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SEMILINEAR ELLIPTIC SYSTEM WITH BOUNDARY SINGULARITY

Yimei Li and Jiguang Bao

School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, China

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Abstract. In this paper, we investigate the asymptotic behavior of local solutions for the semilinear elliptic system $-\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u}$ with boundary isolated singularity, where $1 < p < \frac{n+2}{n-2}$, $n \geq 2$ and $\mathbf{u}$ is a C^2 nonnegative vector-valued function defined on the half space. This work generalizes the correspondence results of Bidaut-Véron-Ponce-Véron on the scalar case, and Ghergu-Kim-Shahgholian on the internal singularity case.

1. **Introduction.**

The semilinear elliptic system

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} \tag{1.1}$$

has attracted a lot of attention, where $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator with $n \geq 2$. $\mathbf{u} := (u_1, u_2, \cdots, u_m)$ is a nonnegative vector-valued function defined on a domain in $\mathbb{R}^n$, $m \geq 1$, and $p > 1$. Now coupled systems of nonlinear Schrödinger equations like (1.1) are parts of several important branches of mathematical physics. They appear in the Hartree-Fock theory for Bose-Einstein double condensates, the fiber-optic theory, the langmuir waves theory for plasma physics, and in studying the behavior of deep water waves and freak waves in the ocean. A general reference in book form on such systems and their role in physics is by Ablowitz-Prinari-Trubatch [1]. Our interests is to obtain the asymptotic behavior of local solutions near the boundary singularity for the semilinear elliptic systems (1.1).

The corresponding internal isolated singularity for the systems had been very well understood. It is worth mentioning that the classification of the entire solutions plays an important part in the study of the internal isolated singularity. Using the method of moving spheres, Druet-Hebey-Vetóis [13] had proved that any nonnegative C^2 solutions of the strongly coupled critical elliptic system

$$-\Delta \mathbf{u} = |\mathbf{u}|^{\frac{4}{n-2}}\mathbf{u} \quad \text{in} \quad \mathbb{R}^n,$$

is of the form $\mathbf{u} = u\mathbf{c}$, where $\mathbf{c} \in \mathbb{S}^+ := \{\theta = (\theta_1, \theta_2, \cdots, \theta_m) \in \mathbb{S}^{m-1} : \theta_i \geq 0, \quad i = 1, 2, \cdots, m\}$, $\mathbb{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$, and u is the positive solution of the

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Corresponding author Jiguang Bao.

scalar Yamabe equations

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \ \ \mathbb{R}^n.$$

Caju-do Ó-Silva Santos [9] obtained that if $\mathbf{u} \in C^2(\mathbb{R}^n \backslash \{0\})$ is the positive solution of

$$-\Delta \mathbf{u} = |\mathbf{u}|^{\frac{4}{n-2}} \mathbf{u} \quad \text{in} \ \ \mathbb{R}^n \backslash \{0\},$$

then $\mathbf{u} = u\mathbf{c}$, where $\mathbf{c} \in \mathbb{S}^+$, and u is a positive solution of

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \ \ \mathbb{R}^n \backslash \{0\}.$$

Recently, Ghergu-Kim-Shahgholian [14] obtained that $\mathbf{u} = 0$ is the only nonnegative C^2 solution of

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1} \mathbf{u} \quad \text{in} \ \ \mathbb{R}^n$$

for $1 < p < \frac{n+2}{n-2}$. Furthermore, they also classified the solutions in the punctured space, and proved that if $\mathbf{u} \in C^2(\mathbb{R}^n \backslash \{0\})$ is the positive solutions of

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1} \mathbf{u} \quad \text{in} \ \ \mathbb{R}^n \backslash \{0\},$$

with $\limsup_{x \to 0} |\mathbf{u}(x)| = +\infty$, then $\mathbf{u}$ is radially symmetric.

In the same paper [14], they derived the priori estimates that there exists a constant C depending only on n, m such that

$$|\mathbf{u}(x)| \le C|x|^{-\frac{2}{p-1}} \quad \text{near} \ \ x = 0,$$

for the local positive solutions in $C^2(B_1 \backslash \{0\})$ of

$$-\Delta \mathbf{u} = |\mathbf{u}|^{p-1} \mathbf{u} \quad \text{in} \ \ B_1 \backslash \{0\},$$

where $B_1 := \{x \in \mathbb{R}^n : |x| < 1\}$, $1 < p \le \frac{n+2}{n-2}$. And they got the asymptotic radial symmetry,

$$\mathbf{u}(x) = (1 + O(|x|))\overline{\mathbf{u}}(|x|) \quad \text{near} \ \ x = 0,$$

where $\overline{\mathbf{u}}(r)$ is the average of $\mathbf{u}$ over ∂B_r. Utilizing the classification of solutions in the punctured space and the above asymptotic radial symmetry, they further studied in [14] the exact asymptotic behavior of local solutions around the singularity. In precise, either $\mathbf{u}$ can be continuously extended at the origin, or there exists a lower bound around the origin.

Especially, for the internal isolated singularity of the scalar case, see [3, 8, 16, 20, 22]. See also Li [21] and Han-Li-Teixeira [19] for conformally invariant fully nonlinear elliptic equations. The Sobolev critical exponent case $p = \frac{n+2}{n-2}$ is of particular interest, because the equation connects to the Yamabe problem and the conformal invariance, which leads to a richer isolated singularity structure.

Another motivation stems from that the scalar case of system (1.1) with a boundary singularity had been considered by a series of seminal papers. The authors proved in [10, 11, 18] respectively that the nonexistence of positive bounded solutions in $C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ for

$$\begin{cases} -\Delta u = u^p & \text{in} \ \ \mathbb{R}^n_+, \\ u = 0 & \text{on} \ \ \partial\mathbb{R}^n_+, \end{cases}$$

where $\mathbb{R}^n_+$ stands for the half space and $p \ge 1$. Xiong lately removed in [25] the condition on boundness of the solutions. Based on the classification of the solution on the half space, the asymptotic behavior of the positive singular solutions in $C^2(B_1^+)\cap C(\overline{B_1^+}\backslash\{0\})$ for

$$\begin{cases} -\Delta u = u^p & \text{in } \ B_1^+, \\ u = 0 & \text{on } \ \partial' B_1^+\backslash\{0\} \end{cases}$$

has been established by many works, where $B_1^+ := B_1 \cap \mathbb{R}^n_+$ and $\partial' B_1^+ := \overline{B_1^+} \cap \partial\mathbb{R}^n_+$. See Bidaut-Véron-Vivier [6] for $1 < p < \frac{n+1}{n-1}$, Bidaut-Véron-Ponce-Véron [4, 5] for $\frac{n+1}{n-1} \le p < \frac{n+2}{n-2}$ and Xiong [25] for $p = \frac{n+2}{n-2}$. Under a blow up rate assumption: $|x|^{\frac{2}{p-1}}u(x)$ is bounded in B_1^+, then Bidaut-Véron-Ponce-Véron [4, 5] obtained refined asymptotic behaviors for the supercritical case $\frac{n+2}{n-2} < p < \frac{n+1}{n-3}$. We refer to [4] and references therein for related results on boundary singularity. The fact that the exponent $\frac{n+1}{n-1}$ corresponds to $\frac{n}{n-2}$ for the internal singularity was discovered by Brézis-Turner [7].

In the present paper, our primary interest is to analyze the behavior of the singular positive solutions in $C^2(B_1^+)\cap C(\overline{B_1^+}\backslash\{0\})$ for the semilinear elliptic system with Dirichlet boundary value conditions

$$\begin{cases} -\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} & \text{in } \ B_1^+, \\ \mathbf{u} = 0 & \text{on } \ \partial' B_1^+\backslash\{0\}, \end{cases} \tag{1.2}$$

where $\mathbf{u} := (u_1, u_2, \cdots, u_m)$, $m \ge 1$ and $p > 1$.

Via the method of moving spheres, we first classify the solutions on the half space, which will be used in the blow up analysis and is consistent with the work of [10, 11, 18].

Theorem 1.1. *Let $1 < p \le \frac{n+2}{n-2}$, $n \ge 3$ and $\mathbf{u} \in C^2(\mathbb{R}^n_+)\cap C(\overline{\mathbb{R}^n_+})$ be a nonnegative solution of*

$$\begin{cases} -\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} & \text{in } \ \mathbb{R}^n_+, \\ \mathbf{u} = 0 & \text{on } \ \partial\mathbb{R}^n_+, \end{cases} \tag{1.3}$$

then $\mathbf{u} = 0$.

The description of the boundary behavior of positive solutions of (1.2) is greatly helped by using a specific separable solutions of the same equation. This was early performed by Gmira-Véron [17] in 1991, and recently Porretta-Véron [24] also use the method for quasilinear Lane-Emden equations. Motivated by these, our next work is to look for the special positive solutions in $C^2(\mathbb{R}^n_+)\cap C(\overline{\mathbb{R}^n_+}\backslash\{0\})$ for

$$\begin{cases} -\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} & \text{in } \ \mathbb{R}^n_+, \\ \mathbf{u} = 0 & \text{on } \ \partial\mathbb{R}^n_+\backslash\{0\} \end{cases} \tag{1.4}$$

with the form

$$\mathbf{u} = |x|^{-\frac{2}{p-1}}\mathbf{w}\left(\frac{x}{|x|}\right).$$

By a direct calculation, $\mathbf{w} \in C^2(\mathbb{S}^{n-1}_+)\cap C(\overline{\mathbb{S}^{n-1}_+})$ must satisfy

$$\begin{cases} -\Delta_s \mathbf{w} = l_{n,p}\mathbf{w} + |\mathbf{w}|^{p-1}\mathbf{w} & \text{in } \ \mathbb{S}^{n-1}_+, \\ \mathbf{w} = 0 & \text{on } \ \partial\mathbb{S}^{n-1}_+, \end{cases} \tag{1.5}$$

where Δ_s denotes the Laplace-Beltrami operator in the unit sphere $\mathbb{S}^{n-1}$, $\mathbb{S}^{n-1}_+ := \mathbb{S}^{n-1}\cap\mathbb{R}^n_+$, and

$$l_{n,p} = \frac{2(n-p(n-2))}{(p-1)^2}.$$

Indeed, we need to obtain the existence of solutions for the Dirichlet problem (1.5) on semisphere. On the technical level, we shall transform (1.5) to a similar problem in an Euclidean space by stereographic projection. Inspired by the result of [14], we shall obtain a similar description Theorem 1.2 in this case, which is consistent with the work of [4].

Theorem 1.2. *(i) Let $1 < p \leq \frac{n+1}{n-1}$ for $n \geq 2$; and let $p = \frac{n+1}{n-3}$ for $n \geq 4$, then (1.5) admits no positive solution.*

(ii) Let $\frac{n+1}{n-1} < p \leq \frac{n+3}{n-1}$, then (1.5) admits positive solution of the form $\mathbf{w} := w\mathbf{c}$, where $\mathbf{c} \in \mathbb{S}^+$, and w is the positive solution of

$$\begin{cases} -\Delta_s w = l_{n,p} w + w^p & \text{in } \ \mathbb{S}^{n-1}_+, \\ w = 0 & \text{on } \ \partial\mathbb{S}^{n-1}_+. \end{cases} \tag{1.6}$$

Remark that for the case $m = 1$, (1.5) admits a unique positive solution for $\frac{n+1}{n-1} < p < \frac{n+1}{n-3}$, and no positive solution for $1 < p \leq \frac{n+1}{n-1}$ or $p \geq \frac{n+1}{n-3}$. So far, we have no idea whether this conclusion holds or not for $m \geq 2$, $p > \frac{n+3}{n-1}$.

