $A_{ij}^{\alpha\beta}$ are piecewise constants and $G = F \equiv 0$ we have*

$$\|v\|_{C^k(\overline{Q_{1/2}^\pm(0)})} \leq C\|v\|_{L^2(Q_1(0))}, \tag{3.2}$$

where $C > 0$ depends on n, N, Λ, k.

As in classical Campanato method, the following inequalities will play crucial roles in our procedure.

Lemma 3.1 *Let $v \in H^1(Q_1(0); \mathbb{R}^N)$ be a weak solution to system*

$$-\partial_\alpha \left(A_{ij}^{\alpha\beta}(x)\partial_\beta v^j \right) = 0 \ \textit{in} \ Q_1(0) \tag{3.3}$$

where $\{A_{ij}^{\alpha\beta}\}$ satisfies Eq. 1.2 and $A_{ij}^{\alpha\beta}\big|_{Q_1^+(0)}$, $A_{ij}^{\alpha\beta}\big|_{Q_1^-(0)}$ are constants. Then

(i) *For any $x_0 \in \Gamma_{1/2}(0)$, $0 < \rho \leq R < dist(x_0, \partial Q_1(0))$, we have*

$$\int_{Q_\rho(x_0)} |Dv|^2 \,\mathrm{d}x \leq C\left(\frac{\rho}{R}\right)^n \int_{Q_R(x_0)} |Dv|^2 \,\mathrm{d}x \tag{3.4}$$

and

$$\int_{Q_\rho(x_0)} |D_{x'}v - (D_{x'}v)_{x_0,\rho}|^2 \,\mathrm{d}x \leq C\left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(x_0)} |D_{x'}v - (D_{x'}v)_{x_0,R}|^2 \,\mathrm{d}x, \tag{3.5}$$

where $C > 0$ depends on n, N, Λ.

(ii) *For any $x_0 \in Q_{1/2}^\pm$, $0 < \rho \leq R < dist(x_0, \partial Q_1^\pm(0))$, then Eqs. 3.4 and 3.5 holds.*

Proof We only prove *(i)* and assume $x_0 = 0$. Let $\rho < 1/2$. By Proposition 3.1 and the Poincaré inequality, we have

$$\begin{aligned}\int_{Q_\rho^+(0)} |Dv|^2 \,\mathrm{d}x &\leq C\rho^n \|Dv\|^2_{C^0(Q_\rho^+(0))} \\ &\leq C\rho^n \int_{Q_1(0)} |v - (v)_{0,1}|^2 \,\mathrm{d}x \\ &\leq C\rho^n \int_{Q_1(0)} |Dv|^2 \,\mathrm{d}x,\end{aligned}$$

where $(v)_{0,1} = \fint_{Q_1(0)} v \, dx$. For $\rho \geq 1/2$, above inequality holds by choosing $C \geq 2^n$. Similarly, we have

$$\int_{Q_\rho^-(0)} |Dv|^2 \, dx \leq C\rho^n \int_{Q_1(0)} |Dv|^2 \, dx.$$

By rescaling, then we complete the proof of Eq. 3.4.

On the other hand, Proposition 3.1 informs us that $D_{x'}v$ is continuous in $Q_1(0)$. By mean value theorem and noting that $D_{x'}v - (D_{x'}v)_{0,1}$ are still solutions to Eq. 3.3, we have, for $0 < \rho < 1/2$

$$\begin{aligned}\int_{Q_\rho^+(0)} |D_{x'}v - (D_{x'}v)_{0,\rho}|^2 \, dx &\leq \int_{Q_\rho^+(0)} |D_{x'}v - D_{x'}v(0)|^2 \, dx \\ &\leq C\rho^{n+2} \|DD_{x'}v\|^2_{C^0(Q_\rho^+(0))} \\ &\leq C\rho^{n+2} \int_{Q_1(0)} |D_{x'}v - (D_{x'}v)_{0,1}|^2 \, dx.\end{aligned}$$

Similarly, for $\rho \geq 1/2$ and $Q_\rho^-(0)$ the above inequality is till valid. By rescaling, then we complete the proof of Eq. 3.5. □

In order to handle problems in generalized Morrey spaces and Campanato–John–Nirenberg spaces, we need generalized iteration lemma, see [11] for proof.

Lemma 3.2 *Let H be a nonnegative almost increasing function on $[0, R_0]$ and F a positive function on $(0, R_0]$. Suppose that*

(i) *There exist $A, B, \varepsilon, \beta > 0$ such that*

$$H(\rho) \leq \left(A\left(\frac{\rho}{R}\right)^\beta + \varepsilon\right) H(R) + B \cdot F(R) \ \text{ for } \ 0 < \rho \leq R \leq R_0.$$

(ii) *There exists $\gamma \in (0, \beta)$ such that $\rho^\gamma / F(\rho)$ is almost increasing in $(0, R_0]$.*
Then there exist ε_0 and C such that if $\varepsilon < \varepsilon_0$ then

$$H(\rho) \leq C\frac{F(\rho)}{F(R)} H(R) + CB \cdot F(\rho) \ \text{ for } \ 0 < \rho \leq R \leq R_0,$$

where ε_0 depends only on A, β and γ; C depends only on A, β, γ K_H and $K_{\rho^\gamma/F(\rho)}$.

Lemma 3.3 *Let $u \in H^1(Q_1(0); \mathbb{R}^N)$ be a weak solution to*

$$-\partial_\alpha \left(a_{ij}^{\alpha\beta} \partial_\beta u^j\right) = 0 \ \text{in} \ Q_1(0), \tag{3.6}$$

where $\{a_{ij}^{\alpha\beta}\}$ satisfies Eq. 1.2 and $a_{ij}^{\alpha\beta}|_{Q_1^\pm(0)} \in VMO(Q_1^\pm(0))$. Then for $0 < \mu < n$, there exist $R_0 \leq 1/4$ and C depending only on n, N, μ, Λ and $[a_{ij}^{\alpha\beta}]_{BMO(Q_1^\pm(0);\sigma)}$ such that for any $x_0 \in Q_{1/2}(0)$ and $0 < \rho \leq R \leq R_0$,

$$\int_{Q_\rho(x_0)} |Du|^2 \, dx \leq C\left(\frac{\rho}{R}\right)^\mu \int_{Q_R(x_0)} |Du|^2 \, dx. \tag{3.7}$$

Proof The proof is similar to Lemma 3.1 of [11], based on Campanato's method and reverse Hölder inequality. We only prove the case $x_0 \in \Gamma_{1/2}(0)$. Let $0 < \rho \leq R \leq R_0 \leq 1/4$, where R_0 will be determined later. By the Lax-Milgram theorem, let v and w be the solutions to

$$\begin{cases} -\partial_\alpha \left(\left(a_{ij}^{\alpha\beta} \right)^{\pm}_{x_0, R} \partial_\beta v^j \right) = 0 \text{ in } Q_R(x_0), \\ v - u \in H_0^1 \left(Q_R(x_0); \mathbb{R}^N \right), \end{cases} \tag{3.8}$$

and

$$\begin{cases} -\partial_\alpha \left(\left(a_{ij}^{\alpha\beta} \right)^{\pm}_{x_0, R} \partial_\beta w^j \right) = -\partial_\alpha \left\{ \left(\left(a_{ij}^{\alpha\beta} \right)^{\pm}_{x_0, R} - a_{ij}^{\alpha\beta}(x) \right) \partial_\beta u^j \right\} \text{ in } Q_R(x_0), \\ w \in H_0^1 \left(Q_R(x_0); \mathbb{R}^N \right), \end{cases} \tag{3.9}$$

respectively. Obviously, $u = v + w$ by uniqueness. Taking into account Eq. 3.4 in Lemma 3.1, we have

$$\begin{aligned} \int_{Q_\rho(x_0)} |Du|^2 &\leq C \left(\frac{\rho}{R} \right)^n \int_{Q_R(x_0)} |Dv|^2 + 2 \int_{Q_\rho(x_0)} |Dw|^2 \\ &\leq C \left(\frac{\rho}{R} \right)^n \int_{Q_R(x_0)} |Du|^2 + C \int_{Q_R(x_0)} |Dw|^2. \end{aligned} \tag{3.10}$$

