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Optimal boundary gradient estimates for Lamé systems with partially infinite coefficients

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A B S T R A C T

In this paper, we derive the pointwise upper bounds and lower bounds on the gradients of solutions to the Lamé systems with partially infinite coefficients as the surface of discontinuity of the coefficients of the system is located very close to the boundary. When the distance tends to zero, the optimal blow-up rates of the gradients are established for inclusions with arbitrary shapes and in all dimensions.

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1. Introduction and main results

It is a common phenomenon that high concentration of extreme mechanical loads occurs in high-contrast fiber-reinforced composites in the zones that include the narrow regions between two adjacent inclusions and the thin gaps between the inclusions and the exterior boundary of the background medium. Extreme loads are always amplified by such composite microstructure, which will cause failure or fracture initiation. Stimulated by the well-known work of Babuška et al. [12], where computational analysis of damage and fracture in fiber composite systems is investigated, we consider the Lamé system in linear elasticity with partially infinite coefficients to characterize the high-contrast composites. This paper is a continuation of [15,16], where the upper bound of the gradient estimate for two adjacent inclusions is established, which can be regarded as interior estimates for this problem.

Due to the interaction from the boundary data, solutions of these systems become more irregular near the boundary. In this paper, we mainly investigate the boundary gradient estimates for the Lamé system with partially infinite coefficients when the inclusion is spaced very close to the matrix exterior boundary. The novelty of these estimates is that they give not only the pointwise upper bounds but also lower bounds of the gradient, which shows that the blow-up rate of the gradient with respect to the distance between the inclusion and the matrix exterior boundary that we obtain is optimal. The role of the boundary data is embodied in these estimates. Especially, an explicit factor that determines whether the blow-up occurs or not is singled out in the lower bound estimates. We would like to emphasize that the gradient estimates obtained in this paper hold for inclusions with arbitrary convex shapes and in all dimensions.

Let $D \subset \mathbb{R}^d (d \geq 2)$ be a bounded open set with $C^{2,\gamma}$ boundary, and D_1 be a strictly convex open set in D with $C^{2,\gamma}$ boundary, $0 < \gamma < 1$, and spaced very close to the boundary ∂D. More precisely,

$$\begin{aligned} &\overline{D}_1 \subset D, \quad \text{the principle curvatures of } \ \partial D, \partial D_1 \geq \kappa_0 > 0, \\ &\varepsilon := dist(D_1, \partial D) > 0, \end{aligned} \tag{1.1}$$

where κ_0 is constant independent of ε. We also assume that the $C^{2,\gamma}$ norms of ∂D_1 are bounded by some constant independent of ε. This implies that D_1 contains a ball of radius r_0^* for some constant $r_0^* > 0$ independent of ε. See Fig. 1.

Denote

$$\Omega := D \setminus \overline{D}_1.$$

We assume that Ω and D_1 are occupied, respectively, by two different isotropic and homogeneous materials with different Lamé constants (λ, μ) and (λ_1, μ_1). Then the elasticity tensors for the background and the inclusion can be written, respectively, as $\mathbb{C}^0$ and $\mathbb{C}^1$, with

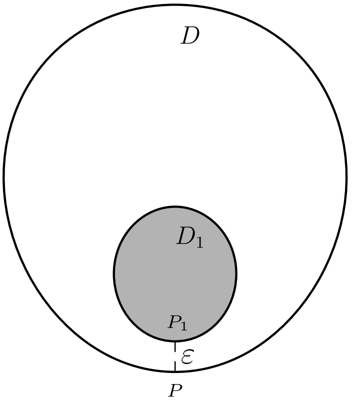

Fig. 1. One inclusion close to the boundary.

$$C^0_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

and

$$C^1_{ijkl} = \lambda_1\delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $i, j, k, l = 1, 2, \cdots, d$ and δ_{ij} is the Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$.

Let $u = (u^1, u^2, \cdots, u^d)^T : D \to \mathbb{R}^d$ denote the displacement field. For a given vector valued function $\varphi = (\varphi^1, \varphi^2, \cdots, \varphi^d)^T$, we consider the following Dirichlet problem for the Lamé system:

$$\begin{cases} \nabla \cdot \big((\chi_\Omega \mathbb{C}^0 + \chi_{D_1}\mathbb{C}^1)e(u)\big) = 0, & \text{in } D, \\ u = \varphi, & \text{on } \partial D, \end{cases} \tag{1.2}$$

where χ_Ω is the characteristic function of $\Omega \subset \mathbb{R}^d$,

$$e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

is the strain tensor.

Assume that the standard ellipticity condition holds for (1.2), that is,

$$\mu > 0, \quad d\lambda + 2\mu > 0, \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0.$$

For $\varphi \in H^1(D; \mathbb{R}^d)$, it is well known that there exists a unique solution $u \in H^1(D; \mathbb{R}^d)$ to the Dirichlet problem (1.2), which is also the minimizer of the energy functional

$$J_1[u] := \frac{1}{2}\int_\Omega \big((\chi_\Omega \mathbb{C}^0 + \chi_{D_1}\mathbb{C}^1)e(u), e(u)\big)\, dx$$

on

$$H^1_\varphi(D;\mathbb{R}^d) := \left\{u \in H^1(D;\mathbb{R}^d) \;\middle|\; u-\varphi \in H^1_0(D;\mathbb{R}^d)\right\}.$$

We introduce the linear space of rigid displacement in $\mathbb{R}^d$:

$$\Psi := \{\psi \in C^1(\mathbb{R}^d;\mathbb{R}^d) \mid \nabla\psi + (\nabla\psi)^T = 0\}.$$

With $e_1, \cdots, e_d$ denoting the standard basis of $\mathbb{R}^d$,

$$\left\{\; e_i,\; x_j e_k - x_k e_j \;\middle|\; 1 \le i \le d,\; 1 \le j < k \le d \;\right\}$$

is a basis of Ψ. Denote this basis of Ψ as $\left\{\psi_\alpha \;\middle|\; \alpha = 1,2,\cdots,\frac{d(d+1)}{2}\right\}$.

For fixed λ and μ satisfying $\mu > 0$ and $d\lambda + 2\mu > 0$, denote u_{λ_1,μ_1} as the solution of (1.2). Then similarly as in the Appendix of [15], we also have

$$u_{\lambda_1,\mu_1} \to u \quad \text{in } H^1(D;\mathbb{R}^d), \quad \text{as } \min\{\mu_1, d\lambda_1 + 2\mu_1\} \to \infty,$$

where u is a $H^1(D;\mathbb{R}^d)$ solution of

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u := \nabla\cdot(\mathbb{C}^0 e(u)) = 0, & \text{in } \Omega, \\ u|_+ = u|_-, & \text{on } \partial D_1, \\ e(u) = 0, & \text{in } D_1, \\ \int_{\partial D_1} \frac{\partial u}{\partial \nu_0}\Big|_+ \cdot \psi_\alpha = 0, & \alpha = 1,2,\cdots,\frac{d(d+1)}{2}, \\ u = \varphi, & \text{on } \partial D, \end{cases} \tag{1.3}$$

where

$$\frac{\partial u}{\partial \nu_0}\Big|_+ := (\mathbb{C}^0 e(u))\vec{n} = \lambda(\nabla\cdot u)\vec{n} + \mu(\nabla u + (\nabla u)^T)\vec{n},$$

and $\vec{n}$ is the unit outer normal of D_1. Here and throughout this paper the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of weak solutions to (1.3) are proved in the Appendix of [15], where multiple inclusions case is studied. In particular, the H^1 weak solution to (1.3) is in $C^1(\overline{\Omega};\mathbb{R}^d) \cap C^1(\overline{D}_1;\mathbb{R}^d)$. The solution is also the unique function which has the least energy in appropriate functional spaces, characterized by

$$I_\infty[u] = \min_{v\in\mathcal{A}} I_\infty[v], \qquad I_\infty[v] := \frac{1}{2}\int_\Omega (\mathbb{C}^0 e(v), e(v))dx,$$

where

$$\mathcal{A} := \left\{v \in H^1_\varphi(D;\mathbb{R}^d) \;\middle|\; e(v) = 0 \;\text{ in }\; D_1\right\}. \tag{1.4}$$

It is well known that for any open set O and $u, v \in C^2(O)$,

$$\int_O (\mathbb{C}^0 e(u), e(v))dx = -\int_O (\mathcal{L}_{\lambda,\mu} u) \cdot v + \int_{\partial O} \frac{\partial u}{\partial \nu_0}\Big|_+ \cdot v. \tag{1.5}$$

A calculation gives

$$(\mathcal{L}_{\lambda,\mu} u)_k = \mu \Delta u_k + (\lambda + \mu)\partial_{x_k}(\nabla \cdot u), \quad k = 1, \cdots, d.$$

We assume that for some $\delta_0 > 0$,

$$\delta_0 \leq \mu, d\lambda + 2\mu \leq \frac{1}{\delta_0}. \tag{1.6}$$

It is clear that there exist two points $P_1 \in \partial D_1$ and $P \in \partial D$, such that

$$\mathrm{dist}(P, P_1) = \mathrm{dist}(D, \partial D) = \varepsilon.$$

We use $\overline{P_1 P}$ to denote the line segment connecting P_1 and P. Denote

$$\rho_d(\varepsilon) = \begin{cases} \sqrt{\varepsilon}, & \text{if} \quad d = 2, \\ \frac{1}{|\log \varepsilon|}, & \text{if} \quad d = 3, \\ 1, & \text{if} \quad d \geq 4. \end{cases}$$

The first of our results concerns an upper bound of the gradient of solutions to (1.3). In brief, this result asserts that the blow up rate of $|\nabla u|$ is, respectively, $\varepsilon^{-1/2}$ in dimension $d = 2$, $(\varepsilon|\log \varepsilon|)^{-1}$ in dimension $d = 3$, and ε^{-1} in dimension $d \geq 4$, which is exactly the same as the perfect conductivity problem, see e.g. [13].