We next establish a universal upper estimate near the singularity for (1.2) using doubling property (see [23, Lemma 5.1]). It is consistent with the work of [4] for the equation.

Theorem 1.3. *Let $1 < p < \frac{n+2}{n-2}$ and $\mathbf{u}$ be a positive solution of (1.2), then there exists a positive constant C independent of the solution such that*

$$|\mathbf{u}(x)| \leq C|x|^{-\frac{2}{p-1}} \quad \text{near } x = 0. \tag{1.7}$$

In the upcoming sections of this paper, we focus on the exact asymptotic behavior of the local solutions of (1.2) for the subcritical case $1 < p < \frac{n+2}{n-2}$, which generalizes the work of [4] for the equation and [14] for internal isolated singularity of the same systems. In [14], the upper bound and the classification of solutions in $\mathbb{R}^n\backslash\{0\}$ play a key role in the asymptotic analysis. Next, we devote to studying the lower critical exponent case $p = \frac{n+1}{n-1}$. Due to the multiplicity of components $|\mathbf{u}|^{p-1}\mathbf{u}$, the lower critical exponent case $p = \frac{n}{n-2}$ is very different from the situation $\frac{n}{n-2} < p < \frac{n+2}{n-2}$ for the internal isolated singularity. To overcome this problem, the authors in [14] first give a more precise upper estimates and then obtain a asymptotic near the singularity. The similar problem also happens for the scalar case of system (1.1) with a boundary singularity. Motivated by the precise work, we also use a similar method to obtain the following theorem.

Theorem 1.4. *Let $p = \frac{n+1}{n-1}$ and $\mathbf{u}$ be a positive solution of (1.2), then either $\mathbf{u}$ can be continuously extended at 0, or*

$$\lim_{x\to 0} \left| |x|^{n-1} \left(\log \frac{1}{|x|}\right)^{\frac{n-1}{2}} \mathbf{u}(x) - \frac{k x_n}{|x|}\widetilde{\mathbf{e}} \right| = 0, \tag{1.8}$$

where $\widetilde{\mathbf{e}} := (1, 1, \cdots, 1) \in \mathbb{R}^m$ and k is a positive constant depending only on n.

As for the case $1 < p < \frac{n+1}{n-1}$, we study a more general case involving boundary measures in a subsequent work by a very different method.

We note that $\mathbf{u} \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ if $u_i \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ for any $i \in \{1, 2, \cdots, m\}$, $m \geq 1$, and $\mathbf{u}$ is nonnegative if u_i is nonnegative for any $i \in$

$\{1, 2, \cdots, m\}$. In a word, we say a vector $\mathbf{u}$ has some properties means that every component of $\mathbf{u}$ has the same properties. Moreover, if for a fixed $i \in \{1, 2, \cdots, m\}$, u_i vanishes somewhere, then since u_i is super-harmonic and nonnegative, we know that u_i vanishes everywhere on $\mathbb{R}^n_+$. As a result, if $\mathbf{u} \not\equiv 0$, then there is $k \in \{1, 2, \cdots, m\}$ such that after suitable rearrangement in the components of $\mathbf{u}$, $u_i > 0$ if and only if $i = 1, 2, \cdots, k$. Without loss of generality, in our paper, $\mathbf{u}$ is nonnegative either u_i is positive for any $i \in \{1, 2, \cdots, m\}$ or $\mathbf{u} = 0$.

Our paper is organized as follows. Section 2 includes one proposition to prove Theorem 1.1. Section 3 is devoted to obtain the existence of solutions for (1.5). The upper bound for solutions of (1.2) will be provided in Section 4. In Section 5, we shall show the removability under some blow up assumption. Finally, we obtain the asymptotic symmetry in Section 6, including Theorem 1.4.

2. **Nonexistence of entire solutions.**

We now give the following proposition, which is about the monotonicity of positive solutions, that is, the positive solutions is monotone increasing in x_n direction.

Proposition 2.1. *Let* $\mathbf{u}$ *be a positive solution of* (1.3), *then* $\frac{\partial \mathbf{u}}{\partial x_n} > 0$.

Then we can start to prove Theorem 1.1, and we shall give a proof for Proposition 2.1 later.

Proof of Theorem 1.1. Proposition 2.1 implies that the monotonicity of $\mathbf{u}$ with respect to the variable x_n. If we also have proved that $\mathbf{u}$ is bounded in $\mathbb{R}^n_+$, then

$$\mathbf{u}_\infty(x') := \lim_{x_n \to +\infty} \mathbf{u}(x', x_n) > \mathbf{u}(x', 0) = 0.$$

Moreover, the positive vector-valued function $\mathbf{u}_\infty \in C^2(\mathbb{R}^{n-1})$ satisfies

$$-\Delta \mathbf{u}_\infty = |\mathbf{u}_\infty|^{p-1}\mathbf{u}_\infty \quad \text{in} \;\; \mathbb{R}^{n-1}.$$

Together with $1 < p \leq \frac{n+2}{n-2} < \frac{n+1}{n-3}$ and the Liouville Theorem [14], we derive that $\mathbf{u}_\infty = 0$. It is a contradiction. Then we complete the proof of Theorem 1.1.

Now, we shall prove that $\mathbf{u}$ is bounded in $\mathbb{R}^n_+$. If not, then there exist $\overline{x}_k \in \mathbb{R}^n_+$, $k = 1, 2, \cdots$, such that

$$|\mathbf{u}(\overline{x}_k)| \to +\infty \quad \text{as} \;\; k \to +\infty.$$

By Proposition 2.1 the monotonicity of $\mathbf{u}$ in the x_n direction, we may assume that

$$(\overline{x}_k)_n \to \infty \quad \text{as} \;\; k \to +\infty.$$

Consider

$$v_k(x) := (1 - |x - \overline{x}_k|)^{\frac{2}{p-1}} |\mathbf{u}(x)| \quad \text{in} \;\; B_1(\overline{x}_k),$$

Let $|x^k - \bar{x}_k| < 1$ satisfy

$$v_k(x^k) = \max_{|x - \overline{x}_k| \leq 1} v_k(x),$$

and denote

$$\mu_k := \frac{1}{2}(1 - |x^k - \bar{x}_k|).$$

Then

$$0 < 2\mu_k \leq 1 \quad \text{and} \quad 1 - |x - \bar{x}_k| \geq \mu_k \quad \text{in } B_{\mu_k}(x^k). \tag{2.1}$$

By the definition of v_k, we have

$$(2\mu_k)^{\frac{2}{p-1}} |\mathbf{u}(x^k)| = v_k(x^k) \geq v_k(x) \geq (\mu_k)^{\frac{2}{p-1}} |\mathbf{u}(x)| \text{ in } B_{\mu_k}(x^k).$$

Thus, we have

$$2^{\frac{2}{p-1}}|\mathbf{u}(x^k)| \geq |\mathbf{u}(x)| \quad \text{in } B_{\mu_k}(x^k). \tag{2.2}$$

We also have

$$(2\mu_k)^{\frac{2}{p-1}}|\mathbf{u}(x^k)| = v_k(x^k) \geq v_k(\overline{x}_k) = |\mathbf{u}(\overline{x}_k)| \to +\infty \quad \text{as } k \to +\infty.$$

Now, consider

$$\mathbf{w}_k(y) := \frac{1}{|\mathbf{u}(x^k)|}\mathbf{u}\left(x^k + \frac{y}{|\mathbf{u}(x^k)|^{\frac{p-1}{2}}}\right) \quad \text{in } B_{R_k},$$

where $R_k := \mu_k|\mathbf{u}(x^k)|^{\frac{p-1}{2}} \to +\infty$ as $k \to +\infty$. It follows from (2.2) that

$$|\mathbf{w}_k(0)| = 1, \quad |\mathbf{w}_k| \leq 2^{\frac{2}{p-1}} \quad \text{in } B_{R_k}. \tag{2.3}$$

Using the equations satisfied by $\mathbf{u}$, a direct calculation gives that

$$-\Delta\mathbf{w}_k = |\mathbf{w}_k|^{p-1}\mathbf{w}_k \quad \text{in } B_{R_k}. \tag{2.4}$$

By (2.3), (2.4) and standard elliptic estimates, after extracting a subsequence, we have

$$\mathbf{w}_k \to \mathbf{w}_\infty \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n),$$

and conclude that $\mathbf{w}_\infty$ is a classical solution of

$$-\Delta\mathbf{w}_\infty = |\mathbf{w}_\infty|^{p-1}\mathbf{w}_\infty \quad \text{in } \mathbb{R}^n$$

and $|\mathbf{w}_\infty(0)| = 1$. By the Liouville Theorem [14], we have $\mathbf{w}_\infty = 0$ for $1 < p < \frac{n+2}{n-2}$, and for the critical case $p = \frac{n+2}{n-2}$, we obtain that

$$\mathbf{w}_\infty = (n(n-2))^{\frac{n-2}{4}}\left(\frac{r}{r^2 + |x-z|^2}\right)^{\frac{n-2}{2}}\mathbf{e}$$

for some $z \in \mathbb{R}^n$, $r \geq 0$, and a unit nonnegative vector $\mathbf{e} \in \mathbb{R}^m$. But we know from the monotonicity of $\mathbf{w}_k$ that $\mathbf{w}_\infty$ must be non-decreasing in x_n direction. This is a contradiction and the claim is proved. □

For any $R > 0$, $\lambda > 0$, and $x_R := (0, 0, \cdots, 0, -R)$, define

$$\mathbf{u}_{x_R,\lambda}(y) := \left(\frac{\lambda}{|y - x_R|}\right)^{n-2}\mathbf{u}\left(x_R + \frac{\lambda^2(y - x_R)}{|y - x_R|^2}\right) \quad \text{in } B_\lambda(x_R),$$

the Kelvin transformation of $\mathbf{u}$ with respect to the ball $B_\lambda(x_R)$, where $B_\lambda(x_R) := \{x \in \mathbb{R}^n : |x - x_R| < \lambda\}$. Next we shall prove the monotonicity, that is Proposition 2.1.

Proof of Proposition 2.1. Suppose that for any $\lambda > R$, we have

$$\mathbf{u} \leq \mathbf{u}_{x_R,\lambda} \quad \text{in } B^+_\lambda(x_R), \tag{2.5}$$

where $B^+_\lambda(x_R) := B_\lambda(x_R) \cap \mathbb{R}^n_+$. Then for any $y \in \mathbb{R}^n_+$, and every $a > 0$, it follows that

$$\mathbf{u}(y) \leq \mathbf{u}_{x_R, R+y_n+a/2}(y).$$

Let $R \to +\infty$, we obtain by the above inequality that

$$\mathbf{u}(y) \leq \lim_{R\to+\infty}\mathbf{u}_{x_R,R+y_n+a/2}(y) = \mathbf{u}(y_1, y_2, \cdots, y_{n-1}, y_n + a),$$

which implies that

$$\frac{\partial\mathbf{u}}{\partial x_n} \geq 0 \quad \text{in } \mathbb{R}^n_+.$$

It is a straightforward computation to show that

$$-\Delta\left(\frac{\partial \mathbf{u}}{\partial x_n}\right) = (p-1)|\mathbf{u}|^{p-3}\mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial x_n}\mathbf{u} + |\mathbf{u}|^{p-1}\frac{\partial \mathbf{u}}{\partial x_n} \quad \text{in } \mathbb{R}^n_+.$$

Applying the Strong Maximum principle, we conclude that $\frac{\partial \mathbf{u}}{\partial x_n}$ is always zero or strictly positive in $\mathbb{R}^n_+$. If $\frac{\partial \mathbf{u}}{\partial x_n} \equiv 0$, together with the boundary condition, we conclude that $\mathbf{u} \equiv 0$. It is a contradiction with the positive of the solution. Then we finish the proof. □

In order to prove Proposition 2.1, it suffices to obtain (2.5). For the purpose, we introduce

$$\bar{\lambda}(R) := \sup\left\{\mu > R \;\middle|\; \mathbf{u} \le \mathbf{u}_{x_R,\lambda} \quad \text{in } B^+_\lambda(x_R),\ \forall\, \lambda \in (R,\mu)\right\}.$$

First, we need the following lemma to guarantee that the set over which we are taking the supremum is non-empty such that $\bar{\lambda}(R)$ is well defined.