In view of Eq. 3.6, we have the reverse Hölder inequality, i.e., there exists some $p \in (2, \infty)$ depending only on n, N, Λ such that

$$\left(⨍_{Q_R(x_0)} |Du|^p \right)^{2/p} \leq C ⨍_{Q_{2R}(x_0)} |Du|^2, \text{ if } R < \frac{1}{2\sqrt{n}} dist(x_0, \partial Q_1(0)), \tag{3.11}$$

where $C > 0$ depending only on n, N, Λ and p. From now on, p is fixed. We refer to [10] for the proof of reverse Hölder inequality. Multiplying Eq. 3.9 by w and integrating by parts, we have

$$\begin{aligned} \int_{Q_R(x_0)} |Dw|^2 &= \int_{Q_R(x_0)} \left(\left(a_{ij}^{\alpha\beta} \right)^{\pm}_{x_0, R} - a_{ij}^{\alpha\beta}(x) \right) \partial_\beta u^j \partial_\beta w^i \\ &\leq \left(\int_{Q_R(x_0)} \left| \left(a_{ij}^{\alpha\beta} \right)^{\pm}_{x_0, R} - a_{ij}^{\alpha\beta}(x) \right|^2 |Du|^2 \right)^{1/2} \left(\int_{Q_R(x_0)} |Dw|^2 \right)^{1/2}, \end{aligned}$$

where we used Hölder inequality in the last inequality. Therefore,

$$\begin{aligned}\int_{Q_R(x_0)} |Dw|^2 &\leq \int_{Q_R(x_0)} \left| \left(a_{ij}^{\alpha\beta}\right)^{\pm}_{x_0,R} - a_{ij}^{\alpha\beta}(x)\right|^2 |Du|^2 \\ &\leq |Q_R(x_0)| \left(\fint_{Q_R(x_0)} \left| \left(a_{ij}^{\alpha\beta}\right)^{\pm}_{x_0,R} - a_{ij}^{\alpha\beta}(x)\right|^{2p/(p-2)} \right)^{(p-2)/p} \\ &\quad \times \left(\fint_{Q_R(x_0)} |Du|^p \right)^{2/p} \\ &\leq C \left(\fint_{Q_R(x_0)} \left| \left(a_{ij}^{\alpha\beta}\right)^{\pm}_{x_0,R} - a_{ij}^{\alpha\beta}(x)\right|^{2p/(p-2)} \right)^{(p-2)/p} \int_{Q_{2R}(x_0)} |Du|^2, \quad (3.12)\end{aligned}$$

where p, C is the same as the ones in Eq. 3.11. Since $a_{ij}^{\alpha\beta}|_{Q_1^{\pm}(0)} \in L^\infty(Q_1^{\pm}(0)) \cap VMO(Q_1^{\pm}(0))$, we have, for any $\varepsilon > 0$, there exists $R_0 > 0$ depending on ε such that

$$\max\left\{ \left[a_{ij}^{\alpha\beta}\right]^2_{BMO(Q_1^+(0);R)}, \left[a_{ij}^{\alpha\beta}\right]^2_{BMO(Q_1^-(0);R)} \right\} < \varepsilon, \tag{3.13}$$

once $R \leq R_0$. By the John–Nirenberg inequality, see [13] or Proposition 1.13 in [1], we have

$$\begin{aligned}&\left(\fint_{Q_R(x_0)} \left| \left(a_{ij}^{\alpha\beta}\right)^{\pm}_{x_0,R} - a_{ij}^{\alpha\beta}(x)\right|^{2p/(p-2)} \right)^{(p-2)/p} \\ &\quad \leq C \max\left\{ \left[a_{ij}^{\alpha\beta}\right]^2_{BMO(Q_1^+(0);R)}, \left[a_{ij}^{\alpha\beta}\right]^2_{BMO(Q_1^-(0);R)} \right\}, \qquad (3.14)\end{aligned}$$

where $C > 0$ depending only on p, n. It follows from Eqs. 3.12, 3.13 and 3.14 that

$$\int_{Q_R(x_0)} |Dw|^2 \leq C\varepsilon \int_{Q_{2R}(x_0)} |Du|^2, \tag{3.15}$$

for all $R < \min\{R_0, \frac{1}{2\sqrt{n}} dist(x_0, \partial Q_1(0))\}$. Since p is fixed, the positive constant C in Eq. 3.15 can be chosen to depend only on n, N, Λ. It follows from Eq. 3.10 that

$$\int_{Q_\rho(x_0)} |Du|^2 \leq C\left(\left(\frac{\rho}{R}\right)^n + \varepsilon \right) \int_{Q_{2R}(x_0)} |Du|^2,$$

for $\rho < R < \min\{R_0, \frac{1}{2\sqrt{n}} dist(x_0, \partial Q_1(0))\}$, where $C > 0$ depends only on n, N, Λ. Applying Lemma 3.2, we obtain

$$\int_{Q_\rho(x_0)} |Du|^2 \leq C\left(\frac{\rho}{R}\right)^{\mu} \int_{Q_{2R}(x_0)} |Du|^2, \tag{3.16}$$

with $F = \rho^{\mu}$.

Hence, the Lemma follows immediately. □

Proof of Theorem 2.1 First, we prove Theorem 2.1 in $Q_1(0)$ for the case that $\{a_{ij}^{\alpha\beta}\}$ is as in the Lemma 3.3, and

$$g \in L_\varphi^{2n/(n+2),\mu n/(n+2)}(Q_1(0)), \quad f \in L_\varphi^{2,\mu}(Q_1(0)). \tag{3.17}$$

(i) Let $x_0 \in \Gamma_{1/2}(0)$, $0 < \rho < R \leq R_0$, where R_0 is given in Lemma 3.3. Let v and w be weak solutions to

$$\begin{cases} -\partial_\alpha \left(a_{ij}^{\alpha\beta}(x)\partial_\beta v^j\right) = 0 \text{ in } Q_R(x_0) \\ v - u \in H_0^1\left(Q_R(x_0); \mathbb{R}^N\right) \end{cases} \tag{3.18}$$

and

$$\begin{cases} -\partial_\alpha \left(a_{ij}^{\alpha\beta}(x)\partial_\beta w^j\right) = g_i - \partial_\beta f_i^\beta \text{ in } Q_R(x_0) \\ w \in H_0^1\left(Q_R(x_0); \mathbb{R}^N\right), \end{cases} \tag{3.19}$$

respectively. By uniqueness, $u = v + w$. Let $0 < \mu < \gamma < n$. Applying Eq. 3.7 to v with respect to the index γ, we obtain

$$\int_{Q_\rho(x_0)} |Du|^2 \leq C\left(\frac{\rho}{R}\right)^\gamma \int_{Q_R(x_0)} |Du|^2 + C\int_{Q_R(x_0)} |Dw|^2. \tag{3.20}$$

From Eq. 3.19, it follows that

$$\int_{Q_R(x_0)} |Dw|^2 \leq C\int_{Q_R(x_0)} |g|\cdot|w| + C\int_{Q_R(x_0)} |f|\cdot|Dw|.$$

By Hölder and Sobolev inequalities

$$\int_{Q_R(x_0)} |Dw|^2 \leq C\left(\int_{Q_R(x_0)} |g|^{2n/(n+2)}\right)^{1+2/n} + C\int_{Q_R(x_0)} |f|^2. \tag{3.21}$$

Combining Eqs. 3.17, 3.20 and 3.21, we have

$$\int_{Q_\rho(x_0)} |Du|^2 \leq C\left(\frac{\rho}{R}\right)^\gamma \int_{Q_R(x_0)} |Du|^2 + CR^\mu\varphi^2(R)\left(\|g\|^2_{L_\varphi^{2n/(n+2),\mu n/(n+2)}(Q_1(0))} + \|f\|^2_{L_\varphi^{2,\mu}(Q_1(0))}\right).$$

It follows Lemma 32 that

$$\int_{Q_\rho(x_0)} |Du|^2 \leq C\frac{\rho^\mu\varphi^2(\rho)}{R^\mu\varphi^2(R)} \int_{Q_R(x_0)} |Du|^2 + C\rho^\mu\varphi^2(\rho)\left(\|g\|^2_{L_\varphi^{2n/(n+2),\mu n/(n+2)}(Q_1(0))} + \|f\|^2_{L_\varphi^{2,\mu}(Q_1(0))}\right). \tag{3.22}$$

(ii) Let $x_0 \in \Gamma_{1/2}(0) \times (-R_1, R_1)$ with $R_1 = \frac{1}{2}R_0$, $0 < \rho < R \leq R_1$, and $Q_\rho(x_0) \cap \Gamma(0, \frac{1}{2}) \neq \emptyset$, then one can find a point $x_1 \in \Gamma(0, \frac{1}{2})$ such that $Q_\rho(x_0) \subset Q_{2\rho}(x_1)$. Consequently, it is easy to see that Eq. 3.22 holds in this case.

(iii) For $x_0 \in \Gamma_{1/2}(0) \times (-R_1, R_1)$ satisfying $Q_\rho(x_0) \cap \Gamma_{1/2}(0) = \emptyset$, or $x_0 \in \{\Gamma_{1/2}(0) \times (-1/2, -R_1]\} \cup \{\Gamma_{1/2}(0) \times [R_1, 1/2)\}$, then one can establish Eq. 3.22 as the procedure in case (i) but simpler since standard Campanato method can be applied.