Theorem 1.1. *(Upper bound). Assume that $\Omega, D \subset \mathbb{R}^d$, ε are defined in (1.1), $\varphi \in C^2(\partial D; \mathbb{R}^d)$. Let $u \in H^1(D; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$ be a solution to (1.3). Then for $0 < \varepsilon < 1/2$, we have*

$$|\nabla u(x)| \leq \frac{C\rho_d(\varepsilon)}{\varepsilon} \|\varphi\|_{C^2(\partial D; \mathbb{R}^d)}, \quad x \in \Omega, \tag{1.7}$$

and

$$|\nabla u(x)| \leq C\|\varphi\|_{C^2(\partial D; \mathbb{R}^d)}, \quad x \in D_1, \tag{1.8}$$

where C depends only on κ_0, δ_0, d, the $C^{2,\gamma}$ norm of ∂D_1 and ∂D, but not on ε.

Remark 1.2. Actually, for $d \geq 2$, we have the following pointwise upper bound of $|\nabla u|$ in Ω:

$$|\nabla u(x)| \leq C\left[\frac{\rho_d(\varepsilon)}{\varepsilon+\mathrm{dist}^2(x,\overline{P_1P})}+\left(\frac{\mathrm{dist}(x,\overline{P_1P})}{\varepsilon+\mathrm{dist}^2(x,\overline{P_1P})}+1\right)\right]\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)}. \quad (1.9)$$

This shows that the right hand side archives its maximum at $\overline{P_1P}$, with value $\frac{C\rho_d(\varepsilon)}{\varepsilon}\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)}$ for ε sufficiently small.

In order to show that the blow-up rate of the gradients obtained in Theorem 1.1 is optimal, we need to investigate its lower bound. Denote $D_1^* := \{\, x\in\mathbb{R}^d \mid x+P_1\in D_1 \,\}$. Set $\Omega^* := D\setminus\overline{D_1^*}$. Let u_0^* be the solution of the boundary value problem:

$$\begin{cases}\mathcal{L}_{\lambda,\mu}u_0^*=0, & \text{in } \Omega^*,\\ u_0^*=0, & \text{on } \partial D_1^*,\\ u_0^*=\varphi(x)-\varphi(P), & \text{on } \partial D.\end{cases} \quad (1.10)$$

Define

$$b_\alpha^* := \int_{\partial D_1^*}\frac{\partial u_0^*}{\partial\nu_0}\Big|_+\cdot\psi_\alpha, \quad \alpha=1,2,\cdots,\frac{d(d+1)}{2},$$

which is a functional of φ, playing an important role in the following establishment of lower bounds of $|\nabla u|$ on the segment $\overline{P_1P}$.

Theorem 1.3. *(Lower bound). Under the assumption as in Theorem 1.1, let $u\in H^1(D;\mathbb{R}^d)\cap C^1(\overline{\Omega};\mathbb{R}^d)$ be a solution to (1.3). Then*

- *(i) for $d=2$, if there exists some integer $1\leq k_0\leq d$ such that $b_{k_0}^*\neq 0$ and $\nabla_{x'}\varphi^{k_0}(P)=0$;*
- *(ii) for $d=3$, if there exists some integer $1\leq k_0\leq d$ such that $b_{k_0}^*\neq 0$;*
- *(iii) for $d\geq 4$, if there exists some integer $1\leq k_0\leq d$ such that $b_{k_0}^*\neq 0$ and $b_\alpha^*=0$ for all $\alpha\neq k_0$,*

then for sufficiently small $0<\varepsilon<1/2$,

$$\big|\nabla u(x)\big| \geq \frac{\rho_d(\varepsilon)}{C\varepsilon}, \quad x\in\overline{P_1P},$$

where C depends only on κ_0,δ_0,d, the $C^{2,\gamma}$ norm of ∂D_1 and the C^2 norm of ∂D, but not on ε.

Remark 1.4. In Theorem 1.3 we do not try to find the most general assumptions to guarantee blow-up occur, but instead give simple conditions (i)–(iii), which show, however, the essential role of the boundary data in this problem. Since u_0 is uniquely determined by (1.10) with given data $\varphi(x)-\varphi(P)$, Theorem 1.3 shows that whether $|\nabla u|$ blows up or not totally depends only on the boundary data $\varphi(x) - \varphi(P)$. Furthermore, if the blow-up occurs, then from Theorem 1.1 and 1.3, we know that it may occur only on the segment $\overline{P_1 P}$.

Remark 1.5. Theorem 1.1 and 1.3 give not only the upper bound but also a lower bound of the blow-up rate of the strain tensor in all dimensions, which shows the optimality of our estimates. Especially for the lower bound, new difficulties need to be overcome and a number of refined estimates are used in our proof. More important, a blow-up factor, totally depending on the given boundary data, is captured.

Remark 1.6. The strict convexity assumption on ∂D and ∂D_1 in Theorem 1.1 and 1.3 can be extended to a weaker relative strict convexity assumption, see (2.6)–(2.8) below.

The organization of this paper is as follows. In Section 2 we first decompose the solution u of (1.3) as a linear combination of u_α, $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$, defined by (2.3) and (2.4) below, and then deduce the proof of Theorem 1.1 to two aspects: the estimates of $|\nabla u_\alpha|$ and those of the coefficients C^α and $C^\alpha - \varphi^\alpha(0)$. In Section 3 we establish an upper bound of the gradient of solutions to a boundary problem of Lamé system on Ω with general Dirichlet boundary data in Theorem 2.1, of independent interest, and then obtain the estimates of $|\nabla u_\alpha|$ as a consequence of Theorem 2.1. In Section 4 we present the estimates of the coefficients C^α and $C^\alpha - \varphi^\alpha(0)$. Theorem 1.3 on the lower bound of ∇u on the segment $\overline{P_1 P}$ is proved by studying the functional b_α^* of boundary data φ in Section 5. In the rest of the introduction we review some earlier results on interior gradient estimates for high contrast composites.

As mentioned before, Babuška, Andersson, Smith and Levin [12] computationally analyzed the damage and fracture in composite materials and observed numerically that the size of the strain tensor remains bounded when the distance ε, between two inclusions, tends to zero. This was proved by Li and Nirenberg in [31]. Indeed such ε-independent gradient estimates was established there for solutions of divergence form second order elliptic systems, including linear systems of elasticity, with piecewise Hölder continuous coefficients in all dimensions. See Bonnetier and Vogelius [19] and Li and Vogelius [32] for responding results on divergence form elliptic equations.

The estimates in [31] and [32] depend on the ellipticity of the coefficients. If ellipticity constants are allowed to deteriorate, the situation is very different. Consider the simplified scalar model, also called as conductivity problem,

$$\begin{cases} \nabla \cdot \Big(a_k(x) \nabla u_k \Big) = 0, & \text{in } \Omega, \\ u_k = \varphi, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^d$, $d \geq 2$, containing two ε-apart convex inclusions D_1 and D_2, $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0,\infty), & \text{in } D_1 \cup D_2, \\ 1, & \text{in } \Omega \setminus \overline{D_1 \cup D_2}. \end{cases}$$

When $k = \infty$, the L^∞-norm of $|\nabla u_\infty|$ for the solutions u_∞ of the following perfect conductivity problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \setminus \overline{D_1 \cup D_2}, \\ u|_+ = u|_-, & \text{on } \partial D_1 \cup \partial D_2, \\ \nabla u = 0, & \text{in } \ D_1 \cup D_2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \vec{n}}\Big|_+ = 0, & i = 1, 2, \\ u = \varphi, & \text{on } \partial\Omega \end{cases} \tag{1.11}$$

generally becomes unbounded as ε tends to 0. There have been much more important progress on the interior gradient estimate of the solution of (1.11), in contrast to the elasticity vector case. The blow up rate of $|\nabla u_\infty|$ is respectively $\varepsilon^{-1/2}$ in dimension $d = 2$, $(\varepsilon|\ln\varepsilon|)^{-1}$ in dimension $d = 3$, and ε^{-1} in dimension $d \geq 4$. See Bao, Li and Yin [13], as well as Budiansky and Carrier [20], Markenscoff [34], Ammari, Kang and Lim [7], Ammari, Kang, Lee, Lee and Lim [9], Yun [37,38] in $\mathbb{R}^2$, and Lim and Yun [33] in $\mathbb{R}^3$. Further, more detailed, characterizations of the singular behavior of ∇u_∞ have been obtained by Ammari, Ciraolo, Kang, Lee and Yun [4], Ammari, Kang, Lee, Lim and Zribi [11], Bonnetier and Triki [17,18], Gorb and Novikov [25] and Kang, Lim and Yun [26,27]. For more related works, see [3,5,6,8,10,14,17,21–23,28–30,33] and the references therein.

2. Outline of the proof of Theorem 1.1 (Upper bound)

We now describe our methods of proof. By a translation and rotation of the coordinates if necessary, we may assume without loss of generality that

$$P_1 = (0, \varepsilon) \in \partial D_1, \quad P = (0, 0) \in \partial D.$$

In order to prove Theorem 1.1, it suffices to consider the following problem, by replacing u by $u - \varphi(0)$,

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = 0, & \text{in } \Omega, \\ u|_+ = u|_-, & \text{on } \partial D_1, \\ e(u) = 0, & \text{in } \ D_1, \\ \int_{\partial D_1} \frac{\partial u}{\partial \nu_0}\Big|_+ \cdot \psi_\alpha = 0, & \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \\ u = \varphi(x) - \varphi(0), & \text{on } \partial D. \end{cases} \tag{2.1}$$

By the third line of (2.1) and the definition of Ψ, u is a linear combination of $\{\psi_\alpha\}$ in D_1. Since it is clear that $\mathcal{L}_{\lambda,\mu}\xi = 0$ in Ω and $\xi = 0$ on $\partial\Omega$ imply that $\xi = 0$ in Ω, we decompose the solution of (2.1), in the spire of [13], as follows:

$$u = \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C^\alpha \psi_\alpha - \varphi(0) = \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0))\psi_\alpha + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha \psi_\alpha, \qquad \text{in } \overline{D}_1,$$

for some constants C^α, $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$, (to be determined by the forth line in (2.1)) and

$$u = \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0))u_\alpha + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha u_\alpha + u_0, \qquad \text{in } \Omega, \tag{2.2}$$

where $u_\alpha \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^2(\Omega; \mathbb{R}^d)$, $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$, respectively, satisfy

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u_\alpha = 0, & \text{in } \Omega, \\ u_\alpha = \psi_\alpha, & \text{on } \partial D_1, \\ u_\alpha = 0, & \text{on } \partial D; \end{cases} \tag{2.3}$$

and $u_0 \in C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^2(\Omega; \mathbb{R}^d)$ satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u_0 = 0, & \text{in } \Omega, \\ u_0 = 0, & \text{on } \partial D_1, \\ u_0 = \varphi(x) - \varphi(0), & \text{on } \partial D. \end{cases} \tag{2.4}$$

By the decomposition (2.2), we write

$$\nabla u = \sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0))\nabla u_\alpha + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha \nabla u_\alpha + \nabla u_0, \quad \text{in } \Omega. \tag{2.5}$$

To estimate $|\nabla u|$, two ingredients are in order: (i) estimates of $|\nabla u_\alpha|$, $\alpha = 0, 1, \cdots, \frac{d(d+1)}{2}$; (ii) estimates of $C^\alpha - \varphi^\alpha(0)$, $\alpha = 1, \cdots, d$ and C^α, $\alpha = 1, \cdots, \frac{d(d+1)}{2}$. Since the singular behavior of ∇u may occur only in the narrow region between D_1 and ∂D, we are particularly interested in such narrow region. See Fig. 2.

