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REGULARITY OF VERY WEAK SOLUTIONS FOR NONHOMOGENEOUS ELLIPTIC EQUATION

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In this paper, we study the local regularity of very weak solution $u \in L^1_{\text{loc}}(\Omega)$ of the elliptic equation $D_j(a_{ij}(x)D_iu) = f - D_ig_i$. Using the bootstrap argument and the difference quotient method, we obtain that if $a_{ij} \in C^{0,1}_{\text{loc}}(\Omega)$, $g_i \in L^p_{\text{loc}}(\Omega)$ and $f \in L^{O_p}_{\text{loc}}(\Omega)$ with $1 < p < \infty$, then $u \in W^{1,p}_{\text{loc}}(\Omega)$. Furthermore, we consider the higher regularity of u.

Keywords: Very weak solution; regularity; linear elliptic equation; Sobolev space; Orlicz space.

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1. Introduction

The simplest kind of linear elliptic equations in divergence form is

$$D_j(a_{ij}(x)D_iu) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geq 2$, and the coefficients $a_{ij}(x)$ are bounded measurable functions satisfying the ellipticity condition, i.e.

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall\, x \in \Omega,\ \ \xi \in \mathbb{R}^N, \tag{1.2}$$

with $0 < \lambda \leq \Lambda < \infty$. $u \in W^{1,p}_{\text{loc}}(\Omega)$ for $p \in [1, \infty)$ is called a weak solution of (1.1) over Ω if

$$\int_\Omega a_{ij}D_iuD_j\varphi = 0, \quad \forall\, \varphi \in C^\infty_c(\Omega). \tag{1.3}$$

A fundamental result of De Giorge [7] states that if $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution of (1.1), then u is locally bounded and then locally Hölder continuous. Meyers [17] also proved that $u \in W^{1,p}_{\text{loc}}(\Omega)$ for some $p > 2$.

Serrin in [20] showed by a counterexample that in general the solutions of (1.1) in $W^{1,p}_{\text{loc}}(\Omega)$ for $p \in (1,2)$ need not be locally bounded only under the assumption (1.2). He proposed a conjecture that if the coefficients a_{ij} are locally Hölder continuous, then any weak solution $u \in W^{1,1}_{\text{loc}}(\Omega)$ of (1.1) must be in $W^{1,2}_{\text{loc}}(\Omega)$. Hager and Ross [12] proved that the conjecture is true for the weak solutions in $W^{1,p}_{\text{loc}}(\Omega)$ for $p \in (1,2)$. In 2008, a celebrated theorem was established by Brezis (see [3], a full proof can be found in [2]).

Theorem 1.1. *Assume that a_{ij} are Dini continuous in Ω, and let $u \in \mathrm{BV}_{\text{loc}}(\Omega)$ be a weak solution of* (1.1), *then $u \in W^{1,2}_{\text{loc}}(\Omega)$.*

Here the coefficients a_{ij} are Dini continuous in Ω, i.e. $a_{ij} \in C^0(\Omega)$, and for any subdomain $\Omega' \Subset \Omega$, there exists a function φ, such that

$$|a_{ij}(x) - a_{ij}(y)| \le \varphi(|x-y|),\ x,\ y \in \Omega', \quad \text{where } \int_0^{\operatorname{diam}\Omega'} \frac{\varphi(r)}{r} dr < \infty.$$

And $u \in \mathrm{BV}_{\text{loc}}(\Omega)$ means $u \in L^1_{\text{loc}}(\Omega)$ and

$$\int_{\Omega'} |Du| = \sup\left\{ \int_{\Omega'} u \cdot \operatorname{div}\overrightarrow{v} : \overrightarrow{v} \in C^1_0(\Omega', \mathbb{R}^N), |\overrightarrow{v}| \le 1 \right\} < \infty, \quad \forall\, \Omega' \Subset \Omega.$$

Theorem 1.1 confirmed completely Serrin's conjecture in the case of less smooth given coefficients and solutions, since Hölder continuity on a_{ij} were replaced by Dini continuity, and u was extended from $W^{1,1}_{\text{loc}}(\Omega)$ to $\mathrm{BV}_{\text{loc}}(\Omega)$.

For merely continuity on a_{ij}, Brezis obtained the following result.

Theorem 1.2. *Assume that $a_{ij} \in C^0(\Omega)$. If $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution of* (1.1) *for some $p > 1$, then $u \in W^{1,q}_{\text{loc}}(\Omega)$ for every $q < \infty$.*

There are the counterexamples to show Theorem 1.2 is not true in the cases $p = 1$ or $q = \infty$. Therefore Theorem 1.2 is optimal in some sense. For the unit ball B_1 and the continuous coefficients a_{ij}, Jin, Maz'ya and Schaftingen [14] constructed a weak solution $u \in W^{1,1}_{\text{loc}}(B_1)\backslash W^{1,p}_{\text{loc}}(B_1)$ for every $p > 1$. They also gave a function $u \in W^{1,q}_{\text{loc}}(B_1)\backslash W^{1,\infty}_{\text{loc}}(B_1)$ for every $q < \infty$, satisfying (1.1).

Recently, in [23], we have considered a very weak solution $u \in L^1_{\text{loc}}(\Omega)$ of (1.1), namely

$$\int_\Omega u D_i(a_{ij} D_j \varphi) = 0, \quad \forall\, \varphi \in C^\infty_c(\Omega).$$

Because of the very weak assumptions made on the solutions it is natural that the coefficients should be interpreted as the local Lipschitz functions. And we have the following theorem.

Theorem 1.3. *Assume that $a_{ij} \in C^{0,1}_{\text{loc}}(\Omega)$. If $u \in L^1_{\text{loc}}(\Omega)$ is a very weak solution of* (1.1), *then $u \in W^{2,q}_{\text{loc}}(\Omega)$ for any $q \in [1, \infty)$.*

In fact, we give a positive answer as the above theorem to the question, raised by Brezis, whether any very weak solution $u \in L^1_{\text{loc}}(\Omega)$ of (1.1) is in $W^{1,2}_{\text{loc}}(\Omega)$.

