Optimal Regularity for Convex Strong Solutions of Special Lagrangian Equations in Dimension 3

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ABSTRACT. By means of the Reilly formula, we obtain a weighted iterative inequality and the local $C^{1,1}$ estimates of the $W^{2,p}$ convex strong solutions to the special Lagrangian equations in dimension 3 for $p > 3$, and then prove that these solutions are smooth. The regularity result fails if $p < 3$.

1. INTRODUCTION

In this paper we consider local $C^{1,1}$ estimates and regularity for the convex strong solutions of the special Lagrangian equations

$$\det(D^2u) = \Delta u, \quad \text{a.e. in } \Omega, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^3$.

For a smooth function on Ω, the graph of its gradient is automatically a Lagrangian submanifold in $\mathbb{C}^3 = \mathbb{R}^3 \times \mathbb{R}^3$ with the standard complex structure. If the function further satisfies equation (1.1), then the mean curvature of the graph vanishes, hence the graph is a minimal submanifold. Minimal Lagrangian submanifolds in $\mathbb{C}^3$, more generally in a Calabi-Yau 3-fold, are called special Lagrangian submanifolds (cf. [10]).

Equation (1.1) is elliptic at its solution u (cf. [10, Theorem 2.13]). It is a classical result that, when $p > \frac{3}{2}$, the functions in $W^{2,p}_{\text{loc}}(\Omega)$ are pointwise twice differentiable almost everywhere. A function $u \in W^{2,p}_{\text{loc}}(\Omega)$ for $p > \frac{3}{2}$ is called a strong solution of (1.1) in Ω if it satisfies (1.1) almost everywhere in Ω (cf. [6], [7] for more general cases).

The main result in this paper is a regularity result for convex strong solutions of equation (1.1), which is optimal in the Sobolev exponent. More precisely, we prove the following result.

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Theorem 1.1. *Let u be a convex strong solution of equation (1.1) in $W^{2,p}_{\mathrm{loc}}(\Omega)$ with $p > 3$. Then $u \in C^{1,1}(\Omega)$ and for any compact sub-domain Ω' of Ω*

$$\sup_{\Omega'} |D^2 u| \le C, \tag{1.2}$$

where C depends only on p, Ω', $\operatorname{dist}(\Omega', \partial\Omega)$, and $\|\Delta u\|_{L^p(\Omega')}$. In particular, for any $p > 3$, the convex strong solutions of equation (1.1) in $W^{2,p}_{\mathrm{loc}}(\Omega)$ are smooth.

Remark 1.2. There is an example to show that Theorem 1.1 is false if $p < 3$.

It is well known that convexity and $C^{1,1}$ estimate together imply smoothness of solutions to (1.1) by the regularity theorem of Evans-Krylov ([5], [8], [12]) and the standard elliptic regularity result. Without assumption on convexity, it is shown in [17] that $C^{2,\alpha}$-norm of a solution to equation (1.1) can be bounded by its $C^{1,1}$-norm.

There are interesting results on regularity for the Hessian equations (cf. [16]) and optimal regularity for the Monge-Ampère equations (cf. [3], [4]). It was proved that the $W^{2,p}$ k-convex solutions with $p > k(n-1)/2$ belong to $C^{1,1}$ for the Hessian equations $S_k(D^2u) = 1$ in dimension n case. Here $S_k(D^2u)$ is the k-th elementary symmetric function of the eigenvalues of the Hessian of u.

In [2] smoothness results for the $W^{2,p}$ convex solutions with $p > (n-1)\max(n-k, 2)$, among other things, are obtained for a Hessian quotient equation

$$\frac{S_n(D^2u)}{S_k(D^2u)} = 1,$$

which includes the special Lagrangian equation (1.1) as a special case. Theorem 1.1 improves Theorem 1.1 in [2], which is the analogous result with $p > 4$ if $n = 3$ and $k = 1$. Our main theorem will be proved by showing that if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ with some $p > 3$, then we have $u \in W^{2,\bar{p}}_{\mathrm{loc}}(\Omega)$ for some $\bar{p} > 4$ (in fact, for any $\bar{p} < \infty$), and therefore the conclusion follows from [2].

In order to prove Theorem 1.1, we use two kinds of approximations of the solution u. One is the mollification u_ε of u, which allows us to apply Reilly's formula to obtain integral estimates. Another approximation is using the second difference quotient $\Delta^h_{\xi\xi} u$ to replace D^2u, which avoids dealing directly with the fourth order weak derivatives of u.

The rest part of the paper is organized as follows: In the next section we set up notations and establish preliminary uniform estimates for various quantities involving u and its approximations. Section 3 is devoted to the main contribution of our article, that is, the local $W^{2,\bar{p}}$ estimate (for any $\bar{p} < \infty$) of the $W^{2,3+}$ solutions to equation (1.1), see Proposition 3.5. We also complete the proof of Theorem 1.1 by using the $W^{2,4+}$ estimate and [2]. In the last section, a counterexample is given to show $3+$ is the optimal exponent for the Sobolev space.

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2. Preliminary Estimates

We define a function F by

$$F(r) := \log(\det r) - \log(\operatorname{tr} r),$$

for any 3×3 positive matrix $r = (r_{ij})$. Now we can write equation (1.1) in the form

$$F(D^2 u) = 0, \quad \text{a.e. in } \Omega. \tag{2.1}$$

Furthermore, for $i, j, k, \ell = 1, 2, 3$ we denote the first derivatives of F by

$$F^{ij}(r) := \frac{\partial F}{\partial r_{ij}}(r)$$

and the second derivatives of F by

$$F^{ij,k\ell}(r) := \frac{\partial^2 F}{\partial r_{ij} \partial r_{k\ell}}(r).$$

By a standard calculation for determinants (cf. [9, Section 17.6])

$$\begin{aligned} F^{ij}(r) &= r^{ij} - \frac{\delta^{ij}}{\operatorname{tr} r}, \quad i, j = 1, 2, 3, \\ F^{ij,k\ell}(r) &= -r^{ik} r^{j\ell} + \frac{\delta^{ij}\delta^{k\ell}}{(\operatorname{tr} r)^2}, \quad i, j, k, \ell = 1, 2, 3, \end{aligned} \tag{2.2}$$

where (r^{ij}) denotes the inverse matrix of $r = (r_{ij})$. Consequently equation (2.1) is elliptic and the function F is concave with respect to any $W^{2,p}$ convex function (cf. [15]).