Fix a small constant $0 < R < 1$, independent of ε, such that the portions of ∂D_1 near P_1 and ∂D near P can be represented, respectively, by

$$x_d = \varepsilon + h_1(x'), \quad \text{and} \quad x_d = h(x'), \quad \text{for } |x'| < 2R.$$

Moreover, in view of the assumptions of ∂D_1 and ∂D, h_1 and h satisfy

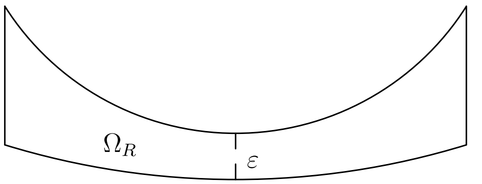

Fig. 2. The narrow region between ∂D_1 and ∂D.

$$\varepsilon + h_1(x') > h(x'), \quad \text{for } |x'| < 2R, \tag{2.6}$$

$$h_1(0') = h(0') = 0, \quad \nabla_{x'} h_1(0') = \nabla_{x'} h(0') = 0, \tag{2.7}$$

$$\nabla^2_{x'} h_1(0'), \nabla^2_{x'} h(0') \geq \kappa_0 I, \quad \nabla^2_{x'}(h_1 - h)(0') \geq \kappa_1 I, \tag{2.8}$$

and

$$\|h_1\|_{C^{2,\gamma}(\overline{B_{2R}(0')})} + \|h\|_{C^{2,\gamma}(\overline{B_{2R}(0')})} \leq \kappa_2, \tag{2.9}$$

where κ_0, κ_1 and κ_2 are some positive constants. Throughout the paper, unless otherwise stated, we use C to denote some positive constant, whose values may vary from line to line, which depend only on $\delta_0, \kappa_0, \kappa_1$ and κ_2, but not on ε. Also, we call a constant having such dependence a *universal constant*.

For $0 < r < 2R$, we denote

$$\Omega_r := \left\{ x = (x', x_d) \in \mathbb{R}^d \;\middle|\; h(x') < x_d < \varepsilon + h_1(x'),\ |x'| < r \right\}.$$

The top and bottom boundaries of Ω_r are

$$\Gamma_r^+ = \{x \in \mathbb{R}^d \,|\, x_d = \varepsilon + h_1(x'), |x'| < r\},\ \Gamma_r^- = \{x \in \mathbb{R}^d \,|\, x_d = h(x'), |x'| < r\},$$

respectively.

To estimate $|\nabla u_\alpha|$, we consider the following general boundary value problems:

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v := \nabla \cdot (\mathbb{C}^0 e(v)) = 0, & \text{in } \Omega, \\ v = \psi(x), & \text{on } \partial D_1, \\ v = 0, & \text{on } \partial D, \end{cases} \tag{2.10}$$

where $\psi(x) = (\psi^1(x), \psi^2(x), \cdots, \psi^d(x))^T \in C^2(\partial D_1; \mathbb{R}^d)$ is given vector-valued functions. Locally pointwise gradient estimates for problem (2.10) is as follows:

Theorem 2.1. *Assume that hypotheses (2.6)–(2.9) are satisfied, and let $v \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution of problem (2.10). Then for $0 < \varepsilon < 1/2$,*

$$|\nabla v(x', x_d)| \leq \frac{C}{\varepsilon + |x'|^2} \left| \psi(x', \varepsilon + h_1(x')) \right| + C\|\psi\|_{C^2(\partial D_1; \mathbb{R}^d)}, \quad \forall x \in \Omega_R, \tag{2.11}$$

and

$$|\nabla v(x)| \leq C\|\psi\|_{C^2(\partial D_1;\mathbb{R}^d)} \quad \forall\, x \in \Omega \setminus \Omega_R.$$

Remark 2.2. Theorem 2.1 is of independent interest. We also can deal with more general case when $v = \phi(x)$ on ∂D, instead of the condition $v = 0$ there. The proof of Theorem 2.1 is given in Section 3.

Without loss of generality, we only need to prove Theorem 1.1 for $\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)} = 1$, and for general case by considering $u/\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)}$ if $\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)} > 0$. If $\varphi|_{\partial D} = 0$, then $u \equiv 0$. First, the estimates of $|\nabla u_\alpha|$ are some immediate consequences of Theorem 2.1, only taking $\psi = \psi_\alpha, \alpha = 1, \cdots, \frac{d(d+1)}{2}$, respectively, or $\psi = \varphi(x) - \varphi(0)$ with minor modifications.

Corollary 2.3. *Under the hypotheses of Theorem 1.1 and with the normalization $\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)} = 1$. Then for $0 < \varepsilon < 1/2$,*

$$|\nabla u_\alpha(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad \alpha = 1, 2, \cdots, d, \quad \forall\, x \in \Omega_R; \tag{2.12}$$

$$|\nabla u_\alpha(x)| \leq \frac{C(\varepsilon + |x'|)}{\varepsilon + |x'|^2}, \quad \alpha = d+1, \cdots, \frac{d(d+1)}{2}, \quad \forall\, x \in \Omega_R; \tag{2.13}$$

$$|\nabla u_0(x)| \leq \frac{C|\nabla\varphi(0)||x'|}{\varepsilon + |x'|^2} + C, \quad \forall\, x \in \Omega_R; \tag{2.14}$$

and

$$|\nabla u_\alpha(x)| \leq C, \quad \alpha = 0, 1, 2, \cdots, \frac{d(d+1)}{2}, \quad \forall x \in \Omega \setminus \Omega_R.$$

On the other hand, we need the following estimates on C^α and $|C^\alpha - \varphi^\alpha(0)|$. The proof is given in Section 4.

Proposition 2.4. *Under the hypotheses of Theorem 1.1 and with the normalization $\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)} = 1$. Then*

$$|C^\alpha| \leq C, \quad \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \tag{2.15}$$

and

$$|C^\alpha - \varphi^\alpha(0)| \leq C\rho_d(\varepsilon), \quad \alpha = 1, 2, \cdots, d. \tag{2.16}$$

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Since

$$\nabla u=\sum_{\alpha=d+1}^{\frac{d(d+1)}{2}}C^{\alpha}\nabla\psi_{\alpha}=\begin{pmatrix}0 & C^{d+1} & C^{d+2} & \cdots & C^{2d-1}\\ -C^{d+1} & 0 & C^{2d} & \cdots & C^{3d-3}\\ -C^{d+2} & -C^{2d} & 0 & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & C^{\frac{d(d+1)}{2}}\\ -C^{2d-1} & -C^{3d-3} & \cdots & -C^{\frac{d(d+1)}{2}} & 0\end{pmatrix}\quad\text{in}\ \ D_1,$$

The estimate (1.8) immediately follows from (2.15).

By (2.16), Corollary 2.3 and Proposition 2.4, we have, for $x\in\Omega_R$,

$$\begin{aligned}|\nabla u(x)|&\le\sum_{\alpha=1}^{d}|C^{\alpha}-\varphi^{\alpha}(0)|\,|\nabla u_{\alpha}|+\sum_{\alpha=d+1}^{\frac{d(d+1)}{2}}C^{\alpha}|\nabla u_{\alpha}|+|\nabla u_0|\\&\le C\left(\frac{\rho_d(\varepsilon)}{\varepsilon+|x'|^2}+\frac{|x'|}{\varepsilon+|x'|^2}+1\right).\end{aligned}$$

Thus, (1.9) is proved, so (1.7). □

To complete this section, we recall some properties of the tensor $\mathbb{C}$. For the isotropic elastic material, let

$$\mathbb{C}:=(C_{ijkl})=(\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})),\quad\mu>0,\quad d\lambda+2\mu>0.$$

The components C_{ijkl} satisfy the following symmetric condition:

$$C_{ijkl}=C_{klij}=C_{klji},\quad i,j,k,l=1,2,\cdots,d. \tag{2.17}$$

We will use the following notations:

$$(\mathbb{C}A)_{ij}=\sum_{k,l=1}^{d}C_{ijkl}A_{kl},\quad\text{and}\quad(A,B)\equiv A:B=\sum_{i,j=1}^{d}A_{ij}B_{ij},$$

for every pair of $d\times d$ matrices $A=(A_{ij})$, $B=(B_{ij})$. Clearly,

$$(\mathbb{C}A,B)=(A,\mathbb{C}B).$$

If A is symmetric, then, by the symmetry condition (2.17), we have that

$$(\mathbb{C}A,A)=C_{ijkl}A_{kl}A_{ij}=\lambda A_{ii}A_{kk}+2\mu A_{kj}A_{kj}.$$

Thus $\mathbb{C}$ satisfies the following ellipticity condition: For every $d\times d$ real symmetric matrix $\eta=(\eta_{ij})$,

$$\min\{2\mu, d\lambda+2\mu\}|\eta|^2 \le (\mathbb{C}\eta,\eta) \le \max\{2\mu, d\lambda+2\mu\}|\eta|^2, \tag{2.18}$$

where $|\eta|^2 = \sum_{ij}\eta_{ij}^2$. In particular,

$$\min\{2\mu, d\lambda+2\mu\}|A+A^T|^2 \le (\mathbb{C}(A+A^T),(A+A^T)). \tag{2.19}$$

3. Proof of Theorem 2.1 and estimates of $|\nabla u_\alpha|$

In this section, we first prove Theorem 2.1, then give some much finer estimates on $|\nabla u_\alpha|$, which will be useful for the establishment of the low bound estimates in Section 4 and Section 5.