In combination, we have

$$\|Du\|_{L_\varphi^{2,\mu}(Q_{1/2}(0))} \leq C\left(\|Du\|_{L^2(Q_1(0))} + \|f\|_{L_\varphi^{2,\mu}(Q_1(0))} + \|g\|_{L_\varphi^{2n/(n+2),\mu n/(n+2)}(Q_1(0))}\right).$$

Let $x_0 \in \partial\Omega_m$ $(m = 1, \cdots, L)$. Since $\partial\Omega_m \in C^{1,VMO}$, there exist a neighborhood N_{x_0} and a Lipschitz transform $\mathcal{T}$ such that

$$\mathcal{T}(N_{x_0} \cap \Omega_m) = B_1^+,$$
$$\mathcal{T}(N_{x_0} \cap \Omega_m^c) = B_1^-,$$

$$D\mathcal{T}\big|_{N_{x_0}\cap\Omega_m} \in VMO(N_{x_0} \cap \Omega_m),\ \ D\mathcal{T}^{-1}\big|_{B_1^+} \in VMO(B_1^+)$$
$$D\mathcal{T}\big|_{N_{x_0}\cap\Omega_m^c} \in VMO(N_{x_0} \cap \Omega_m^c),\ \ D\mathcal{T}^{-1}\big|_{B_1^-} \in VMO(B_1^-).$$

Set $y = \mathcal{T}x,\ \overline{u}(y) = u(\mathcal{T}y)$. Let $\phi \in C_0^\infty(N_{x_0}; \mathbb{R}^N)$, then

$$\int_{N_{x_0}} a_{ij}^{\alpha\beta}(x)\frac{\partial u^j}{\partial x_\beta}\frac{\partial \phi^i}{\partial x_\alpha}\,\mathrm{d}x = \int_{B_1} a_{ij}^{\alpha\beta}(x)\frac{\partial y_\iota}{\partial x_\beta}\frac{\partial y_\vartheta}{\partial x_\alpha}\frac{\partial \overline{u}^j}{\partial y_\iota}\frac{\partial \overline{\phi}^i}{\partial y_\vartheta}|D\mathcal{T}^{-1}(y)|\,\mathrm{d}y,$$

where $\overline{\phi} = \phi\circ\mathcal{T}^{-1}$ and $|A| = |\det A|$. Therefore, $\overline{u}$ satisfies

$$-\partial_\vartheta\left(\overline{a}_{ij}^{\vartheta\iota}(y)\partial_\iota\overline{u}^j\right) = \overline{g}_i(y) - \partial_\iota\overline{f}_i^\iota(y) \text{ in } B_1,$$

where

$$\overline{a}_{ij}^{\vartheta\iota} = a_{ij}^{\alpha\beta}\circ\mathcal{T}^{-1}\frac{\partial y_\iota}{\partial x_\beta}\frac{\partial y_\vartheta}{\partial x_\alpha}|D\mathcal{T}^{-1}|,$$

$$\overline{g}_i = g_i\circ\mathcal{T}^{-1}|D\mathcal{T}^{-1}|,$$

$$\overline{f}_i^\iota = f_i^\beta\circ\mathcal{T}^{-1}\frac{\partial y_\iota}{\partial x_\beta}\left|D\mathcal{T}^{-1}\right|.$$

Arguing as in [11], we conclude that there exists a constant $\delta > 0$ such that

$$\|Du\|_{L_\varphi^{2,\mu}(\Omega_m^\delta)} \leq C\left(\|Du\|_{L^2(\Omega)} + \|f\|_{L_\varphi^{2,\mu}(\Omega)} + \|g\|_{L_\varphi^{2n/(n+2),\mu n/(n+2)}(\Omega)}\right), \tag{3.23}$$

where $\Omega_m^\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega_m) < \delta\},\ \ m = 1, \cdots, L$. We complete the proof by taking into account the proof in step (iii) and Eq. 3.23. □

4 Estimates in Generalized Campanato–John–Nirenberg Space

In the section we shall prove Theorem 2.2.

Let $u \in H^1(Q_1(0); \mathbb{R}^N)$ be a weak solution to

$$-\partial_\alpha\left(a_{ij}^{\alpha\beta}(x)\partial_\beta u^j\right) = g_i(x) - \partial_\beta f_i^\beta(x) \text{ in } Q_1(0), \tag{4.1}$$

where $a_{ij}^{\alpha\beta}(x)$ satisfies Eq. 1.2 in $Q_1(0)$ and

$$a_{ij}^{\alpha\beta}\big|_{Q_1^\pm(0)} \in VMO_\omega\left(Q_1^\pm(0)\right), \tag{4.2}$$

and

$$g \in L_{\psi}^{2n/(n+2),n^2/(n+2)}(Q_1(0)), \;\; f\big|_{Q_1^{\pm}(0)} \in BMO_{\psi}\left(Q_1^{\pm}(0)\right), \tag{4.3}$$

where $g = (g_1, \cdots, g_N)$, $f = \{f_i^{\beta}\}$ and

$$\omega(r) = \psi(r)\left(\int_r^d \frac{\psi(t)}{t}\right)^{-1}$$

with $d = diam\, Q_1(0)$.

Lemma 4.1 (Approximation) *Let* $x_0 \in \Gamma_{1/2}(0)$, $0 < \rho \le 1/2$ *and* $h \in H^1(Q_\rho(x_0); \mathbb{R}^N)$ *be a weak solution to*

$$\begin{cases} -\partial_\alpha\left(\left(a_{ij}^{\alpha\beta}\right)_{x_0,\rho}^{\pm} \partial_\beta h^j\right) = -\partial_\beta\left(f_i^{\beta}\right)_{x_0,\rho}^{\pm} & in\ Q_\rho(x_0) \\ h - u \in H_0^1\left(Q_\rho(x_0); \mathbb{R}^N\right). \end{cases} \tag{4.4}$$

Then

$$\int_{Q_\rho(x_0)} |Du - Dh|^2 \le C\int_{Q_\rho(x_0)} \left|\left(a_{ij}^{\alpha\beta}\right)_{x_0,\rho}^{\pm} - a_{ij}^{\alpha\beta}\right|^2 |Du|^2 + C\rho^n\psi^2(\rho)L(f,g), \tag{4.5}$$

and

$$\int_{Q_{\rho/2}(x_0)} \left|Dh - (Dh)_{x_0,\rho/2}^{\pm}\right|^2 \le C\int_{Q_\rho(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,\rho}\right|^2 + C\int_{Q_\rho(x_0)} |Du - Dh|^2, \tag{4.6}$$

where

$$L(f,g) = \|g(x)\|^2_{L_{\psi}^{2n/(n+2),n^2/(n+2)}(Q_1(0))} + \|f\|^2_{BMO_\psi(Q_1^+(0))} + \|f\|^2_{BMO_\psi(Q_1^-(0))},$$

and $C > 0$ *depends only on* n, N, Λ.

Proof By Eqs. 4.1 and 4.4, we have

$$-\partial_\alpha\left(\left(a_{ij}^{\alpha\beta}\right)_{x_0,\rho}^{\pm} \partial_\beta\left(h^j - u^j\right)\right) = -\partial_\alpha\left(\left(\left(a_{ij}^{\alpha\beta}\right)_{x_0,\rho}^{\pm} - a_{ij}^{\alpha\beta}(x)\right)u^j\right) - g + \partial_\beta f_i^{\beta} - \partial_\beta\left(f_i^{\beta}\right)_{x_0,\rho}^{\pm}. \tag{4.7}$$

Multiplying Eq. 4.7 by $h - u$, then Eq. 4.5 follows from integrating by parts and the Sobolev and Hölder inequalities.