In this paper, we consider more general nonhomogeneous linear elliptic equations having principal part in divergence form

$$D_j(a_{ij}(x)D_iu) = f - D_ig_i \quad \text{in } \Omega, \tag{1.4}$$

where $a_{ij} \in C^{0,1}_{\text{loc}}(\Omega)$, $f,\ g_i \in L^1_{\text{loc}}(\Omega)$ for $i, j = 1, 2, \ldots, N$. $u \in L^1_{\text{loc}}(\Omega)$ is said to be a very weak solution of Eq. (1.4)

$$\int_\Omega uD_i(a_{ij}D_j\varphi) = \int_\Omega f\varphi + g_iD_i\varphi, \quad \forall\, \varphi \in C^\infty_c(\Omega). \tag{1.5}$$

A very weak solution in $W^{1,p}_{\text{loc}}(\Omega)$, $p \in [1, \infty)$, of (1.4) must be a usual weak solution, that is

$$-\int_\Omega a_{ij}D_iuD_j\varphi = \int_\Omega f\varphi + g_iD_i\varphi, \quad \forall\, \varphi \in C^\infty_c(\Omega).$$

Throughout the paper, we always assume that the coefficients $a_{ij} \in C^{0,1}_{\text{loc}}(\Omega)$ are elliptic, i.e. for any subdomain $\Omega' \subset\subset \Omega$, there exist the constants K, λ, Λ, depending only on Ω', such that

$$|a_{ij}(x) - a_{ij}(y)| \le K|x - y|, \quad \forall\, x, y \in \Omega',\ \ i, j = 1, 2, \ldots, N, \tag{1.6}$$

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall\, x \in \Omega',\ \ \xi \in \mathbb{R}^N. \tag{1.7}$$

Suppose that $1 < p < \infty$ and

$$f \in L^{O_p}_{\text{loc}}(\Omega), \quad g_i \in L^p_{\text{loc}}(\Omega),\ i = 1, 2, \ldots, N, \tag{1.8}$$

where

$$O_p = \begin{cases} 1, & 1 < p < \dfrac{N}{N-1}, \\[2ex] A = A(s) =: \max\limits_{t \ge 0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\},\ s > 0, & p = \dfrac{N}{N-1}, \\[2ex] \dfrac{Np}{N+p}, & \dfrac{N}{N-1} < p < \infty. \end{cases}$$

Here $L^{O_p}_{\text{loc}}(\Omega) = L^A_{\text{loc}}(\Omega)$ is a local Orlicz space in the case of $p = \frac{N}{N-1}$. The Orlicz space is defined as

$$L^A(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega A\left(\frac{|f(x)|}{k}\right) dx < \infty \text{ for some } k > 0 \right\}.$$

The Luxemburg norm $\|f\|_{L^A(\Omega)}$ is defined as

$$\|f\|_{L^A(\Omega)} = \inf\left\{ k > 0 : \int_\Omega A\left(\frac{|f(x)|}{k}\right) dx \le 1 \right\}.$$

The space $L^A(\Omega)$, equipped with the norm $\|\cdot\|_{L^A(\Omega)}$, is a Banach space. $f \in L^A_{\text{loc}}(\Omega)$ means $\|f\|_{L^A(\Omega')} < \infty$ for any $\Omega' \subset\subset \Omega$.

Our main results are the L^p, $W^{1,p}$ and $W^{n,p}$ regularity of the very weak solutions. In fact, under weaker integrable assumptions on f and g_i, $i = 1, 2, \ldots, N$, L^p regularity is obtained.

Theorem 1.4. *Assume that the conditions* (1.6), (1.7) *hold,* $f \in L^{Q_p}_{\mathrm{loc}}(\Omega)$ *and* $g_i \in L^{O_p}_{\mathrm{loc}}(\Omega)$ *with* $1 < p < \infty$, $i = 1, 2, \ldots, N$. *If* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is a very weak solution of* (1.4), *then* $u \in L^p_{\mathrm{loc}}(\Omega)$. *Moreover*

$$\|u\|_{L^p(\omega)} \le C\left(\|u\|_{L^1(\Omega')} + \|f\|_{L^{Q_p}(\Omega')} + \sum_{i=1}^{N} \|g_i\|_{L^{O_p}(\Omega')}\right),$$

for every $\omega \subset\subset \Omega' \subset\subset \Omega$, *where the constant* C *depends only on* $N, p, \lambda, \Lambda, K, \omega$ *and* Ω', *and*

$$Q_p = \begin{cases} 1, & 1 < p < \dfrac{N}{N-2}, \\ A = A(s) =: \max\limits_{t \ge 0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\},\ s > 0, & p = \dfrac{N}{N-2}, \\ \dfrac{Np}{N+2p}, & \dfrac{N}{N-2} < p < \infty. \end{cases}$$

Corollary 1.1. *Assume that the conditions* $(1.6), (1.7)$ *hold,* $f, g_i \in L^1_{\mathrm{loc}}(\Omega)$, $i = 1, 2, \ldots, N$. *If* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is a very weak solution of* (1.4), *then* $u \in L^p_{\mathrm{loc}}(\Omega)$ *for any* $p \in [1, \frac{N}{N-1})$.

Theorem 1.5. *Assume that the conditions* (1.6), (1.7) *and* (1.8) *hold,* $1 < p < \infty$. *If* $u \in L^p_{\mathrm{loc}}(\Omega)$ *is a very weak solution of* (1.4), *then* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. *Moreover*

$$\|u\|_{W^{1,p}(\Omega'')} \le C\left(\|u\|_{L^p(\Omega')} + \|f\|_{L^{O_p}(\Omega')} + \sum_{i=1}^{N} \|g_i\|_{L^p(\Omega')}\right),$$

for every $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, *where the constant* C *depends only on* N, p, λ, Λ, K, Ω'' *and* Ω'.

Remark 1.1. In fact, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a weak solution of (1.4), then a fine priori estimate is obtained from the above theorem.

At the same time, we have the following result by combining Theorems 1.4 and 1.5.

Corollary 1.2. *Assume that* $a_{ij} \in C^{0,1}_{\mathrm{loc}}(\Omega)$, $f \in L^1_{\mathrm{loc}}(\Omega)$ *and* $g_i \in L^p_{\mathrm{loc}}(\Omega)$, $1 < p < \frac{N}{N-1}$, $i, j = 1, 2, \ldots, N$. *If* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is a very weak solution of* (1.4), *then* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$.

In the case of $g_i = 0$, $i, j = 1, 2, \ldots, N$, if we strengthen the integrable condition on f by assuming $f \in L^p$, then we have the following proposition.

Proposition 1.1. *Assume that* $a_{ij} \in C^{0,1}_{\mathrm{loc}}(\Omega)$, $f \in L^p_{\mathrm{loc}}(\Omega)$ *with* $1 < p < \infty$ *and* $g_i = 0$, $i, j = 1, 2, \ldots, N$. *If* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is a very weak solution of* (1.4), *then* $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$.