We always assume that $u \in W^{2,p}_{\text{loc}}(\Omega)$ with $p > 3$ is a convex strong solution of equation (1.1) or equation (2.1), and Ω' is a compact domain of Ω in $\mathbb{R}^3$. Let φ be a mollifier, that is, φ is a non-negative function in $C^\infty(\mathbb{R}^3)$ vanishing outside the unit ball $B_1(0)$ and satisfying

$$\int_{\mathbb{R}^3} \varphi(x)\, \mathrm{d}x = 1.$$

For $\varepsilon > 0$, the regularization of u is defined by the convolution

$$u_\varepsilon(x) = \varepsilon^{-3} \int_\Omega \varphi\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y = \int_{B_1(0)} \varphi(y) u(x - \varepsilon y)\,\mathrm{d}y. \tag{2.3}$$

Then u_ε is convex in Ω' and u_ε belongs to $C^\infty(\Omega')$ provided $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, and

$$u_\varepsilon \to u \quad \text{in } W^{2,p}(\Omega') \tag{2.4}$$

as $\varepsilon \to 0$, by [9, Lemmas 7.2 and 7.3]. Moreover, by the arithmetic-geometric mean inequality and equation (1.1), we have

$$\begin{gathered} \Delta u \geq 3(\det D^2 u)^{1/3} = 3(\Delta u)^{1/3}, \\ \Delta u \geq 3\sqrt{3}, \text{ a.e. in } \Omega, \ \Delta u_\varepsilon = (\Delta u)_\varepsilon \geq 3\sqrt{3} \text{ in } \Omega'. \end{gathered} \tag{2.5}$$

Now we regard u_ε as a smooth convex solution of

$$F(D^2 u_\varepsilon) = f_\varepsilon(x), \quad \text{in } \Omega', \tag{2.6}$$

where

$$f_\varepsilon(x) = \log \frac{\det(D^2 u_\varepsilon)}{\Delta u_\varepsilon}. \tag{2.7}$$

For later use, denote the algebraic co-factors of $\det D^2 u_\varepsilon$ with respect to $D_{ij} u_\varepsilon$ by $A^{ij}(D^2 u_\varepsilon)$ and set

$$a^{ij}(D^2 u_\varepsilon) = A^{ij}(D^2 u_\varepsilon) - \delta^{ij}.$$

Let λ_ε and Λ_ε be the minimal and maximal eigenvalues and $\mathcal{T}_\varepsilon$ be the trace of the matrix $(a_\varepsilon^{ij}(x)) := (a^{ij}(D^2 u_\varepsilon(x)))$ respectively. Also, let $\mathcal{T}$ be the trace of $(a^{ij}(D^2 u))$.

With the notations established above, we reveal some useful properties of the mollified solution u_ε in the following three lemmas.

Lemma 2.1. *Let $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ be a convex strong solution to equation (1.1) with $p > 3$; then*

$$\det(D^2 u_\varepsilon) \geq \Delta u_\varepsilon \quad \text{and } f_\varepsilon \geq 0 \text{ in } \Omega', \tag{2.8}$$

if $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, and

$$\mathcal{T}_\varepsilon \to \mathcal{T} \quad \text{in } L^{p/2}(\Omega'), \tag{2.9}$$

$$\frac{\det(D^2 u_\varepsilon)}{\Delta u_\varepsilon} \to 1 \quad \text{in } L^{p/2}(\Omega'), \tag{2.10}$$

$$f_\varepsilon \to 0 \quad \text{in } L^{p/(p-3)}(\Omega'), \tag{2.11}$$

as $\varepsilon \to 0$.

Proof. Using equation (2.1) and the concavity of F at convex functions, we have for $x \in \Omega'$ and a.e. $y \in \Omega$

$$0 = F(D^2u(y)) \le F(D^2u_\varepsilon(x)) + F^{ij}(D^2u_\varepsilon(x))(D_{ij}u(y) - D_{ij}u_\varepsilon(x)).$$

Therefore by [9, Lemma 7.3]

$$\begin{aligned} 0 &\le \varepsilon^{-3}\int_\Omega \varphi\left(\frac{x-y}{\varepsilon}\right)(F(D^2u_\varepsilon(x)) + F^{ij}(D^2u_\varepsilon(x))(D_{ij}u(y) - D_{ij}u_\varepsilon(x)))\,\mathrm{d}y \\ &= F(D^2u_\varepsilon(x)) + F^{ij}(D^2u_\varepsilon(x))(D_{ij}u_\varepsilon(x) - D_{ij}u_\varepsilon(x)) = F(D^2u_\varepsilon(x)), \end{aligned}$$

if $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$. This implies from (2.6) and (2.7) that (2.8) holds.

On the other hand, noting that $|D^2u| \le 9\Delta u$ and $|D^2u_\varepsilon| \le 9\Delta u_\varepsilon$ by the convexity of u and u_ε, a direct calculation yields

$$\begin{aligned} |\mathcal{T}_\varepsilon - \mathcal{T}| &\le 6(|D^2u_\varepsilon| + |D^2u|)|D^2u_\varepsilon - D^2u|, \\ \left|\frac{\det(D^2u_\varepsilon)}{\Delta u_\varepsilon} - \frac{\det(D^2u)}{\Delta u}\right| &\le 18(|D^2u_\varepsilon| + |D^2u|)|D^2u_\varepsilon - D^2u|. \end{aligned}$$

So (2.9) and (2.10) follow from (2.4), (1.1) and Hölder's inequality. Using the elementary inequality

$$\log t \le s(t-1)^{1/s}, \quad t \ge 1,\ s \ge 1,$$

and taking

$$t = \frac{\det(D^2u_\varepsilon)}{\Delta u_\varepsilon},\ s = \frac{2}{p-3}, \quad \text{if } p \le 5$$

we have from (2.8) that

$$0 \le f_\varepsilon \le \frac{2}{p-3}\left(\frac{\det(D^2u_\varepsilon)}{\Delta u_\varepsilon} - 1\right)^{(p-3)/2}.$$

Combining the above estimate with (2.10), we get (2.11) if $p \le 5$.

