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Positive solution for semilinear elliptic equation on general domain

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Abstract

We give the comparison principle, the uniqueness theorem, and the necessary and sufficient conditions for the solvability with positive solutions in $W^{2,p}_{\text{loc}} \cap L^\infty$ of the Dirichlet problems for semilinear elliptic equations on general bounded domains. The domains may be irregular while the equations are in general form. The resultant theorem includes previous work either in sublinear and some superlinear cases on the smooth domains or in linear case on the general domains. Our methods are the refined a priori estimates and the degree theory.

Keywords: Positive solution; Existence; Uniqueness; Necessary and sufficient condition; Semilinear equation; General domain; First eigenvalue

1. Introduction

Let Ω be a general bounded domain in $\mathbf{R}^n$, which may be with irregular boundary points. In this paper, we are concerned with the problem of finding a positive function u in $W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$, $1 < p < +\infty$, satisfying the semilinear elliptic equations

$$-Mu = f(x,u) \quad \text{in } \Omega \tag{1}$$

with zero Dirichlet boundary condition in some weak sense, where

$$M = \sum_{i,j=1}^{n} a^{ij}(x)D_{ij} + \sum_{i=1}^{n} b^i(x)D_i, \qquad a^{ij} \in C(\Omega),\ \ b^i \in L^\infty(\Omega),$$

$$c_0|\xi|^2 \leqslant \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leqslant c_0^{-1}|\xi|^2$$

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for $x \in \Omega$, $\xi \in \mathbf{R}^n$, c_0 is some positive constant, and $f(x,u): \Omega \times [0,+\infty) \to \mathbf{R}$ is nonlinear.

The semilinear elliptic boundary value problems of the type

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{3}$$

arise in a variety of situations in the theory of nonlinear diffusion, thermal ignition of gases, quantum field, mechanical statistics, gravitational equilibrium of stars and elsewhere. Therefore, (2), (3) have received considerable study both in the sublinear case and in the superlinear case. A survey concerning the problems is given by Lions, see [9] and references therein.

We are interested here in the sublinear case, in which $f(u)$ is assumed to grow slower than linearly in u. But our results also contain the linear case and some superlinear case.

If $\partial\Omega$ is smooth, Berestycki [1] proved the existence and uniqueness theorem of the positive solutions for (2), (3), using the variational method. Under the similar conditions, Brezis and Oswald [3] gave a necessary and sufficient condition of the existence for the more general equations $\Delta u = f(x,u)$. By making good use of the super–subsolution method, Taira and Umezu [14] had a generalization of [3] to a class of sublinear equations having principal part in divergence form

$$-\sum_{i,j=1}^{n} D_i(a^{ij}(x)D_j u) + c(x)u = f(x,u).$$

We also refer to a number of earlier results, which are closely related to ours, for instance, [5, Chapter 2], [6,8,11].

However in our situation, Eq. (1) is not a Euler equation of some functional. Thus the variational methods cannot be applied, even if Ω is a smooth domain.

For the irregular domain Ω, boundary condition (3) is too strong a hypothesis. In general, one cannot prescribe boundary values of solutions at every point of $\partial\Omega$. In such a case, it is difficult to estimate bounds of the solutions for the usual maximum principles are invalid. On the other hand, it is now well-understood that a priori bounds in the sup norm of positive solutions for (2), (3) provide information about the existence of positive solutions.

We notice that Berestycki et al. [2] worked with a refined version for the linear elliptic equations in general bounded domains. They introduced a function $u_0 \in W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$, for all $p > 1$, satisfying

$$Mu_0 = -1, \quad u_0 > 0 \quad \text{in } \Omega \tag{4}$$

and used the following notion to describe zero Dirichlet boundary condition in some weak sense.

Definition 1. For a sequence $x^l \to \partial\Omega$, we say $x^l \overset{u_0}{\to} \partial\Omega$ if $u_0(x^l) \to 0$. Given $u \in C(\Omega)$, the notation $u \overset{u_0}{=} 0$ on $\partial\Omega$ means: along any sequence $x^l \overset{u_0}{\to} \partial\Omega$, we have $u(x^l) \to 0$.