We decompose the solution of (2.10) as follows:

$$v = v_1 + v_2 + \cdots + v_d,$$

where $v_l = (v_l^1, v_l^2, \cdots, v_l^d)^T$, $l = 1,2,\cdots,d$, with $v_l^j = 0$ for $j \ne l$, and v_l satisfy the following boundary value problem, respectively,

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_l := \nabla\cdot(\mathbb{C}^0 e(v_l)) = 0, & \text{in } \Omega,\\ v_l = (0,\cdots,0,\psi^l,0,\cdots,0)^T, & \text{on } \partial D_1,\\ v_l = 0, & \text{on } \partial D. \end{cases} \tag{3.1}$$

Then

$$\nabla v = \sum_{l=1}^{d} \nabla v_l. \tag{3.2}$$

In order to estimate $|\nabla v_l|$ one by one, we first introduce a scalar auxiliary function $\bar{v} \in C^2(\mathbb{R}^n)$ such that $\bar{v} = 1$ on ∂D_1, $\bar{v} = 0$ on ∂D and

$$\bar{v}(x) = \frac{x_d - h(x')}{\varepsilon + h_1(x') - h(x')}, \quad \text{in } \Omega_{2R},$$

and

$$\|\bar{v}\|_{C^2(\Omega\setminus\Omega_{R/2})} \le C. \tag{3.3}$$

By a direct calculation, we obtain that for $k, j = 1,\cdots,d-1$, and $x \in \Omega_{2R}$,

$$|\partial_{x_k}\bar{v}(x)| \le \frac{C|x'|}{\varepsilon+|x'|^2}, \qquad \frac{1}{C(\varepsilon+|x'|^2)} \le |\partial_{x_d}\bar{v}(x)| \le \frac{C}{\varepsilon+|x'|^2}, \tag{3.4}$$

and

$$|\partial_{x_kx_j}\bar{v}(x)| \le \frac{C}{\varepsilon+|x'|^2}, \quad |\partial_{x_kx_d}\bar{v}(x)| \le \frac{C|x'|}{(\varepsilon+|x'|^2)^2}, \quad \partial_{x_dx_d}\bar{v}(x)=0. \tag{3.5}$$

Extend $\psi \in C^2(\partial D;\mathbb{R}^d)$ to $\psi \in C^2(\overline{\Omega};\mathbb{R}^d)$ such that $\|\psi^l\|_{C^2(\overline{\Omega\setminus\Omega_R})} \le C\|\psi^l\|_{C^2(\partial D_1)}$, for $l=1,2,\cdots,d$. We can find $\rho \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} &0\le\rho\le1,\ |\nabla\rho|\le C,\ \text{on } \overline{\Omega},\\ &\rho=1 \text{ on } \overline{\Omega}_{\frac{3}{2}R},\ \text{and } \rho=0 \text{ on } \overline{\Omega}\setminus\Omega_{2R}. \end{aligned} \tag{3.6}$$

Define

$$\tilde{v}_l(x)=(0,\cdots,0,\left[\rho(x)\psi^l(x',\varepsilon+h_1(x'))+(1-\rho(x))\psi^l(x)\right]\bar{v}(x),0,\cdots,0)^T \quad \text{in } \Omega. \tag{3.7}$$

In particular,

$$\tilde{v}_l(x)=(0,\cdots,0,\psi^l(x',\varepsilon+h_1(x'))\bar{v}(x),0,\cdots,0)^T \quad \text{in } \Omega_R, \tag{3.8}$$

and in view of (3.3),

$$\|\tilde{v}_l\|_{C^2(\Omega\setminus\Omega_{R/2})} \le C\|\psi^l\|_{C^2(\partial D_1)}. \tag{3.9}$$

Due to (3.4), and (3.5), for $l=1,2,\cdots,d$, and $k,j=1,2,\cdots,d-1$, for $x\in\Omega_R$,

$$|\partial_{x_k}\tilde{v}_l(x)| \le \frac{C|x'||\psi^l(x',\varepsilon+h_1(x'))|}{\varepsilon+|x'|^2}+C\|\nabla\psi^l\|_{L^\infty}, \tag{3.10}$$

$$\frac{|\psi^l(x',\varepsilon+h_1(x'))|}{C(\varepsilon+|x'|^2)} \le |\partial_{x_d}\tilde{v}_l(x)| \le \frac{C|\psi^l(x',\varepsilon+h_1(x'))|}{\varepsilon+|x'|^2}; \tag{3.11}$$

and

$$\begin{aligned} &|\partial_{x_kx_j}\tilde{v}_l(x)|\\ &\le \frac{C|\psi^l(x',\varepsilon+h_1(x'))|}{\varepsilon+|x'|^2}+C\left(\frac{|x'|}{\varepsilon+|x'|^2}+1\right)\|\nabla\psi^l\|_{L^\infty}+C\|\nabla^2\psi^l\|_{L^\infty}, \end{aligned} \tag{3.12}$$

$$|\partial_{x_kx_d}\tilde{v}_l(x)| \le \frac{C|x'|}{(\varepsilon+|x'|^2)^2}|\psi^l(x',\varepsilon+h_1(x'))|+\frac{C}{\varepsilon+|x'|^2}\|\nabla\psi^l\|_{L^\infty}, \tag{3.13}$$

$$\partial_{x_dx_d}\tilde{v}_l(x)=0. \tag{3.14}$$

Here and throughout this section, for simplicity we use $\|\nabla\psi\|_{L^\infty}$ and $\|\nabla^2\psi\|_{L^\infty}$ to denote $\|\nabla\psi\|_{L^\infty(\partial D_1)}$ and $\|\nabla^2\psi\|_{L^\infty(\partial D_1)}$, respectively.

Let

$$w_l := v_l-\tilde{v}_l, \qquad l=1,2,\cdots,d. \tag{3.15}$$

Lemma 3.1. *Let $v_l \in H^1(\Omega;\mathbb{R}^d)$ be a weak solution of (3.1). Then*

$$\int\limits_{\Omega} |\nabla w_l|^2 dx \leq C\|\psi^l\|_{C^2(\partial D_1)}, \qquad l = 1, 2, \cdots, d. \tag{3.16}$$

Proof. For simplicity, we denote

$$w := w_l, \quad \text{and} \quad \tilde{v} := \tilde{v}_l.$$

Thus, w satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu} w = -\mathcal{L}_{\lambda,\mu}\tilde{v}, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

Multiplying the equation in (3.17) by w and applying integration by parts, we get

$$\int\limits_{\Omega} \left(\mathbb{C}^0 e(w), e(w)\right) dx = \int\limits_{\Omega} w \left(\mathcal{L}_{\lambda,\mu}\tilde{v}\right) dx. \tag{3.18}$$

By the Poincaré inequality,

$$\|w\|_{L^2(\Omega\setminus\Omega_R)} \leq C\|\nabla w\|_{L^2(\Omega\setminus\Omega_R)}. \tag{3.19}$$

Note that the above constant C is independent of ε. Using the Sobolev trace embedding theorem,

$$\int\limits_{\substack{|x'|=R,\\ h(x')<x_d<\varepsilon+h_1(x')}} |w| dx \leq C \left(\int\limits_{\Omega\setminus\Omega_R} |\nabla w|^2 dx \right)^{\frac{1}{2}}. \tag{3.20}$$

According to (3.10), we have

$$\begin{aligned} &\int\limits_{\Omega_R} |\nabla_{x'}\tilde{v}|^2 dx \\ &\leq C \int\limits_{|x'|<R} (\varepsilon + h_1(x') - h(x')) \left(\frac{|x'|^2|\psi^l(x', \varepsilon + h_1(x'))|^2}{(\varepsilon + |x'|^2)^2} + \|\nabla\psi^l\|_{L^\infty}^2 \right) dx' \\ &\leq C\|\psi^l\|_{C^1(\partial D_1)}^2, \end{aligned} \tag{3.21}$$

where C depends only on d and κ_0.

The first Korn's inequality together with (2.19), (3.18), (3.9) and (3.19) implies

$$\begin{aligned}\int_\Omega |\nabla w|^2 dx &\leq 2\int_\Omega |e(w)|^2 dx\\ &\leq C\left|\int_{\Omega_R} w(\mathcal{L}_{\lambda,\mu}\tilde{v})dx\right| + C\left|\int_{\Omega\setminus\Omega_R} w(\mathcal{L}_{\lambda,\mu}\tilde{v})dx\right|\\ &\leq C\left|\int_{\Omega_R} w(\mathcal{L}_{\lambda,\mu}\tilde{v})dx\right| + C\|\psi^l\|_{C^2(\partial D_1)}\int_{\Omega\setminus\Omega_R}|w|dx\\ &\leq C\left|\int_{\Omega_R} w(\mathcal{L}_{\lambda,\mu}\tilde{v})dx\right| + C\|\psi^l\|_{C^2(\partial D_1)}\left(\int_{\Omega\setminus\Omega_R}|\nabla w|^2\right)^{1/2},\end{aligned}$$

while, due to (3.14), (3.20) and (3.21),

$$\begin{aligned}&\left|\int_{\Omega_R} w(\mathcal{L}_{\lambda,\mu}\tilde{v})dx\right| \leq C\sum_{k+l<2d}\left|\int_{\Omega_R} w\partial_{x_kx_l}\tilde{v}dx\right|\\ &\leq C\int_{\Omega_R}|\nabla w||\nabla_{x'}\tilde{v}|dx + \int_{\substack{|x'|=R,\\ h(x')<x_d<\varepsilon+h_1(x')}} C|\nabla_{x'}\tilde{v}||w|dx\\ &\leq C\left(\int_{\Omega_R}|\nabla w|^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega_R}|\nabla_{x'}\tilde{v}|^2dx\right)^{\frac{1}{2}} + C\|\psi^l\|_{C^1(\partial D_1)}\left(\int_{\Omega\setminus\Omega_R}|\nabla w|^2dx\right)^{\frac{1}{2}}\\ &\leq C\|\psi^l\|_{C^1(\partial D_1)}\left(\int_\Omega|\nabla w|^2dx\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\int_\Omega|\nabla w|^2dx \leq C\|\psi^l\|_{C^2(\partial D_1)}\left(\int_\Omega|\nabla w|^2dx\right)^{\frac{1}{2}}.$$

The proof of Lemma 3.1 is completed. □

For convenience, we denote

$$\delta(z) := \varepsilon + h_1(z') - h(z'), \quad \text{for} \;\; z=(z',z_d)\in\Omega_R.$$

By (2.8), we have

$$\frac{1}{C}(\varepsilon+|z'|^2)\leq\delta(z)\leq C(\varepsilon+|z'|^2).$$

Set

$$\widehat{\Omega}_s(z):=\left\{\ x\in\Omega_{2R}\ \middle|\ |x'-z'|<s\ \right\},\quad \forall\, 0\leq s\leq R.$$

It follows from (3.7) and (3.10)–(3.14) that for $x\in\Omega_R$, $l=1,2,\cdots,d$,

$$\begin{aligned}|\mathcal{L}_{\lambda,\mu}\tilde{v}_l|\leq C|\nabla^2\tilde{v}_l|\leq&\left(\frac{C}{\varepsilon+|x'|^2}+\frac{C|x'|}{(\varepsilon+|x'|^2)^2}\right)|\psi^l(x',\varepsilon+h_1(x'))|\\&+\frac{C}{\varepsilon+|x'|^2}\|\nabla\psi^l\|_{L^\infty}+C\|\nabla^2\psi^l\|_{L^\infty},\end{aligned}\tag{3.22}$$

where C is independent of ε.