Proof. For $B_R = B_R(x_0) \subset B_{2R}(x_0) = B_{2R} \subset \Omega$, we consider the Dirichlet problem

$$\begin{cases} D_j(a_{ij}(x)D_i v) = f(x), & x \in B_{2R}, \\ v = 0, & x \in \partial B_{2R}. \end{cases} \tag{1.9}$$

From [6, Theorems 6.3 and 6.4], (1.9) exists a unique solution

$$v \in W^{2,p}(B_{2R}) \cap W^{1,p}_0(B_{2R})$$

satisfying

$$\|v\|_{W^{2,p}(B_{2R})} \le C\|f\|_{L^p(B_{2R})}. \tag{1.10}$$

Let $w = u - v$, then w is a very weak solution of (1.1). By Theorem 1.3, we have

$$\|u - v\|_{W^{2,p}(B_R)} \le C\|u - v\|_{L^1(B_{2R})}.$$

From Minkowski inequality, Hölder inequality and (1.10), we get

$$\|u\|_{W^{2,p}(B_R)} \le C(\|u\|_{L^1(B_{2R})} + \|f\|_{L^p(B_{2R})}).$$

Now, using finite covering theorem, we obtain the result. □

The very weak solution has been studied by many authors. In [4], Brezis, Cazenave, Martel and Ramiandrisoa proved the existence and uniqueness theorem for a very weak solution in $L^1(\Omega)$ of the Poisson equations $\Delta u = f(x)$ with zero boundary value. They also established the estimate

$$\|u\|_{L^1(\Omega)} \le \|f \cdot \mathrm{dist}(x, \partial\Omega)\|_{L^1(\Omega)}.$$

Later, Cabré and Martel [5] showed the very weak solution is in $L^q(\Omega)$ for any $1 \le q < \frac{N}{N-2}$.

Therefore, the question of the integrability of the weak derivative of the very weak solution arises in a natural way.

Recently, Diaz and Rakotoson [9] extended the results of Brezis *et al.* to $Lu = f(x)$, where L is a linear second-order elliptic operator with variable coefficients. They obtained if $f \cdot \mathrm{dist}^\alpha(x, \partial\Omega) \in L^1(\Omega)$, $0 \le \alpha < 1$, then Du belongs to the Lorentz space $L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$, where

$$L^{\frac{N}{N-1+\alpha},\infty}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable and } \sup_{t \le |\Omega|} t^{\frac{\alpha-1}{N}} \int_0^t |f|_* ds < \infty \right\},$$

$|f|_*(s) = \inf\{t \in \mathbb{R} : \mathrm{meas}\{|f| > t\} \le s\}$ for $0 \le s \le |\Omega|$. In particular, for Poisson equation (see [8, Lemma 6]), if $f \in L^1(\Omega)$, then $Du \in L^q(\Omega) \subset L^{\frac{N}{N-1},\infty}(\Omega)$, where $1 \le q < \frac{N}{N-1}$. It is almost the same to Corollary 1.2, and a special case of Theorems 1.4 and 1.5.

Further differentiability of very weak solutions can be deduced readily from the proof of Theorem 1.5. Suppose that we strengthen the smoothness conditions on the coefficients by assuming

$$a_{ij} \in C^{n,1}_{\text{loc}}(\Omega), \quad i, j = 1, 2, \ldots, N, \tag{1.11}$$

together with

$$g_i \in W^{n,p}_{\text{loc}}(\Omega), \quad 1 < p < \infty,\ i = 1, 2, \ldots, N, \tag{1.12}$$

and

$$f \in W^{n,O_p}_{\text{loc}}(\Omega), \quad 1 < p < \infty, \tag{1.13}$$

where $n = 1, 2, \ldots,$ and

$$W^{n,O_p} = \begin{cases} W^{n,1}, & 1 < p < \dfrac{N}{N-1}, \\ W^{n,A}, & p = \dfrac{N}{N-1}, \\ W^{n,\frac{Np}{N+p}}, & \dfrac{N}{N-1} < p < \infty. \end{cases}$$

We can then conclude the following extension of Theorems 1.4 and 1.5.

Theorem 1.6. *Assume that the conditions* (1.7), (1.11), (1.12) *and* (1.13) *hold. If* $u \in L^1_{\text{loc}}(\Omega)$ *is a very weak solution of* (1.4), *then*

$$u \in W^{n+1,p}_{\text{loc}}(\Omega). \tag{1.14}$$

Moreover

$$\|u\|_{W^{n+1,p}(\omega)} \le C \left(\|u\|_{L^1(\Omega')} + \|f\|_{W^{n,O_p}(\Omega')} + \sum_{i=1}^{N} \|g_i\|_{W^{n,p}(\Omega')} \right),$$

for every $\omega \subset\subset \Omega' \subset\subset \Omega$, *where* C *is a constant depending only on* $n, N, p, \lambda, \Lambda, \omega, \Omega'$ *and the norms of* $a_{ij}(x)$ *in* $C^{n,1}(\overline{\Omega'})$.

To our knowledge, Theorem 1.6 is new in the case of $n = 1$. At this time, the very weak solution has become not only a weak solution, but also a strong solution. The $W^{2,p}$ regularity in (1.14) is local and independent of the uniqueness of solutions, which is different from [6]. Once $u \in W^{2,p}_{\text{loc}}(\Omega)$, Eq. (1.4) can be written in the general form

$$a_{ij}(x) D_{ij} u + D_j a_{ij}(x) D_i u = f(x) - D_i g_i(x),$$

so that the usual regularity theory of strong solutions would apply. By (1.11), (1.12) and (1.13), we have

$$a_{ij}, D_j a_{ij} \in C^{n-1,1}_{\text{loc}}(\Omega) \subset C^{n-2,1}_{\text{loc}}(\Omega), \quad n = 2, 3, \ldots,$$

$$f - D_i g_i \in W^{n-1,p}_{\text{loc}}(\Omega), \quad 1 < p < \infty.$$

Here we have used the fact that $W^{n,O_p}_{\text{loc}}(\Omega) \subset W^{n-1,p}_{\text{loc}}(\Omega)$. It follows from the higher-order regularity theorem (see [11, Theorem 9.19]) that $u \in W^{n+1,p}_{\text{loc}}(\Omega)$.

There are other results on the very weak solutions, such as [8, 10, 13, 16, 21] for semilinear elliptic equations, [19, 22] for elliptic systems, [18] for Neumann problems.

The rest part of the paper is organized as follows: In the next section we present some preliminary facts which will be used later. Sections 3–5 are devoted to the proof of Theorems 1.4–1.6, respectively. We obtain the L^p, $W^{1,p}$ and $W^{n,p}$ regularity of the very weak solutions, by using the bootstrap argument and the difference quotient method.

2. Some Preliminary Facts

In this section, we list some preliminary facts that will be needed in our proof.

For convenience, we abbreviate a ball with center x_0 and radius R as B_R, and then consider the Dirichlet problem

$$\begin{cases} a_{ij}(x)D_{ij}v = f(x), & x \in B_R, \\ v = 0, & x \in \partial B_R. \end{cases} \tag{2.1}$$

Lemma 2.1. *Suppose that $a_{ij} \in C^0(\overline{B_R})$ satisfy* (1.2) *in B_R and $f \in L^p(B_R)$ with $1 < p < \infty$. Then* (2.1) *exists a unique solution $v \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$ satisfying*

$$\|v\|_{W^{2,p}(B_R)} \le C\|f\|_{L^p(B_R)},$$

where C depends only on N, p, λ, Λ, R and the modulus of continuity of a_{ij} on $\overline{B_R}$.

This lemma is the direct conclusion of Theorems 6.3 and 6.4 in [6, Chap. 3].