Padilla [13] presented an extension of their results in [2] to the Riemann manifolds.

It is claimed in [2] that u_0 vanishes on the boundary in the following sense: u_0 can be extended as a continuous function to every point y of $\partial\Omega$ admitting a strong barrier function by setting $u_0(y) = 0$. Miller [12] has shown that every point $y \in \partial\Omega$, where $\partial\Omega$ satisfies an exterior cone condition, admits a strong barrier function. Consequently, for a regular domain Ω, which means for each boundary point, there is a strong barrier function, $u \overset{u_0}{=} 0$ on $\partial\Omega$ is the same as u continuous on $\partial\Omega$ and $u(x) = 0$ for $x \in \partial\Omega$, see [4].

Recently, Liu shows using the probability method

$$u_0(x) = E_x[\tau_\Omega], \quad x \in \Omega,$$

the expectation of the first exit time from Ω

$$\tau_\Omega = \inf\{t \geqslant 0 \,|\, X(t) \notin \Omega\},$$

where $(X(t), P)$ is Markov process corresponding to the linear elliptic operator M, see [10, Theorem 3.1].

2. Main results

The goal of this paper is to provide the comparison principle, prove the uniqueness theorem, and give a necessary and sufficient condition for the existence of positive solutions to the generalized Dirichlet problem

$$-Mu = f(x,u) \quad \text{in } \Omega, \tag{5}$$

$$u \geqslant 0, \quad u \not\equiv 0 \quad \text{in } \Omega, \tag{6}$$

$$u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega. \tag{7}$$

The essential point here is that we do not assume the domains are regular and the equations are in divergence form. In our proof, we will use the refined ABP maximum principle, the property of first eigenvalue for elliptic operators and the degree theory.

On the nonlinearity f, we make the following assumptions:

(f1) for a.e. $x \in \Omega$ the function $u \to f(x,u)$ is continuous on $[0, +\infty)$, and the function $u \to f(x,u)/u$ is decreasing in $(0, +\infty)$;
(f2) for each $u \geqslant 0$ the function $x \to f(x,u)$ belongs to $L^\infty(\Omega)$;
(f3) there is a constant $C > 0$ such that

$$f(x,u) \leqslant C(u+1) \quad \text{for a.e. } x \in \Omega, \quad \text{any } u \geqslant 0.$$

For later applications we quote the definition of the first eigenvalue of the linear elliptic operator $-M - c(x)$ on the domain Ω in [2, (1.13)]

$$\lambda_1(-M - c(x), \Omega) = -\inf_{\varphi} \sup_{\Omega} \left\{ \frac{M\varphi}{\varphi} + c(x) \right\}, \tag{8}$$

where $c(x)$ is any semi-bounded function in Ω, and the inf is taken over all positive functions φ in $W^{2,n}_{\rm loc}(\Omega)$. We will sometimes denote $\lambda_1(-M-c(x),\Omega)$ by $\lambda_1(-M-c(x))$ if Ω is fixed and the coefficient $c(x)$ is allowed to vary.

We also introduce the functions

$$c_0(x) = \lim_{u\to 0+0} \frac{f(x,u)}{u}, \tag{9}$$

$$c_\infty(x) = \lim_{u\to +\infty} \frac{f(x,u)}{u}, \tag{10}$$

uniformly in $x\in\Omega$. Note that $c_0(x), c_\infty(x)$ may take the infinite values, which importance will be seen in the latter half of this section.

Our main result is the following:

Theorem 2. *Under the conditions* (f1)–(f3), *for* $p>1$,

(a) *a* $W^{2,p}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$ *solution of the generalized Dirichlet problem* (5)–(7) *exists if*

$$\lambda_1(-M-c_0(x))<0, \tag{11}$$

and

$$\lambda_1(-M-c_\infty(x))>0. \tag{12}$$

(b) *If* $c_0\in L^\infty(\Omega)$, *then conditions* (11) *and* (12) *are necessary and sufficient for the existence of* $W^{2,p}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$ *solutions of the generalized Dirichlet problem* (5)–(7), *and the solution* u *is unique in the space* $W^{2,n}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$.