If $p > 5$, $W^{2,p}_{\text{loc}}(\Omega) \subset W^{2,5}_{\text{loc}}(\Omega)$. By (2.11) for $p = 5$,

$$f_\varepsilon \to 0 \quad \text{in } L^{5/(5-3)}(\Omega'),$$

as $\varepsilon \to 0$. Since $p/(p-3) < 5/(5-3)$ for $p > 5$, (2.11) also holds for $p > 5$. □

Lemma 2.2. *Let $u \in W^{2,p}_{\text{loc}}(\Omega)$ be a convex strong solution to equation (1.1) with $p > 3$. If $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, then λ_ε, Λ_ε and $\mathcal{T}_\varepsilon$ satisfy the following estimates* *in* Ω'

$$\frac{2}{\Delta u_\varepsilon} \le \lambda_\varepsilon, \tag{2.12}$$

$$\frac{2}{\sqrt{3}} \le \Lambda_\varepsilon \le (\Delta u_\varepsilon)^2, \tag{2.13}$$

$$\frac{2}{\sqrt{3}} \le \mathcal{T}_\varepsilon \le 3(\Delta u_\varepsilon)^2. \tag{2.14}$$

Proof. Let λ_1, λ_2, λ_3 be the eigenvalues of $D^2 u_\varepsilon$, here the ε-dependence of the λ_j's is omitted for simplicity. Without losing any generality we may assume $\lambda_1 \ge \lambda_2 \ge \lambda_3$, and $D^2 u_\varepsilon$ is in diagonal form at the point under consideration. Therefore, the algebraic cofactors $A^{ij}(D^2 u_\varepsilon)$ are in diagonal form as well, and

$$(a_\varepsilon^{ij}) = \operatorname{diag}(\lambda_2\lambda_3 - 1, \lambda_3\lambda_1 - 1, \lambda_1\lambda_2 - 1).$$

We rewrite equation (2.6) as

$$\lambda_1\lambda_2\lambda_3 = e^{f_\varepsilon}(\lambda_1 + \lambda_2 + \lambda_3). \tag{2.15}$$

Combining with (2.8) we observe that

$$\lambda_2\lambda_3 - 1 \ge \frac{1}{\lambda_1}(\lambda_1\lambda_2\lambda_3 - \lambda_1 e^{f_\varepsilon}) = \frac{1}{\lambda_1}(\lambda_2 + \lambda_3)e^{f_\varepsilon} > 0,$$

which implies the matrix (a_ε^{ij}) is positive definite. Moreover, it follows from (2.15) that $\lambda_1\lambda_3^2 \le 3\lambda_1 e^{f_\varepsilon}$; in turn

$$\lambda_3 \le \sqrt{3} e^{f_\varepsilon/2}. \tag{2.16}$$

Therefore we have by (2.8), (2.15)

$$\lambda_\varepsilon = \lambda_2\lambda_3 - 1 \ge \lambda_2 \cdot \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2 - 1} - 1 = \frac{\lambda_2^2 + 1}{\lambda_1\lambda_2 - 1} \ge \frac{2\lambda_2}{\lambda_1\lambda_2} = \frac{2}{\lambda_1},$$

and by (2.8), (2.15), (2.16)

$$\begin{aligned} \Lambda_\varepsilon = \lambda_1\lambda_2 - 1 &= \lambda_1 \cdot \frac{e^{f_\varepsilon}(\lambda_1 + \lambda_3)}{\lambda_1\lambda_3 - e^{f_\varepsilon}} - 1 \\ &= \frac{e^{f_\varepsilon}(\lambda_1\lambda_3 + \lambda_1^2 + 1) - \lambda_1\lambda_3}{\lambda_1\lambda_3 - e^{f_\varepsilon}} \ge \frac{2\lambda_1 e^{f_\varepsilon}}{\lambda_1\lambda_3} \ge \frac{2}{\sqrt{3}}. \end{aligned}$$

From the convexity of u_ε, $\Lambda_\varepsilon \le \mathcal{T}_\varepsilon \le 3\Lambda_\varepsilon$ and $\lambda_1 \le \Delta u_\varepsilon$. It is then clear that (2.12), (2.13) and (2.14) hold. □

Lemma 2.3. *Let $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ be a convex strong solution to equation (1.1) with $p > 3$. Then*

$$\limsup_{\varepsilon \to 0} \int_{\Omega'} (\Delta u_\varepsilon)^{p-2} \mathcal{T}_\varepsilon \, dx \le 2^{p-3} \int_{\Omega'} (\Delta u)^{p-2} \mathcal{T} \, dx. \tag{2.17}$$

Proof. Using the inequality

$$|\Delta u_\varepsilon|^{p-2} \le 2^{p-3}(|\Delta u_\varepsilon - \Delta u|^{p-2} + |\Delta u|^{p-2})$$

and Hölder's inequality, we have

$$\begin{aligned}
&\int_{\Omega'} ((\Delta u_\varepsilon)^{p-2} \mathcal{T}_\varepsilon - 2^{p-3} (\Delta u)^{p-2} \mathcal{T}) \, dx \\
&\quad \le 2^{p-3} \int_{\Omega'} (|\Delta u_\varepsilon - \Delta u|^{p-2} \mathcal{T}_\varepsilon + |\Delta u|^{p-2} \, |\mathcal{T}_\varepsilon - \mathcal{T}|) \, dx \\
&\quad \le 2^{p-3} (||\Delta u_\varepsilon - \Delta u||^{p-2}_{L^p(\Omega')} ||\mathcal{T}_\varepsilon||^2_{L^{p/2}(\Omega')} + ||\Delta u||^{p-2}_{L^p(\Omega')} \|\mathcal{T}_\varepsilon - \mathcal{T}\|_{L^{p/2}(\Omega')}),
\end{aligned}$$

if $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$. By (2.4) and (2.9) we therefore obtain (2.17). □

3. Local $W^{2,\bar{p}}$ Estimates

We derive in this section the local $W^{2,\bar{p}}$ estimates for the convex $W^{2,p}$ strong solution of equation (1.1), where $\bar{p}$ is any positive constant and $p > 3$.