Lemma 2.2. *Let u be a $W^{2,p}(B_R)$ solution of* (2.1) *with $1 < p < \infty$. If $a_{ij} \in C^{0,1}(\overline{B_R})$ are uniformly elliptic, and $f \in W^{1,q}(B_R)$ with $1 < q < \infty$, then $u \in W^{3,q}(B_R)$.*

This lemma is a special case of [11, Theorem 9.19].

Lemma 2.3 ([1, Theorem 8.27]) (Trudinger's Theorem). *Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying the cone condition. Let $mp = N$, m be a positive integer and $p > 1$. Set*

$$B(t) = \exp(t^{\frac{N}{N-m}}) - 1 = \exp(t^{\frac{p}{p-1}}) - 1.$$

Then there exists the imbedding

$$W^{m,p}(\Omega) \hookrightarrow L^B(\Omega).$$

Moreover,

$$\|u\|_{L^B(\Omega)} \le C\|u\|_{W^{m,p}(\Omega)}$$

for $u \in W^{m,p}(\Omega)$, where C is a constant depending only on N, p, $|\Omega|$ and the cone condition.

We recall some basic definitions about N-function.

Definition 2.1. A function $M(u)$ is called an N-function if it admits of the representation

$$M(u) = \int_0^{|u|} p(t)dt,$$

where the function $p(t)$ is right-continuous for $t \geq 0$, positive and nondecreasing for $t > 0$ which satisfies the conditions $p(0) = 0$, $p(\infty) = \lim_{t\to\infty} p(t) = \infty$.

Definition 2.2. Let $M_1(u)$ and $M_2(u)$ be two N-functions, and we write $M_1 u \prec M_2 u$ if there exist positive constants u_0 and k such that $M_1(u) \leq M_2(ku)$ $(u \geq u_0)$. And we say that the N-functions $M_1(u)$ and $M_2(u)$ are equivalent and write

$$M_1(u) \sim M_2(u)$$

if $M_1 u \prec M_2 u$ and $M_2 u \prec M_1 u$.

Definition 2.3. A convex function $Q(u)$ will be called the principal part of the N-function $M(u)$ if $Q(u) = M(u)$ for large values of the argument.

Definition 2.4. We will say that the N-function $M(u)$ satisfies the $\triangle_3$-condition if it is equivalent to the N-function $|u|M(u)$.

Lemma 2.4 ([1, Theorem 8.11]) (A Generalized Hölder Inequality). *If $B(t)$ and $\tilde{B}(s)$ are complementary N-functions, that is to say,*

$$\tilde{B}(s) = \max_{t\geq 0}\{st - B(t)\},$$

and $u \in L^B(\Omega)$, $v \in L^{\tilde{B}}(\Omega)$, we have

$$\int_\Omega |uv| \leq 2\|u\|_{L^B(\Omega)}\|v\|_{L^{\tilde{B}}(\Omega)}.$$

Lemma 2.5 ([15, Chap. 1, Theorem 6.25]). *Suppose that the N-function $M(u)$, which is complementary to N-function $N(v)$, satisfies the $\triangle_3$ condition and let the function*

$$Q(v) = |v|M^{-1}(|v|)$$

be the principal part of an N-function $\Psi(v)$, where $M^{-1}(|v|)$ is the function inverse to $M(u)$. Then $\Psi(v) \sim N(v)$.

Remark 2.1. When Ω is a bounded domain,

$$A(|v|) = \max_{|u|\geq 0}\{uv - \exp(|u|^{\frac{N}{N-1}}) + 1\},$$

which is complementary to $\exp(|u|^{\frac{N}{N-1}}) - 1$, is equivalent to the N-function

$$|v|(\ln(|v| + 1))^{\frac{N-1}{N}}.$$

And we see

$$L^p(\Omega) \subset L^A(\Omega) \subset L^1(\Omega)$$

for any $p > 1$, since $L^p(\Omega) \subset L(\ln(L+1))^{\frac{N-1}{N}}(\Omega) \subset L^1(\Omega)$ for any $p > 1$.

3. L^p Regularity

In this section, we will prove Theorem 1.4.

Proof of Theorem 1.4. We first get the estimate for a 2δ ball, and choose δ small enough, such that $0 < 2\delta \le 1$. For $\Omega' \subset\subset \Omega$, $x_0 \in \Omega_{2\delta} := \{x \in \Omega' | d(x, \partial\Omega') > 2\delta\}$, let $\eta(x) \in C_c^\infty(B_{2\delta}(x_0))$ be a cut-off function:

$$\eta(x) = \begin{cases} 1, & |x - x_0| \le \delta, \\ 0, & |x - x_0| \ge 2\delta, \end{cases}$$

such that $0 \le \eta(x) \le 1$, $|D\eta| \le \frac{M_1}{\delta}$ and $|D^2\eta| \le \frac{M_2}{\delta^2}$, where M_1, M_2 are positive constants.

For the sake of clarity, we divide the estimate in Theorem 1.4 into five steps.

Step 1: $1 < p < \frac{N}{N-1}$. Let w_1 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_1 \in W^{2,q_1}(B_{2\delta}) \cap W_0^{1,q_1}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x) D_{ij} v_1 = w_1, & x \in B_{2\delta}, \\ v_1 = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_1\|_{W^{2,q_1}(B_{2\delta})} \le C \|w_1\|_{L^{q_1}(B_{2\delta})}, \tag{3.1}$$

where $q_1 \in (N, \infty)$. Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_1 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,q_1}(B_{2\delta})$, by Lemma 2.2, we have $v_1 \in W^{3,q_1}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we get $v_1 \in C^{2,\alpha}(\overline{B_{2\delta}})$.

From (1.5) and a density argument, we have

$$\int_\Omega u D_i(a_{ij} D_j v) = \int_\Omega f v + g_i D_i v, \quad \forall\, v \in C_0^2(\Omega). \tag{3.2}$$

Now we choose $v = \eta^2 v_1$ in (3.2), and get

$$\begin{aligned} \int_\Omega f\eta^2 v_1 + g_i D_i(\eta^2 v_1) &= \int_\Omega u D_i(a_{ij} D_j(\eta^2 v_1)) \\ &= \int_{B_{2\delta}} u D_i a_{ij} D_j(\eta^2 v_1) + u a_{ij} D_{ij}(\eta^2 v_1) \\ &= \int_{B_{2\delta}} 2\eta u v_1 D_i a_{ij} D_j \eta + u\eta^2 D_i a_{ij} D_j v_1 \\ &\quad + \int_{B_{2\delta}} 2u v_1 a_{ij} D_i \eta D_j \eta + 2\eta u v_1 a_{ij} D_{ij} \eta \\ &\quad + \int_{B_{2\delta}} 4\eta u a_{ij} D_i \eta D_j v_1 + u\eta^2 a_{ij} D_{ij} v_1. \end{aligned}$$