Lemma 2.2. *For $R > 0$ there exists $\lambda_0(R) \in (R, 2R)$ such that for any $\lambda \in (R, \lambda_0(R))$,*

$$\mathbf{u} \le \mathbf{u}_{x_R,\lambda} \quad \text{in } B^+_\lambda(x_R). \tag{2.6}$$

Next we shall prove

Lemma 2.3. *$\bar{\lambda}(R) = +\infty$ for all $R > 0$.*

Equivalent to (2.5), the following job gives the proof of Lemma 2.2 and Lemma 2.3.

Proof of Lemma 2.2. A direct calculation gives that

$$\Delta \mathbf{u}_{x_R,\lambda} = \left(\frac{\lambda}{|y-x_R|}\right)^{n+2} \Delta \mathbf{u}\left(x_R + \frac{\lambda^2(y-x_R)}{|y-x_R|^2}\right),$$

$$-\Delta \mathbf{u}_{x_R,\lambda} = \left(\frac{\lambda}{|y-x_R|}\right)^{n+2-p(n-2)} |\mathbf{u}_{x_R,\lambda}|^{p-1}\mathbf{u}_{x_R,\lambda},$$

$$-\Delta(\mathbf{u} - \mathbf{u}_{x_R,\lambda}) = |\mathbf{u}|^{p-1}\mathbf{u} - \left(\frac{\lambda}{|y-x_R|}\right)^{n+2-p(n-2)} |\mathbf{u}_{x_R,\lambda}|^{p-1}\mathbf{u}_{x_R,\lambda}.$$

We will make use of the narrow domain technique to conclude (2.6). Denote

$$\mathbf{w}_\lambda := \mathbf{u}_{x_R,\lambda} - \mathbf{u}, \quad \mathbf{w}^-_\lambda := \max\{0, -\mathbf{w}_\lambda\}.$$

Multiplying both sides of the equation by $\mathbf{w}^-_\lambda$ and integrating by parts in $B^+_\lambda(x_R)$, it follows that

$$\begin{aligned}
&\int_{B^+_\lambda(x_R)} |\nabla \mathbf{w}^-_\lambda|^2 \\
&= \int_{B^+_\lambda(x_R)} \left(|\mathbf{u}|^{p-1}\mathbf{u} - \left(\frac{\lambda}{|y-x_R|}\right)^{n+2-p(n-2)} |\mathbf{u}_{x_R,\lambda}|^{p-1}\mathbf{u}_{x_R,\lambda}\right)\cdot \mathbf{w}^-_\lambda \\
&\le \int_{B^+_\lambda(x_R)} (|\mathbf{u}|^{p-1}\mathbf{u} - |\mathbf{u}_{x_R,\lambda}|^{p-1}\mathbf{u}_{x_R,\lambda})\cdot \mathbf{w}^-_\lambda \\
&= \int_{B^+_\lambda(x_R)} |\mathbf{u}|^{p-1}|\mathbf{w}^-_\lambda|^2 + \int_{B^+_\lambda(x_R)} (|\mathbf{u}|^{p-1} - |\mathbf{u}_{x_R,\lambda}|^{p-1})\mathbf{u}_{x_R,\lambda}\cdot \mathbf{w}^-_\lambda \\
&=: I_1 + I_2.
\end{aligned} \tag{2.7}$$

For any $\lambda \in (R, 2R)$ and $y \in B_\lambda^+(x_R)$, we have

$$\left| x_R + \frac{\lambda^2 (y - x_R)}{|y - x_R|^2} \right| \le |x_R| + \frac{\lambda^2}{R} \le 5R.$$

With the help of Hölder inequality and Sobolev inequality, we obtain that

$$\begin{aligned} I_1 =& \int_{B_\lambda^+(x_R)} |\mathbf{u}|^{p-1} |\mathbf{w}_\lambda^-|^2 \\ \le& \sup_{B_{5R}^+} |\mathbf{u}|^{p-1} |B_\lambda^+(x_R)|^{\frac{2}{n}} \left(\int_{B_\lambda^+(x_R)} \left|\mathbf{w}_\lambda^-\right|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ \le& S(n) \sup_{B_{5R}^+} |\mathbf{u}|^{p-1} |B_\lambda^+(x_R)|^{\frac{2}{n}} \int_{B_\lambda^+(x_R)} |\nabla \mathbf{w}_\lambda^-|^2, \end{aligned}$$

where $S(n)$ is a constant depending only on n. By the Mean Value Theorem, there exists a $\theta \in (0, 1)$ such that

$$\begin{aligned} |\mathbf{u}|^{p-1} - |\mathbf{u}_{x_R,\lambda}|^{p-1} =& (p-1) |\widetilde{\mathbf{u}}|^{p-3} \widetilde{\mathbf{u}} \cdot (\mathbf{u} - \mathbf{u}_{x_R,\lambda}) \\ \le& (p-1) |\mathbf{u}|^{p-2} |\mathbf{w}_\lambda^-|, \end{aligned}$$

where $\widetilde{\mathbf{u}} := \theta \mathbf{u} + (1-\theta) \mathbf{u}_{x_R,\lambda}$. Using the Hölder inequality and Sobolev inequality again,

$$\begin{aligned} I_2 \le& (p-1) \int_{B_\lambda^+(x_R)} |\mathbf{u}|^{p-1} |\mathbf{w}_\lambda^-|^2 \\ \le& (p-1) S(n) \sup_{B_{5R}^+} |\mathbf{u}|^{p-1} |B_\lambda^+(x_R)|^{\frac{2}{n}} \int_{B_\lambda^+(x_R)} |\nabla \mathbf{w}_\lambda^-|^2. \end{aligned}$$

From the above argument, it follows that (2.7) implies

$$\int_{B_\lambda^+(x_R)} |\nabla \mathbf{w}_\lambda^-|^2 \le S(n) p \sup_{B_{5R}^+} |\mathbf{u}|^{p-1} |B_\lambda^+(x_R)|^{\frac{2}{n}} \int_{B_\lambda^+(x_R)} |\nabla \mathbf{w}_\lambda^-|^2.$$

Choose $\lambda_0(R) > R$ but very close to R, then $|B_\lambda^+(x_R)|^{\frac{2}{n}}$ is small such that

$$S(n) p \sup_{B_{5R}^+} |\mathbf{u}|^{p-1} |B_\lambda^+(x_R)|^{\frac{2}{n}} \le \frac{1}{2}.$$

It follows that

$$|\nabla \mathbf{w}_\lambda^-| = 0 \quad \text{in} \quad B_\lambda^+(x_R).$$

Together with

$$\mathbf{w}_\lambda^- = 0 \quad \text{on} \quad \partial B_\lambda^+(x_R),$$

we conclude that

$$\mathbf{w}_\lambda^- = 0 \quad \text{in} \quad B_\lambda^+(x_R).$$

Hence, we complete the proof. □

Proof of Lemma 2.3. We establish Lemma 2.3 by contradiction. If $\bar{\lambda}(R) < +\infty$ for some R, we shall prove that there exists a positive constant ε such that for all $\lambda \in (\bar{\lambda}(R), \bar{\lambda}(R) + \varepsilon)$,

$$\mathbf{u} \le \mathbf{u}_{x_R,\lambda} \quad \text{in} \quad B_\lambda^+(x_R), \tag{2.8}$$

which contradicts with the definition of $\bar{\lambda}(R)$.

It is clearly to see by the definition of $\bar{\lambda}(R)$ that

$$\mathbf{u} \leq \mathbf{u}_{x_R,\bar{\lambda}(R)} \quad \text{in} \ \ B^+_{\bar{\lambda}(R)}(x_R).$$

Combining with

$$\mathbf{u} < \mathbf{u}_{x_R,\bar{\lambda}(R)} \quad \text{on} \quad \partial B^+_{\bar{\lambda}(R)}(x_R) \cap \partial\mathbb{R}^n_+,$$

we have, by the Strong Maximum principle,

$$\mathbf{u} < \mathbf{u}_{x_R,\bar{\lambda}(R)} \quad \text{in} \ \ B^+_{\bar{\lambda}(R)}(x_R).$$

In order to obtain (2.8), we divided the region $B^+_\lambda(x_R)$ into two parts,

$$K_1 := \left\{ x \in B^+_{\bar{\lambda}(R)}(x_R) \middle| \operatorname{dist}(x, \ \partial B^+_{\bar{\lambda}(R)}(x_R)) \geq \delta \right\};$$
$$K_2 := B^+_\lambda(x_R) \backslash K_1,$$

where δ is a small positive constant will be fixed later.

Since K_1 is compact,

$$b_i := \min_{K_1}(u_{i,x_R,\bar{\lambda}(R)} - u_i) > 0.$$

From the fact that the uniform continuity of $\mathbf{u}$ on compact sets, we can choose $\varepsilon < \delta$ sufficient small such that for any $\lambda \in (\bar{\lambda}(R), \bar{\lambda}(R) + \varepsilon)$,

$$\mathbf{u}_{x_R,\lambda} - \mathbf{u}_{x_R,\bar{\lambda}(R)} \geq -\frac{\mathbf{b}}{2} \quad \text{in} \ \ K_1,$$

where $\mathbf{b} := (b_1, b_2, \cdots, b_m)$. Consequently, in view of the above argument, we obtain that for any $\lambda \in (\bar{\lambda}(R), \bar{\lambda}(R) + \varepsilon)$,

$$\mathbf{u}_{x_R,\lambda} - \mathbf{u} \geq \frac{\mathbf{b}}{2} \quad \text{in} \ \ K_1.$$

Now let us focus on the region K_2. Using the narrow domain technique as that in Lemma 2.2, we can fix the value of δ small such that for any $\lambda \in (\bar{\lambda}(R), \bar{\lambda}(R) + \varepsilon)$,

$$\mathbf{u}_{x_R,\lambda} \geq \mathbf{u} \quad \text{in} \ \ K_2.$$

Together with the above argument, we can see that the moving spheres procedure may continue beyond $\bar{\lambda}(R)$ where we reach a contradiction. And we complete the proof of Lemma 2.3. □

3. **Existence of solutions of the PDES in $\mathbb{S}^{n-1}_+$.** First, by applying stereographic projection, the upper semisphere $\mathbb{S}^{n-1}_+$ is mapped into the unit ball of $\mathbb{R}^{n-1}$. Then as for (1.5), the Laplace-Beltrami operator in $\mathbb{S}^{n-1}_+$ can be reduced to the Euclidean Laplace operator, which is convenient to study. Then, we shall show that

$$\mathbf{w} = w\mathbf{c},$$

where w is a positive solution of (1.6), $\mathbf{c} \in \mathbb{S}^+$. Then we can complete the proof.