Lemma 3.2. *For $\delta=\delta(z)\leq R$, $z\in\Omega_R$, and $l=1,2,\cdots,d$,*

$$\int\limits_{\widehat{\Omega}_\delta(z)}|\nabla w_l|^2dx\leq C\delta^{d+1}\left(|\psi^l(z',\varepsilon+h_1(z'))|^2+\delta(\|\psi^l\|^2_{C^2(\partial D_1)}+1)\right).\tag{3.23}$$

Proof. Still denote $w:=w_l$, and $\tilde{v}:=\tilde{v}_l$. For $0<t<s<1$, let $\eta(x')$ be a smooth cutoff function satisfying $0\leq\eta(x')\leq1$, $\eta(x')=1$ if $|x'-z'|<t$, $\eta(x')=0$ if $|x'-z'|>s$ and $|\nabla\eta(x')|\leq\frac{2}{s-t}$. Multiplying η^2w on both side of the equation in (3.17) and applying integration by parts leads to

$$\int\limits_{\widehat{\Omega}_s(z)}(\mathbb{C}^0e(w),e(\eta^2w))dx=\int\limits_{\widehat{\Omega}_s(z)}(\eta^2w)\mathcal{L}_{\lambda,\mu}\tilde{v}dx.\tag{3.24}$$

By the first Korn's inequality and the standard arguments, we have

$$\int\limits_{\widehat{\Omega}_s(z)}(\mathbb{C}^0e(w),e(\eta^2w))dx\geq\frac{1}{C}\int\limits_{\widehat{\Omega}_s(z)}|\eta\nabla w|^2dx-C\int\limits_{\widehat{\Omega}_s(z)}|\nabla\eta|^2|w|^2dx.\tag{3.25}$$

For the right hand side of (3.24), in view of Hölder inequality and Cauchy inequality,

$$\begin{aligned}\left|\int\limits_{\widehat{\Omega}_s(z)}(\eta^2w)\mathcal{L}_{\lambda,\mu}\tilde{v}dx\right|&\leq\left(\int\limits_{\widehat{\Omega}_s(z)}|w|^2dx\right)^{\frac12}\left(\int\limits_{\widehat{\Omega}_s(z)}|\mathcal{L}_{\lambda,\mu}\tilde{v}|^2dx\right)^{\frac12}\\&\leq\frac{1}{(s-t)^2}\int\limits_{\widehat{\Omega}_s(z)}|w|^2dx+(s-t)^2\int\limits_{\widehat{\Omega}_s(z)}|\mathcal{L}_{\lambda,\mu}\tilde{v}|^2dx.\end{aligned}$$

This, together with (3.24) and (3.25), implies that

$$\int_{\widehat{\Omega}_t(z)} |\nabla w|^2 dx \le \frac{C}{(s-t)^2} \int_{\widehat{\Omega}_s(z)} |w|^2 dx + C(s-t)^2 \int_{\widehat{\Omega}_s(z)} |\mathcal{L}_{\lambda,\mu}\tilde{v}|^2 dx. \tag{3.26}$$

We know that $w = 0$ on Γ_R^-. By using (2.6)–(2.9) and Hölder inequality, we obtain

$$\begin{aligned} \int_{\widehat{\Omega}_s(z)} |w|^2 dx &= \int_{\widehat{\Omega}_s(z)} \left| \int_{h(x')}^{x_d} \partial_{x_d} w(x',\xi) d\xi \right|^2 dx \\ &\le \int_{\widehat{\Omega}_s(z)} (\varepsilon + h_1(x') - h(x')) \int_{h(x')}^{\varepsilon + h_1(x')} |\nabla w(x',\xi)|^2 d\xi \, dx \\ &\le C\left(\varepsilon + (|z'| + s)^2\right)^2 \int_{\widehat{\Omega}_s(z)} |\nabla w|^2 dx. \end{aligned} \tag{3.27}$$

It follows from (3.22) and the mean value theorem that

$$\begin{aligned} &\int_{\widehat{\Omega}_s(z)} |\mathcal{L}_{\lambda,\mu}\tilde{v}|^2 dx \\ &\le |\psi^l(z', \varepsilon + h_1(z'))|^2 \int_{\widehat{\Omega}_s(z)} \left(\frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2} \right)^2 dx \\ &\quad + \|\nabla\psi^l\|_{L^\infty}^2 \int_{\widehat{\Omega}_s(z)} \left(\frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2} \right)^2 |x' - z'|^2 dx \\ &\quad + \|\nabla\psi^l\|_{L^\infty}^2 \int_{\widehat{\Omega}_s(z)} \left(\frac{C}{\varepsilon + |x'|^2} \right)^2 dx + C\delta(z) s^{d-1} \|\nabla^2\psi^l\|_{L^\infty}^2 \\ &\le C|\psi^l(z', \varepsilon + h_1(z'))|^2 \int_{|x'-z'|<s} \frac{dx'}{(\varepsilon + |x'|^2)^2} \\ &\quad + C\|\nabla\psi^l\|_{L^\infty}^2 \int_{|x'-z'|<s} \left(\frac{1}{\varepsilon + |x'|^2} + \frac{s^2}{(\varepsilon + |x'|^2)^2} \right) dx' + C\delta(z) s^{d-1} \|\nabla^2\psi^l\|_{L^\infty}^2. \end{aligned} \tag{3.28}$$

Case 1. For $0 \le |z'| \le \sqrt{\varepsilon}$, (i.e. $\varepsilon \le \delta(z) \le C\varepsilon$), and $0 < t < s < \sqrt{\varepsilon}$, by means of (3.27) and (3.28), we have

$$\int\limits_{\widehat{\Omega}_s(z)} |w|^2 dx \le C\varepsilon^2 \int\limits_{\widehat{\Omega}_s(z)} |\nabla w|^2 dx, \tag{3.29}$$

and

$$\begin{aligned} &\int\limits_{\widehat{\Omega}_s(z)} |\mathcal{L}_{\lambda,\mu}\tilde{v}|^2 dx \\ &\le C|\psi^l(z',\varepsilon+h_1(z'))|^2 \frac{s^{d-1}}{\varepsilon^2} + C\|\nabla\psi^l\|^2_{L^\infty} \frac{s^{d-1}}{\varepsilon} + C\varepsilon\, s^{d-1}\|\nabla^2\psi^l\|^2_{L^\infty}. \end{aligned} \tag{3.30}$$

Denote

$$F(t) := \int\limits_{\widehat{\Omega}_t(z)} |\nabla w|^2 dx.$$

By (3.26), (3.29) and (3.30), for some universal constant $c_1 > 0$, we get for $0 < t < s < \sqrt{\varepsilon}$,

$$\begin{aligned} F(t) \le \left(\frac{c_1\varepsilon}{s-t}\right)^2 F(s) + C(s-t)^2 s^{d-1}\cdot \\ \left(\frac{|\psi^l(z',\varepsilon+h_1(z'))|^2}{\varepsilon^2} + \frac{\|\nabla\psi^l\|^2_{L^\infty}}{\varepsilon} + \varepsilon\|\nabla^2\psi^l\|^2_{L^\infty}\right). \end{aligned} \tag{3.31}$$

Let $t_i = \delta + 2c_1 i\varepsilon$, $i = 0, 1, \cdots$ and $k = \left[\frac{1}{4c_1\sqrt{\varepsilon}}\right] + 1$, then

$$\frac{c_1\varepsilon}{t_{i+1} - t_i} = \frac{1}{2}.$$

Using (3.31) with $s = t_{i+1}$ and $t = t_i$, we obtain

$$\begin{aligned} F(t_i) \le \frac{1}{4}F(t_{i+1}) + C(i+2)^{d-1}\varepsilon^{d+1}\cdot \\ \left(|\psi^l(z',\varepsilon+h_1(z'))|^2 + \varepsilon(\|\nabla\psi^l\|^2_{L^\infty} + \|\nabla^2\psi^l\|^2_{L^\infty})\right), \quad i = 0, 1, 2, \cdots, k. \end{aligned}$$

After k iterations, making use of (3.16), we have, for sufficiently small ε,

$$\begin{aligned} F(t_0) &\le \left(\frac{1}{4}\right)^k F(t_k) + C\varepsilon^{d+1}\sum_{i=1}^{k}\left(\frac{1}{4}\right)^{i-1}(i+1)^{d-1}\cdot \\ &\qquad \left(|\psi^l(z',\varepsilon+h_1(z'))|^2 + \varepsilon(\|\nabla\psi^l\|^2_{L^\infty} + \|\nabla^2\psi^1\|^2_{L^\infty})\right) \\ &\le \left(\frac{1}{4}\right)^k F(\sqrt{\varepsilon}) + C\varepsilon^{d+1}\left(|\psi^l(z',\varepsilon+h_1(z'))|^2 + \varepsilon(\|\nabla\psi^l\|^2_{L^\infty} + \|\nabla^2\psi^l\|^2_{L^\infty})\right) \\ &\le C\varepsilon^{d+1}\left(|\psi^l(z',\varepsilon+h_1(z'))|^2 + \varepsilon\|\psi^l\|^2_{C^2(\partial D_1)}\right), \end{aligned}$$

here we used the fact that $\left(\frac{1}{4}\right)^k \leq \left(\frac{1}{4}\right)^{\frac{1}{4c_1\sqrt{\varepsilon}}} \leq \varepsilon^{d+1}$ if ε sufficiently small. This implies that for $0 \leq |z'| \leq \sqrt{\varepsilon}$,

$$\|\nabla w\|^2_{L^2(\widehat{\Omega}_\delta(z))} \leq C\varepsilon^{d+1}\left(|\psi^l(z', \varepsilon + h_1(z'))|^2 + \varepsilon\|\psi^l\|^2_{C^2(\partial D_1)}\right).$$