By the properties of the cut-off function and a_{ij}, we have

$$\begin{aligned}\left|\int_{B_{2\delta}} u\eta^2 a_{ij} D_{ij} v_1\right| &\le \frac{2KM_1}{\delta}\int_{B_{2\delta}}|uv_1| + K\int_{B_{2\delta}}|uD_j v_1| \\ &\quad + \frac{2\Lambda {M_1}^2}{\delta^2}\int_{B_{2\delta}}|uv_1| + \frac{2\Lambda M_2}{\delta^2}\int_{B_{2\delta}}|uv_1| \\ &\quad + \frac{4\Lambda M_1}{\delta}\int_{B_{2\delta}}|uD_j v_1| + \int_{B_{2\delta}}|fv_1| \\ &\quad + \frac{2M_1}{\delta}\int_{B_{2\delta}}|g_i v_1| + \int_{B_{2\delta}}|g_i D_i v_1| \\ &\le C(\|v_1\|_{W^{1,\infty}(B_{2\delta})}\|u\|_{L^1(B_{2\delta})} \\ &\quad + \|v_1\|_{L^\infty(B_{2\delta})}\|f\|_{L^1(B_{2\delta})} \\ &\quad + \|v_1\|_{W^{1,\infty}(B_{2\delta})}\|g_i\|_{L^1(B_{2\delta})}).\end{aligned}$$

By the Sobolev imbedding theorem and (3.1), we have

$$\|v_1\|_{W^{1,\infty}(B_{2\delta})} \le C\|v_1\|_{W^{2,q_1}(B_{2\delta})} \le C\|w_1\|_{L^{q_1}(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} \eta^2 u w_1\right| \le C\|w_1\|_{L^{q_1}(B_{2\delta})}(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^1(B_{2\delta})}).$$

Since w_1 is an arbitrary smooth function in $\overline{B_{2\delta}}$, we conclude

$$\|\eta^2 u\|_{L^{p_1}(B_{2\delta})} \le C(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^1(B_{2\delta})}), \tag{3.3}$$

where $p_1 := \frac{q_1}{q_1-1} \in (1, \frac{N}{N-1})$.

Now using finite covering theorem, we obtain $u \in L^p_{\mathrm{loc}}(\Omega)$, $\forall\, p \in (1, \frac{N}{N-1})$.

Step 2: $p = \frac{N}{N-1}$. Let w_2 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_2 \in W^{2,N}(B_{2\delta}) \cap W_0^{1,N}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x)D_{ij}v_2 = w_2, & x \in B_{2\delta}, \\ v_2 = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_2\|_{W^{2,N}(B_{2\delta})} \le C\|w_2\|_{L^N(B_{2\delta})}. \tag{3.4}$$

Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_2 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,N}(B_{2\delta})$, by Lemma 2.2, we have $v_2 \in W^{3,N}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we get $v_2 \in W^{2,r_2}(B_{2\delta})$, $\forall\, r_2 < \infty$.

From (1.5) and a density argument, we have

$$\int_\Omega uD_i(a_{ij}D_jv) = \int_\Omega fv + g_iD_iv, \quad \forall\, v \in W_0^{2,l_2}(B_{2\delta}),\ l_2 \in (N,\infty). \tag{3.5}$$

Now we choose $v = \eta^4 v_2$ in (3.5), use Lemma 2.4, the properties of the cut-off function and a_{ij}, and have

$$\begin{aligned}\left|\int_{B_{2\delta}} \eta^4 u a_{ij} D_{ij} v_2\right| \le C\,\big(&\|v_2\|_{W^{1,B}(B_{2\delta})}\|\eta^2 u\|_{L^A(B_{2\delta})}\\ &+ \|v_2\|_{L^\infty(B_{2\delta})}\|f\|_{L^1(B_{2\delta})}\\ &+ \|v_2\|_{W^{1,B}(B_{2\delta})}\|g_i\|_{L^A(B_{2\delta})}\big),\end{aligned}$$

where $B = B(t) = \exp(t^{\frac{N}{N-1}}) - 1$, $A = A(s) = \max_{t\ge0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\}, s > 0$.

By Lemma 2.3 and (3.4), we obtain

$$\|v_2\|_{W^{1,B}(B_{2\delta})} \le C\|v_2\|_{W^{2,N}(B_{2\delta})} \le C\|w_2\|_{L^N(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} \eta^4 u w_2\right| \le C\|w_2\|_{L^N(B_{2\delta})}(\|\eta^2 u\|_{L^A(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^A(B_{2\delta})}).$$

From a duality argument, Remark 2.1 and (3.3), we conclude

$$\|\eta^4 u\|_{L^{\frac{N}{N-1}}(B_{2\delta})} \le C(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^A(B_{2\delta})}). \tag{3.6}$$

Now using finite covering theorem, we have $u \in L^p_{\mathrm{loc}}(\Omega)$, $p = \frac{N}{N-1}$.

Step 3: $\frac{N}{N-1} < p < \frac{N}{N-2}$. Let w_3 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_3 \in W^{2,q_3}(B_{2\delta}) \cap W_0^{1,q_3}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x)D_{ij}v_3 = w_3, & x \in B_{2\delta},\\ v_3 = 0, & x \in \partial B_{2\delta}.\end{cases}$$

Moreover

$$\|v_3\|_{W^{2,q_3}(B_{2\delta})} \le C\|w_3\|_{L^{q_3}(B_{2\delta})}, \tag{3.7}$$

where $q_3 \in (\frac{N}{2}, N)$. Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_3 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,q_3}(B_{2\delta})$, by Lemma 2.2, we have $v_3 \in W^{3,q_3}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we obtain $v_3 \in W^{2,r_3}(B_{2\delta})$, $r_3 = \frac{Nq_3}{N-q_3} \in (N,\infty)$.

From (1.5) and a density argument, we have

$$\int_\Omega uD_i(a_{ij}D_jv) = \int_\Omega fv + g_iD_iv, \quad \forall\, v \in W_0^{2,N}(B_{2\delta}). \tag{3.8}$$

Now we choose $v = \eta^6 v_3$ in (3.8), use the properties of the cut-off function and a_{ij}, and we obtain

$$\left|\int_{B_{2\delta}} u\eta^6 a_{ij} D_{ij} v_3\right| \le C\Big(\|v_3\|_{W^{1,\frac{Nq_3}{N-q_3}}(B_{2\delta})}\|\eta^4 u\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})} + \|v_3\|_{L^\infty(B_{2\delta})}\|f\|_{L^1(B_{2\delta})} + \|v_3\|_{W^{1,\frac{Nq_3}{N-q_3}}(B_{2\delta})}\|g_i\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})}\Big).$$

Recall that $q_3 \in (\frac{N}{2}, N)$. So we have $\frac{Nq_3}{Nq_3-N+q_3} \in (1, \frac{N}{N-1})$, and

$$\|\eta^4 u\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})} \le \|\eta^4 u\|_{L^{\frac{N}{N-1}}(B_{2\delta})}.$$