For the purpose, let us summarize some well-known properties of this transformation: For any point $\xi \in \mathbb{S}^{n-1}\backslash\{S\}$, S is the south pole. Let g_ξ be the straight line through the points ξ and S, and let $g_\xi \cap \{X \in \mathbb{R}^n | X_n = 0\} := (x, 0)$, $x \in \mathbb{R}^{n-1}$. The mapping $\xi \to x$ is conformal and satisfies:

$$\xi = \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right).$$

Proof of Theorem 1.2. By applying stereographic projection $\xi \mapsto x$, $\mathbb{S}_+^{n-1}$ is transformed into $B_1 \subset \mathbb{R}^{n-1}$, and Δ_s is transformed into

$$Lv := \frac{1}{4}(1+r^2)^2\Delta v + \frac{1}{8}(n-3)(1+r^2)^3\nabla\left(\frac{2}{1+r^2}\right)\cdot\nabla v,$$

where Δ now is the Euclidean Laplace operator in $\mathbb{R}^{n-1}$. Suppose $\mathbf{w}$ is the solution of (1.5) and set

$$\mathbf{W}(x) := \frac{\mathbf{w}(\xi)}{(1+r^2)^{(n-3)/2}},$$

it follows that

$$-\Delta\mathbf{W} = \frac{(n-1)(n-3)+4l_{n,p}}{(1+r^2)^2}\mathbf{W} + \frac{4|\mathbf{W}|^{p-1}\mathbf{W}}{(1+r^2)^{(n+1-p(n-3))/2}} \quad \text{in } \ B_1. \tag{3.1}$$

Then we can obtain that $\mathbf{W}(x) = \mathbf{W}(|x|)$, and $\mathbf{W}_r(x) \le 0$ using the method of moving plane as [15], which turned out to be a very powerful technique in proving symmetry results for positive solutions of semilinear elliptic problems in symmetric domains. Then the symmetry proofs in this work depend on a number of technical steps. In order to guarantee the method of moving plane is effective, we need that the right-hand side of (3.1) is non-increasing in r. It is sufficient that

$$(n-1)(n-3)+4l_{n,p} = (n-1)(n-3) + \frac{8(n-p(n-2))}{(p-1)^2} \ge 0,$$

$$n+1-p(n-3) \ge 0.$$

That is, $1 < p \le \frac{n+3}{n-1}$ for $n \ge 2$; and $p = \frac{n+1}{n-3}$ for $n \ge 4$.

Define

$$t := -\ln|x| = -\ln r \quad \text{and} \quad \mathbf{U}(t) := |x|^{\frac{n-3}{2}}\mathbf{W}(x),$$

it follows that

$$-\mathbf{U}'' + \frac{(n-3)^2}{4}\mathbf{U} = \frac{(n-1)(n-3)+4l_{n,p}}{(r^{-1}+r)^2}\mathbf{U} + \frac{4|\mathbf{U}|^{p-1}\mathbf{U}}{(r^{-1}+r)^{(n+1-p(n-3))/2}} \quad \text{in} \quad \mathbb{R}_+.$$

In particular, for any $i \in \{1,2,\cdots,m\}$, we have

$$-U_i'' + \frac{(n-3)^2}{4}U_i = \frac{(n-1)(n-3)+4l_{n,p}}{(r^{-1}+r)^2}U_i + \frac{4|\mathbf{U}|^{p-1}U_i}{(r^{-1}+r)^{(n+1-p(n-3))/2}} \quad \text{in} \quad \mathbb{R}_+.$$

Hence, we obtain that for any $i,\ j \in \{1,2,\cdots,m\}$,

$$U_i''U_j = U_j''U_i,$$

which implies that

$$(U_i'U_j - U_j'U_i)' = 0 \Rightarrow U_i'U_j - U_j'U_i = c,$$

where c is a constant. The following we shall show that $c = 0$. Suppose that $c \ne 0$. Without loss of generality, we can assume that $c > 0$. A direct calculation gives that

$$-\left(\frac{U_j}{U_i}\right)' = \frac{U_i'U_j - U_j'U_i}{U_i^2} = \frac{c}{U_i^2}.$$

Since there exist a positive constant M such that $|\mathbf{U}| \le M$, integrating from 0 to t, we have

$$-\int_0^t \left(\frac{U_j}{U_i}\right)' \ge \int_0^t \frac{c}{M^2}.$$

It follows that

$$-\frac{U_j(t)}{U_i(t)} \geq \frac{ct}{M^2} - \frac{U_j(0)}{U_i(0)} > 0$$

if t sufficiently large. It is a contradiction. Hence, from the above argument, we have

$$\left(\frac{U_j}{U_i}\right)' = 0.$$

Therefore, we conclude that there exist constants c_{ji} such that

$$\frac{U_j}{U_i} = c_{ji}.$$

Fixed any $i \in \{1, 2, \cdots, m\}$, define $\mathbf{c}_i := (c_{1i}, c_{2i}, \cdots, c_{mi})$, we have

$$\mathbf{U} = (c_{1i}, c_{2i}, \cdots, c_{mi})U_i = |\mathbf{c}_i| \left(\frac{c_{1i}}{|\mathbf{c}_i|}, \frac{c_{2i}}{|\mathbf{c}_i|}, \cdots, \frac{c_{mi}}{|\mathbf{c}_i|}\right) U_i.$$

Back to the definition of $\mathbf{U}$, we have

$$\mathbf{w} = (c_{1i}, c_{2i}, \cdots, c_{mi})w_i = |\mathbf{c}_i| \left(\frac{c_{1i}}{|\mathbf{c}_i|}, \frac{c_{2i}}{|\mathbf{c}_i|}, \cdots, \frac{c_{mi}}{|\mathbf{c}_i|}\right) w_i.$$

Since $|\mathbf{c}_i|w_i$ is a solution of (1.6), we know by the work of [4] that if $1 < p \leq \frac{n+1}{n-1}$ for $n \geq 2$, and $p = \frac{n+1}{n-3}$ for $n \geq 4$, then (1.6) admits no positive solution; if $n \geq 2$, $\frac{n+1}{n-1} < p \leq \frac{n+3}{n-1}$, then (1.6) admits a unique positive solution. Therefore, we complete the proof of this theorem. □

4. **Proof of Theorem 1.3.** To prove Theorem 1.3, we introduce two useful propositions, the first one is the doubling property [23, Lemma 5.1], and the second one is Proposition 4.2, which plays a key role in the proof of Theorem 1.3.

Proposition 4.1. *Suppose that $\emptyset \neq D \subset \Sigma \subset \mathbb{R}^n$, Σ is closed and $\Gamma = \Sigma \setminus D$. Let $M : D \to (0, \infty)$ be bounded on compact subset of D. If for a fixed positive constant k, there exists $y \in D$ satisfying*

$$M(y)\text{dist}(y, \Gamma) > 2k,$$

then there exists $x \in D$ such that

$$M(x) \geq M(y), \qquad M(x)\text{dist}(x, \Gamma) > 2k,$$

and for all $z \in D \cap B_{k/M(x)}(x)$,

$$M(z) \leq 2M(x).$$

Proposition 4.2. *Suppose that $1 < p < \frac{n+2}{n-2}$, $0 < r < \frac{1}{2}$ and $\mathbf{u} \in C^2(B_{2r}^+ \backslash \overline{B_r^+}) \cap C(\overline{B_{2r}^+} \backslash B_r^+)$ is a positive solution of*

$$\begin{cases} -\Delta \mathbf{u} = |\mathbf{u}|^{p-1}\mathbf{u} & \text{in } \; B_{2r}^+ \backslash \overline{B_r^+}, \\ \mathbf{u} = 0 & \text{on } \; \partial' B_{2r}^+ \setminus \partial' B_r^+, \end{cases} \tag{4.1}$$

then there exists a positive constant C which is independent of the solution such that

$$|\mathbf{u}(x)| \leq C[\text{dist}(x, \partial'' B_{2r}^+ \cup \partial'' B_r^+)]^{-\frac{2}{p-1}} \quad \text{in } \; B_{2r}^+ \backslash \overline{B_r^+}, \tag{4.2}$$

where $\partial'' B_{2r}^+ \cup \partial'' B_r^+ := \partial(B_{2r}^+ \backslash B_r^+) \cap \mathbb{R}_+^n$.

Proof. Assume by contradiction that (4.2) is false. Then, for every integer $k \geq 1$, there exist $0 < r_k < \frac{1}{2}$, a solution $\mathbf{u}_k$ of (4.1) with $r = r_k$, and $y_k \in B^+_{2r_k} \backslash \overline{B^+_{r_k}}$ such that

$$|\mathbf{u}_k(y_k)| > (2k)^{\frac{2}{p-1}} [\mathrm{dist}(y_k, \partial'' B^+_{2r_k} \cup \partial'' B^+_{r_k})]^{-\frac{2}{p-1}} \quad \text{in} \quad B^+_{2r_k} \backslash \overline{B^+_{r_k}}.$$

Applying the previous Proposition 4.1 with

$$M_k(x) = |\mathbf{u}_k(x)|^{\frac{p-1}{2}}, \quad D_k = B^+_{2r_k} \backslash \overline{B^+_{r_k}}, \quad \Gamma_k = \partial'' B^+_{2r_k} \cup \partial'' B^+_{r_k},$$

there exists $x_k \in D_k$ such that

$$M_k(x_k) \geq M_k(y_k), \quad M_k(x_k) > 2k[\mathrm{dist}(x_k, \Gamma_k)]^{-1} \geq 2k, \tag{4.3}$$

and for any $z \in D_k$ and $|z - x_k| \leq k/M_k(x_k)$,

$$M_k(z) \leq 2M_k(x_k). \tag{4.4}$$

It follows from (4.3) that for any $k \in \mathbb{N}^+$,

$$\mathrm{dist}(x_k, \Gamma_k) > 2k/M_k(x_k), \tag{4.5}$$

and

$$1/M_k(x_k) \to 0 \qquad \text{as } k \to +\infty, \tag{4.6}$$

Consider

$$\mathbf{w}_k(y) := M_k^{\frac{-2}{p-1}}(x_k)\mathbf{u}_k\left(x_k + y/M_k(x_k)\right) \qquad \text{in} \quad \widetilde{B_k},$$

where $\widetilde{B_k} := B_k \cap \{y \in \mathbb{R}^n : y_n > -M_k(x_k)(x_k)_n\}$. Combining (4.5), we obtain that for any $y \in \widetilde{B_k}$,

$$|x_k + y/M_k(x_k) - x_k| = |y|/M_k(x_k) \leq k/M_k(x_k) < \frac{1}{2}\mathrm{dist}(x_k, \Gamma_k),$$

that is,

$$x_k + y/M_k(x_k) \in B_{\frac{1}{2}\mathrm{dist}(x_k,\Gamma_k)}(x_k) \subset D_k.$$

Therefore, $\mathbf{w}_k$ is well defined in $\widetilde{B_k}$ and a calculation gives that $\mathbf{w}_k$ satisfies

$$\begin{cases} -\Delta \mathbf{w}_k = |\mathbf{w}_k|^{p-1}\mathbf{w}_k & \text{in} \quad \widetilde{B_k}, \\ \mathbf{w}_k = 0 & \text{on} \quad B_k \cap \{y \in \mathbb{R}^n : y_n = -M_k(x_k)(x_k)_n\}, \end{cases}$$

and $|\mathbf{w}_k(0)| = 1$. Moreover, from (4.4), we find that for all $y \in \widetilde{B_k}$,

$$|\mathbf{u}_k(x_k + y/M_k(x_k))|^{\frac{p-1}{2}} \leq 2|\mathbf{u}_k(x_k)|^{\frac{p-1}{2}} = 2M_k(x_k).$$

Since

$$|\mathbf{w}_k(y)|^{\frac{p-1}{2}} = \left(|\mathbf{u}_k(x_k + y/M_k(x_k))|^{\frac{p-1}{2}}\right)/M_k(x_k),$$

it implies that

$$|\mathbf{w}_k(y)|^{\frac{p-1}{2}} \leq 2M_k(x_k)/M_k(x_k) = 2. \tag{4.7}$$

Hence, the sequence $\mathbf{w}_k$ is uniformly bounded, it follows that $-\Delta \mathbf{w}_k$ is also uniformly bounded. Passing to a subsequence if necessary, we may assume that either $-M_k(x_k)(x_k)_n \to -\infty$ or $-M_k(x_k)(x_k)_n \to -c \leq 0$.