For any sufficiently small $h > 0$, $\varepsilon > 0$, $x \in \Omega' \Subset \Omega$, and any given unit vector ξ in $\mathbb{R}^3$, we use the concavity of F to conclude that

$$\begin{aligned}
F(D^2 u_\varepsilon(x \pm h\xi)) \le F(D^2 u_\varepsilon(x)) \\
+ F^{ij}(D^2 u_\varepsilon(x))(D_{ij} u_\varepsilon(x \pm h\xi) - D_{ij} u_\varepsilon(x)).
\end{aligned} \tag{3.1}$$

Introduce the second order difference quotients

$$\Delta^h_{\xi\xi} u_\varepsilon(x) = \frac{u_\varepsilon(x + h\xi) - 2u_\varepsilon(x) + u_\varepsilon(x - h\xi)}{h^2}.$$

We see by (3.1) and (2.6) that $F^{ij}(D^2 u_\varepsilon(x)) D_{ij}(\Delta^h_{\xi\xi} u_\varepsilon(x)) \ge \Delta^h_{\xi\xi} f_\varepsilon(x)$, that is,

$$a^{ij}_\varepsilon(x) D_{ij} v_\varepsilon \ge g_\varepsilon(x), \quad x \in \Omega', \tag{3.2}$$

where we set

$$v_\varepsilon = \Delta^h_{\xi\xi} u_\varepsilon, \tag{3.3}$$

$$g_\varepsilon = \det(D^2 u_\varepsilon) \Delta^h_{\xi\xi} f_\varepsilon + \left(\frac{\det(D^2 u_\varepsilon)}{\Delta u_\varepsilon} - 1 \right) \Delta^h_{\xi\xi}(\Delta u_\varepsilon). \tag{3.4}$$

Here we have used the fact that, by recalling (2.2) and the definition of $a_\varepsilon^{ij}(x)$,

$$F^{ij}(D^2u_\varepsilon) = \frac{A^{ij}(D^2u_\varepsilon)}{\det(D^2u_\varepsilon)} - \frac{\delta^{ij}}{\Delta u_\varepsilon} = \frac{a_\varepsilon^{ij}(x)}{\det(D^2u_\varepsilon)} + \left(\frac{1}{\det(D^2u_\varepsilon)} - \frac{1}{\Delta u_\varepsilon}\right)\delta^{ij}.$$

We immediately obtain the following convergence on g_ε.

Lemma 3.1. *For fixed* $h < \operatorname{dist}(\Omega', \partial\Omega)$, *we have*

$$g_\varepsilon \to 0 \quad \text{in } L^1(\Omega'), \tag{3.5}$$

as $\varepsilon \to 0$.

Proof. Using Hölder inequality and Young's inequality we have

$$\begin{aligned} \|g_\varepsilon\|_{L^1(\Omega')} \le{}& \|\det(D^2u_\varepsilon)\|_{L^{p/3}(\Omega')}\,\|\Delta_{\xi\xi}^h f_\varepsilon\|_{L^{p/(p-3)}(\Omega')} \\ &+ \left\|\frac{\det(D^2u_\varepsilon)}{\Delta u_\varepsilon} - 1\right\|_{L^{p/2}(\Omega')} \|\Delta_{\xi\xi}^h(\Delta u_\varepsilon)\|_{L^p(\Omega')}\,\|1\|_{L^{p/(p-3)}(\Omega')} \\ \le{}& C(h)(\|D^2u_\varepsilon\|_{L^p(\Omega_h')} + 1) \\ &\times \left(\|f_\varepsilon\|_{L^{p/(p-3)}(\Omega_h')} + \left\|\frac{\det(D^2u_\varepsilon)}{\Delta u_\varepsilon} - 1\right\|_{L^{p/2}(\Omega_h')}\right), \end{aligned}$$

where $\Omega_h' = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega') < h\}$ and $C(h)$ is a positive constant depending on h, p and Ω'. The lemma is then proved by means of (2.10) and (2.11). □

To establish the $W_{\mathrm{loc}}^{2,\bar p}(\Omega)$ estimates, we shall also require the following result in [14].

Proposition 3.2. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and* $u \in C^3(\Omega)$. *Then*

$$\sum_{j=1}^n \frac{\partial}{\partial x_j} S^{ij}(D^2u) = 0, \quad \text{in } \Omega, \tag{3.6}$$

for each $i = 1, 2, \ldots, n$, *where* $S_k(D^2u)$ *is the Hessian operator, defined by*

$$S_k(D^2u) = \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

$k = 1, 2, \ldots, n$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* D^2u.

Next we shall show that integrations on the boundary of a ball dominate integrations on the ball.

Lemma 3.3. *For all $q > 1$ and $\rho > 0$ such that $B_{2\rho}(y) \subset \Omega' \Subset \Omega$, we have*

$$\int_{B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dx \le \rho \int_{\partial B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dS + q\rho^2 \int_{B_\rho(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, dx, \tag{3.7}$$

$$\int_{B_\rho(y)} a_\varepsilon^{ij} D_{ij} v_\varepsilon^q \, dx \le \frac{2}{\rho^2} \int_{B_{2\rho}(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dx + 2q \int_{B_{2\rho}(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, dx. \tag{3.8}$$

Proof. By (3.2) and $v_\varepsilon \ge 0$ we have in Ω'

$$\begin{aligned} a_\varepsilon^{ij} D_{ij} v_\varepsilon^q &= q v_\varepsilon^{q-2} a_\varepsilon^{ij} (v_\varepsilon D_{ij} v_\varepsilon + (q-1) D_i v_\varepsilon D_j v_\varepsilon) \\ &\ge q v_\varepsilon^{q-1} g_\varepsilon + q(q-1) v_\varepsilon^{q-2} a_\varepsilon^{ij} D_i v_\varepsilon D_j v_\varepsilon. \end{aligned} \tag{3.9}$$