Case 2. For $\sqrt{\varepsilon} \leq |z'| < R$, (i.e. $C|z'|^2 \leq \delta(z) \leq (C+1)|z'|^2$), $0 < t < s < \frac{2|z'|}{3}$, by using (3.27) and (3.28) again, we have

$$\int\limits_{\widehat{\Omega}_s(z)} |w|^2 dx \leq C|z'|^4 \int\limits_{\widehat{\Omega}_s(z)} |\nabla w|^2 dx,$$

$$\begin{aligned}
&\int\limits_{\widehat{\Omega}_s(z)} |\mathcal{L}_{\lambda,\mu}\tilde{v}|^2 dx \\
&\leq C|\psi^l(z', \varepsilon + h_1(z'))|^2 \frac{s^{d-1}}{|z'|^4} + C\|\nabla\psi^l\|^2_{L^\infty}\frac{s^{d-1}}{|z'|^2} + C|z'|^2 s^{d-1}\|\nabla^2\psi^l\|^2_{L^\infty}.
\end{aligned}$$

Thus, for $0 < t < s < \frac{2|z'|}{3}$,

$$\begin{aligned}
F(t) \leq \left(\frac{c_2|z'|^2}{s-t}\right)^2 F(s) + C(s-t)^2 s^{d-1}\cdot \\
\left(\frac{|\psi^l(z', \varepsilon + h_1(z'))|^2}{|z'|^4} + \frac{\|\nabla\psi^l\|^2_{L^\infty}}{|z'|^2} + |z'|^2\|\nabla^2\psi^l\|^2_{L^\infty}\right),
\end{aligned} \tag{3.32}$$

where c_2 is another universal constant. Taking the same iteration procedure as in Case 1, setting $t_i = \delta + 2c_2 i|z'|^2$, $i = 0, 1, \cdots$ and $k = \left[\frac{1}{4c_2|z'|}\right] + 1$, by (3.32) with $s = t_{i+1}$ and $t = t_i$, we have, for $i = 0, 1, 2, \cdots, k$,

$$\begin{aligned}
F(t_i) \leq \frac{1}{4}F(t_{i+1}) + C(i+2)^{d-1}|z'|^{2(d+1)}\cdot \\
\left(|\psi^l(z', \varepsilon + h_1(z'))|^2 + |z'|^2(\|\nabla\psi^l\|^2_{L^\infty} + \|\nabla^2\psi^l\|^2_{L^\infty})\right).
\end{aligned}$$

Similarly, after k iterations, we have

$$\begin{aligned}
F(t_0) &\leq \left(\frac{1}{4}\right)^k F(t_k) + C\sum_{i=1}^{k}\left(\frac{1}{4}\right)^{i-1}(i+1)^{d-1}|z'|^{2(d+1)}\cdot \\
&\qquad \left(|\psi^l(z', \varepsilon + h_1(z'))|^2 + |z'|^2(\|\nabla\psi^l\|^2_{L^\infty} + \|\nabla^2\psi^l\|^2_{L^\infty})\right) \\
&\leq \left(\frac{1}{4}\right)^k F(|z'|) \\
&\quad + C|z'|^{2(d+1)}\left(|\psi^l(z', \varepsilon + h_1(z'))|^2 + |z'|^2(\|\nabla\psi^l\|^2_{L^\infty} + \|\nabla^2\psi^1\|^2_{L^\infty})\right) \\
&\leq C|z'|^{2(d+1)}\left(|\psi^l(z', \varepsilon + h_1(z'))|^2 + |z'|^2\|\psi^l\|^2_{C^2(\partial D_1)}\right),
\end{aligned}$$

which implies that, for $\sqrt{\varepsilon} \leq |z'| < R$,

$$\|\nabla w\|^2_{L^2(\widehat{\Omega}_\delta(z))} \leq C|z'|^{2(d+1)} \left(|\psi^l(z', \varepsilon + h_1(z'))|^2 + |z'|^2 \|\psi^l\|^2_{C^2(\partial D_1)} \right).$$

The proof of Lemma 3.2 is completed. □

Lemma 3.3. *For $l = 1, 2, \cdots, d$,*

$$|\nabla w_l(x)| \leq \frac{C|\psi^l(x', \varepsilon + h_1(x'))|}{\sqrt{\delta(x)}} + C\|\psi^l\|_{C^2(\partial D_1)}, \quad \forall\, x \in \Omega_R. \tag{3.33}$$

Consequently, by (3.10), (3.11) and (3.15), we have for sufficiently small ε and $x \in \Omega_R$,

$$\frac{|\psi^l(x', \varepsilon + h_1(x'))|}{C(\varepsilon + |x'|^2)} \leq |\nabla v_l(x', x_d)| \leq \frac{C|\psi^l(x', \varepsilon + h_1(x'))|}{\varepsilon + |x'|^2} + C\|\psi^l\|_{C^2(\partial D_1)}. \tag{3.34}$$

Proof. Take $w := w_l$ and $\tilde{v} := \tilde{v}_l$ for simplicity. Given $z = (z', z_d) \in \Omega_R$, making a change of variables

$$\begin{cases} x' - z' = \delta y', \\ x_d = \delta y_d, \end{cases}$$

where $\delta = \delta(z)$. Define

$$\hat{h}_1(y') := \frac{1}{\delta}\left(\varepsilon + h_1(\delta y' + z')\right), \quad \hat{h}(y') := \frac{1}{\delta} h(\delta y' + z').$$

Then, the region $\widehat{\Omega}_\delta(z)$ becomes Q_1, where

$$Q_r = \{y \in \mathbb{R}^d \mid \hat{h}(y') < y_d < \hat{h}_1(y'),\ |y'| < r\}, \quad 0 < r \leq 1,$$

and the top and bottom boundaries of Q_r become

$$\widehat{\Gamma}^+_r := \{y \in \mathbb{R}^d \mid y_d = \hat{h}_1(y'),\ |y'| \leq r\}$$

and

$$\widehat{\Gamma}^-_r := \{y \in \mathbb{R}^d \mid y_d = \hat{h}(y'),\ |y'| \leq r\},$$

respectively. From (2.6)–(2.9) and the definition of $\hat{h}_1$ and $\hat{h}$, it follows that

$$\hat{h}_1(0') - \hat{h}(0') = 1,$$

and for $|y'| < 1$,

$$|\nabla \hat{h}_1(y')| + |\nabla \hat{h}(y')| \leq C(\delta + |z'|), \quad |\nabla^2 \hat{h}_1(y')| + |\nabla^2 \hat{h}(y')| \leq C\delta.$$

Since R is small, $\|\hat{h}_1\|_{C^{1,1}(\overline{B_1(0')})}$ and $\|\hat{h}\|_{C^{1,1}(\overline{B_1(0')})}$ are small and Q_1 is approximately a unit square (or a cylinder-shaped domain) as far as applications of the Sobolev embedding theorems and classical L^p estimates for elliptic systems are concerned.

Let

$$\hat{v}(y', y_d) := \tilde{v}(\delta y' + z', \delta y_d), \qquad \hat{w}(y', y_d) := w(\delta y' + z', \delta y_d).$$

Thus, $\hat{w}(y)$ satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\hat{w} = -\mathcal{L}_{\lambda,\mu}\hat{v} & \text{in } Q_1, \\ \hat{w} = 0, & \text{on } \widehat{\Gamma}_1^{\pm}. \end{cases}$$

In view of $\hat{w} = 0$ on the upper and lower boundaries of Q_1, we have, by Poincaré inequality, that

$$\|\hat{w}\|_{H^1(Q_1)} \leq C\|\nabla \hat{w}\|_{L^2(Q_1)}.$$

Using the Sobolev embedding theorem and classical $W^{2,p}$ estimates for elliptic systems (see e.g. [2], or Theorem 2.5 in [24]), we have, for some $p > n$,

$$\|\nabla \hat{w}\|_{L^\infty(Q_{1/2})} \leq C\|\hat{w}\|_{W^{2,p}(Q_{1/2})} \leq C\left(\|\nabla \hat{w}\|_{L^2(Q_1)} + \|\mathcal{L}_{\lambda,\mu}\hat{v}\|_{L^\infty(Q_1)}\right).$$

Since

$$\|\nabla \hat{w}\|_{L^\infty(Q_{1/2})} = \delta\|\nabla w\|_{L^\infty(\widehat{\Omega}_{\delta/2}(z))}, \quad \|\nabla \hat{w}\|_{L^2(Q_1)} = \delta^{1-\frac{d}{2}}\|\nabla w\|_{L^2(\widehat{\Omega}_\delta(z))}$$

and

$$\|\mathcal{L}_{\lambda,\mu}\hat{v}\|_{L^\infty(Q_1)} = \delta^2\|\mathcal{L}_{\lambda,\mu}\tilde{v}\|_{L^\infty(\widehat{\Omega}_\delta(z))}$$

Tracking back to w through the transforms, we have

$$\|\nabla w\|_{L^\infty(\widehat{\Omega}_{\delta/2}(z))} \leq \frac{C}{\delta}\left(\delta^{1-\frac{d}{2}}\|\nabla w\|_{L^2(\widehat{\Omega}_\delta(z))} + \delta^2\|\mathcal{L}_{\lambda,\mu}\tilde{v}\|_{L^\infty(\widehat{\Omega}_\delta(z))}\right). \tag{3.35}$$