(c) *If the nonlinear term* $f(x,u)$ *is independent of* x, *then the generalized Dirichlet problem* (5)–(7) *has a solution* $u\in W^{2,p}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$ *if and only if*

$$\lambda_1(-M-c_0)<0<\lambda_1(-M-c_\infty).$$

If $M=\Delta$, the Laplace operator, and $\partial\Omega$ is smooth, then Theorem 2(a) is indeed [3, Theorem 2]. Their argument makes use of energy associated with Δ. Since our operator M has nondivergence form, arguments involving energy cannot be used.

If $f(x,u)$ is linear in u, then Theorem 2 can be regarded as an extension of [2, Theorem 1.2] for the positive solutions.

We observe from (f1) that

$$c_\infty(x)\leqslant f(x,1)\leqslant c_0(x) \quad \text{for a.e. } x\in\Omega,$$

and (f2) implies that there is a constant $C>0$ such that

$$c_0(x)\geqslant -C, \qquad c_\infty(x)\leqslant C \quad \text{for a.e. } x\in\Omega.$$

Therefore,

$$\lambda_1(-M-c_0(x))\in[-\infty,+\infty),$$

$$\lambda_1(-M-c_\infty(x))\in(-\infty,+\infty].$$

Here we have used $\lambda_1(-M)$ is bounded, see [2, Lemma 1.1].

As four simple cases of Theorem 2, the followings show the various situation that $c_0(x)$ and $c_\infty(x)$ take the finite or infinite values.

Corollary 3 ($c_0(x)=+\infty, c_\infty(x)=0$). *For any* $p>1$, $0<\alpha<1$ *and* $0\leqslant f\in L^\infty(\Omega)$, *there is a positive solution* $u\in W^{2,p}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)$ *of the generalized Dirichlet problem*

$$-\Delta u = u^\alpha + f(x) \quad \text{in } \Omega, \qquad u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega. \tag{13}$$

Furthermore, $u\in C^{2,\alpha}(\Omega)$ *if* $f\in C^\alpha(\Omega)$.

Corollary 4 ($c_0(x)=+\infty, c_\infty(x)=-\infty$). *For any* $p>1$, $\beta>1$ *and* $f\in L^\infty(\Omega)$. *If* $\inf_\Omega f>0$, *then there is a positive solution* $u\in W^{2,p}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)$ *of the generalized Dirichlet problem*

$$-\Delta u = -u^\beta + f(x) \quad \text{in } \Omega, \qquad u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega. \tag{14}$$

Corollary 5 ($c_0(x)=-c(x), c_\infty(x)=-\infty$). *For any* $p>1$, $c\in L^\infty(\Omega)$, *and* $\inf_\Omega c>0$, *there is no positive solution* $u\in W^{2,p}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)$ *of the generalized Dirichlet problem*

$$-\Delta u = c(x)(1-e^u) \quad \text{in } \Omega, \qquad u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega. \tag{15}$$

Corollary 6 ($c_0(x)=\lambda, c_\infty(x)=0$). *For any* $p>1$ *and* $\lambda\in\mathbf{R}$, *there is a unique positive solution* $u\in W^{2,p}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)$ *of the generalized eigenvalue problem*

$$-\Delta u = \lambda\log(1+u) \quad \text{in } \Omega, \qquad u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega, \tag{16}$$

if only if $\lambda>\lambda_1(-\Delta)$.

Sublinear problem (13) is the most typical case in our consideration. (14) and (15) are superlinear (from below) problems, and (16) is an asymptotic linear problem near $u=0$.

As we have seen, $c_0(x)$ may take $+\infty$ and $c_\infty(x)$ may take $-\infty$, which seem to be even more important cases. At the same time, their infinite values bring about much difficulty in our discussion.

The rest part of the paper is organized as follows. In the following section we state some known facts to be used. Section 4 is devoted to the comparison principle and the uniqueness theorem for the generalized Dirichlet problems (5)–(7), and the necessity of conditions (11) and (12) for the existence. In Section 5 we derive the supnorm estimates, employing the refined ABP maximum principle. In the last section, the existence theorem is obtained, where the degree theory is used.