Multiplying (3.9) by the cutoff function

$$\eta(x) = \rho^2 - |x-y|^2, \quad x \in B_\rho(y),$$

then integrating by parts over $B_\rho(y)$ twice, and using the formula

$$\sum_{j=1}^{3} \frac{\partial a_\varepsilon^{ij}}{\partial x_j} = 0, \quad \text{for each } i = 1, 2, 3, \tag{3.10}$$

from Proposition 3.2, we have

$$\begin{aligned} q \int_{B_\rho(y)} & \eta v_\varepsilon^{q-1} g_\varepsilon \, dx \\ &\le \int_{B_\rho(y)} \eta a_\varepsilon^{ij} D_{ij} v_\varepsilon^q \, dx = -\int_{B_\rho(y)} a_\varepsilon^{ij} D_i v_\varepsilon^q D_j \eta \, dx \\ &= \int_{B_\rho(y)} v_\varepsilon^q a_\varepsilon^{ij} D_{ij} \eta \, dx - \int_{\partial B_\rho(y)} v_\varepsilon^q a_\varepsilon^{ij} \frac{x_i - y_i}{\rho} D_j \eta \, dS \\ &= -2 \int_{B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dx + \frac{2}{\rho} \int_{\partial B_\rho(y)} v_\varepsilon^q a_\varepsilon^{ij} (x_i - y_i)(x_j - y_j) \, dS \\ &\le -2 \int_{B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dx + 2\rho \int_{\partial B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dS, \end{aligned}$$

which implies (3.7).

Next we choose $\chi \in C_0^2(B_{2\rho}(y))$ such that

$$\chi = 1 \text{ in } B_\rho(y), \text{ and } 0 \le \chi \le 1, \quad |D^2\chi| \le \frac{2}{\rho^2} \text{ in } B_{2\rho}(y).$$

From (3.9) and (3.10) we see that

$$\begin{aligned} 0 &\le \int_{B_\rho(y)} (a_\varepsilon^{ij} D_{ij} v_\varepsilon^q - q v_\varepsilon^{q-1} g_\varepsilon)\, \mathrm{d}x \\ &\le \int_{B_{2\rho}(y)} \chi (a_\varepsilon^{ij} D_{ij} v_\varepsilon^q - q v_\varepsilon^{q-1} g_\varepsilon)\, \mathrm{d}x \\ &= \int_{B_{2\rho}(y)} v_\varepsilon^q a_\varepsilon^{ij} D_{ij} \chi \, \mathrm{d}x - q \int_{B_{2\rho}(y)} \chi v_\varepsilon^{q-1} g_\varepsilon \, \mathrm{d}x \\ &\le \frac{2}{\rho^2} \int_{B_{2\rho}(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, \mathrm{d}x + q \int_{B_{2\rho}(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, \mathrm{d}x. \end{aligned}$$

So we complete the proof of the lemma. □

We are also going to use the following Sobolev inequality on the manifolds (see [13, Theorem 2.1] or [1]).

Proposition 3.4. *Let w be a nonnegative $C^1(U)$ function which vanishes outside a compact subset of U. Then*

$$\left(\int_M w^{mr/(m-r)} \right)^{(m-r)/mr} \le C(n,m,r) \left(\int_M (|\delta w|^r + (w|\mathcal{H}|)^r) \right)^{1/r}, \tag{3.11}$$

where $1 \le r < m$, M is a m-dimensional C^2 sub-manifold of $\mathbb{R}^n$, U is an open subset of $\mathbb{R}^n$ which contains M, δ is the tangential gradient operator on M, and $\mathcal{H}$ is the mean curvature vector of M.

The remaining part of this section consists of the proof of the following result.

Proposition 3.5. *Let $p > 3$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ be a convex strong solution of (1.1). Then we have $u \in W^{2,\bar{p}}_{\mathrm{loc}}(\Omega)$ for any $\bar{p} < \infty$, and for $\Omega' \Subset \Omega$ there exists a positive constant C, depending only on p, $\bar{p}$, Ω', $\mathrm{dist}(\Omega', \partial\Omega)$, and the local L^p norm of Δu in Ω, such that*

$$\|D^2 u\|_{L^{\bar{p}}(\Omega')} \le C.$$

Proof. The proof of this proposition is divided into three steps.

Step 1. we derive a suitable integral estimate on v_ε.

Let $B_{4R}(y) \subset \Omega'$. We begin with some integral estimates on the sphere $\partial B_\rho(y)$, where $\rho \in [R, 2R]$. Take

$$q = \frac{3(p-2)^2}{p}.$$

It is clear that $q > p-2 > 1$ since $p > 3$. By Young's inequality and (2.14), we have

$$\int_{\partial B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dS \le \int_{\partial B_\rho(y)} (v_\varepsilon^{3(p-2)} + \mathcal{T}_\varepsilon^{p/2}) \, dS \tag{3.12}$$
$$\le \int_{\partial B_\rho(y)} (v_\varepsilon^{3(p-2)} + 3^{(p-2)/2}(\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon) \, dS.$$

Applying Proposition 3.4 with $w = (v_\varepsilon)^{(p-2)/2}$, $M = \partial B_\rho(y)$, $m = 2$, and $r = \frac{3}{2}$, we have

$$\left(\int_{\partial B_\rho(y)} ((v_\varepsilon)^{(p-2)/2})^6 \, dS \right)^{1/6}$$
$$\le C\left(\int_{\partial B_\rho(y)} (|D(v_\varepsilon)^{(p-2)/2}|^{3/2} + ((v_\varepsilon)^{(p-2)/2}|\mathcal{H}|)^{3/2}) \, dS \right)^{2/3},$$

that is,

$$\left(\int_{\partial B_\rho(y)} v_\varepsilon^{3(p-2)} \, dS \right)^{1/4} \tag{3.13}$$
$$\le C \int_{\partial B_\rho(y)} (|D(v_\varepsilon)^{(p-2)/2}|^{3/2} + \rho^{-3/2}(v_\varepsilon)^{3(p-2)/4}) \, dS,$$

where C is a universal constant.