By (3.22) and (3.23), we have

$$\delta^{-\frac{d}{2}}\|\nabla w\|_{L^2(\widehat{\Omega}_\delta(z))} \leq \frac{C}{\sqrt{\delta}}|\psi^l(z', \varepsilon + h_1(z'))| + C\|\psi^l\|_{C^2(\partial D_1)},$$

and

$$\delta\|\mathcal{L}_{\lambda,\mu}\tilde{v}\|_{L^\infty(\widehat{\Omega}_\delta(z))} \leq \frac{C}{\sqrt{\delta}}|\psi^l(z', \varepsilon + h_1(z'))| + C(\|\nabla\psi^1\|_{L^\infty} + \|\nabla^2\psi^l\|_{L^\infty}).$$

Plugging these estimates above into (3.35) yields (3.33). The proof of Lemma 3.3 is finished. □

Proof of Theorem 2.1. By using (3.34) and the decomposition of ∇u, (3.2),

$$|\nabla v(x)| \leq \sum_{l=1}^{d} |\nabla v_l| \leq \frac{C|\psi(x', \varepsilon + h_1(x'))|}{\varepsilon + |x'|^2} + C\|\psi\|_{C^2(\partial D_1)}, \quad x \in \Omega_R.$$

Note that for any $x \in \Omega \setminus \Omega_R$, by using the standard interior estimates and boundary estimates for elliptic systems (2.10) (see Agmon et al. [1] and [2]), we have

$$\|\nabla v\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C\|\psi\|_{C^2(\partial D_1)}.$$

The proof of Theorem 2.1 is completed. □

The following finer estimates in Ω_R will be useful in Section 4 and Section 5. We assume that $\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)} = 1$ without loss of generality. For problem (2.3), taking

$$\psi = \psi_\alpha, \quad \text{and} \quad \tilde{u}_\alpha := \bar{v}\psi_\alpha, \ \alpha = 1, \cdots, d$$

in the proof of Lemma 3.1–Lemma 3.3, respectively, we have

Corollary 3.4. *For $\alpha = 1, 2, \cdots, d$,*

$$|\nabla(u_\alpha - \tilde{u}_\alpha)(x)| \leq \frac{C}{\sqrt{\delta(x)}}, \qquad \forall \ \ x \in \Omega_R. \tag{3.36}$$

Consequently, by the definition of $\tilde{u}_\alpha$ and (3.4), we have, for $\alpha = 1, \cdots, d$,

$$|\nabla_{x'} u_\alpha(x)| \leq \frac{C}{\sqrt{\delta(x)}}, \qquad \forall \ \ x \in \Omega_R, \tag{3.37}$$

and

$$\frac{1}{C\delta(x)} \leq |\partial_{x_d} u_\alpha(x)| \leq \frac{C}{\delta(x)}, \quad x \in \Omega_R. \tag{3.38}$$

Proof. According to the definition of $\tilde{u}_\alpha$ and (3.4), we have

$$|\nabla_{x'} \tilde{u}_\alpha(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2}, \quad x \in \Omega_R,$$

$$\frac{1}{C(\varepsilon + |x'|^2)} \leq |\partial_{x_d} \tilde{u}_\alpha(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_R;$$

and

$$\begin{aligned} |\mathcal{L}_{\lambda,\mu} \tilde{u}_\alpha(x)| &\leq C \sum_{k+l<2d} |\partial_{x_k x_l} \tilde{u}_\alpha(x)| \\ &\leq \left(\frac{C}{\varepsilon + |x'|^2} + \frac{C|x'|}{(\varepsilon + |x'|^2)^2} \right) |\psi^l(x', \varepsilon + h_1(x'))|, \quad x \in \Omega_R. \end{aligned}$$

Clearly, (3.36) follows from the proof of Lemma 3.1–Lemma 3.3. □

For problem (2.4), we decompose the solution u_0 as

$$u_0 = u_{01} + u_{02} + \cdots + u_{0d},$$

where u_{0l}, $l = 1, 2, \cdots, d$, satisfy, respectively,

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u_{0l} = 0, & \text{in } \Omega, \\ u_{0l} = 0, & \text{on } \partial D_1, \\ u_{0l} = (0, \cdots, 0, \varphi^l(x) - \varphi^l(0), 0, \cdots, 0)^T, & \text{on } \partial D. \end{cases} \tag{3.39}$$

Similar as (3.7), we define

$$\begin{aligned} \tilde{u}_{0l}(x) := (0, \cdots, 0, \big[\, \rho(x)(\varphi^l(x', h(x')) - \varphi^l(0)) \\ + (1 - \rho(x))(\varphi^l(x) - \varphi^l(0))\big](1 - \bar{v})(x), 0, \cdots, 0)^T, \quad \forall\, x \in \Omega. \end{aligned} \tag{3.40}$$

where $\rho \in C^2(\overline{\Omega})$ is a cutoff function satisfying (3.6) as before. In particular,

$$\tilde{u}_{0l} = (0, \cdots, 0, (\varphi^l(x', h(x')) - \varphi^l(0))(1 - \bar{v})(x), 0, \cdots, 0)^T, \quad \forall \;\; x \in \Omega_R.$$

Adapting the proofs of Lemma 3.1–Lemma 3.3 to the equation (3.39), we obtain the following corollary.

Corollary 3.5. *For $l = 1, 2, \cdots, d$,*

$$|\nabla(u_{0l} - \tilde{u}_{0l})(x)| \le C\|\varphi^l\|_{C^2(\partial D)}, \qquad x \in \Omega_R. \tag{3.41}$$

Consequently,

$$|\nabla_{x'} u_{0l}(x)| \le \; C\|\varphi^l\|_{C^2(\partial D)}, \qquad x \in \Omega_R, \tag{3.42}$$

and

$$\frac{|\varphi^l(x', h(x')) - \varphi^l(0)|}{C\delta(x)} \le |\partial_{x_d} u_{0l}(x)| \le \frac{C|\nabla_{x'}\varphi^l(0)||x'|}{\delta(x)} + C\|\varphi^l\|_{C^2(\partial D)}, \; x \in \Omega_R. \tag{3.43}$$

Proof. For (3.39), it is clear from (3.40) that $\tilde{u}_{0l} = u_{0l} = 0$ on ∂D_1, $\tilde{u}_{0l} = u_{0l}$ on ∂D. Note that $\tilde{u}_{0l}^k = 0$, if $k \ne l$, and for $x \in \Omega_R$,

$$\begin{aligned} \nabla_{x'} \tilde{u}_{0l}^l &= -\left(\varphi^l(x', h(x')) - \varphi^l(0)\right) \nabla_{x'} \bar{v}(x) \\ &\quad + \Big[\nabla_{x'}\varphi^l(x', h(x')) + \partial_{x_d}\varphi^l(x', h(x'))\nabla_{x'} h(x')\Big](1 - \bar{v})(x), \\ \partial_{x_d} \tilde{u}_{0l}^l &= -\left(\varphi^l(x', h(x')) - \varphi^l(0)\right) \partial_{x_d} \bar{v}(x). \end{aligned}$$

By the Taylor expansion and (2.7)–(2.8),

$$\begin{aligned}\varphi^l(x', h(x')) = &\varphi^l(0) + \nabla_{x'}\varphi^l(0)x' \\ &+ \frac{1}{2}x'^T\left[\nabla^2_{x'}\varphi^l(0) + \partial_{x_d}\varphi^l(0)\nabla^2_{x'}h(0')\right]x' + O(|x'|^{2+\gamma}).\end{aligned} \tag{3.44}$$

Hence, using (3.4), we have

$$|\nabla_{x'}\tilde{u}_{0l}(x)| \leq \frac{C|\nabla_{x'}\varphi^l(0)||x'|^2}{\varepsilon + |x'|^2} + C\|\varphi^l\|_{C^2(\partial D)} \leq C\|\varphi^l\|_{C^2(\partial D)}, \ x \in \Omega_R, \tag{3.45}$$

and

$$\frac{|\varphi^l(x', h(x')) - \varphi^l(0)|}{C(\varepsilon + |x'|^2)} \leq |\partial_{x_d}\tilde{u}_{0l}(x)| \leq \frac{C|\nabla_{x'}\varphi^l(0)||x'|}{\varepsilon + |x'|^2} + C\|\varphi^l\|_{C^2(\partial D)}, \ x \in \Omega_R. \tag{3.46}$$

Adapting the proof of Lemma 3.1–Lemma 3.2 and using (3.44), we obtain

$$|\nabla(u_{0l} - \tilde{u}_{0l})(x)| \leq \frac{C|\nabla_{x'}\varphi^l(0)||x'|}{\sqrt{\varepsilon + |x'|^2}} + C\|\varphi^l\|_{C^2(\partial D)} \leq C\|\varphi^l\|_{C^2(\partial D)}, \quad x \in \Omega_R,$$

which, together with (3.45) and (3.46), implies that (3.42) and (3.43). □

4. Proof of Proposition 2.4 and estimates of C^α

In this Section, we are devoted to prove Proposition 2.4 under the normalization $\|\varphi\|_{C^2(\partial D;\mathbb{R}^d)} = 1$.