3. Preliminaries

We give three preliminary results. The first lemma was proved in [2, Theorem 2.4 and Propositioon 2.1]. Lemmas 8 and 9 are [2, Theorems 1.3 and 1.2], respectively.

Lemma 7 (Property of the first eigenvalue). (i) *If* $c_1, c_2 \in L^\infty(\Omega)$ *and* $c_1 \geqslant c_2, c_1 \not\equiv c_2$, *then*

$$\lambda_1(-M - c_1(x)) < \lambda_1(-M - c_2(x)).$$

(ii) *If* Ω_1 *and* Ω_2 *are two bounded domains with* $\Omega_1 \supset \Omega_2$, *and* $\Omega_1 \neq \Omega_2$, *then*

$$\lambda_1(-M - c(x), \Omega_1) < \lambda_1(-M - c(x), \Omega_2)$$

for any $c \in L^\infty(\Omega_1)$.

(iii) $\lambda_1(-M - c(x))$ *is Lipschitz continuous with respect to* $c(x)$ *in the* $L^\infty(\Omega)$ *norm, with Lipschitz constant* 1.

Lemma 8 (Refined ABP Theorem). *Assume* $c \in L^\infty(\Omega)$ *and* $\lambda_1(-M - c(x)) > 0$. *Suppose* $u \in W^{2,n}_{\text{loc}}(\Omega)$ *is a function, bounded from above, satisfying*

$$-(M + c(x))u \leqslant f(x) \quad \text{in } \Omega,$$

$$\limsup_{l \to \infty} u(x^l) \leqslant \beta \quad if\ x^l \stackrel{u_0}{\to} \partial\Omega,$$

where $0 \leqslant f \in L^n(\Omega)$ *and* $\beta \geqslant 0$. *Then*

$$\sup_\Omega u \leqslant C \left(\|f\|_{L^n(\Omega)} + \beta|\Omega|^{1/n} \sup_\Omega c^+ \right) + \beta.$$

Here C *is the constant depending only on* $c_0, \Omega, \|b\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}$ *and* $\lambda_1(-M - c(x))$.

Lemma 9 (Existence for linear equations). *Assume* $c \in L^\infty(\Omega)$ *and* $\lambda_1(-M - c(x)) > 0$. *Then, given* $f \in L^n(\Omega)$, *there is a unique solution in* $W^{2,n}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ *of*

$$-(M + c(x))u = f(x) \quad \text{in } \Omega,$$

$$u \stackrel{u_0}{=} 0 \quad \text{on } \partial\Omega.$$

4. Uniqueness and necessity

We start with the following lemmas.

Lemma 10 (Positivity). *Assume* (f 1), (f 2) *hold, and let* $u \in W^{2,n}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ *be a solution of* (5), (6). *Then we have* $u > 0$ *in* Ω.

Proof. Since (f1), (f2) it follows that

$$\frac{f(x,u)}{u} \geqslant \frac{f(x, 1 + \|u\|_{L^\infty(\Omega)})}{1 + \|u\|_{L^\infty(\Omega)}} := \bar{c}(x) \in L^\infty(\Omega), \tag{17}$$

and, therefore,

$$-Mu + Cu \geqslant 0 \quad \text{in } \Omega,$$

for some constant $C > 0$. From the weak Harnack inequality [7, Theorem 9.22]

$$\left(\frac{1}{|B_R|}\int_{B_R} u^p\,\mathrm{d}x\right)^{1/p} \leqslant C\inf_{B_R} u \tag{18}$$

if $B_{2R} \subset \Omega$, where p and C are the positive constants depending only on n, c_0, R, $\|b\|_{L^\infty(\Omega)}$ and $\|\bar{c}\|_{L^\infty(\Omega)}$.