According to Proposition 3.1, $h \in C^\infty(Q_\rho^\pm(0))$. Note that $D_{x'}h - M$ (M is a constant matrix) satisfies

$$-\partial_\alpha \left(\left(a_{ij}^{\alpha\beta} \right)_{x_0,\rho}^{\pm} \partial_\beta \left(D_{x'}h^j - M^j \right) \right) = 0.$$

It follows from Proposition 3.1 that

$$\begin{aligned} \|D^2 h\|_{C^0(Q_{\rho/2}^\pm(x_0))} &\le \frac{C}{\rho^{(n+2)/2}} \|D_{x'}h - (D_{x'}u)_{x_0,\rho}\|_{L^2(Q_\rho(x_0))} \\ &\le \frac{C}{\rho^{(n+2)/2}} \{\|D_{x'}h - D_{x'}u\|_{L^2(Q_\rho(x_0))} \\ &\qquad + \|D_{x'}u - (D_{x'}u)_{x_0,\rho}\|_{L^2(Q_\rho(x_0))}\}. \end{aligned} \tag{4.8}$$

Note that

$$\begin{aligned} &\int_{Q_{\rho/2}(x_0)} \left| Dh - (Dh)_{x_0,\rho/2}^{\pm} \right|^2 \\ &\quad \le C\rho^{n+2} \left(\|D^2 h\|_{C^0(Q_{\rho/2}^+(x_0))}^2 + \|D^2 h\|_{C^0(Q_{\rho/2}^-(x_0))}^2 \right), \end{aligned} \tag{4.9}$$

thus Eq. 4.6 follows. □

From the proof above, it is easy to see that if $x_0 \in Q_{1/2}^\pm(0)$ and $\rho \le dist(x_0, \partial Q_1^\pm(0))$, then Eq. 4.5 is valid by replacing $(a_{ij}^{\alpha\beta})_{x_0,\rho}^\pm$ with $(a_{ij}^{\alpha\beta})_{x_0,\rho}$ and Eq. 4.6 is valid by replacing $(Dh)_{x_0,\rho/2}^\pm$ with $(Dh)_{x_0,\rho/2}$.

Corollary 4.1 *Assume the above, then for* $x_0 \in \Gamma_{1/2}(0)$ *and* $0 < \rho < \frac{1}{2}$,

$$\begin{aligned} \int_{Q_{\rho/2}(x_0)} \left| Du - (Du)_{x_0,\rho/2}^{\pm} \right|^2 &\le C \int_{Q_\rho(x_0)} \left| D_{x'}u - (D_{x'}u)_{x_0,\rho} \right|^2 \\ &\quad + C \int_{Q_\rho(x_0)} \left| \left(a_{ij}^{\alpha\beta} \right)_{x_0,\rho}^{\pm} - a_{ij}^{\alpha\beta} \right|^2 |Du|^2 \\ &\quad + C\rho^n \psi^2(\rho) L(f,g). \end{aligned} \tag{4.10}$$

Proof The corollary follows from direct computation and Lemma 4.1.

$$\begin{aligned} \int_{Q_{\rho/2}(x_0)} \left| Du - (Du)_{x_0,\rho/2}^{\pm} \right|^2 &\le \int_{Q_{\rho/2}(x_0)} \left| Du - (Dh)_{x_0,\rho/2}^{\pm} \right|^2 \\ &\le 2\int_{Q_{\rho/2}(x_0)} |D(u-h)|^2 + 2\int_{Q_{\rho/2}(x_0)} \left| Dh - (Dh)_{x_0,\rho/2}^{\pm} \right|^2 \\ &\le C\int_{Q_\rho(x_0)} |D(u-h)|^2 + C\int_{Q_\rho(x_0)} \left| D_{x'}u - (D_{x'}u)_{x_0,\rho} \right|^2. \end{aligned}$$

By Eq. 4.5, we complete the proof. □

Similar to Lemma 4.1, if $x_0 \in Q^{\pm}_{1/2}(0)$ and $\rho \leq dist(x_0, \partial Q^{\pm}_1(0))$, then Eq. 4.10 is valid by replacing $(a^{\alpha\beta}_{ij})^{\pm}_{x_0,\rho}$ and $(Du)^{\pm}_{x_0,\rho/2}$ with $(a^{\alpha\beta}_{ij})_{x_0,\rho}$ and $(Du)_{x_0,\rho/2}$, respectively.

The following proposition is proved in [1] and [11].

Proposition 4.1

(i) *Let D be a bounded open subset of $\mathbb{R}^n$. Let $D' \subset\subset D$ and $\delta = dist(D', \partial D)/\sqrt{n}$. For $x_0 \in \overline{D'}$, $\sigma \in (0, \delta]$ and $f \in BMO_\psi$, we have*

$$\left| f_{Q_\sigma(x_0)} \right| \leq C \left\{ \int_\sigma^\delta \frac{\psi(t)}{t} \cdot [f]_{BMO_\psi(D)} + \delta^{-n/2} \| f \|_{L^2(D)} \right\}, \tag{4.11}$$

where $C > 0$ depends only on n, K_ψ.

(ii) *Let $x_0 \in \{x \in \mathbb{R}^n : x_n = 0\}$ and fix $r > r' > 0$. For $x \in \Gamma_{r'}(x_0)$, $\sigma \in (0, r - r')$ and $f \in BMO_\psi(Q^+_r(x_0))$ we have*

$$\left| f_{Q^+_\sigma(x)} \right| \leq C \left\{ \int_\sigma^{r-r'} \frac{\psi(t)}{t} \cdot [f]_{BMO_\psi(Q^+_r(x_0))} + \left(r - r'\right)^{-n/2} \| f \|_{L^2\left(Q^+_r(x_0)\right)} \right\}, \tag{4.12}$$

where $C > 0$ depends only on n, K_ψ.

Corollary 4.2 *Let $u \in H^1(\mathbb{R}^n; \mathbb{R}^N)$ be a weak solution of Eq. 4.1 in $\mathbb{R}^n$. Suppose ψ and $r/\psi(r)$ are almost increasing. Assume $supp(u) \subset Q_{R^*}(0)$ for some $R^* > 0$, $D_{x'}u \in BMO_\psi(\mathbb{R}^n)$ and $Du \in BMO_\psi(\mathbb{R}^n_\pm)$. Suppose $a^{\alpha\beta}_{ij}$, g and f satisfy Eqs. 4.2 and 4.3 by replacing $\mathbb{R}^n$ and $\mathbb{R}^n_\pm$ into $Q_1(0)$ and $Q^\pm_1(0)$, respectively. Then we have*

$$[Du]_{BMO_\psi(\mathbb{R}^n_+)} + [Du]_{BMO_\psi(\mathbb{R}^n_-)} \leq C \left([D_{x'}u]_{BMO_\psi(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} + L^{1/2}(f, g) \right), \tag{4.13}$$

where $C > 0$ depends only on n, N, Λ, K_ψ, $K_{r/\psi(r)}$ and $[a^{\alpha\beta}_{ij}]_{BMO_\omega(\mathbb{R}^n_\pm;\sigma)}$.

Furthermore, if ψ satisfies Eq. 2.1, then $a^{\alpha\beta}_{ij}$ can be reduced to be in $BMO_\psi(\mathbb{R}^n_\pm)$.

Proof

(i) Let $x_0 \in \Gamma_{R^*}(0)$, $0 < \rho \leq \overline{R}$, where $\overline{R}$ will be determined later. By Proposition 4.1 and Hölder inequality, we have

$$\begin{aligned}
&\int_{Q^+_\rho(x_0)} \left| \left(a^{\alpha\beta}_{ij}\right)^+_{x_0,\rho} - a^{\alpha\beta}_{ij} \right|^2 |Du|^2 \\
&\leq C \int_{Q^+_\rho(x_0)} \left| \left(a^{\alpha\beta}_{ij}\right)^+_{x_0,\rho} - a^{\alpha\beta}_{ij} \right|^2 |Du - (Du)^+_{x_0,\rho}|^2 \\
&\quad + C \int_{Q^+_\rho(x_0)} \left| \left(a^{\alpha\beta}_{ij}\right)^+_{x_0,\rho} - a^{\alpha\beta}_{ij} \right|^2 |(Du)^+_{x_0,\rho}|^2 \\
&\leq C\rho^n \psi^2(\rho)\omega^2(\rho) \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} [Du]^2_{BMO_\psi(\mathbb{R}^n_+)} \\
&\quad + C\rho^n \omega^2(\rho) \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})}
\end{aligned}$$

$$\times\left(\int_\rho^d \frac{\psi(t)}{t}[Du]_{BMO_\psi(\mathbb{R}^n_+)} + \|Du\|_{L^2(Q^+_{2\rho}(x_0))}\right)^2$$
$$\leq C\rho^n\psi^2(\rho)[a^{\alpha\beta}_{ij}]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})}$$
$$\times\left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + \|u\|^2_{L^2(\mathbb{R}^n)} + L(f,g)\right), \tag{4.14}$$

where $d = diam Q_{R^*}(0)$. In the last inequality we have used the definition of $\omega(r)$ and $L^{2,0}_\psi$ estimates of Du. Similarly,

$$\int_{Q^-_\rho(x_0)}\left|\left(a^{\alpha\beta}_{ij}\right)^-_{x_0,\rho} - a^{\alpha\beta}_{ij}\right|^2|Du|^2$$
$$\leq C\rho^n\psi^2(\rho)\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\left([Du]^2_{BMO_\psi(\mathbb{R}^n_-)} + \|u\|^2_{L^2(\mathbb{R}^n)} + L(f,g)\right). \tag{4.15}$$

In view of Eq. 4.10, we have

$$\int_{Q^+_{\rho/2}(x_0)}\left|Du - (Du)^+_{x_0,\rho/2}\right|^2$$
$$\leq C\rho^n\psi^2(\rho)\,[D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)}$$
$$+ C\rho^n\psi^2(\rho)\left(\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right)$$
$$\times\left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right)$$
$$+ C\rho^n\psi^2(\rho)(L(f,g) + \|u\|^2_{L^2(\mathbb{R}^n)}). \tag{4.16}$$

Clearly, Eq. 4.16 holds for $Q^-_{\rho/2}(x_0)$.