We combine (3.7), (3.12) and (3.13) to obtain

$$\left(\int_{B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dx \right)^{1/4} \le C\left(\int_{\partial B_\rho(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \, dS + \int_{B_\rho(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, dx \right)^{1/4}$$
$$\le C\left(\int_{\partial B_\rho(y)} (v_\varepsilon^{3(p-2)} + (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon) \, dS + \int_{B_\rho(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, dx \right)^{1/4}$$
$$\le C\left(\int_{\partial B_\rho(y)} v_\varepsilon^{3(p-2)} \, dS \right)^{1/4}$$
$$+ C\left(\int_{\partial B_\rho(y)} (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon \, dS + \int_{B_\rho(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, dx \right)^{1/4}$$
$$\le C \int_{\partial B_\rho(y)} (|D(v_\varepsilon)^{(p-2)/2}|^{3/2} + (v_\varepsilon)^{3(p-2)/4}) \, dS$$
$$+ C\left(\int_{\partial B_\rho(y)} (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon \, dS + \int_{B_\rho(y)} v_\varepsilon^{q-1} |g_\varepsilon| \, dx \right),$$

where C stands for constants depending only on p and R, and we have used (2.5) and (2.14) to conclude

$$(\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon \geq 6. \tag{3.14}$$

Integrating the above inequality over $\rho \in [R, 2R]$, we arrive at

$$\Big(\int_{B_R(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \,\mathrm{d}x\Big)^{1/4} \leq C\int_{B_{2R}(y)} (|D(v_\varepsilon)^{(p-2)/2}|^{3/2} + (v_\varepsilon)^{3(p-2)/4} + (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon + v_\varepsilon^{q-1}|g_\varepsilon|)\,\mathrm{d}x.$$

Next we estimate the first integral in above inequality. It follows from (2.12) and (3.9)

$$\begin{aligned}
\frac{2\left|D(v_\varepsilon)^{(p-2)/2}\right|^2}{\Delta u_\varepsilon} &\leq a_\varepsilon^{ij} D_i(v_\varepsilon)^{(p-2)/2} D_j(v_\varepsilon)^{(p-2)/2} \\
&= \frac{(p-2)^2}{4} v_\varepsilon^{p-4} a_\varepsilon^{ij} D_i v_\varepsilon D_j v_\varepsilon \\
&\leq \frac{p-2}{4(p-3)}(a_\varepsilon^{ij} D_{ij} v_\varepsilon^{p-2} - q v_\varepsilon^{p-3} g_\varepsilon).
\end{aligned}$$

By Hölder's inequality and (3.8), we have

$$\begin{aligned}
&\int_{B_{2R}(y)} |D(v_\varepsilon)^{(p-2)/2}|^{3/2}\,\mathrm{d}x \\
&\quad\leq \Big(\int_{B_{2R}(y)} \frac{|D(v_\varepsilon)^{(p-2)/2}|^2}{\Delta u_\varepsilon}\,\mathrm{d}x\Big)^{3/4}\Big(\int_{B_{2R}(y)} (\Delta u_\varepsilon)^3\,\mathrm{d}x\Big)^{1/4} \\
&\quad\leq C\Big(\int_{B_{2R}(y)} (a_\varepsilon^{ij} D_{ij} v_\varepsilon^{p-2} + v_\varepsilon^{p-3}|g_\varepsilon|)\,\mathrm{d}x\Big)^{3/4} \\
&\quad\leq C\Big(\int_{B_{4R}(y)} (v_\varepsilon^{p-2}\mathcal{T}_\varepsilon + v_\varepsilon^{p-3}|g_\varepsilon|)\,\mathrm{d}x\Big)^{3/4},
\end{aligned}$$

where C depends on p, R and $\|\Delta u\|_{L^3(B_{2R}(y))}$, and is bounded when p is bounded away from 3. Finally we obtain that

$$
\begin{aligned}
(3.15)\qquad &\left(\int_{B_R(y)} v_\varepsilon^q \mathcal{T}_\varepsilon \,\mathrm{d}x\right)^{1/4}\\
&\le C\left(\int_{B_{4R}(y)} (v_\varepsilon^{p-2}\mathcal{T}_\varepsilon + v_\varepsilon^{p-3}|g_\varepsilon|)\,\mathrm{d}x\right)^{3/4}\\
&\quad + C\int_{B_{2R}(y)} ((v_\varepsilon)^{3(p-2)/4} + (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon + v_\varepsilon^{q-1}|g_\varepsilon|)\,\mathrm{d}x\\
&\le C\int_{B_{4R}(y)} (v_\varepsilon^{p-2}\mathcal{T}_\varepsilon + v_\varepsilon^{p-3}|g_\varepsilon|)\,\mathrm{d}x\\
&\quad + C\int_{B_{2R}(y)} (v_\varepsilon^{p-2} + (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon + v_\varepsilon^{q-1}|g_\varepsilon|)\,\mathrm{d}x\\
&\le C\int_{B_{4R}(y)} (v_\varepsilon^{p-2}\mathcal{T}_\varepsilon + (\Delta u_\varepsilon)^{p-2}\mathcal{T}_\varepsilon + (v_\varepsilon^{q-1}+1)|g_\varepsilon|)\,\mathrm{d}x,
\end{aligned}
$$

here we have used Young's inequality and (3.14), and C denote universal constants depending only on p, R and $\|\Delta u\|_{L^3(B_{2R}(y))}$.

Step 2. By letting $\varepsilon \to 0$ and then $h \to 0$, we establish an iteration formula on Δu in the integral norm weighted by $\mathcal{T}$.

By the definition (3.3) of v_ε and Sobolev imbedding theorem, we have

$$v_\varepsilon \to \Delta^h_{\xi\xi} u, \quad \text{uniformly in } B_{4R}(y).$$

Letting $\varepsilon \to 0$ for fixed $h < \operatorname{dist}(B_{4R}(y), \partial\Omega)$, using (2.9), (2.17), (3.5),and (3.15) we obtain

$$(3.16)\qquad \left(\int_{B_R(y)} (\Delta^h_{\xi\xi} u)^q \mathcal{T}\,\mathrm{d}x\right)^{1/4} \le C\int_{B_{2R}(y)} ((\Delta^h_{\xi\xi} u)^{p-2} + (\Delta u)^{p-2})\mathcal{T}\,\mathrm{d}x.$$

Now we choose ξ to be the coordinate directions e_ℓ, $\ell = 1, 2, 3$. By [9, Lemma 7.23] we have

$$\|\Delta^h_{e_\ell e_\ell} u\|_{L^p(B_{4R}(y))} \le \|D_{\ell\ell} u\|_{L^p(B_{4R+h})}.$$