For any two points $x_1, x_2 \in \Omega$, let Γ be a closed arc joining x_1 and x_2 and choose R so that $2R < \mathrm{dist}(\Gamma, \partial\Omega)$. By virtue of the Heine–Borel theorem, Γ can be covered by a finite number of balls of radius R. Applying estimate (18) in each ball and combining the resulting inequalities, we obtain $u(x_1) = 0$ if $u(x_2) = 0$. Hence $u(x) > 0$ in Ω since $0 \leqslant u \not\equiv 0$ in Ω. □

Lemma 11. *Let* $c_0 \in L^\infty(\Omega)$, $f(x,u)$ *satisfy* (f 1) *and* (f 2). *If* $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ *is a positive solution of* (5), (7), *then*

$$\lambda_1\left(-M - \frac{f(x,u)}{u}\right) = 0.$$

Proof. First we see that

$$\bar{c}(x) < \frac{f(x,u(x))}{u(x)} < c_0(x),$$

for a.e. $x \in \Omega$, where $\bar{c}(x)$ is defined in (17). This implies that

$$\frac{f(x,u(x))}{u(x)} \in L^\infty(\Omega).$$

Writing Eq. (5) in the form

$$-\left(M + \frac{f(x,u)}{u}\right)u = 0 \quad \text{in } \Omega,$$

we then have by Lemma 8

$$\lambda_1\left(-M - \frac{f(x,u)}{u}\right) \leqslant 0.$$

On the other hand, by the definition of $\lambda_1(-M - \frac{f(x,u)}{u})$, and since $u > 0$ in Ω, we have

$$\lambda_1\left(-M - \frac{f(x,u)}{u}\right) \geqslant -\sup_{\Omega}\frac{Mu + f(x,u)}{u} \geqslant 0. \tag{19}$$

This lemma is proved. □

In the proof of (19), we only require the one-sided inequality $-Mu \geqslant f(x,u)$ in Ω, and the condition $c_0 \in L^\infty(\Omega)$ is not used.

The refined comparison principle has the following extension to the semilinear operators for the supersolutions.

Theorem 12 (Comparison). *Let $c_0 \in L^\infty(\Omega)$, (f 1) and (f 2) hold, and $u, v \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ are positive and satisfy*

$$0 \leqslant -Mu - f(u,x) \leqslant -Mv - f(v,x) \quad \text{in } \Omega, \tag{20}$$

$$\limsup_{l\to\infty} (u-v)(x^l) \leqslant 0 \quad \text{if } x^l \overset{u_0}{\to} \partial\Omega. \tag{21}$$

It then follows that $u \leqslant v$ in Ω.

Proof. We write

$$L = M + \frac{f(x,u)}{u}, \qquad D = \{x \in \Omega \mid u(x) > v(x)\},$$

and have by (f1), (20) and (21)

$$-L(u-v) = -Mu - f(x,u) + Mv + \frac{v}{u} f(x,u) < 0 \quad \text{in } D,$$

$$\limsup_{l\to\infty} (u-v)(x^l) \leqslant 0 \quad \text{if } x^l \overset{u_0}{\to} \partial D.$$

If $D = \Omega$, then $u > v$ in Ω, and consequently

$$\bar{c}(x) < \frac{f(x,u)}{u} < \frac{f(x,v)}{v} < c_0(x) \quad \text{in } \Omega.$$

It follows from Lemma 7(i) and (19) that

$$\lambda_1(-L,D) = \lambda_1\left(-M - \frac{f(x,u)}{u}, \Omega\right) > \lambda_1\left(-M - \frac{f(x,v)}{v}, \Omega\right) \geqslant 0.$$

If $D \neq \Omega$, then we also have by applying Lemma 7(ii) and (19),

$$\lambda_1(-L,D) > \lambda_1(-L,\Omega) \geqslant 0.$$

We deduce from Lemma 8, $u - v \leqslant 0$ in D, which contradict the definition of D, and conclude that $u \leqslant v$ in Ω. □

A uniqueness theorem for the generalized Dirichlet problems (5)–(7) follows immediately from Theorem 12.

Theorem 13 (Uniqueness). *Let $u, v \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfy (5)–(7). If $c_0 \in L^\infty(\Omega)$, (f 1) and (f 2) hold, then $u = v$ in Ω.*

Next, we turn to the necessity of conditions (11) and (12) for the existence to the generalized Dirichlet problems (5)–(7).