If $-M_k(x_k)(x_k)_n \to -\infty$, from interior elliptic estimates, we have for every $q \in (1, \infty)$,

$$\|\mathbf{w}_k\|_{W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n)} \leq C_q.$$

Then up to a subsequences $\mathbf{w}_k$ converges locally uniformly in $\mathbb{R}^n$ to some smooth function $\mathbf{w}_\infty$ such that $\mathbf{w}_\infty$ satisfies

$$-\Delta \mathbf{w}_\infty = |\mathbf{w}|^{p-1}\mathbf{w}_\infty \quad \text{in} \quad \mathbb{R}^n,$$

and $|\mathbf{w}_\infty(0)| = 1$, from the Liouville Theorem [14] that $\mathbf{w}_\infty = 0$, it is a contradiction.

If $-M_k(x_k)(x_k)_n \to -c \le 0$, it follows from interior-boundary elliptic estimates that

$$\|\mathbf{w}_k\|_{W^{2,q}_{\text{loc}}(\{y\in\mathbb{R}^n : y_n \ge -c\})} \le C_q.$$

Then up to a subsequence $\mathbf{w}_k$ converges locally uniformly in $\{y \in \mathbb{R}^n : y_n \ge -c\}$ to some smooth function $\mathbf{w}_\infty$ such that $\mathbf{w}_\infty$ satisfies

$$\begin{cases} -\Delta \mathbf{w}_\infty = |\mathbf{w}|^{p-1}\mathbf{w}_\infty & \text{in } \{y \in \mathbb{R}^n : y_n > -c\}, \\ \mathbf{w}_\infty = 0 & \text{on } \{y \in \mathbb{R}^n : y_n = -c\}, \end{cases}$$

and $|\mathbf{w}_\infty(0)| = 1$, from the Theorem 1.1 that $\mathbf{w}_\infty = 0$, it is impossible. Hence, we finish the proof. □

Proof of Theorem 1.3. For any $x \in B^+_{1/2}$, we apply Proposition 4.2 with $r = \frac{2}{3}|x|$. Since $\text{dist}(x, \partial'' B^+_{2r} \cup \partial'' B^+_r) = \frac{r}{2}$, we deduce that

$$|\mathbf{u}(x)| \le C[\text{dist}(x, \partial'' B^+_{2r} \cup \partial'' B^+_r)]^{-\frac{2}{p-1}} = C\left(\frac{r}{2}\right)^{-\frac{2}{p-1}} = \widetilde{C}|x|^{-\frac{2}{p-1}}.$$

This establishes the result. □

5. **Removability.**

5.1. **The case for $p > \frac{n+1}{n-1}$.**

Theorem 5.1. *Suppose that $p > \frac{n+1}{n-1}$, and $\mathbf{u} \in C^2(B^+_1) \cap C(\overline{B^+_1}\setminus\{0\})$ is a positive solution of (1.2). If*

$$\lim_{x\to 0} |x|^{\frac{2}{p-1}}|\mathbf{u}(x)| = 0, \tag{5.1}$$

then $\mathbf{u} \in C^\alpha(\overline{B^+_{1/2}})$ for any $\alpha \in (0,1)$ and $\mathbf{u}(0) = 0$.

Proof. In terms of spherical coordinates, we can write (1.2) as

$$\begin{cases} \mathbf{u}_{rr} + \dfrac{n-1}{r}\mathbf{u}_r + \dfrac{1}{r^2}\Delta_s \mathbf{u} + |\mathbf{u}|^{p-1}\mathbf{u} = 0 & \text{in } (0,1) \times \mathbb{S}^{n-1}_+, \\ \mathbf{u} = 0 & \text{on } (0,1] \times \partial\mathbb{S}^{n-1}_+. \end{cases}$$

Let

$$t := -\ln r, \quad \mathbf{v}(t,\sigma) := r^{\frac{2}{p-1}}\mathbf{u}(r,\sigma). \tag{5.2}$$

Moreover, $\mathbf{v}$ satisfies

$$\begin{cases} \mathbf{v}_{tt} + \Delta_s \mathbf{v} - \left(n - \dfrac{2(p+1)}{p-1}\right)\mathbf{v}_t + l_{n,p}\mathbf{v} + |\mathbf{v}|^{p-1}\mathbf{v} = 0 & \text{in } (0,+\infty) \times \mathbb{S}^{n-1}_+, \\ \mathbf{v} = 0 & \text{on } (0,+\infty) \times \partial\mathbb{S}^{n-1}_+ \end{cases} \tag{5.3}$$

it follows by assumption (5.1) that there exists $T_0 > 0$ such that $\mathbf{v}$ is bounded on $[T_0, +\infty) \times \mathbb{S}^{n-1}_+$. Then by the Agmon-Douglis-Nirenberg estimates (see [2]) we have for any $q \in (1,+\infty)$,

$$\|\mathbf{v}\|_{W^{2,q}((t-1,t+1)\times\mathbb{S}^{n-1}_+)} \le C\|\mathbf{v}\|_{L^q((t-2,t+2)\times\mathbb{S}^{n-1}_+)},$$

where $t \in [T_0 + 3, +\infty)$, and C depends on q but not on t. Together with $\mathbf{v}$ is bounded on $[T_0, +\infty) \times \mathbb{S}^{n-1}_+$, we have

$$\|\mathbf{v}\|_{W^{2,q}((t-1,t+1)\times\mathbb{S}^{n-1}_+)} \le C\|\mathbf{v}\|_{L^2((t-3,t+3)\times\mathbb{S}^{n-1}_+)}, \tag{5.4}$$

for $t \in [T_0+3, +\infty)$, q is any number in $(1, +\infty)$. Multiplying both sides of the i-th components of the system (5.3) by v_i and integrating by parts in $\mathbb{S}_+^{n-1}$, it follows that

$$\int_{\mathbb{S}_+^{n-1}} v_i \left[(v_i)_{tt} + \Delta_s v_i - \left(n - \frac{2(p+1)}{p-1}\right)(v_i)_t + l_{n,p} v_i + |\mathbf{v}|^{p-1} v_i\right] d\sigma = 0.$$

Then in order to obtain the theorem, for any $t > 0$, we define

$$X_i(t) := \|v_i(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}.$$

A direct calculation gives that for every $t \in (0, +\infty)$,

$$X_i(X_i)_t = \int_{\mathbb{S}_+^{n-1}} v_i(v_i)_t d\sigma. \tag{5.5}$$

Using Hölder's inequality we have

$$|X_i(X_i)_t| \le \|v_i(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})} \|(v_i)_t(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}.$$

Thus,

$$|(X_i)_t| \le \|(v_i)_t(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}. \tag{5.6}$$

Computing the derivative with respect to t on both sides of identity (5.5), we get

$$\begin{aligned}(X_i)_t^2 + X_i(X_i)_{tt} &= \int_{\mathbb{S}_+^{n-1}} (v_i)_t^2 d\sigma + \int_{\mathbb{S}_+^{n-1}} v_i(v_i)_{tt} d\sigma \\ &= \|(v_i)_t(t,\cdot)\|^2_{L^2(\mathbb{S}_+^{n-1})} + \int_{\mathbb{S}_+^{n-1}} v_i(v_i)_{tt} d\sigma.\end{aligned}$$

From this identity and estimate (5.6), we deduce that

$$X_i(X_i)_{tt} \ge \int_{\mathbb{S}_+^{n-1}} v_i(v_i)_{tt} d\sigma.$$

On the other hand, since the first eigenvalue of the Laplace-Beltrami operator $-\Delta_s$ in $W_0^{1,2}(\mathbb{S}_+^{n-1})$ is $n-1$,

$$(n-1)(X_i)^2 \le \int_{\mathbb{S}_+^{n-1}} |\nabla_\sigma v_i|^2 d\sigma = -\int_{\mathbb{S}_+^{n-1}} v_i \Delta_s v_i d\sigma.$$

The Hölder inequality gives that

$$\int_{\mathbb{S}_+^{n-1}} |\mathbf{v}|^{p-1} v_i^2 \le (X_i)^2 \|\mathbf{v}(t,\cdot)\|^{p-1}_{L^\infty(\mathbb{S}_+^{n-1})}.$$

Combining with the above estimates, we have

$$(X_i)_{tt} - \left(n - \frac{2(p+1)}{p-1}\right)(X_i)_t + \left(l_{n,p} - n + 1 + \|\mathbf{v}(t,\cdot)\|^{p-1}_{L^\infty(S_+^{n-1})}\right) X_i \ge 0. \tag{5.7}$$

From the condition we have

$$\lim_{t\to+\infty} |\mathbf{v}(t,\cdot)| = 0 \quad \text{uniformly} \quad \text{in} \quad \mathbb{S}_+^{n-1}.$$

it follows that for a given sufficient small $\varepsilon > 0$ there exists $t_0 > T_0 + 10$ such that for any $t \in (t_0, +\infty)$,

$$\|\mathbf{v}(t,\cdot)\|^{p-1}_{L^\infty(\mathbb{S}_+^{n-1})} < \varepsilon.$$

We deduce that for every $t \in (t_0, +\infty)$,

$$(X_i)_{tt} - \left(n - \frac{2(p+1)}{p-1}\right)(X_i)_t + (l_{n,p} - n + 1 + \varepsilon)X_i \geq 0. \tag{5.8}$$

On the other hand, consider

$$Z(t) := C_0 e^{-\left(\frac{p+1}{p-1} + \frac{\sqrt{n^2-4\varepsilon}-n}{2}\right)t},$$

where $0 < \varepsilon < \min\left\{\frac{n^2}{4}, \frac{p+1}{p-1}\left(n - \frac{p+1}{p-1}\right)\right\}$, C_0 is a positive constant C_0 such that

$$X_i(t_0) \leq Z(t_0),$$

and it follows that $Z(t)$ satisfies

$$Z_{tt} - \left(n - \frac{2(p+1)}{p-1}\right)Z_t + (l_{n,p} - n + 1 + \varepsilon)Z = 0,$$

and

$$\lim_{t \to +\infty} Z(t) = 0.$$

Moreover, if $\varepsilon > 0$ sufficient small, the fact $p > \frac{n+1}{n-1}$ gives

$$l_{n,p} - n + 1 + \varepsilon < 0.$$

Hence, using the Maximum principle for X_i and Z, we obtain that for any $t \in (t_0, +\infty)$,

$$X_i(t) \leq C_0 e^{-\left(\frac{p+1}{p-1} + \frac{\sqrt{n^2-4\varepsilon}-n}{2}\right)t}.$$