By the weak compactness of bounded sets in $L^p(B_{4R}(y))$, there exists a sequence $\{h_j\}$ tending to zero, such that (also cf. the proof of [9, Lemma 7.24])

$$(3.17)\qquad \Delta^{h_j}_{e_\ell e_\ell} u \rightharpoonup D_{\ell\ell} u, \quad \text{weakly in } L^p(B_{4R}(y)).$$

Using the weak lower semi-continuity in $L^p(B_{4R}(y))$, we get

$$\begin{aligned}\int_{B_{4R}(y)} |D_{\ell\ell}u|^p \,\mathrm{d}x &\le \liminf_{j\to\infty}\int_{B_{4R}(y)} |\Delta^{h_j}_{e_\ell e_\ell}u|^p \,\mathrm{d}x\\ &\le \liminf_{j\to\infty}\int_{B_{4R+h_j}(y)} |D_{\ell\ell}u|^p \,\mathrm{d}x = \int_{B_{4R}(y)} |D_{\ell\ell}u|^p \,\mathrm{d}x.\end{aligned}$$

Consequently,

$$\lim_{j\to\infty} \|\Delta^{h_j}_{e_\ell e_\ell}u\|_{L^p(B_{4R}(y))} = \|D_{\ell\ell}u\|_{L^p(B_{4R}(y))}. \tag{3.18}$$

Therefore, applying Radon-Riesz Theorem [11], we have

$$\Delta^{h_j}_{e_\ell e_\ell}u \to D_{\ell\ell}u, \quad \text{in } L^p(B_{4R}(y)). \tag{3.19}$$

It follows from Fatou Lemma, (3.16) and (3.19) that

$$\begin{aligned}\left(\int_{B_R(y)} |D_\ell u|^q \mathcal{T}\,\mathrm{d}x\right)^{1/4} &\le \liminf_{j\to\infty}\left(\int_{B_{4R}(y)} |\Delta^{h_j}_{e_\ell e_\ell}u|^q \mathcal{T}\,\mathrm{d}x\right)^{1/4}\\ &\le C\liminf_{j\to\infty}\int_{B_{4R}(y)} ((\Delta^{h_j}_{e_\ell e_\ell}u)^{p-2} + (\Delta u)^{p-2})\mathcal{T}\,\mathrm{d}x\\ &\le C\int_{B_{4R}(y)} (\Delta u)^{p-2}\mathcal{T}\,\mathrm{d}x.\end{aligned}$$

Thus we arrive at

$$\left(\int_{B_R(y)} (\Delta u)^q \mathcal{T}\,\mathrm{d}x\right)^{1/4} \le C\int_{B_{4R}(y)} (\Delta u)^{p-2}\mathcal{T}\,\mathrm{d}x, \tag{3.20}$$

where C is a constant depending only on p, R and $\|\Delta u\|_{L^3(B_{2R}(y))}$.

Step 3. We use the iteration formula (3.20) to complete the proof.

Noting (3.20) holds for any $p > 3$ and $q > p - 2$, it can be iterated finitely many times to yield the desired estimates. In fact, for any preassigned number $\bar{p} < \infty$, choose N such that $\bar{p} \ge \kappa^N(p-2)$, where $\kappa = q/(p-2) > 1$. Let $R < \operatorname{dist}(\Omega', \partial\Omega)/4^N$ for $\Omega' \Subset \Omega$. By using (2.5) and iterating (3.20) N times, we obtain

$$\begin{aligned}\int_{B_R(y)} (\Delta u)^{\bar{p}} \mathcal{T}\, dx &\le \int_{B_R(y)} (\Delta u)^{\kappa^N (p-2)} \mathcal{T}\, dx \\ &\le \left(C \int_{B_{4R}(y)} (\Delta u)^{\kappa^{N-1}(p-2)} \mathcal{T}\, dx \right)^4 \\ &\le C^4 \left(C \int_{B_{4^2 R}} (\Delta u)^{\kappa^{N-2}(p-2)} \mathcal{T}\, dx \right)^{4^2} \\ &\le \cdots \\ &\le C^{4+4^2+\cdots+4^N} \left(\int_{B_{4^N R}} (\Delta u)^{p-2} \mathcal{T}\, dx \right)^{4^N}.\end{aligned}$$

Therefore, we obtain by (2.14)

$$\begin{aligned}\int_{B_R(y)} (\Delta u)^{\bar{p}}\, dx &\le \frac{\sqrt{3}}{2} \int_{B_R(y)} (\Delta u)^{\bar{p}} \mathcal{T}\, dx \\ &\le C \left(\int_{B_{4^N R}} (\Delta u)^{p-2} \mathcal{T}\, dx \right)^{4^N} \\ &\le C \left(\int_{\Omega'} (\Delta u)^{p}\, dx \right)^{4^N},\end{aligned}$$

where C depends only on p, $\bar{p}$ and $\operatorname{dist}(\Omega', \partial\Omega)$. This completes the proof of Proposition 3.5 by applying the finite covering theorem. ◻

We would like to point out that the constant C above, hence in Proposition 3.5, depends on $\bar{p}$. Therefore L^∞ estimates for Δu do not follow directly by letting $\bar{p} \to \infty$.

As discussed in Introduction, Theorem 1.1 now follows from the $L^{\bar{p}}$ ($\bar{p} > 4$) estimates for $D^2 u$ in Proposition 3.5 and Theorem 1.1 in [2].