Theorem 14 (Necessity). *Let $f(x,u)$ satisfy (f 1) and (f 2), $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ be a positive solution of (5), (7), then condition (12) holds. In addition, if $c_0 \in L^\infty(\Omega)$ or $f(x,u) = f(u)$, we have (11) holds.*

Proof. From (f1), it is very easy to obtain

$$c_\infty(x) < \bar{c}(x) < \frac{f(x,u(x))}{u(x)} < c_0(x),$$

for a.e. $x\in\Omega$. Combining Lemma 7(i) with Lemma 11, we get

$$\lambda_1(-M-c_\infty(x)) \geqslant \lambda_1(-M-\bar{c}(x)) > \lambda_1\left(-M-\frac{f(x,u)}{u}\right) = 0 > \lambda_1(-M-c_0(x))$$

in the case $c_0\in L^\infty(\Omega)$.

If $f(x,u)=f(u)$ and $c_0=+\infty$, then the conclusion of this theorem is clearly. □

Remark 15. From the proof of Theorem 14, we see

$$0<\lambda_1(-M-c_m(x))<+\infty, \qquad \|u\|_{L^\infty(\Omega)}<m<+\infty,$$

where

$$c_m(x)=\frac{f(x,m)}{m}, \quad \text{a.e. } x\in\Omega.$$

5. Refined estimate

First, we establish the refinement of the supnorm estimates of the positive subsolutions to the generalized Dirichlet problems (5)–(7).

Theorem 16 (Refined estimate). *Suppose $u\in W^{2,n}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ is a positive function, satisfying*

$$-Mu \leqslant f(x,u) \quad \text{in } \Omega,$$

$$\limsup_{l\to\infty} u(x^l) \leqslant \beta \quad \text{if } x^l \xrightarrow{u_0} \partial\Omega,$$

where β is some nonnegative constant. If (f1), (f2) *hold, then we have*

$$u(x)\leqslant C, \quad x\in\Omega,$$

where the constant C depends only on $n, c_0, \beta, \mathrm{diam}\,\Omega, \|b\|_{L^\infty(\Omega)}, \lambda_1(-M-c_\infty(x))$ and the behavior of f in the limit arising in (10).

Proof. From Remark 15 and (f2), we can find $m>0$, such that $0<\lambda_1(-M-c_m(x)) <+\infty$ and $c_m(x)\in L^\infty(\Omega)$. Because of (f1), we have

$$-Mu-c_m(x)u\leqslant 0 \quad \text{in } \Omega_m,$$

$$\limsup_{l\to\infty} u(x^l)\leqslant \max(\beta,m) \quad \text{if } x^l \xrightarrow{u_0} \partial\Omega_m,$$

where $\Omega_m = \{x \in \Omega \mid u(x) > m\}$. Applying Lemma 8 we find

$$\sup_{\Omega_m} u \leqslant \max(\beta, m)\left(C|\Omega_m|^{1/n} \sup_{\Omega_m} f_m^+ + 1\right) \leqslant C.$$

Therefore,

$$\sup_{\Omega} u \leqslant C. \qquad \square$$

Now, we can give the interior $W^{2,p}$ estimates for the positive solutions of Eq. (5).

Theorem 17 (Interior $W^{2,p}$ estimate). *Suppose $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ is a positive solution of (5). If (f 1)–(f 3) hold, then for any $\Omega' \subset\subset \Omega$ and $p > 1$, we have $u \in W^{2,p}(\Omega')$. Furthermore, for a constant C depending only on $n, p, c_0, \|b\|_{L^\infty(\Omega)}, \Omega', \Omega, \lambda_1(-M - c_\infty(x))$, the moduli of continuity of a^{ij} on $\bar{\Omega}'$ and the behavior of f in the limit arising in (10)*

$$\|u\|_{W^{2,p}(\Omega')} \leqslant C.$$

Proof. It follows from (f1)–(f3)

$$-|f(x, \|u\|_{L^\infty(\Omega)})| \leqslant \frac{u}{\|u\|_{L^\infty(\Omega)}} f(x, \|u\|_{L^\infty(\Omega)}) \leqslant f(x,u) \leqslant C(\|u\|_{L^\infty(\Omega)} + 1)$$

and $f(x, u(x)) \in L^\infty(\Omega)$. By means of the interior $W^{2,p}$ regularity [7, Lemma 9.16] we obtain $u \in W^{2,p}(\Omega')$. It follows from the interior $W^{2,p}$ estimates [7, Theorem 9.11] and Theorem 16 that

$$\|u\|_{W^{2,p}(\Omega')} \leqslant C(\|u\|_{L^p(\Omega)} + \|f(x,u)\|_{L^p(\Omega)}) \leqslant C. \qquad \square$$