By the Sobolev imbedding theorem and (3.7), we obtain

$$\|v_3\|_{W^{1,\frac{Nq_3}{N-q_3}}(B_{2\delta})} \le C\|v_3\|_{W^{2,q_3}(B_{2\delta})} \le C\|w_3\|_{L^{q_3}(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} \eta^6 u w_3\right| \le C\|w_3\|_{L^{q_3}(B_{2\delta})}\Big(\|\eta^4 u\|_{L^{\frac{N}{N-1}}(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})}\Big).$$

From a duality argument, Remark 2.1 and (3.6), we conclude

$$\|\eta^6 u\|_{L^{\frac{q_3}{q_3-1}}(B_{2\delta})} \le C\Big(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^{\frac{Nq_3}{Nq_3-N+q_3}}(B_{2\delta})}\Big). \tag{3.9}$$

Since $q_3 \in (\frac{N}{2}, N)$, we have $\frac{q_3}{q_3-1} \in (\frac{N}{N-1}, \frac{N}{N-2})$. By taking $p_3 = \frac{q_3}{q_3-1}$, it follows that $q_3 = \frac{p_3}{p_3-1}$. So we get

$$\frac{Nq_3}{Nq_3 - N + q_3} = \frac{Np_3}{N+p_3} \in \left(1, \frac{N}{N-1}\right),$$

and

$$\|\eta^6 u\|_{L^{p_3}(B_{2\delta})} \le C\Big(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^1(B_{2\delta})} + \|g_i\|_{L^{\frac{Np_3}{N+p_3}}(B_{2\delta})}\Big), \tag{3.10}$$

for $\frac{N}{N-1} < p_3 < \frac{N}{N-2}$.

Now using finite covering theorem, we have $u \in L^p_{\text{loc}}(\Omega)$, $p = p_3 \in (\frac{N}{N-1}, \frac{N}{N-2})$.

Step 4: $p = \frac{N}{N-2}$ $(N \ge 3)$. Let w_4 be a smooth function in $\overline{B_{2\delta}}$. According to Lemma 2.1, there must be a unique function $v_4 \in W^{2,\frac{N}{2}}(B_{2\delta}) \cap W_0^{1,\frac{N}{2}}(B_{2\delta})$, such that

$$\begin{cases} a_{ij}(x)D_{ij}v_4 = w_4, & x \in B_{2\delta}, \\ v_4 = 0, & x \in \partial B_{2\delta}. \end{cases}$$

Moreover

$$\|v_4\|_{W^{2,\frac{N}{2}}(B_{2\delta})} \le C\|w_4\|_{L^{\frac{N}{2}}(B_{2\delta})}. \tag{3.11}$$

Since $a_{ij} \in C^{0,1}(\overline{B_{2\delta}})$, $w_4 \in C^\infty(\overline{B_{2\delta}}) \subset W^{1,\frac{N}{2}}(B_{2\delta})$, by Lemma 2.2, we have $v_4 \in W^{3,\frac{N}{2}}(B_{2\delta})$. Then using the Sobolev imbedding theorem, we get $v_2 \in W^{2,N}(B_{2\delta})$.

From (1.5) and a density argument, we have

$$\int_\Omega u D_i(a_{ij}D_j v) = \int_\Omega fv + g_i D_i v, \quad \forall\, v \in W_0^{2,l_4}(B_{2\delta}),\ \ l_4 \in \left(\frac{N}{2}, \infty\right). \tag{3.12}$$

Now we choose $v = \eta^8 v_4$ in (3.12), use Lemma 2.4, the properties of the cut-off function and a_{ij}, and have

$$\begin{aligned}\left|\int_{B_{2\delta}} u\eta^8 a_{ij} D_{ij} v_4\right| \le C\Big(&\|v_4\|_{W^{1,N}(B_{2\delta})}\|\eta^6 u\|_{L^{\frac{N}{N-1}}(B_{2\delta})}\\ &+ \|v_4\|_{L^B(B_{2\delta})}\|f\|_{L^A(B_{2\delta})}\\ &+ \|v_4\|_{W^{1,N}(B_{2\delta})}\|g_i\|_{L^{\frac{N}{N-1}}(B_{2\delta})}\Big),\end{aligned}$$

where $B = B(t) = \exp(t^{\frac{N}{N-1}}) - 1$, $A = A(s) = \max_{t\ge 0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\}, s > 0$.

By Lemma 2.3, the Sobolev imbedding theorem and (3.11), we obtain

$$\|v_4\|_{L^B(B_{2\delta})} \le C\|v_4\|_{W^{1,N}(B_{2\delta})},$$

$$\|v_4\|_{W^{1,N}(B_{2\delta})} \le C\|v_4\|_{W^{2,\frac{N}{2}}(B_{2\delta})} \le C\|w_4\|_{L^{\frac{N}{2}}(B_{2\delta})}.$$

So we get

$$\left|\int_{B_{2\delta}} \eta^8 u w_4\right| \le C\|w_4\|_{L^{\frac{N}{2}}(B_{2\delta})}\Big(\|\eta^6 u\|_{L^{\frac{N}{N-1}}(B_{2\delta})} + \|f\|_{L^A(B_{2\delta})} + \|g_i\|_{L^{\frac{N}{N-1}}(B_{2\delta})}\Big).$$

From a duality argument, Remark 2.1 and (3.10), we conclude

$$\|\eta^8 u\|_{L^{\frac{N}{N-2}}(B_{2\delta})} \le C\Big(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^A(B_{2\delta})} + \|g_i\|_{L^{\frac{Np}{N+p}}(B_{2\delta})}\Big), \tag{3.13}$$

for $p = \frac{N}{N-2}$.

Now using finite covering theorem, we have $u \in L^p_{\mathrm{loc}}(\Omega)$, $p = \frac{N}{N-2}$.

Step 5: $p \ge \frac{N}{N-2}$ $(N \ge 3)$. From Step 2 to Step 3, we have $\eta^6 u \in L^p(B_{2\delta})$ for any $p \in [\frac{N}{N-1}, \frac{N}{N-2})$. Likewise, for any given positive integer $k = 3, 4, \ldots, N$, we obtain that $\eta^{2(k+1)}u \in L^p(B_{2\delta})$, $\forall\, p \in [\frac{N}{N+1-k}, \frac{N}{N-k})$. Moreover,

$$\|\eta^{2(k+1)}u\|_{L^p(B_{2\delta})} \le C(\|u\|_{L^1(B_{2\delta})} + \|f\|_{L^{Q_p}(B_{2\delta})} + \|g_i\|_{L^{O_p}(B_{2\delta})}), \tag{3.14}$$

for all $p \geq \frac{N}{N-2}$, where

$$Q_p = \begin{cases} 1, & 1 < p < \dfrac{N}{N-2}, \\ A = A(s) =: \max\limits_{t \geq 0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\},\ s > 0, & p = \dfrac{N}{N-2}, \\ \dfrac{Np}{N+2p}, & \dfrac{N}{N-2} < p < \infty, \end{cases}$$

$$O_p = \begin{cases} 1, & 1 < p < \dfrac{N}{N-1}, \\ A = A(s) =: \max\limits_{t \geq 0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\},\ s > 0, & p = \dfrac{N}{N-1}, \\ \dfrac{Np}{N+p}, & \dfrac{N}{N-1} < p < \infty. \end{cases}$$

Finally, using finite covering theorem, we have

$$\|u\|_{L^p(\omega)} \leq C(\|u\|_{L^1(\Omega')} + \|f\|_{L^{Q_p}(\Omega')} + \|g_i\|_{L^{O_p}(\Omega')}),$$

where the constant C depends only on N, p, λ, Λ, K, ω and Ω'. □

4. $W^{1,p}$ Regularity

In this section, we will prove Theorem 1.5.