4. A Counterexample for $p < 3$

In this section, we shall give an example to show that there is a convex strong solution of the equation (1.1), which is in $W^{2,p}_{\text{loc}}(\mathbb{R}^3)$ with $p < 3$, but not in $W^{2,3}_{\text{loc}}(\mathbb{R}^3)$. In fact, the solution u we are going to construct is convex and satisfies

$$u \in C^\infty(\mathbb{R}^3 \setminus \{0\}) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^3) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}^3),$$

and has global quadratic growth

$$|u(x)| \le C(1 + |x|^2), \quad x \in \mathbb{R}^3.$$

In order to get this solution, we write equation (1.1) in the spherical coordinates system of $\mathbb{R}^3$

$$u''\left(\frac{u'}{\rho}\right)^2 = u'' + \frac{2}{\rho}u',$$

where $\rho = |x|$. Therefore $((u')^2 - \rho^2)\,\mathrm{d}u' - 2\rho u'\,\mathrm{d}\rho = 0$, which can be written as

$$\mathrm{d}((u')^3 - 3\rho^2 u') = 0,$$

and integration leads to

$$(u')^3 - 3\rho^2 u' = 2a^3, \tag{4.1}$$

where a is an arbitrary positive constant. Solving the cubic polynomial equation (4.1) for u', we have

$$u'(\rho) = (a^3 - \sqrt{a^6 - \rho^6})^{1/3} + (a^3 + \sqrt{a^6 - \rho^6})^{1/3}, \tag{4.2}$$

which leads to

$$u''(\rho) = \frac{\rho^5}{\sqrt{a^6 - \rho^6}}((a^3 - \sqrt{a^6 - \rho^6})^{-2/3} - (a^3 + \sqrt{a^6 - \rho^6})^{-2/3}) \tag{4.3}$$

for $\rho \neq a$, and

$$u''(a) = \frac{4}{3}.$$

Clearly, if $\rho \leq a$, $u'(\rho)$ and $u''(\rho)$ are real. For $\rho > a$, we can rewrite u' and u'' as follows

$$\begin{aligned} u'(\rho) &= \rho((a/\rho)^3 - \sqrt{(a/\rho)^6 - 1}\,)^{1/3} + \rho((a/\rho)^3 + \sqrt{(a/\rho)^6 - 1}\,)^{1/3} \\ &= \rho((\cos\theta - i\sin\theta)^{1/3} + (\cos\theta + i\sin\theta)^{1/3}) \\ &= 2\rho\cos\frac{\theta}{3} \end{aligned}$$

and

$$\begin{aligned} u''(\rho) &= \frac{((a/\rho)^3 - \sqrt{(a/\rho)^6 - 1})^{-2/3} - ((a/\rho)^3 + \sqrt{(a/\rho)^6 - 1})^{-2/3}}{\sqrt{(a/\rho)^6 - 1}} \\ &= \frac{(\cos\theta - i\sin\theta)^{-2/3} - (\cos\theta + i\sin\theta)^{-2/3}}{i\sin\theta} \\ &= \frac{2\sin(2\theta/3)}{\sin\theta}, \end{aligned}$$

where $\theta = \theta(\rho) = \arccos(a/\rho)^3$. Therefore the solution to (4.1), given by (4.2), is well defined and real for all $\rho \in [0,\infty)$, and $u''(\rho)$ in (4.3) is also real for all $\rho \in [0,\infty)$.

A direct calculation yields

$$\lim_{\rho\to 0} u'(\rho) = 2^{1/3}a, \tag{4.4}$$

$$\lim_{\rho\to 0} \frac{u''(\rho)}{\rho} = \frac{2^{2/3}}{a}, \tag{4.5}$$

$$\lim_{\rho\to\infty} \frac{u'(\rho)}{\rho} = \lim_{\rho\to\infty} u''(\rho) = \sqrt{3},$$

$$u'' \in C^0[0,\infty) \cap L^\infty[0,\infty),$$

and

$$u'' > 0, \ |u(\rho)| \le C(1+\rho^2) \quad \text{on } (0,\infty).$$

It is clear that u is convex in $\mathbb{R}^3$.

Now we claim that u has second order weak derivatives. Let $R > r > 0$ and $\varphi \in C_0^\infty(B_R(y))$. By Stokes theorem, we have that for $i = 1, 2, 3$

$$\begin{aligned}\int_{B_R(y)\setminus B_r(y)} uD_i\varphi \,\mathrm{d}x &= \int_{\partial(B_R(y)\setminus B_r(y))} u\varphi n_i \,\mathrm{d}S - \int_{B_R(y)\setminus B_r(y)} \varphi D_i u \,\mathrm{d}x \\ &= -\frac{1}{r}\int_{\partial B_r(y)} u\varphi x_i \,\mathrm{d}S - \int_{B_R(y)\setminus B_r(y)} \varphi u' \frac{x_i}{\rho}\,\mathrm{d}x,\end{aligned}$$

where n is the unit outer normal to $\partial(B_R(y)\setminus B_r(y))$. Letting $r \to 0$ and using (4.4), we have

$$\int_{B_R(y)} uD_i\varphi\,\mathrm{d}x = \int_{B_R(y)} \varphi u' \frac{x_i}{\rho}\,\mathrm{d}x,$$

that is, u is weakly differentiable, and

$$D_i u = u'\frac{x_i}{\rho} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3).$$

Similarly, applying Stokes theorem twice leads to

$$\begin{aligned}&\int_{B_R(y)\setminus B_r(y)} uD_{ij}^2\varphi\,\mathrm{d}x \\ &\quad= \int_{\partial(B_R(y)\setminus B_r(y))} uD_j\varphi n_i\,\mathrm{d}S - \int_{B_R(y)\setminus B_r(y)} D_j\varphi D_i u\,\mathrm{d}x \\ &\quad= -\frac{1}{r}\int_{\partial B_r(y)} uD_j\varphi x_i\,\mathrm{d}S + \frac{1}{r^2}\int_{\partial B_r(y)} \varphi u' x_i x_j\,\mathrm{d}S \\ &\qquad\qquad + \int_{B_R\setminus B_r(y)} \varphi\left(u''\frac{x_i x_j}{|x|^2} + u'\left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}\right)\right)\mathrm{d}x.\end{aligned}$$

Now by letting $r \to 0$ and using (4.5) we see that u is twice weakly differentiable, and for any $p < 3$

$$D_{ij}u = u'' \frac{x_i x_j}{\rho^2} + \frac{u'}{\rho}\left(\delta^{ij} - \frac{x_i x_j}{\rho^2}\right) \in L^p_{\text{loc}}(\mathbb{R}^3) \setminus L^3_{\text{loc}}(\mathbb{R}^3)$$

with $i, j = 1, 2, 3$.

In conclusion, we have constructed a convex strong solution u to the special Lagrangian equation (1.1) which belongs to $W^{2,p}_{\text{loc}}(\mathbb{R}^3)$ for any $p < 3$ but it is not in $W^{2,3}_{\text{loc}}(\mathbb{R}^3)$, in fact the radially symmetric solution u is not smooth at the origin.

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