6. Sufficiency

Having established the a priori estimates, in this section we derive the existence theorem of positive solutions for the generalized Dirichlet problems (5)–(7) on the general domains via the topological degree.

Theorem 18 (Existence). *Assume that* (f 1)–(f 3), (11) *and* (12) *hold. Then for any $p > 1$ there is a positive solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ of* (5)–(7).

Proof. In the following we suppose $f(x,u) = 0$ for $u < 0$. Thus the solutions of (5)–(7) must be nonnegative. By Lemma 10 we see that a non-zero solution is a positive solution. So we can use the Leray–Schauder degree directly in

$$X = \left\{u \in C(\Omega) \cap L^\infty(\Omega) \mid u \overset{u_0}{=} 0 \text{ on } \partial\Omega\right\}.$$

Define by Lemma 9 the mapping T_t of $[0,2] \times X \to X$ by $u = T_t v$ being the solution of the problem

$$-t\Delta u - (1-t)Mu = t|v|^{1/2} + (1-t)f(x,v) \quad \text{in } \Omega,$$

$$u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega,$$

if $t \in [0,1]$; or the solution of problem

$$-\Delta u = |v|^{1/2} + (t-1)(\lambda_1 v + 1) \quad \text{in } \Omega,$$

$$u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega,$$

if $t \in [1,2]$, where λ_1 is the first eigenvalue of $-\Delta$.

According to the regularity theory, the solvability of (5)–(7) in the space $W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ is equivalent to the existence of the fixed points of T_0. With the aid of Lemma 8 and the Hölder interior estimates [7, Corollary 9.24], we get

$$\|u\|_{C^\alpha_*(\Omega)} \leqslant C(\|v\|_{L^\infty(\Omega)}),$$

where $C^\alpha_*(\Omega)$ is a subspace of $C^\alpha(\Omega)$, which is a Banach space endowed with the interior Hölder norm

$$\|u\|_{C^\alpha_*(\Omega)} = \sup_\Omega |u| + \sup_{x,y\in\Omega} d^\alpha_{x,y} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

where $0 < \alpha < 1, d_x = \text{dist}(x, \partial\Omega)$, and $d_{x,y} = \min(d_x, d_y)$. Applying the compactness results [7, Lemma 6.33], we have T_t is compact.

When $t \in [0,2]$, from Theorem 16 we have established a uniform bound in X for the fixed points of T_t. Hence for R sufficiently large, T_t has no fixed point on ∂B_R, and therefore the topological degree $\deg(I - T_t, B_R, 0)$ is well-defined and independent of $t \in [0,2]$ by the homotopy invariance, where B_R is the ball in X of radius R with the center at $u \equiv 0$.

For $t = 2$,

$$c_\infty(x) = \lambda_1, \quad \lambda_1(-\Delta - c_\infty(x)) = 0.$$

By Theorem 14, T_2 has no fixed point. Hence

$$\deg(I - T_0, B_R, 0) = \deg(I - T_2, B_R, 0) = 0.$$

By (9) and Lemma 7(iii), we can fix $r > 0$ small such that

$$\lambda_1(-M - c_r(x)) < 0. \tag{22}$$

Define again a compact map S_s from $[0,1] \times X$ into X: $u = S_s v$ is the solution of

$$-Mu = sf(x,v) + (1-s)c_r(x)v \quad \text{in } \Omega,$$

$$u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega.$$

If $u \in \partial B_r \subset X$ satisfies

$$-Mu = sf(x,u) + (1-s)c_r(x)u \quad \text{in } \Omega,$$

$$u \overset{u_0}{=} 0 \quad \text{on } \partial\Omega,$$

with $s \in [0,1]$, then by (22) we have

$$\sup_{\Omega}\left\{\frac{Mu}{u}+c_r(x)\right\} \geqslant -\lambda_1(-M-c_r(x))>0.$$