Applying estimates (5.4) with $q > \frac{n}{2}$, we have

$$\|\mathbf{v}\|_{W^{2,q}((t-1,t+1)\times\mathbb{S}^{n-1}_+)} \leq C e^{-\left(\frac{p+1}{p-1} + \frac{\sqrt{n^2-4\varepsilon}-n}{2}\right)t}.$$

The Morrey's inequality implies that

$$\|\mathbf{v}\|_{L^\infty((t-1,t+1)\times\mathbb{S}^{n-1}_+)} \leq C e^{-\left(\frac{p+1}{p-1} + \frac{\sqrt{n^2-4\varepsilon}-n}{2}\right)t}.$$

The estimate above implies that for $|x|$ small,

$$|\mathbf{u}| \leq C|x|^{\frac{\sqrt{n^2-4\varepsilon}-(n-2)}{2}}.$$

Together with $\varepsilon > 0$ is sufficient small, we conclude that $\mathbf{u}$ is Hölder continuous up to $x = 0$ and $u(0) = 0$. □

5.2. The case for $p = \frac{n+1}{n-1}$.

Theorem 5.2. *For $p = \frac{n+1}{n-1}$, if $\mathbf{u} \in C^2(B_1^+) \cap C(\overline{B_1^+}\backslash\{0\})$ is a positive solution of (1.2) and*

$$\lim_{x \to 0} |\mathbf{u}(x)||x|^{n-1}\left(\log\frac{1}{|x|}\right)^{\frac{n-1}{2}} = 0, \tag{5.9}$$

then $\mathbf{u}$ can be continuously extended to 0.

Proof. Since $p = \frac{n+1}{n-1}$, then (5.3) can be written as

$$\begin{cases} \mathbf{v}_{tt} + \Delta_s \mathbf{v} + n\mathbf{v}_t + (n-1)\mathbf{v} + |\mathbf{v}|^{\frac{2}{n-1}}\mathbf{v} = 0 & \text{in } (0,+\infty)\times \mathbb{S}_+^{n-1}, \\ \mathbf{v} = 0 & \text{on } (0,+\infty)\times \partial\mathbb{S}_+^{n-1}, \end{cases} \tag{5.10}$$

where $\mathbf{v}$ defined as (5.2). For any $t \in (0,+\infty)$, $i \in \{1,2,\cdots,m\}$, let

$$X_i(t) := t^{\frac{n-1}{2}} \|v_i(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}.$$

Multiplying both sides of the i-th components of the system (5.10) by $t^{n-1}v_i$ and integrating by parts in $\mathbb{S}_+^{n-1}$ and by the fact that

$$t^{n-1}\int_{\mathbb{S}_+^{n-1}} v_i (v_i)_t d\sigma = X_i (X_i)_t - \frac{n-1}{2t} X_i^2,$$

and

$$t^{n-1}\int_{\mathbb{S}_+^{n-1}} v_i (v_i)_{tt} d\sigma$$
$$\leq X_i (X_i)_{tt} + \frac{n^2-1}{4t^2} X_i^2 - \frac{n-1}{t} X_i (X_i)_t,$$

then together with (5.9), we conclude that for any $\varepsilon > 0$ there exists t_0 sufficiently large such that for every $t \in (t_0,+\infty)$,

$$(X_i)_{tt} + \left(n - \frac{n-1}{t}\right)(X_i)_t + \frac{1}{t}\left(\frac{n^2-1}{4t} - \frac{n(n-1)}{2} + \varepsilon\right) X_i \geq 0.$$

On the other hand, applying Lemma A.2 in [4] we know that one solution of the equation

$$Z_{tt} + \left(n - \frac{n-1}{t}\right) Z_t + \frac{1}{t}\left(\frac{n^2-1}{4t} - \frac{n(n-1)}{2} + \varepsilon\right) Z = 0$$

satisfies the following asymptotic behaviors as $t \to +\infty$,

$$Z(t) = t^{\frac{n-1}{2}+\frac{\varepsilon}{n}} e^{-nt}(1+o(1)).$$

Let $\varepsilon > 0$ small enough and $T_0 \geq t_0$ large enough such that

$$\frac{n^2-1}{4t} - \frac{n(n-1)}{2} + \varepsilon < 0 \quad \text{in} \quad [T_0,+\infty),$$

Choose a positive constant $C_0 \in \mathbb{R}$ such that

$$X_i(T_0) \leq C_0 Z(T_0).$$

Then from

$$X_i(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Using the Maximum principle for X_i and $C_0 Z$, we deduce that for $t \in (T_0,+\infty)$,

$$X_i(t) \leq C_0 Z(t).$$

In particular, for t large,

$$X_i(t) \leq C_0 t^{\frac{n-1}{2}} e^{-(n-1)t}.$$

By the same argument as Theorem 5.1, we conclude that $\mathbf{u}$ is bounded and $\mathbf{u}$ can be continuously extended to 0. $\square$

6. **Asymptotic.**

6.1. **The case for $p = \frac{n+1}{n-1}$.**
In this part, with a blow up rate assumption, we shall show some asymptotic symmetry.

Theorem 6.1. *Let $p = \frac{n+1}{n-1}$, and $\mathbf{u}$ be a positive solution of (1.2). If*

$$|x|^{\frac{2}{p-1}}|\mathbf{u}(x)| \quad \textit{is bounded}, \tag{6.1}$$

then

$$\lim_{x\to 0} |x|^{\frac{2}{p-1}}\,|\mathbf{u}(x)| = 0.$$

Proof. From our assumption (6.1), it follows that there exists $T_0 > 0$ such that $\mathbf{v}$ is bounded $[T_0, +\infty) \times \mathbb{S}^{n-1}_+$, where $\mathbf{v}$ is defined as (5.2) in Theorem 5.1. It follows from the estimates (5.4) and the Morrey's inequality that for any $\gamma \in (0, 1)$, $t \in [T_0 + 3, +\infty)$,

$$\|\mathbf{v}\|_{C^{1,\gamma}((t-1,t+1)\times\mathbb{S}^{n-1}_+)} \le C(\gamma).$$

Furthermore, we also have by elliptic estimates that

$$\|\mathbf{v}\|_{C^{2,\gamma}((t-1,t+1)\times\mathbb{S}^{n-1}_+)} \le C(\gamma) \tag{6.2}$$

for any $\gamma \in (0, 1)$, $t \in [T_0+3, +\infty)$. With the above estimates, to prove the theorem, we first to show that

$$\mathbf{v}_t(t,\cdot) \to 0 \quad \text{uniformly in } \mathbb{S}^{n-1}_+ \quad \text{as} \quad t \to +\infty. \tag{6.3}$$

Next we can prove that

$$\mathbf{v}(t,\cdot) \to 0 \quad \text{uniformly in } \mathbb{S}^{n-1}_+ \quad \text{as} \quad t \to +\infty. \tag{6.4}$$

As the first step, we shall show (6.3). Multiplying the system (5.3) by $\mathbf{v}_t$ and integrating over $\mathbb{S}^{n-1}_+$ yields

$$\int_{\mathbb{S}^{n-1}_+} \mathbf{v}_t \cdot \mathbf{v}_{tt} d\sigma + \int_{\mathbb{S}^{n-1}_+} \mathbf{v}_t \cdot \Delta_s \mathbf{v} d\sigma - \left(n - \frac{2(p+1)}{p-1}\right) \int_{\mathbb{S}^{n-1}_+} |\mathbf{v}_t|^2 d\sigma$$
$$+ l_{n,p} \int_{\mathbb{S}^{n-1}_+} \mathbf{v}_t \cdot \mathbf{v} d\sigma + \int_{\mathbb{S}^{n-1}_+} |\mathbf{v}|^{p-1}\mathbf{v}_t \cdot \mathbf{v} d\sigma = 0.$$

Combining $\mathbf{v}_t$ vanishes on the boundary $[T_0, +\infty) \times \partial\mathbb{S}^{n-1}_+$, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{S}^{n-1}_+} \left(\frac{|\mathbf{v}_t|^2}{2} - \frac{|\nabla_\sigma \mathbf{v}|^2}{2} + \frac{l_{n,p}|\mathbf{v}|^2}{2} + \frac{|\mathbf{v}|^{p+1}}{p+1}\right) d\sigma \\ &= \left(n - \frac{2(p+1)}{p-1}\right) \int_{\mathbb{S}^{n-1}_+} |\mathbf{v}_t|^2 d\sigma. \end{aligned} \tag{6.5}$$

Since (6.2) gives that

$$\left|\int_{\mathbb{S}^{n-1}_+} \left(\frac{|\mathbf{v}_t|^2}{2} - \frac{|\nabla_\sigma \mathbf{v}|^2}{2} + \frac{l_{n,p}|\mathbf{v}|^2}{2} + \frac{|\mathbf{v}|^{p+1}}{p+1}\right) d\sigma\right| \le C \quad \text{in } [T_0 + 3, +\infty)$$

for some constant $C > 0$. Hence, integrating (6.5) on $(T_0 + 3, +\infty)$, we obtain

$$\left|\left(n - \frac{2(p+1)}{p-1}\right) \int_{T_0+3}^{+\infty} \int_{\mathbb{S}^{n-1}_+} |\mathbf{v}_t|^2 d\sigma ds\right| < +\infty. \tag{6.6}$$

On the other hand, since $p \neq \frac{n+2}{n-2}$, $n - \frac{2(p+1)}{p-1} \neq 0$, we conclude that (6.3) follows. Indeed, by (6.2) we obtain that $\mathbf{v}_t$, $\mathbf{v}_{tt}$ is uniformly bounded in $(T_0+3, +\infty) \times \mathbb{S}_+^{n-1}$. It follows that there exists a constant $M > 0$ such that

$$\left| \frac{d}{dt} \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t|^2 d\sigma \right| \leq 2 \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t||\mathbf{v}_{tt}| d\sigma \leq M$$

for $t \in (T_0 + 3, +\infty)$. If (6.3) not true, for a given $\varepsilon > 0$ there exist a sequences $\{t_l\} \to +\infty$ such that $\int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t(t_l, \cdot)|^2 d\sigma > \varepsilon$ and choose $\eta = \frac{\varepsilon}{4M}$ such that for any $t \in (t_l - \eta, t_l + \eta)$,

$$\begin{aligned}\int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t(t, \cdot)|^2 d\sigma &= \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t(t_l, \cdot)|^2 d\sigma - \int_{t_l}^{t} \frac{d}{dt} \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t(t, \cdot)|^2 d\sigma \\ &\geq \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t(t_l, \cdot)|^2 d\sigma - \frac{\varepsilon}{2} \\ &> \frac{\varepsilon}{2}.\end{aligned}$$

We can assume that $t_l < t_{l+1} - \eta < t_{l+1} < t_{l+1} + \eta < t_{l+2}$, then

$$\int_{T_0+3}^{t_l - \eta} \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}_t|^2 d\sigma ds > \frac{(l-1)M\varepsilon^2}{4} \to +\infty \quad \text{as } l \to +\infty.$$

It is a contradiction with (6.6).