(ii) For $x_0 \in \Gamma_{R^*}(0)\times(-\overline{R}/8, \overline{R}/8)$, $\rho \leq \overline{R}/8$ and $Q_{2\rho}(x_0)\cap\Gamma_{R^*}(0) \neq \emptyset$, then one can find a point $x_1 \in \Gamma_{R^*}(0)$ such that $Q_\rho(x_0) \subset Q_{4\rho}(x_1)$. Without loss of generality, we assume $x_0 \in Q^+_{R^*}(0)$. It follows that

$$\int_{Q^+_\rho(x_0)}\left|Du - (Du)^+_{x_0,\rho}\right|^2$$
$$\leq \int_{Q^+_{2\rho}(x_1)}|Du - (Du)^+_{x_1,4\rho}|^2$$
$$\leq C\rho^n\psi^2(8\rho)\,[D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)}$$
$$+ C\rho^n\psi^2(8\rho)\left(\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right)$$
$$\times\left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right)$$
$$+ C\rho^n\psi^2(8\rho)(L(f,g) + \|u\|^2_{L^2(\mathbb{R}^n)}) \tag{4.17}$$

(iii) For $x_0 \in \Gamma_{R^*}(0)\times(-\overline{R}/8, \overline{R}/8)$, $\rho \leq \overline{R}/8$ and $Q_{2\rho}(x_0)\cap\Gamma_{R^*}(0) \neq \emptyset$, or $x_0 \in \{\Gamma_{R^*}(0)\times(-R^*, -\overline{R}/8)\}\cup\{\Gamma_{R^*}(0)\times(\overline{R}/8, R^*)\}$, $\rho \leq \overline{R}/8$, then Eq. 4.16 holds.

From discussion above and noting that $\psi(2\rho) \leq 2K_{r/\psi(r)}\psi(\rho)$, then for any point $x_0 \in Q^+_{R^*}$ and $0 < \rho < \overline{R}/8$,

$$\begin{aligned}
&\frac{1}{\psi^2(\rho)} \fint_{Q^+_\rho(x_0)} \left|Du - (Du)^+_{x_0,\rho}\right|^2 \\
&\leq C\,[D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)} + C\left(\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right) \\
&\quad \times \left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right) + C(L(f,g) + \|u\|^2_{L^2(\mathbb{R}^n)}). \qquad (4.18)
\end{aligned}$$

Similarly, for any point $x_0 \in Q^-_{R^*}$ and $0 < \rho < \overline{R}/8$,

$$\begin{aligned}
&\frac{1}{\psi^2(\rho)} \fint_{Q^-_\rho(x_0)} \left|Du - (Du)^-_{x_0,\rho}\right|^2 \\
&\leq C\,[D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)} + C\left(\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right) \\
&\quad \times \left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right) + C(L(f,g) + \|u\|^2_{L^2(\mathbb{R}^n)}). \qquad (4.19)
\end{aligned}$$

It follows from Eqs. 4.18 and 4.19 that

$$\begin{aligned}
&[Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)} \\
&\leq C\,[D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)} + C\left(\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\psi(\mathbb{R}^n_-;\overline{R})} + \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\psi(\mathbb{R}^n_+;\overline{R})}\right) \\
&\quad \times \left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right) + C(L(f,g) + \|u\|^2_{L^2(\mathbb{R}^n)}), \qquad (4.20)
\end{aligned}$$

where $C > 0$ is independent of $\overline{R}$. Choosing $\overline{R}$ sufficiently small and noting that if $a^{\alpha\beta}_{ij} \in VMO_\omega(\mathbb{R}^n)$, then Eq. 4.13 follows from Eq. 4.20.

Next, we treat the case ψ satisfying Eq. 2.1 and $a^{\alpha\beta}_{ij} \in BMO_\psi(\mathbb{R}^n)$. Note that if ψ satisfies Eq. 2.1, by Proposition 2.2, then $Du \in L^\infty(\mathbb{R}^n)$ and

$$\begin{aligned}
&\int_{Q^+_\rho(x_0)} |(a^{\alpha\beta}_{ij})^+_{x_0,\rho} - a^{\alpha\beta}_{ij}|^2 |Du|^2 \\
&\leq C\rho^n \psi^2(\rho)[a^{\alpha\beta}_{ij}]_{BMO_\psi(\mathbb{R}^n_+)} \|Du\|^2_{L^\infty(\mathbb{R}^n_+)} \\
&\leq C\rho^n \psi^2(\rho)\left(\int_0^\varepsilon \frac{\psi(t)}{t}\,\mathrm{d}t \cdot [Du]_{BMO_\psi(\mathbb{R}^n_+)} + \varepsilon^{-n/2}\|Du\|_{L^2(\mathbb{R}^n_+)}\right)^2, \qquad (4.21)
\end{aligned}$$

where $C > 0$ is independent of ε. The rest of proof is similar to above by choosing sufficiently small ε if it is necessary. □

Lemma 4.2 *Let $u \in H^1(\mathbb{R}^n; \mathbb{R}^N)$ be a weak solution of Eq. 4.1 in $\mathbb{R}^n$. Assume $supp(u) \subset Q_{R^*}(0)$ for some $R^* > 0$. Suppose $a_{ij}^{\alpha\beta} \in C^{0,1}(\mathbb{R}^n_\pm)$ and satisfies Eq. 1.2, and g, f satisfy Eq. 4.2 by replacing $Q_1(0)$ with $\mathbb{R}^n$. Then $D_{x'}u \in BMO_\psi(\mathbb{R}^n)$ and $Du \in BMO_\psi(\mathbb{R}^n_\pm)$, plus, we have*

$$[D_{x'}u]_{BMO_\psi(\mathbb{R}^n)} \leq C\left(\|u\|_{L^2(\mathbb{R}^n)} + L^{1/2}(f,g)\right), \tag{4.22}$$

and

$$[Du]_{BMO_\psi(\mathbb{R}^n_\pm)} \leq C\left(\|u\|_{L^2(\mathbb{R}^n)} + L^{1/2}(f,g)\right), \tag{4.23}$$

where $C > 0$ depends only on n, N, Λ, K_ψ, $K_{r^\lambda/\psi(r)}$, and $[a_{ij}^{\alpha\beta}]_{BMO_\omega(\mathbb{R}^n_\pm;\sigma)}$.

Furthermore, here the constant C can be required to depend on $[a_{ij}^{\alpha\beta}]_{BMO_\psi(\mathbb{R}^n_\pm)}$ instead of $[a_{ij}^{\alpha\beta}]_{BMO_\omega(\mathbb{R}^n_\pm;\sigma)}$ when ψ satisfies Eq. 2.1.

Proof First of all, we show $D_{x'}u \in BMO_\psi(\mathbb{R}^n)$ and $Du \in BMO_\psi(\mathbb{R}^n_\pm)$. For $x_0 \in \Gamma_{R^*}(0)$, $\rho < R \leq d = diam(Q_{R^*}(0))$, let v solve

$$\begin{cases} -\partial_\alpha\left(\left(a_{ij}^{\alpha\beta}\right)^\pm_{x_0,R}\partial_\beta v^j\right) = 0 \text{ in } Q_R(x_0), \\ v - u \in H_0^1\left(Q_R(x_0); \mathbb{R}^N\right), \end{cases} \tag{4.24}$$

and let $w \in H_0^1(Q_R(x_0); \mathbb{R}^N)$ solve

$$-\partial_\alpha\left(\left(a_{ij}^{\alpha\beta}\right)^\pm_{x_0,R}\partial_\beta w^j\right) = -\partial_\alpha\left\{\left(\left(a_{ij}^{\alpha\beta}\right)^\pm_{x_0,R} - a_{ij}^{\alpha\beta}(x)\right)\partial_\beta u^j\right\} + g_i - \partial_\beta f_i^\beta \text{ in } Q_R(x_0). \tag{4.25}$$

Owing to Eq. 3.5 of Lemma 3.1, for $0 < \rho < R$

$$\int_{Q_\rho(x_0)} \left|D_{x'}v - (D_{x'}v)_{x_0,\rho}\right|^2 \leq C\left(\frac{\rho}{R}\right)^{n+2}\int_{Q_R(x_0)} \left|D_{x'}v - (D_{x'}v)_{x_0,R}\right|^2.$$