Thus at some point $\hat{x}\in\Omega$

$$\begin{aligned}0 &= \frac{Mu+sf(x,u)+(1-s)c_r(x)u}{u}\\ &= \frac{Mu+c_r(x)u}{u}-\frac{s(c_r(x)u-f(x,u))}{u}\\ &\geqslant -\frac{1}{2}\lambda_1(-M-c_r(x))-\left(c_r(x)-\frac{f(x,u)}{u}\right)\\ &\geqslant -\frac{1}{2}\lambda_1(-M-c_r(x))>0\end{aligned}$$

if $u(\hat{x})\leqslant r$. Hence S_s has no fixed point on ∂B_r for all $s\in[0,1]$ provided $r>0$ is sufficiently small. Therefore, by the homotopy invariance

$$\deg(I-S_1,B_r,0)=\deg(I-S_0,B_r,0),$$

that is,

$$\deg(I-T_0,B_r,0)=\deg(I-S_0,B_r,0).$$

Let u is a fixed point of S_0 in B_r, then

$$-Mu=c_r(x)u \quad \text{in } \Omega,$$

$$u\overset{u_0}{=}0 \quad \text{on } \partial\Omega.$$

By the definition of $\lambda_1(-M-c_r(x))$, and since $u>0$ in Ω, we have

$$\lambda_1(-M-c_r(x)) \geqslant -\sup_{\Omega}\frac{Mu+c_r(x)u}{u}=0.$$

Eq. (22) implies that there is no such positive solution. Thus S_0 has no fixed point in $B_r\backslash\{0\}$, and

$$\deg(I-S_0,B_r,0)=1.$$

So we obtain by the additivity

$$\deg(I-T_0,B_R\backslash B_r,0)=-1.$$

This means the existence of positive solution of (5)–(7) by Kronecker theorem. □

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References

[1] H. Berestycki, Le nombre de solutions de certains problémes semi-linèaires elliptiques, J. Funct. Anal. 40 (1981) 1–29.
[2] H. Berestycki, L. Nirenberg, S.V.R. Varadhan, The principal eigenvalue and maximum principle for second order elliptic operators in general domains, Commun. Pure Appl. Math. 47 (1994) 47–92.
[3] H. Brezis, L. Oswald, Remarks of sublinear elliptic problems, Nonlinear Anal. 10 (1986) 55–64.
[4] K.C. Chang, M.Y. Jiang, Parabolic equations and the Feynman–Kac formula on general bounded domains, Sci. China, Ser. A 44 (2001) 311–329.
[5] D.G. de Figueiredo, Positive solutions of semilinear elliptic equations, Lecture Notes in Mathematics, Vol. 957, Springer, Berlin, 1982, pp. 34–87.
[6] D.G. de Figueiredo, I. Massabó, Semilinear elliptic equations with the primitive of the nonlinear interacting with the first eigenvalue, J. Math. Anal. Appl. 156 (1991) 381–394.
[7] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd Edition, Springer, Berlin, 1983.
[8] J.V. Goncalves, On nonresonant sublinear elliptic problems, Nonlinear Anal. 15 (1990) 915–920.
[9] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982) 441–467.
[10] G.Q. Liu, Elliptic equations on general domains and stochastic representation of the solutions, preprint.
[11] J. Mawhin, M. Willem, J. Ward, Variational methods and semilinear elliptic equations, Arch. Ration. Mech. Anal. 95 (1990) 1141–1159.
[12] K. Miller, Barrierd on cones for uniformly elliptic operators, Ann. Math. Pure Appl. 76 (1967) 93–106.
[13] P. Padilla, The principal eigenvalue and maximum principle for second order elliptic operators on Riemann manifolds, J. Math. Anal. Appl. 205 (1997) 285–312.
[14] K. Taira, K. Umezu, Positive solutions of sublinear boundary value problems, Nonlinear Anal. 29 (1997) 761–771.