Proof of Theorem 1.5. For fixed $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, let $2R = \min\{1, \frac{1}{2}d(\Omega'', \partial\Omega')\}$. For any $x_0 \in \Omega''$, we let $\eta(x) \in C_c^\infty(B_{2R})$ be a cut-off function, such that $0 \leq \eta(x) \leq 1$ and

$$\eta(x) = 1 \quad \text{for } x \in B_R,$$

$$|D\eta| \leq \frac{M_1}{R}, \quad |D^2\eta| \leq \frac{M_2}{R^2} \quad \text{for } x \in B_{2R},$$

where M_1, M_2 are positive constants.

For fixed $h < \frac{1}{3}\text{dist}(\text{supp } \eta, \partial B_{2R})$ and $k = 1, 2, \ldots, N$,

$$\triangle_h^k u(x) = \frac{u(x + he_k) - u(x)}{h} \in L^p(\Omega''),$$

we have $|\eta\triangle_h^k u|^{p-1}\text{sign}(\eta\triangle_h^k u) \in L^r(B_{2R})$, where $r = \frac{p}{p-1}$. According to Lemma 2.1, there must be a unique function

$$v_h \in W^{2,r}(B_{2R}) \cap W_0^{1,r}(B_{2R}),$$

such that

$$\begin{cases} a_{ij}(x)D_{ij}v_h = |\eta\triangle_h^k u|^{p-1}\text{sign}(\eta\triangle_h^k u), & x \in B_{2R}, \\ v_h = 0, & x \in \partial B_{2R}. \end{cases}$$

Moreover

$$\|v_h\|_{W^{2,r}(B_{2R})} \le C\||\eta\triangle_h^k u|^{p-1}\|_{L^r(B_{2R})} \le C\|\eta\triangle_h^k u\|_{L^p(B_{2R})}^{p-1}. \tag{4.1}$$

From (1.5) and a density argument, we have

$$\int_\Omega u D_i(a_{ij}D_j w) = \int_\Omega fw + g_i D_i w, \quad \forall\, w \in W_0^{2,r}(\Omega). \tag{4.2}$$

Now we choose $w = \eta\triangle_{-h}^k v_h$ in (4.2), and get

$$\begin{aligned}
&\int_{B_{2R}} f\eta(\triangle_{-h}^k v_h) + g_i D_i(\eta\triangle_{-h}^k v_h) \\
&\quad = \int_{B_{2R}} u D_i(a_{ij}D_j(\eta\triangle_{-h}^k v_h)) \\
&\quad = \int_{B_{2R}} u D_i a_{ij} D_j \eta \triangle_{-h}^k v_h + u\eta D_i a_{ij}\triangle_{-h}^k(D_j v_h) + u a_{ij} D_{ij}\eta(\triangle_{-h}^k v_h) \\
&\qquad + \int_{B_{2R}} 2u a_{ij} D_i \eta \triangle_{-h}^k (D_j v_h) + u a_{ij}\eta\triangle_{-h}^k(D_{ij}v_h).
\end{aligned} \tag{4.3}$$

Meanwhile by the property of difference quotients, we get

$$\begin{aligned}
\int_{B_{2R}} u a_{ij}\eta\triangle_{-h}^k(D_{ij}v_h) = &-\int_{B_{2R}} \eta\triangle_h^k u a_{ij} D_{ij} v_h \\
&- \int_{B_{2R}} (\triangle_h^k \eta) u(x+he_k) a_{ij}(x+he_k) D_{ij} v_h \\
&- \int_{B_{2R}} (\triangle_h^k a_{ij})\eta u(x+he_k) D_{ij} v_h.
\end{aligned} \tag{4.4}$$

From (4.3) and (4.4),

$$\begin{aligned}
\int_{B_{2R}} \eta\triangle_h^k u a_{ij} D_{ij} v_h \le &\int_{B_{2R}} \eta u(x+he_k)(\triangle_h^k a_{ij}) D_{ij} v_h \\
&+ \int_{B_{2R}} (\triangle_h^k \eta) u(x+he_k) a_{ij}(x+he_k) D_{ij} v_h \\
&+ \int_{B_{2R}} |u D_i a_{ij} D_j \eta(\triangle_{-h}^k v_h)| \\
&+ \int_{B_{2R}} |u\eta D_i a_{ij}\triangle_{-h}^k(D_j v_h)| + |u a_{ij} D_{ij}\eta(\triangle_{-h}^k v_h)| \\
&+ \int_{B_{2R}} 2|u a_{ij} D_i \eta\triangle_{-h}^k(D_j v_h)| \\
&+ \int_{B_{2R}} |f\eta(\triangle_{-h}^k v_h)| + |g_i D_i(\eta\triangle_{-h}^k v_h)|.
\end{aligned}$$

By the properties of the cut-off function and a_{ij}, we have

$$\begin{aligned}\int_{B_{2R}} \eta\triangle_h^k u a_{ij} D_{ij} v_h \le{}& K\int_{\operatorname{supp}\eta} |u(x+he_k)D_{ij}v_h| \\ &+ \frac{\Lambda M_1}{R}\int_{\operatorname{supp}\eta} |u(x+he_k)D_{ij}v_h| \\ &+ \frac{KM_1}{R}\int_{\operatorname{supp}\eta} |u\triangle_{-h}^k v_h| \\ &+ K\int_{\operatorname{supp}\eta} |u\triangle_{-h}^k (D_j v_h)| + \frac{\Lambda M_2}{R^2}\int_{\operatorname{supp}\eta} |u\triangle_{-h}^k v_h| \\ &+ \frac{2\Lambda M_1}{R}\int_{\operatorname{supp}\eta} |u\triangle_{-h}^k (D_j v_h)| + \int_{\operatorname{supp}\eta} |f\triangle_{-h}^k v_h| \\ &+ \frac{M_1}{R}\int_{\operatorname{supp}\eta} |g_i\triangle_{-h}^k v_h| + \int_{\operatorname{supp}\eta} |g_i\triangle_{-h}^k D_i v_h|.\end{aligned}$$