Clearly, $u = v + w$. Then we have

$$\begin{aligned} &\int_{Q_\rho(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,\rho}\right|^2 \\ &\quad \leq C\left(\frac{\rho}{R}\right)^{n+2}\int_{Q_R(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,R}\right|^2 + C\int_{Q_R(x_0)} |Dw|^2. \end{aligned} \tag{4.26}$$

From Eq. 4.25, the Sobolev and Hölder inequalities and $a_{ij}^{\alpha\beta} \in C^{0,1}(\mathbb{R}^n_\pm)$

$$\begin{aligned} \int_{Q_R(x_0)} |Dw|^2 &\leq C\int_{Q_R(x_0)} \left|\left(a_{ij}^{\alpha\beta}\right)^\pm_{x_0,R} - a_{ij}^{\alpha\beta}\right|^2 |Du|^2 \\ &\quad + \left(\int_{Q_R(x_0)} |g|^{2n/(n+2)}\right)^{(n+2)/n} + C\int_{Q_R(x_0)} |f - (f)^\pm_{x_0,R}|^2 \\ &\leq CR^2\int_{Q_R(x_0)} |Du|^2 + CR^n\psi^2(R)L(f,g). \end{aligned} \tag{4.27}$$

By the $L^{2,\mu}$ estimates (Theorem 2.2), for $0 < \mu < n$

$$\int_{Q_R(x_0)} |Du|^2 \le CR^{\mu}\|Du\|^2_{L^{2,\mu}(\mathbb{R}^n)}. \tag{4.28}$$

By Eqs. 4.26, 4.27 and 4.28,

$$\int_{Q_\rho(x_0)} |D_{x'}u - (D_{x'}u)_{x_0,\rho}|^2 \le C\left(\frac{\rho}{R}\right)^{n+2}\int_{Q_R(x_0)} |D_{x'}u - (D_{x'}u)_{x_0,R}|^2 + CR^{2+\mu}\|Du\|^2_{L^{2,\mu}(\mathbb{R}^n)} + CR^n\psi^2(R)L(f,g). \tag{4.29}$$

Picking $2+\mu = n+2\lambda$, and using Lemma 3.2, we have

$$\int_{Q_\rho(x_0)} |D_{x'}u - (D_{x'}u)_{x_0,\rho}|^2 \le C\rho^n\psi^2(\rho). \tag{4.30}$$

As in the procedure for proving Corollary 4.2, one can show that $D_{x'}u \in BMO_\psi(\mathbb{R}^n)$. To show $Du \in BMO_\psi(\mathbb{R}^n_\pm)$, we make use of Corollary 4.1. By Eqs. 4.1 and 4.30, and $a^{\alpha\beta}_{ij} \in C^{0,1}(\mathbb{R}^n_\pm)$, we have

$$\int_{Q_{\rho/2}(x_0)} \left|Du - (Du)^{\pm}_{x_0,\rho/2}\right|^2 \le C\int_{Q_\rho(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,\rho}\right|^2 + C\rho^2\int_{Q_\rho(x_0)} |Du|^2 + C\rho^n\psi^2(\rho)L(f,g).$$

By $L^{2,\lambda}$ estimates and $\psi(2\rho) \le 2K_{r/\psi(r)}\psi(\rho)$,

$$\frac{1}{\psi(\rho/2)}\int_{Q_{\rho/2}(x_0)} \left|Du - (Du)^{\pm}_{x_0,\rho/2}\right|^2 \le C\left([D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)} + L(f,g)\right).$$

Then it is not difficult to show $Du \in BMO_\psi(\mathbb{R}^n_\pm)$.

(i) Estimate $[Du]_{BMO_\psi(\mathbb{R}^n_\pm)}$ in terms of $[a^{\alpha\beta}_{ij}]_{BMO_\omega(\mathbb{R}^n_\pm;\sigma)}$. For $x_0 \in \Gamma_{R^*}(0)$, $0 < \rho < R \le \overline{R}$, $\overline{R}$ will be determined later, as before, we have Eq. 4.26. From Eq. 4.25, computing as Eq. 4.14, we obtain

$$\begin{aligned}\int_{Q_R(x_0)} |Dw|^2 &\le C\int_{Q_R(x_0)} \left|\left(a^{\alpha\beta}_{ij}\right)^{\pm}_{x_0,R} - a^{\alpha\beta}_{ij}\right|^2 |Du|^2 + CR^n\psi^2(R)L(f,g)\\ &\le CR^n\psi^2(R)\left(\left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + \left[a^{\alpha\beta}_{ij}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right)\\ &\quad\times\left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right)\\ &\quad + CR^n\psi^2(R)\left(\|u\|_{L^2(\mathbb{R}^n)} + L(f,g)\right).\end{aligned}$$

Therefore, from Eq. 4.26 we have

$$\begin{aligned}
&\int_{Q_\rho(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,\rho}\right|^2 \\
&\leq C\left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,R}\right|^2 + CR^n\psi^2(R) \\
&\times \left(\left[a_{ij}^{\alpha\beta}\right]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + \left[a_{ij}^{\alpha\beta}\right]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right) \\
&\times \left([Du]^2_{BMO_\psi(\mathbb{R}^n_+)} + [Du]^2_{BMO_\psi(\mathbb{R}^n_-)}\right) \\
&+ CR^n\psi^2(R)\left(L(f,g,\mathbb{R}^n) + \|u\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned}$$

By Corollary 4.2 and Lemma 3.2,

$$\begin{aligned}
&\int_{Q_\rho(x_0)} |D_{x'}u - (D_{x'}u)_{x_0,\rho}|^2 \\
&\leq C\frac{\rho^n\psi^2(\rho)}{\overline{R}^n\psi^2(\overline{R})} \int_{Q_{\overline{R}}(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,\overline{R}}\right|^2 \\
&+ CR^n\psi^2(\rho)\left([a_{ij}^{\alpha\beta}]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + [a_{ij}^{\alpha\beta}]^2_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right)[D_{x'}u]^2_{BMO_\psi(\mathbb{R}^n)} \\
&+ CR^n\psi^2(R)\left(L(f,g) + \|u\|^2_{L^2(\mathbb{R}^n)}\right),
\end{aligned} \tag{4.31}$$

where $C > 0$ is independent of $\overline{R}$. With same method used in the proof of Corollary 4.2, for any $x_0 \in \mathbb{R}^n$ and $0 < \rho < R \leq \overline{R}/4$, Eq. 4.31 holds. It implies that

$$\begin{aligned}
[D_{x'}u]_{BMO_\psi(\mathbb{R}^n)} &\leq C(\overline{R})\|Du\|_{L^2} + C\left(L^{1/2}(f,g) + \|u\|_{L^2(\mathbb{R}^n)}\right) \\
&+ C\left([a_{ij}^{\alpha\beta}]_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + [a_{ij}^{\alpha\beta}]_{BMO_\omega(\mathbb{R}^n_-;\overline{R})}\right) \\
&\times [D_{x'}u]_{BMO_\psi(\mathbb{R}^n)}.
\end{aligned} \tag{4.32}$$

Due to $a_{ij}^{\alpha\beta} \in VMO_\omega$ and $L^{2,\mu}$ estimates, by choosing sufficiently small $\overline{R}$, then Eq. 4.22 follows. Taking into account Corollary 4.2, we complete the proof of the first part of the lemma.

(ii) Estimate $[Du]_{BMO_\psi(\mathbb{R}^n_\pm)}$ in terms of $[a_{ij}^{\alpha\beta}]_{BMO_\psi(\mathbb{R}^n_\pm)}$ when ψ satisfies Eq. 2.1. Note that if ψ satisfies Eq. 2.1, by Proposition 2.2, then $Du \in L^\infty(\mathbb{R}^n)$ and

$$\begin{aligned}\int_{Q_R(x_0)} |Dw|^2 &\le C \int_{Q_R(x_0)} |(a_{ij}^{\alpha\beta})^\pm_{x_0,R} - a_{ij}^{\alpha\beta}|^2 |Du|^2 + CR^n\psi^2(R)L(f,g)\\ &\le CR^n\psi^2(R)\left([a_{ij}^{\alpha\beta}]^2_{BMO_\omega(\mathbb{R}^n_+;\overline{R})} + [a_{ij}^{\alpha\beta}]^2_{BMO_\psi(\mathbb{R}^n_-;\overline{R})}\right)\|Du\|^2_{L^\infty(\mathbb{R}^n)}\\ &\quad + CR^n\psi^2(R)L(f,g)\\ &\le CR^n\psi^2(R)\left(\int_0^\varepsilon \frac{\psi(t)}{t}\,\mathrm{d}t\cdot\left([Du]_{BMO_\psi(\mathbb{R}^n_+)} + [Du]_{BMO_\psi(\mathbb{R}^n_-)}\right)\right.\\ &\qquad \left. + \varepsilon^{-n/2}\|Du\|_{L^2(\mathbb{R}^n_+)}\right)^2 + CR^n\psi^2(R)L(f,g),\end{aligned}$$

where $C > 0$ is independent of ε. The rest of proof is similar to *(i)* by choosing sufficiently small ε if it is necessary. □

Lemma 4.3 *Let $u \in H^1(Q_1(0);\mathbb{R}^n)$ be a weak solution to Eq. 4.1 with Eq. 1.2. Suppose that Eq. 4.3 holds and $a_{ij}^{\alpha\beta} \in C^{0,1}(Q_1^\pm(0))$. Then*

$$\begin{aligned}[Du]_{BMO_\psi(Q^\pm_{3/4})(0)} \le C\Big(&\|u\|_{L^2(Q)} + \|g\|_{L^{2n/(n+2),n^2/(n+2)}(Q_1(0))}\\ &+ [f]_{BMO_\psi(Q_1^+)(0)} + [f]_{BMO_\psi(Q_1^-)(0)}\Big),\end{aligned} \tag{4.33}$$

where $C > 0$ depends only on n, N, Λ, K_ψ, $K_{r^\lambda/\psi}$, and $[a_{ij}^{\alpha\beta}]_{BMO_\psi(Q_1^\pm(0);\sigma)}$.