For (6.4), we study the limit set of the trajectories of v_i, $i \in \{1, \cdots, m\}$ and for simplicity, we just consider $i = 1$, namely the set

$$\Gamma = \cap_{\tau \geq 0} \overline{\cup_{t \geq \tau} \{v_1(t, \cdot)\}},$$

where the closure is computed with respect to the usual norm in $C^0(\mathbb{S}_+^{n-1})$. Since Γ is the intersection of a decreasing family of closed connected subsets of $C^0(\mathbb{S}_+^{n-1})$, Γ_i is closed and connected. In addition, from (6.2) and the Arzelà-Ascoli theorem that Γ_i is also compact and nonempty.

For any $w_1 \in \Gamma$, let t_k be a sequence of nonnegative real numbers such that $t_k \to +\infty$ and

$$v_1(t_k, \cdot) \to w_1 \quad \text{uniformly in } \mathbb{S}_+^{n-1}$$

Clearly, w_1 is nonnegative and $w_1 = 0$ on $\partial \mathbb{S}_+^{n-1}$. For each $k \geq 1$, let

$$\mathbf{V}_k : (s, \sigma) \in [0,1] \times \mathbb{S}_+^{n-1} \to \mathbb{R}^m$$

be the function defined by $\mathbf{V}_k(s, \sigma) = \mathbf{v}(t_k + s, \sigma)$. For every $\phi \in C_0^\infty(\mathbb{S}_+^{n-1})$ and for every $\varepsilon \in (0,1)$, from the equation satisfied by $\mathbf{v}$ we have

$$\int_0^\varepsilon \int_{\mathbb{S}_+^{n-1}} \left((\mathbf{V}_k)_{tt} + \Delta_s \mathbf{V}_k - \left(n - \frac{2(p+1)}{p-1} \right) (\mathbf{V}_k)_t + l_{n,p} \mathbf{V}_k + |\mathbf{V_k}|^{p-1} \mathbf{V}_k \right) \phi d\sigma ds$$
$$= 0.$$

Since the sequence $\mathbf{V}_k$ is bounded in $C^1([0,1] \times \mathbb{S}_+^{n-1})$, passing to a subsequence if necessary, we may assume that for some continuous functions $\mathbf{W}$,

$$\mathbf{V}_k \to \mathbf{W} \quad \text{uniformly in } [0,1] \times \mathbb{S}_+^{n-1}.$$

Furthermore, the fact that $\mathbf{v}_t \to 0$ uniformly as $t \to +\infty$ gives

$$\int_0^\varepsilon \int_{\mathbb{S}_+^{n-1}} (\mathbf{V}_k)_{tt} \phi d\sigma ds = \int_{\mathbb{S}_+^{n-1}} (\mathbf{v}_t(t_k + \varepsilon, \sigma) - \mathbf{v}_t(t_k, \sigma)) \phi d\sigma \to 0,$$

and

$$\int_0^\varepsilon \int_{\mathbb{S}_+^{n-1}} (\mathbf{V}_k)_t \phi d\sigma ds \to 0.$$

Therefore, we conclude that for every $\varepsilon \in (0,1)$,

$$\int_0^\varepsilon \int_{\mathbb{S}_+^{n-1}} \left(\mathbf{W}\Delta_s\phi + l_{n,p}\mathbf{W}\phi + |\mathbf{W}|^{p-1}\mathbf{W}\phi\right) d\sigma ds = 0.$$

Dividing both sides by ε, and letting $\varepsilon \to 0$, we get

$$\int_{\mathbb{S}_+^{n-1}} \left(\mathbf{W}(0,\sigma)\Delta_s\phi + l_{n,p}\mathbf{W}(0,\sigma)\phi + |\mathbf{W}(0,\sigma)|^{p-1}\mathbf{W}(0,\sigma)\phi\right) d\sigma = 0.$$

Then (6.4) follows from the fact that $\mathbf{W}(0,\cdot) = 0$ by Theorem 1.2, and we finish the proof. □

Now we establish a more precise estimates near the singularity for $p = \frac{n+1}{n-1}$.

Theorem 6.2. *Let $p = \frac{n+1}{n-1}$ and $\mathbf{u}$ be a positive solution of (1.2), then*

$$|\mathbf{u}(x)| \le C|x|^{1-n}\left(\log\frac{1}{|x|}\right)^{-\frac{n-1}{2}} \qquad \text{near } x = 0$$

for some constant $C > 0$ (possibly depending on the solution).

To obtain Theorem 6.2, we just need to prove the following Theorem.

Theorem 6.3. *Let $p = \frac{n+1}{n-1}$, $E = ker[\Delta_s + (n-1)I]$ and $\mathbf{u}$ be a positive solution of (1.2). $\mathbf{v}$ is defined as (5.2). If $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ is the decomposition of $\mathbf{v}$ as the orthogonal projections in $L^2(\mathbb{S}_+^{n-1})$ onto E and $E^\perp$, respectively, then for t large,*

$$\|\mathbf{v}_1(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})} \le Ct^{-\frac{n-1}{2}}, \tag{6.7}$$

and

$$\|\mathbf{v}_2(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})} \le Ce^{-\frac{t}{2}\left(n+\sqrt{n^2+4n}\right)}. \tag{6.8}$$

Proof. We first prove the estimates (6.7). Denoting by ϕ_1 the positive first eigenfunction of $-\Delta_s$ with $\|\phi_1\|_{L^2(\mathbb{S}_+^{n-1})} = 1$, we have

$$\mathbf{v}_1(t,\sigma) = \mathbf{y}(t)\phi_1(\sigma),$$

where

$$\mathbf{y}(t) := \int_{\mathbb{S}_+^{n-1}} \mathbf{v}(t,\cdot)\phi_1(\cdot) d\sigma.$$

Multiplying (5.10) by ϕ_1 and integrating over $\mathbb{S}_+^{n-1}$, we have

$$\mathbf{y}'' + n\mathbf{y}' + \int_{\mathbb{S}_+^{n-1}} |\mathbf{v}|^{\frac{2}{n-1}}\phi_1\mathbf{v} d\sigma = 0.$$

By Hölder inequality and $\|\phi_1\|_{L^2(\mathbb{S}_+^{n-1})}=1$, it follows that for any $i\in\{1,2,\cdots,m\}$,

$$\begin{aligned} y_i^{\frac{n+1}{n-1}} =&\left(\int_{\mathbb{S}_+^{n-1}} v_i\phi_1^{\frac{n-1}{n+1}}\phi_1^{\frac{2}{n+1}}d\sigma\right)^{\frac{n+1}{n-1}}\\ \leq& C\int_{\mathbb{S}_+^{n-1}} v_i^{\frac{n+1}{n-1}}\phi_1 d\sigma\\ \leq& C\int_{\mathbb{S}_+^{n-1}} |\mathbf{v}|^{\frac{2}{n-1}}v_i\phi_1 d\sigma, \end{aligned}$$

which implies that

$$y_i''+ny_i'+Cy_i^{\frac{n+1}{n-1}}\leq 0.$$

Addition, from Theorem 6.1, we have

$$\lim_{t\to+\infty}|\mathbf{v}(t,\cdot)|=0 \quad \text{uniformly} \quad \text{in} \quad \mathbb{S}_+^{n-1}.$$

It follows that $\lim_{t\to+\infty}y_i(t)=0$. Applying Lemma A.1 in [4] we deduce that

$$y_i\leq Ct^{-\frac{n-1}{2}} \quad \text{as } t\to+\infty.$$

By the definition of $\mathbf{v_1}(t,\cdot)$ and $\|\phi_1\|_{L^2(\mathbb{S}_+^{n-1})}=1$, we conclude that as $t\to+\infty$,

$$\|\mathbf{v_1}(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}=|\mathbf{y}(t)|\|\phi_1(\sigma)\|_{L^2(\mathbb{S}_+^{n-1})}\leq Ct^{-\frac{n-1}{2}}.$$

This proves the first estimate (6.7).

We next prove the estimates (6.8). From $\mathbf{v}(t,\sigma)=\mathbf{y}(t)\phi_1(\sigma)+\mathbf{v}_2(t,\sigma)$, we have

$$\mathbf{v}_t(t,\sigma)=\mathbf{y}_t(t,\sigma)\phi_1(\sigma)+(\mathbf{v}_2)_t(t,\sigma), \quad \mathbf{v}_{tt}=\mathbf{y}_{tt}(t,\sigma)\phi_1(\sigma)+(\mathbf{v}_2)_{tt}(t,\sigma).$$

For $t\in(0,+\infty)$, define

$$Y_i(t):=\|v_{i,2}(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}.$$

By the orthogonality between ϕ_1 and $v_{i,2}$, we have

$$Y_i(Y_i)_t=\int_{\mathbb{S}_+^{n-1}} v_{i,2}(v_{i,2})_t d\sigma=\int_{\mathbb{S}_+^{n-1}} v_{i,2}[(y_i)_t\phi_1+(v_{i,2})_t]d\sigma=\int_{\mathbb{S}_+^{n-1}} v_{i,2}(v_i)_t d\sigma.$$

From the first equality and the Hölder inequality gives that

$$|(Y_i)_t|\leq\|(v_{i,2})_t(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})}.$$

As in the proof of Theorem 5.1, we have

$$Y_i(Y_i)_{tt}\geq\int_{\mathbb{S}_+^{n-1}} v_{i,2}(v_i)_{tt}d\sigma.$$

Since the second eigenvalue of the Laplace-Beltrami operator $-\Delta_s$ in $W_0^{1,2}(\mathbb{S}_+^{n-1})$ is $2n$,

$$2nY_i^2\leq\int_{\mathbb{S}_+^{n-1}}|\nabla v_{i,2}|^2 d\sigma=-\int_{\mathbb{S}_+^{n-1}} v_{i,2}\Delta_s v_{i,2}d\sigma=-\int_{\mathbb{S}_+^{n-1}} v_{i,2}\Delta_s v_i d\sigma.$$

Then multiplying the i-th of (5.10) by $v_{i,2}$ and integrating over $\mathbb{S}_+^{n-1}$, together with $\lim_{t\to+\infty}|\mathbf{v}(t,\cdot)|=0$ uniformly in $\mathbb{S}_+^{n-1}$, we obtain that as in the proof of Theorem 5.1, for every $\varepsilon\in(0,1)$ there exists $t_0>0$ such that for every $t\in(t_0,+\infty)$,

$$(Y_i)_{tt}+n(Y_i)_t-(n+1-\varepsilon)Y_i\geq 0.$$

Define

$$Z(t) := C_0 e^{-\frac{t}{2}\left(n+\sqrt{n^2+4n+4-4\varepsilon}\right)},$$

where C_0 a positive constant such that

$$Y_i(t_0) \le Z(t_0).$$

We also have

$$\lim_{t\to+\infty} Z(t) = 0.$$

It is obviously that Z satisfies

$$Z_{tt} + nZ_t - (n+1-\varepsilon)Z = 0.$$

Since $Y_i(t) \to 0$ as $t \to +\infty$ and $-(n+1-\varepsilon) < 0$, applying the Maximum principle one deduces that for any $t \in (t_0, +\infty)$,

$$Y_i(t) \le Z(t) < Ce^{-\frac{t}{2}\left(n+\sqrt{n^2+4n}\right)}.$$

This gives the estimate for $\mathbf{v_2}$ and we complete the proof of (6.8). □

Proof of Theorem 6.2. By the above theorem, we have as $t \to +\infty$,

$$\|\mathbf{v}(t,\cdot)\|_{L^2(\mathbb{S}^{n-1}_+)} \le Ct^{-\frac{n-1}{2}}.$$

Choosing the q of (5.4) bigger than $\frac{n}{2}$, by the Morrey's inequality the result follows. □

Now we shall begin to prove Theorem 1.4.