By Hölder inequality, (4.1), Sobolev imbedding theorem, the property of difference quotients and Young inequality, we obtain

$$\begin{aligned}\int_{B_{2R}} \eta a_{ij}\triangle_h^k u D_{ij} v_h \le{}& \frac{1}{2}\|\eta\triangle_h^k u\|^p_{L^p(B_{2R})} + \frac{C}{R^{2p}}(\|u\|^p_{L^p(B_{2R})} \\ &+ \|f\|^p_{L^{O_p}(B_{2R})} + \|g_i\|^p_{L^p(B_{2R})}),\end{aligned} \tag{4.5}$$

where

$$O_p = \begin{cases} 1, & 1 < p < \dfrac{N}{N-1}, \\ \max\limits_{t\ge 0}\{st - \exp(t^{\frac{N}{N-1}}) + 1\}, & p = \dfrac{N}{N-1}, \\ \dfrac{Np}{N+p}, & \dfrac{N}{N-1} < p < \infty. \end{cases}$$

Meanwhile

$$\begin{aligned}\int_{B_{2R}} \eta a_{ij}\triangle_h^k u D_{ij} v_h &= \int_{B_{2R}} \eta\triangle_h^k u|\eta\triangle_h^k u|^{p-1}\operatorname{sign}(\eta\triangle_h^k u) \\ &= \|\eta\triangle_h^k u\|^p_{L^p(B_{2R})}.\end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have

$$\|\triangle_h^k u\|_{L^p(B_R)} \le \frac{C}{R^2}\left(\|u\|_{L^p(B_{2R})} + \|f\|_{L^{O_p}(B_{2R})} + \sum_{i=1}^N \|g_i\|_{L^p(B_{2R})}\right).$$

Using the property of difference quotients again, we obtain $D_k u \in L^p(B_R)$, and

$$\|D_k u\|_{L^p(B_R)} \le C\left(\|u\|_{L^p(B_{2R})} + \|f\|_{L^{O_p}(B_{2R})} + \sum_{i=1}^N \|g_i\|_{L^p(B_{2R})}\right).$$

Finally, using finite covering theorem, we have

$$\|u\|_{W^{1,p}(\Omega'')} \le C\left(\|u\|_{L^p(\Omega')} + \|f\|_{L^{O_p}(\Omega')} + \sum_{i=1}^{N}\|g_i\|_{L^p(\Omega')}\right),$$

where C depends only on N, p, λ, Λ, K, Ω'' and Ω'. □

5. Higher Regularity

In this section, we will give the proof of Theorem 1.6.

Proof of Theorem 1.6. We will prove Theorem 1.6 by induction on n, the case $n = 0$ being Theorem 1.4 and Theorem 1.5 above. Assume that Theorem 1.6 holds for some nonnegative integer n. Suppose then

$$a_{ij} \in C^{n+1,1}_{\mathrm{loc}}(\Omega), \quad i, j = 1, 2, \ldots, N, \tag{5.1}$$

$$g_i \in W^{n+1,p}_{\mathrm{loc}}(\Omega), \quad i = 1, 2, \ldots, N, \tag{5.2}$$

$$f \in W^{n+1,O_p}_{\mathrm{loc}}(\Omega) \tag{5.3}$$

with $1 < p < \infty$, and $u \in L^1_{\mathrm{loc}}(\Omega)$ is a very weak solution of (1.4). We want to show that $u \in W^{n+2,p}_{\mathrm{loc}}(\Omega)$. By the induction hypotheses, we have

$$u \in W^{n+1,p}_{\mathrm{loc}}(\Omega), \tag{5.4}$$

with the estimate

$$\|u\|_{W^{n+1,p}(\Omega'')} \le C\left(\|u\|_{L^1(\Omega')} + \|f\|_{W^{n,O_p}(\Omega')} + \sum_{i=1}^{N}\|g_i\|_{W^{n,p}(\Omega')}\right), \tag{5.5}$$

for each $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ and an appropriate C.

Now fix $\omega \subset\subset \Omega''$, and let α be any multiindex with $|\alpha| = n + 1$, and choose any test function $\tilde{\varphi} \in C^\infty_c(\Omega'')$. Insert

$$\varphi := (-1)^{|\alpha|} D^\alpha \tilde{\varphi}$$

into the (1.5), and perform some integrations by parts, eventually to discover

$$\int_\Omega \tilde{u} D_i(a_{ij} D_j \tilde{\varphi}) = \int_\Omega \tilde{f}\tilde{\varphi} + \tilde{g}_i D_i \tilde{\varphi},$$

for

$$\tilde{u} := D^\alpha u \in L^p(\Omega''),$$

$$\tilde{f} := D^\alpha f,$$

and

$$\widetilde{g_i} := D^\alpha g_i + \sum_{\substack{\beta \le \alpha \\ \beta \ne \alpha}} \binom{\alpha}{\beta} D^{\alpha-\beta} a_{ij} D^\beta D_j u.$$

In view of (5.1)–(5.4), we have $\tilde{f} \in L^{O_p}(\Omega'')$, $\widetilde{g_i} \in L^p(\Omega'')$.

In light of Theorem 1.1, we see $\tilde{u} \in W^{1,p}(\Omega')$, with the estimate

$$\|\tilde{u}\|_{W^{1,p}(\omega)} \leq C\left(\|\tilde{u}\|_{L^1(\Omega'')} + \|\tilde{f}\|_{L^{O_p}(\Omega'')} + \sum_{i=1}^{N}\|\tilde{g}_i\|_{L^p(\Omega'')}\right).$$

That is, $D^\alpha u \in W^{1,p}(\omega)$ and

$$\|D^\alpha u\|_{W^{1,p}(\omega)} \leq C\left(\|D^\alpha u\|_{L^1(\Omega'')} + \|D^\alpha f\|_{L^{O_p}(\Omega'')} + \sum_{i=1}^{N}\|D^\alpha g_i\|_{L^p(\Omega'')} + \sum_{\substack{\beta\leq\alpha\\ \beta\neq\alpha}}\|D^\beta Du\|_{L^p(\Omega'')}\right).$$

Meanwhile

$$\|D^\alpha u\|_{L^1(\Omega'')} \leq C\|u\|_{W^{n+1,p}(\Omega'')}$$

and

$$\sum_{\substack{\beta\leq\alpha\\ \beta\neq\alpha}}\|D^\beta Du\|_{L^p(\Omega'')} \leq C\|u\|_{W^{n+1,p}(\Omega'')}.$$

Consequently we have $u \in W^{n+2,p}_{\mathrm{loc}}(\Omega)$, by (5.5), and get

$$\|u\|_{W^{n+2,p}(\omega)} \leq C\left(\|u\|_{L^1(\Omega')} + \|f\|_{W^{n+1,O_p}(\Omega')} + \sum_{i=1}^{N}\|g_i\|_{W^{n+1,p}(\Omega')}\right),$$

where C is a constant depending only on n, N, p, λ, Λ, ω, Ω' and the norms of a_{ij} in $C^{n+1,1}(\overline{\Omega'})$. □

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