Proof Let $\eta \in C_0^\infty(Q_1(0))$, $0 \le \eta \le 0$ and $\eta = 1$ on $Q_{3/4}(0)$. Let $\overline{u} = \eta u$, then

$$\begin{aligned}-\partial_\alpha(a_{ij}^{\alpha\beta}\partial_\beta\overline{u}^j) &= -a_{ij}^{\alpha\beta}\partial_\alpha\eta\partial_\beta u^j - \partial_\alpha\left(a_{ij}^{\alpha\beta}\partial_\beta\eta u^j\right) + \eta g_i + \partial_\beta\eta f_i^\beta - \partial_\beta\left(\eta f_i^\beta\right)\\ &= -a_{ij}^{\alpha\beta}\partial_\alpha\eta\partial_\beta u^j + \eta g_i + \partial_\beta\eta f_i^\beta - \partial_\beta\left(\eta f_i^\beta + a_{ij}^{\alpha\beta}\partial_\alpha\eta u^j\right)\\ &=: \overline{g}_i - \partial_\beta\overline{f}_i^\beta.\end{aligned}$$

According to Lemma 4.2, we have

$$[D\overline{u}]_{BMO_\psi(\mathbb{R}^n_\pm)} \le C\left(\|\overline{u}\|_{L^2} + L^{1/2}\left(\overline{f},\overline{g}\right)\right). \tag{4.34}$$

The rest of argument is similar to Lemma 4.2 of [11], we omit it. □

Proof of Theorem 2.2 In view of Lemma 4.3, we need to find a sequence of approximating systems with smooth coefficients. By Proposition 1.2 of [1], $VMO_\psi(D)$ is identical to the closure of $C^\infty(\overline{D})$ under the norm $\|\cdot\|_{BMO_\psi(D)}$ if ∂D is Lipschitz and $\lim_{t\to 0}\dfrac{t}{\psi(t)} = 0$. Hence, there exists a sequence of $\{a_{ij}^{\alpha\beta}(k)\}$ in $C^\infty(Q_1^\pm(0))$ such that Eq. 1.2 holds, $a_{ij}^{\alpha\beta}(k)$ is uniformly bounded and

$$a_{ij}^{\alpha\beta}(k) \to a_{ij}^{\alpha\beta}, \text{ in } VMO_\omega\left(Q_1^\pm(0)\right).$$

Let $u_k \in H^1(Q_1(0);\mathbb{R}^n)$ be the weak solutions to the following problems

$$\begin{cases} -\partial_\alpha\left(a_{ij}^{\alpha\beta}(k)\partial_\beta u_k^j\right) = g_i - \partial_\beta f_i^\beta \text{ in } Q_1(0) \\ u_k - u \in H_0^1\left(Q_1(0);\mathbb{R}^N\right). \end{cases} \tag{4.35}$$

It is easy to see that

$$\|u_k\|_{H^1(Q_1(0))} \le C\left(\|u\|_{H^1(Q_1(0))} + \|g\|_{L^{2n/(n+2)}(Q_1(0))} + \|f\|_{L^2(Q_1(0))}\right) \tag{4.36}$$

and owing to Lemma 4.3

$$[Du_k]_{H^1(Q_{3/4}(0))} \le C\left(\|u_k\|_{L^2(Q_1(0))} + L^{1/2}(f,g,Q_1(0))\right). \tag{4.37}$$

Since $[a_{ij}^{\alpha\beta}(k)]$ converges to u uniformly, C is independent of k. Because of Eq. 4.36, we assume u_k converges to u in weak topology of $H^1(Q_1(0))$. This implies that for $x_0 \in Q_{3/4}^+(0)$ and $R < 1/4$

$$\begin{aligned} \fint_{Q_R^+(x_0)} |Du - (Du)_{x_0,R}^+|^2 &\le \liminf_{k\to\infty} \fint_{Q_R^+(x_0)} |Du - (Du)_{x_0,R}^+|^2 \\ &\le C\psi^2(R)\liminf_{k\to\infty}[Du_k]^2_{BMO_\psi(Q_1^+(0))}. \end{aligned}$$

Therefore, $Du \in BMO_\psi(Q_{3/4}^+(0))$. By the same procedure, we then have $Du \in BMO_\psi(Q_{3/4}^-(0))$.

On the other hand, if $a_{ij}^{\alpha\beta} \in BMO_\psi$ with ψ satisfying condition (A), i.e., $\psi = \psi_1 \cdot \psi_2$, where $\lim_{r\to 0}\psi_1(r) = 0$, ψ_2 satisfies Eq. 2.1 and ψ_2, $r^\lambda/\psi_2(r)$ are almost increasing, then $a_{ij}^{\alpha\beta} \in VMO_{\psi_2}$. From above proof, we conclude that $Du \in BMO_{\psi_2}(Q_{3/4}^\pm(0))$.

Next, we shall show $Du \in BMO_\psi(Q_{1/4}^\pm(0))$. Since ψ_2 satisfies Eq. 2.1, from Proposition 2.2 we have $Du \in L^\infty(Q_{3/4}(0))$. For $x_0 \in \Gamma_{1/2}(0)$, $0 < \rho < R < 1/4$, let v and ω satisfy Eqs. 4.24 and 4.25, respectively. As Eq. 4.27, we have

$$\begin{aligned} \int_{Q_R(x_0)} |Dw|^2 &\le C\int_{Q_R(x_0)} \left|\left(a_{ij}^{\alpha\beta}\right)_{x_0,R}^\pm - a_{ij}^{\alpha\beta}(x)\right|^2 |Du|^2 + R^n\psi^2(R)L(f,g) \\ &\le C\|Du\|^2_{L^\infty(Q_R(x_0))}R^n\psi^2(R)\left(\left[a_{ij}^{\alpha\beta}\right]^2_{BMO(Q_1^+(0))} + \left[a_{ij}^{\alpha\beta}\right]^2_{BMO(Q_1^-(0))}\right) \\ &\quad + R^n\psi^2(R)L(f,g). \end{aligned} \tag{4.38}$$

Combining Eqs. 4.26 and 4.38, we have

$$\int_{Q_\rho(x_0)} \left|D_{x'}u - (D_{x'}u)_{x_0,\rho}\right|^2 \le C\left(\frac{\rho}{R}\right)^{n+2} \int_{Q_R(x_0)} |D_{x'}u - (D_{x'}u)_{x_0,R}|^2 + CR^n\psi^2(R)\left(\|Du\|^2_{L^\infty(Q_R(x_0))} + L(f,g)\right). \quad (4.39)$$

Similar argument applies to $x_0 \in Q^{\pm}_{1/2}$. By Lemma 3.2, we conclude that $D_{x'}u \in BMO_\psi(Q_{1/2}(0))$. Similarly, due to Corollary 4.1 and making use the similar procedure in the proof of Corollary 4.2, then we have

$$[Du]_{BMO_\psi(Q^+_{1/4})} + [Du]_{BMO_\psi(Q^+_{1/4})} \le C\left([D_{x'}u]_{BMO_\psi(Q^+_{1/2}(0))} + \|Du\|_{L^\infty(Q_{3/4}(0))} + \|u\|_{L^2(\mathbb{R}^n)} + L^{1/2}(f,g)\right). \quad (4.40)$$

The rest of argument is similar to the proof of Theorem 2.1, we omit it here. □

5 High Order Derivatives Estimates

In this section we prove Theorem 2.3 based on the gradient estimates in Section 4.