Proof of Theorem 1.4. Define

$$\mathbf{w}(t,\sigma) := t^{\frac{n-1}{2}}\mathbf{v}(t,\sigma),$$

where $\mathbf{v}$ is defined as (5.2) and Theorem 6.2 implies that $\mathbf{w}$ is bounded in $(T_0, +\infty)\times \mathbb{S}^{n-1}_+$ for some $T_0 > 0$ large enough. By a straightforward computation, $\mathbf{w}$ satisfies

$$\begin{aligned} &\mathbf{w}_{tt} + \Delta_s \mathbf{w} + \left(n-1+\frac{n^2-1}{4t^2}\right)\mathbf{w} \\ &+\left(n-\frac{n-1}{t}\right)\mathbf{w}_t + \frac{1}{t}\left(|\mathbf{w}|^{\frac{2}{n-1}}\mathbf{w} - \frac{n(n-1)}{2}\mathbf{w}\right) = 0. \end{aligned} \tag{6.9}$$

Let $\phi : \mathbb{S}^{n-1}_+ \to \mathbb{R}$ be the function defined by $\phi(\sigma) = \frac{\sigma_n}{|\sigma|}$. We also recall that the first eigenvalue of $-\Delta_s$ in $W^{1,2}_0(\mathbb{S}^{n-1}_+)$ is $n-1$ and the eigenspace associated to this eigenvalue is spanned by the function $\phi(\sigma)$. For $t \in (0, +\infty)$, let

$$\mathbf{z}(t) = \int_{\mathbb{S}^{n-1}_+} \mathbf{w}(t,\cdot)\phi(\cdot)d\sigma,$$

and we know that $\mathbf{z}$ is bounded in $(T_0, +\infty)$. Multiplying (6.9) by ϕ and integrating over $\mathbb{S}^{n-1}_+$, we obtain that

$$\mathbf{z}_{tt} + \frac{n^2-1}{4t^2}\mathbf{z} + \left(n-\frac{n-1}{t}\right)\mathbf{z}_t + \frac{1}{t}\int_{\mathbb{S}^{n-1}_+} |\mathbf{w}|^{\frac{2}{n-1}}\mathbf{w}\phi d\sigma - \frac{n(n-1)}{2t}\mathbf{z} = 0.$$

Thus, for any $i \in \{1, 2, \cdots, m\}$, we have

$$(z_i)_{tt} + \left(n - \frac{n-1}{t}\right)(z_i)_t + \frac{1}{t}\left(\int_{\mathbb{S}_+^{n-1}} z_i^{\frac{n+1}{n-1}} \phi^{\frac{2n}{n-1}} d\sigma - \frac{n(n-1)}{2} z_i\right)$$
$$= -\frac{n^2-1}{4t^2} z_i + \frac{1}{t}\int_{\mathbb{S}_+^{n-1}} (z_i^{\frac{n+1}{n-1}} \phi^{\frac{n+1}{n-1}} - |\mathbf{w}|^{\frac{2}{n-1}} w_i)\phi d\sigma.$$

On the one hand, we obtain by Theorem 6.3 that as $t \to +\infty$,

$$\|v_{i,2}(t,\cdot)\|_{L^2(\mathbb{S}_+^{n-1})} = \left\| v_i(t,\cdot) - \int_{\mathbb{S}_+^{n-1}} v_i(t,\cdot)\phi(\cdot)d\sigma\phi(\cdot)\right\|_{L^2(\mathbb{S}_+^{n-1})} \le Ce^{-\frac{t}{2}\left(n+\sqrt{n^2+4n}\right)}.$$

Hence, we have by the definition of $\mathbf{w}$ and $\mathbf{z}$ that as $t \to +\infty$,

$$\|w_i(t,\cdot) - z_i(t)\phi(\cdot)\|_{L^2(\mathbb{S}_+^{n-1})} \le Ct^{\frac{n-1}{2}} e^{-\frac{t}{2}\left(n+\sqrt{n^2+4n}\right)}. \tag{6.10}$$

It is easy to see that

$$z_i^{\frac{n+1}{n-1}} \phi^{\frac{n+1}{n-1}} - |\mathbf{w}|^{\frac{2}{n-1}} w_i \le (z_i\phi)^{\frac{n+1}{n-1}} - w_i^{\frac{n+1}{n-1}}$$
$$\le \frac{n+1}{n-1}|z_i\phi - w_i|(z_i\phi)^{\frac{2}{n-1}}.$$

From the fact that $\mathbf{z}$ is bounded in $(T_0, +\infty)$, we using the Hölder inequality and (6.10) obtain that as $t \to +\infty$,

$$\int_{\mathbb{S}_+^{n-1}} (z_i^{\frac{n+1}{n-1}} \phi^{\frac{n+1}{n-1}} - |\mathbf{w}|^{\frac{2}{n-1}} w_i)\phi d\sigma$$
$$\le \frac{n+1}{n-1} z_i^{\frac{2}{n-1}} \|w_i(t,\cdot) - z_i(t)\phi(\cdot)\|_{L^2(\mathbb{S}_+^{n-1})} \|\phi^{\frac{2}{n-1}}\|_{L^2(\mathbb{S}_+^{n-1})}$$
$$\le Ct^{\frac{n-1}{2}} e^{-\frac{t}{2}\left(n+\sqrt{n^2+4n}\right)}.$$

Thus, as t large enough, we have

$$\left\| -\frac{n^2-1}{4t^2} z_i + \frac{1}{t}\int_{\mathbb{S}_+^{n-1}} ((z_i\phi)^{\frac{n+1}{n-1}} - w_i^{\frac{n+1}{n-1}})\phi d\sigma \right\|_{L^\infty(\mathbb{S}_+^{n-1})} \le Ct^{-2}.$$

By a straightforward modification of the end of the proof of [7, Corollary 4.2], z_i admits a limit $k \ge 0$ when $t \to +\infty$, where k satisfies

$$\left(\int_{\mathbb{S}_+^{n-1}} \phi^{\frac{2n}{n-1}} d\sigma\right) k^{\frac{n+1}{n-1}} - \frac{n(n-1)}{2} k = 0.$$

Therefore, either $k = 0$ or $k = \left(\frac{n(n-1)}{2\int_{\mathbb{S}_+^{n-1}} \phi^{\frac{2n}{n-1}} d\sigma}\right)^{\frac{n-1}{2}}$. As $t \to +\infty$, we deduce by (6.10) that,

$$w_i(t,\cdot) := t^{\frac{n-1}{2}} v_i(t,\cdot) \to k\phi \quad \text{in} \quad L^2(\mathbb{S}_+^{n-1}).$$

We conclude by (5.4) and Morrey's inequality that

$$t^{\frac{n-1}{2}} v_i(t,\cdot) \to k\phi \quad \text{uniformly in} \quad \mathbb{S}_+^{n-1}.$$

Rewriting the convergence in terms of u, we conclude that either (1.8) holds or

$$|x|^{n-1}\left(\log\frac{1}{|x|}\right)^{\frac{n-1}{2}}\mathbf{u}(x)\to 0 \quad \text{as} \quad x\to 0.$$

If the above estimates holds, then $\mathbf{u}$ must can be continuously extended to 0 in view of Theorem 5.2. □

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REFERENCES

[1] M. J. Ablowitz, B. Prinari and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems*, London Mathematical Society Lecture Note Series, 302. Cambridge University Press, Cambridge, 2004.

[2] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Commun. PureAppl. Math.*, **12** (1959), 623–727.

[3] P. Aviles, Local behavior of solutions of some elliptic equations, *Comm. Math. Phys.*, **108** (1987), 177–192.

[4] M. F. Bidaut-Véron, A. C. Ponce and L. Véron, Isolated boundary singularities of semilinear elliptic equations, *Calc. Var. Partial Differential Equations*, **40** (2011), 183–221.

[5] M. F. Bidaut-Véron, A. C. Ponce and L. Véron, Boundary singularities of positive solutions of some nonlinear elliptic equations, *C. R. Math. Acad. Sci. Paris*, **344** (2007), 83–88.

[6] M. F. Bidaut-Véron and L. Vivier, An elliptic semilinear equation with source term involving boundary measures: The subcritical case, *Rev. Mat. Iberoam.*, **16** (2000), 477–513.

[7] H. Brézis and R. E. L. Turner, On a class of superlinear elliptic problems, *Comm. Partial Differential Equations*, **2** (1977), 601–614.

[8] L. A. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.*, **42** (1989), 271–297.

[9] R. Caju, J. M. do Ó and A. Silva Santos, Qualitative properties of positive singular solutions to nonlinear elliptic systems with critical exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **36** (2019), 1575–1601.

[10] Z. J. Chen, C.-S. Lin and W. M. Zou, Monotonicity and nonexistence results to cooperative systems in the half space, *J. Funct. Anal.*, **266** (2014), 1088–1105.

[11] E. N. Dancer, Some notes on the method of moving planes, *Bull. Aust. Math. Soc.*, **46** (1992), 425–434.

[12] M. del Pino, M. Musso and F. Pacard, Boundary singularities for weak solutions of semilinear elliptic problems, *J. Funct. Anal.*, **253** (2007), 241–272.

[13] O. Druet, E. Hebey and J. Vétois, Bounded stability for strongly coupled critical elliptic systems below the geometric threshold of the conformal Laplacian, *J. Funct. Anal.*, **258** (2010), 999–1059.

[14] M. Ghergu, S. Kim and H. Shahgholian, *Isolated Singularities for Semilinear Elliptic Systems with Power-Law Nonlinearity*, `arXiv:1804.04291`.

[15] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of related properties via the maximum principle, *Comm. Math. Phys.*, **68** (1979), 209–243.

[16] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.*, **34** (1981), 525–598.

[17] A. Gmira and L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations, *Duke Math. J.*, **60** (1991), 271–324.

[18] Y. X. Guo, Non-existence, monotonicity for positive solutions of semilinear elliptic system in $\mathbb{R}^n_+$, *Commun. Contemp. Math.*, **12** (2010), 351–372.

[19] Z. C. Han, Y. Y. Li and E. V. Teixeira, Asymptotic behavior of solutions to the σ_k-Yamabe equation near isolated singularities, *Invent. Math.*, **182** (2010), 635–684.

[20] N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, *Invent. Math.*, **135** (1999), 233–272.

[21] Y. Y. Li, Conformally invariant fully nonlinear elliptic equations and isolated singularities, *J. Funct. Anal.*, **233** (2006), 380–425.

[22] P.-L. Lions, Isolated singularities in semilinear problems, *J. Differential Equations*, **38** (1980), 441–450.

[23] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems, *Duke Math. J.*, **139** (2007), 555–579.

[24] A. Porretta and L. Véron, Separable solutions of quasilinear Lane-Emden equations, *J. Eur. Math. Soc.*, **15** (2011), 755–774.

[25] J. G. Xiong, The critical semilinear elliptic equation with boundary isolated singularities, *J. Differential Equations*, **263** (2017), 1907–1930.

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E-mail address: lym@mail.bnu.edu.cn
E-mail address: jgbao@bnu.edu.cn