Proof of Theorem 2.3 Let $u \in H^1(Q_1(0);\mathbb{R}^N)$ be a weak solution to

$$\partial_\alpha\left(a^{\alpha\beta}_{ij}(x)\partial_\beta u^j\right) = \partial_\beta f^\beta_i(x) \text{ in } Q_1(0), \quad (5.1)$$

where $a^{\alpha\beta}_{ij}$ satisfies Eq. 2.2 and

$$a^{\alpha\beta}_{ij}\Big|_{Q^\pm_1(0)} \in C^{k-1,1}\left(\overline{Q^\pm_1(0)}\right) \text{ and } D^k a^{\alpha\beta}_{ij}\Big|_{Q^\pm_1(0)} \in BMO_\psi\left(Q^\pm_1(0)\right), \quad (5.2)$$

$$f\Big|_{Q^\pm_1(0)} \in C^{k-1,1}\left(\overline{Q^\pm_1(0)}\right) \text{ and } D^k f\Big|_{Q^\pm_1(0)} \in BMO_\psi\left(Q^\pm_1(0)\right), \quad (5.3)$$

Since ψ satisfies condition (A), $D^k a^{\alpha\beta}_{ij}\big|_{Q^\pm_1(0)}, D^k f\big|_{Q^\pm_1(0)} \in VMO_{\psi_2}(Q^\pm_1(0))$. It follows that $D^k a^{\alpha\beta}_{ij}\big|_{Q^\pm_1(0)}, D^k f\big|_{Q^\pm_1(0)}$ can be approximated by smooth functions with the BMO_{ψ_2} norm. Therefore, we can assume $D^k a^{\alpha\beta}_{ij}\big|_{Q^\pm_1(0)}, D^k f\big|_{Q^\pm_1(0)}$ are smooth. By Proposition 3.1, we can also assume $u\big|_{Q^\pm_1(0)}$ is smooth to the boundary $\partial Q^\pm_1(0)$, but our estimates derived in the below is independent of smoothness of them.

If $k = 1$, differentiating system 5.1 with respect to x', we obtain

$$\partial_\alpha\left(a^{\alpha\beta}_{ij}\partial_\beta\left(D_{x'}u^j\right)\right) = -\partial_\beta\left(D_{x'}a^{\alpha\beta}_{ij}\partial_\alpha u^j\right) + \partial_\beta\left(D_{x'}f^\beta_i\right). \quad (5.4)$$

According to Theorem 2.2, we have

$$[DD_{x'}u]_{BMO_{\psi_2}(Q^+_{1/2}(0))} + [DD_{x'}u]_{BMO_{\psi_2}(Q^+_{1/2}(0))} \le C\left(\|D_{x'}u\|_{L^2(Q_{3/4}(0))} + \left[D_{x'}f\right]_{BMO_{\psi_2}(Q^+_1(0))} + \left[D_{x'}f\right]_{BMO_{\psi_2}(Q^+_1(0))}\right),$$

where $C > 0$ depends only on n, N, Λ, K_{ψ_2}, $K_{r/\psi_2(r)}$, $\|D_{x'}a_{ij}^{\alpha\beta}\|_{L^\infty(Q_1(0))}$ and $[D_{x'}a_{ij}^{\alpha\beta}]_{BMO_{\psi_2}Q_1^\pm(0)}$. In order to estimate $[\partial^2_{nn}u]_{BMO_{\psi_2}(Q^\pm_{1/2})}$, we make use of system 5.1

$$\partial_\alpha a_{ij}^{\alpha\beta}(x)\cdot\partial_\beta u^j + a_{ij}^{\alpha\beta}\partial^2_{\alpha\beta}u^j = \partial_\beta f_i^\beta \text{ in } Q_1^+(0),$$

i.e.,

$$a_{ij}^{nn}\partial^2_{nn}u^j = -\sum_{\alpha<n} a_{ij}^{\alpha\beta}\partial^2_{\alpha\beta}u^j - \partial_\alpha a_{ij}^{\alpha\beta}(x)\cdot\partial_\beta u^j + \partial_\beta f_i^\beta \text{ in } Q_1^+(0).$$

By ellipticity condition, then $\partial^2_{nn}u \in BMO_{\psi_2}(Q^+_{1/2})$. Similarly we have $\partial^2_{nn}u \in BMO_{\psi_2}(Q^-_{1/2})$. In combination, we proved $D^2u \in BMO_{\psi_2}(Q^\pm_{1/2}(0))$. As in the proof of Theorem 2.2, then $D^2u \in BMO_\psi(Q^\pm_{1/4}(0))$ and

$$\begin{aligned}&\left[D^2u\right]_{BMO_\psi(Q^+_{1/4}(0))} + \left[D^2u\right]_{BMO_\psi(Q^+_{1/4}(0))}\\ &\quad\le C\left(\|u\|_{L^2(Q_1(0))} + \sum_{|\gamma|\le 1}\left(\left[D^\gamma f\right]_{BMO_\psi(Q_1^+(0))} + \left[D^\gamma f\right]_{BMO_\psi(Q_1^+(0))}\right)\right),\end{aligned}$$

where $C > 0$ depends only on n, N, Λ, K_{ψ_2}, $K_{r/\psi_2(r)}$, $\|Da_{ij}^{\alpha\beta}\|_{L^\infty(Q_1(0))}$ and $[Da_{ij}^{\alpha\beta}]_{BMO_\psi Q_1^\pm(0)}$.

By induction, it is not difficult to complete the proof. □

Acknowledgement We would like to thank Prof. YanYan Li for giving valuable talks on problems in composite materials and transmission in our department, which motivated the present work.

References

1. Acquistapace, P.: On BMO regularity for linear elliptic equation systems. Ann. Math. Pura Appl. **161**(4), 231–269 (1992)
2. Byun, S.-S., Wang, L.: Gradient estimates for elliptic systems in non-smooth domains. Math. Ann. **341**(3), 629–650 (2008)
3. Caffarelli, L.: Interior a priori estimates for solutions of fully non linear equations. Ann. Math. **130**, 189–213 (1989)
4. Caffarelli, L., Huang, Q.: Estimates in the generalized Campanato–John–Nirenberg spaces for fully nonlinear elliptic equations. Duke Math. J. **118**, 1–17 (2003)
5. Caffarelli, L., Peral, I.: On $W^{1,p}$ estimates for elliptic equations in divergence form. Commun. Pure Appl. Math. **51**(1), 1–21 (1998)
6. Chipot, M., Kinderlehrer, D., Vergara-Caffarelli, G.: Smoothness of linear Laminates. Arch. Ration. Mech. Anal. **96**(1), 81–96 (1986)
7. Escauriaza, L., Fabes, E., Verchota, G.: On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries. Proc. Am. Math. Soc. **115**(4), 1069–1076 (1992)
8. Escauriaza, L., Mitrea, M.: Transmission problems and spectral theory for singular integral operators on Lipschitz domains. J. Funct. Anal. **216**, 141–171 (2004)
9. Escauriaza, L., Seo, J.: Regularity properties of solutions to transmission problems. Trans. Am. Math. Soc. **338**(1), 405–430 (1993)
10. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. In: Annals of Mathematics Studies, vol. 105. Princeton University, Princeton (1983)
11. Huang, Q.: Estimates on the generalized Morrey spaces $L^{2,\lambda}_\phi$ and BMO_ψ for linear elliptic systems. Indiana Univ. Math. J. **45**, 397–439 (1996)
12. John, F.: “Quasi-isometric mappings” in Seminari 1962/63 di Analisi Algebra, Geometria e Topologia, vol. II, Ist. Naz. Alta Mat, pp. 462–473. Ediz. Cremonese, Rome (1965)

13. John, F.; Nirenberg, L.: On functions of bounded mean oscillation. Commun. Pure Appl. Math. **14**, 415–426 (1961)
14. Kohr, K., Pintea, C., Wendland, W.: Brinkman-type operators on Riemannian manifolds: transmission problems in Lipschitz and C^1 domains. Potential Anal. **32**, 229–273 (2010)
15. Ladyzenskaya, O.A., Rivkind, J., Ural'ceva, N.: The classical solvability of diffraction problems. Proc. Steklov. Inst. Math. **92**, 132–166 (1966)
16. Li, Y.Y., Nirenberg, L.: Estimates for ellptic system from composition material. Commun. Pure Appl. Math. **LVI**, 892–925 (2003)
17. Li, Y.Y., Vogelius, M.: Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch. Ration. Mech. Anal. **153**(2), 91–151 (2000)
18. Sarason, D.: Functions of vanishing mean oscillation. Trans. Am. Math. Soc. **207**, 391–405 (1975)
19. Spanne, S.: Some function space defined by using the mean oscillation over cubes. Ann. Sc. Norm. Sup. Pisa **19**, 593–608 (1965)