Denote

$$a_{\alpha\beta} := -\int\limits_{\partial D_1} \frac{\partial u_\alpha}{\partial \nu_0}\bigg|_+ \cdot \psi_\beta, \quad b_\beta := \int\limits_{\partial D_1} \frac{\partial u_0}{\partial \nu_0}\bigg|_+ \cdot \psi_\beta, \quad \alpha, \beta = 1, 2, \cdots, \frac{d(d+1)}{2}.$$

Multiplying the first line of (2.3) and (2.4), by u_β, respectively, and applying integration by parts over Ω leads to

$$a_{\alpha\beta} = \int\limits_\Omega (\mathbb{C}^0 e(u_\alpha), e(u_\beta))dx, \quad b_\beta = -\int\limits_\Omega (\mathbb{C}^0 e(u_0), e(u_\beta))dx.$$

By (2.5) and the linearity of $e(u)$,

$$e(u) = \sum_{\alpha=1}^{d}(C^\alpha - \varphi^\alpha(0))e(u_\alpha) + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha e(u_\alpha) + e(u_0), \quad \text{in } \Omega.$$

Then, it follows from the forth line of (2.1) that for $\beta = 1, 2, \cdots, \frac{d(d+1)}{2}$,

$$\sum_{\alpha=1}^{d} (C^\alpha - \varphi^\alpha(0)) a_{\alpha\beta} + \sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha a_{\alpha\beta} = b_\beta. \tag{4.1}$$

Denote

$$X^1 = (C^1 - \varphi^1(0), \cdots, C^d - \varphi^d(0))^T, \quad X^2 = (C^{d+1}, \cdots, C^{\frac{d(d+1)}{2}})^T,$$
$$P^1 = (b_1, \cdots, b_d)^T, \quad P^2 = (b_{d+1}, \cdots, b_{\frac{d(d+1)}{2}})^T,$$

and

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}, \quad B = \begin{pmatrix} a_{1\ d+1} & \cdots & a_{1\ \frac{d(d+1)}{2}} \\ \vdots & \ddots & \vdots \\ a_{d\ d+1} & \cdots & a_{d\ \frac{d(d+1)}{2}} \end{pmatrix},$$
$$D = \begin{pmatrix} a_{d+1\ d+1} & \cdots & a_{d+1\ \frac{d(d+1)}{2}} \\ \vdots & \ddots & \vdots \\ a_{\frac{d(d+1)}{2}\ d+1} & \cdots & a_{\frac{d(d+1)}{2}\ \frac{d(d+1)}{2}} \end{pmatrix}.$$

Thus, by using the symmetry property of $a_{\alpha\beta}$, (4.1) can be rewritten as

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} \tag{4.2}$$

Lemma 4.1. *There exists a positive universal constant C, independent of ε, such that*

$$\sum_{\alpha,\beta=1}^{\frac{d(d+1)}{2}} a_{\alpha\beta}\xi_\alpha\xi_\beta \geq \frac{1}{C}, \quad \forall\ \xi \in \mathbb{R}^{\frac{d(d+1)}{2}},\ |\xi| = 1. \tag{4.3}$$

Proof. To emphasize the dependence on ε, we use $\Omega_\varepsilon := D \setminus \overline{D_1}$ and u_α^ε to denote the corresponding solution of (2.3) with $\alpha = 1, \cdots, d$. For $\xi \in \mathbb{R}^{\frac{d(d+1)}{2}}$ with $|\xi| = 1$, using (1.6), we have

$$\sum_{\alpha,\beta=1}^{\frac{d(d+1)}{2}} a_{\alpha\beta}\xi_\alpha\xi_\beta = \int_\Omega \left(\mathbb{C}^0 e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha u_\alpha^\varepsilon \Big), e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\beta u_\beta^\varepsilon \Big) \right) dx$$
$$\geq \frac{1}{C} \int_\Omega \left| e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha u_\alpha^\varepsilon \Big) \right|^2 dx.$$

We claim that there exists a constant $C > 0$, independent of ε, such that

$$\int_{\Omega} \left| e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha u_\alpha^\varepsilon \Big) \right|^2 dx \geq \frac{1}{C}, \quad \forall\, \xi \in \mathbb{R}^{\frac{d(d+1)}{2}},\ |\xi| = 1.$$

Indeed, if not, then there exist $\varepsilon_i \to 0^+$, $|\xi^i| = 1$, such that

$$\int_{\Omega} \left| e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \xi_\alpha^i u_\alpha^{\varepsilon_i} \Big) \right|^2 dx \to 0, \quad \text{as } i \to \infty. \tag{4.4}$$

Here and in the following proof, we use the notations $D_1^* := \{\ x \in \mathbb{R}^d \mid x + (0', \varepsilon) \in D_1\ \}$, $\Omega^* := D \setminus \overline{D_1^*}$. Since $u_\alpha^{\varepsilon_i} = 0$ on ∂D, it follows from the second Korn's inequality (see Theorem 2.5 in [36]) that there exists a constant C, independent of ε_i, such that

$$\|u_\alpha^{\varepsilon_i}\|_{H^1(\Omega_\varepsilon \setminus B_{\bar{r}};\mathbb{R}^d)} \leq C,$$

where $\bar{r} > 0$ is fixed. Then there exists a subsequence, we still denote $\{u_\alpha^{\varepsilon_i}\}$, such that

$$u_\alpha^{\varepsilon_i} \rightharpoonup \bar{u}_\alpha, \quad \text{in } H^1(\Omega_\varepsilon \setminus B_{\bar{r}}; \mathbb{R}^d), \quad \text{as} \quad i \to \infty.$$

By (4.4), there exists $\bar{\xi}$ such that

$$\xi^i \to \bar{\xi}, \quad \text{as } i \to \infty, \quad \text{with } |\bar{\xi}| = 1,$$

and

$$\int_{\Omega^*} \left| e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha \Big) \right|^2 dx = 0.$$

This implies that

$$e\Big(\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha \Big) = 0, \quad \text{in } \Omega^*.$$

That means that $\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha \in \Psi$ in Ω^*. Hence, there exist some constants c_β, $\beta = 1, 2, \cdots, \frac{d(d+1)}{2}$, such that

$$\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha = \sum_{\beta=1}^{\frac{d(d+1)}{2}} c_\beta \psi_\beta, \quad \text{in } \Omega^*.$$

Since $\sum_{\beta=1}^{\frac{d(d+1)}{2}} c_\beta \psi_\beta = 0$ on ∂D, it follows from Lemma 6.1 in [16] that $c_\beta = 0$, $\beta = 1, \cdots, \frac{d(d+1)}{2}$. Thus,

$$\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \bar{u}_\alpha = 0, \quad \text{in } \Omega^*.$$

Restricted on ∂D_1^*, it says that $\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \bar{\xi}_\alpha \psi_\alpha = 0$ on ∂D_1^*. This yields, using again Lemma 6.1 in [16], $\bar{\xi}_\alpha = 0, \alpha = 1, \cdots, d$, which contradicts with $|\bar{\xi}| = 1$. $\square$

Lemma 4.2. *For $d \geq 2$, we have*

$$\frac{1}{C\rho_d(\varepsilon)} \leq a_{\alpha\alpha} \leq \frac{C}{\rho_d(\varepsilon)}, \quad \alpha = 1, \cdots, d; \tag{4.5}$$

$$\frac{1}{C} \leq a_{\alpha\alpha} \leq C, \quad \alpha = d+1, \cdots, \frac{d(d+1)}{2}; \tag{4.6}$$

$$a_{\alpha\beta} \leq C, \quad \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \ \beta = d+1, \cdots, \frac{d(d+1)}{2}, \alpha \neq \beta; \tag{4.7}$$

and if $d = 2$, then

$$|a_{12}| = |a_{21}| \leq C|\log \varepsilon|; \tag{4.8}$$

if $d \geq 3$, then

$$|a_{\alpha\beta}| = |a_{\beta\alpha}| \leq C, \quad \alpha, \beta = 1, \cdots, d, \ \alpha \neq \beta. \tag{4.9}$$

Consequently,

$$\frac{1}{C(\rho_d(\varepsilon))^d} \leq detA \leq \frac{C}{(\rho_d(\varepsilon))^d}, \quad \frac{1}{C} I \leq D \leq CI. \tag{4.10}$$

Proof. STEP 1. Proof of (4.5). In view of (2.18), (3.37) and (3.38), we have, for $\alpha = 1, \cdots, d$,

$$\begin{aligned} a_{\alpha\alpha} &= \int_\Omega (\mathbb{C}^0 e(u_\alpha), e(u_\alpha)) dx \leq C \int_\Omega |\nabla u_\alpha|^2 dx \\ &\leq C \int_{\Omega_R} \frac{dx}{(\varepsilon + |x'|^2)^2} + C \\ &\leq C \int_{|x'|<R} \frac{dx'}{\varepsilon + |x'|^2} + C \\ &= C \int_0^R \frac{\rho^{d-2}}{\varepsilon + \rho^2} d\rho + C \leq \frac{C}{\rho_d(\varepsilon)}, \end{aligned}$$

and

$$a_{\alpha\alpha} = \int_\Omega (\mathbb{C}^0 e(u_\alpha), e(u_\alpha))dx \geq \frac{1}{C}\int_\Omega |e(u_\alpha)|^2 dx$$
$$\geq \frac{1}{C}\int_\Omega |e_{\alpha d}(u_\alpha)|^2 dx \geq \frac{1}{C}\int_{\Omega_R} |\partial_{x_d} u_\alpha^\alpha|^2 dx.$$

Notice that $u_\alpha^\alpha|_{\partial D_1} = \bar{v}|_{\partial D_1} = 1, u_\alpha^\alpha|_{\partial D} = \bar{v}|_{\partial D} = 0$, and recalling the definition of $\bar{v}$, $\bar{v}(x', x_d)$ is linear in x_d for fixed x', so $\bar{v}(x', \cdot)$ is harmonic, hence its energy is minimal, that is

$$\int_{h(x')}^{h_1(x')+\varepsilon} |\partial_{x_d} u_\alpha^\alpha|^2 dx_d \geq \int_{h(x')}^{h_1(x')+\varepsilon} |\partial_{x_d} \bar{v}|^2 dx_d = \frac{1}{\varepsilon + h_1(x') - h(x')}.$$

Integrating on $B_R(0')$ for x', we obtain

$$\int_{\Omega_R} |\partial_{x_d} u_\alpha^\alpha|^2 dx = \int_{|x'|<R} \int_{h(x')}^{h_1(x')+\varepsilon} |\partial_{x_d} u_\alpha^\alpha|^2 dx_d dx'$$
$$\geq \frac{1}{C}\int_{|x'|<R} \frac{dx'}{\varepsilon + |x'|^2} \geq \frac{1}{C\rho_d(\varepsilon)}.$$

Estimate (4.5) is proved.

STEP 2. Proof of (4.6) and (4.7). By means of (2.13), for $\alpha, \beta = d+1, \cdots, \frac{d(d+1)}{2}$, we have

$$a_{\alpha\beta} = \int_\Omega (\mathbb{C}^0 e(u_\alpha), e(u_\beta))dx \leq C\int_\Omega |\nabla u_\alpha||\nabla u_\beta| dx$$
$$\leq C\int_{\Omega_R} \frac{(\varepsilon + |x'|)^2}{(\varepsilon + |x'|^2)^2} dx + C \leq C.$$

On the other hand, it follows immediately from Lemma 4.1 that there exists a universal constant C such that

$$a_{\alpha\alpha} \geq \frac{1}{C}, \quad \alpha = d+1, \cdots, \frac{d(d+1)}{2}.$$

We now consider the elements for $\alpha = 1, 2, \cdots, d$, $\beta = d+1, d+2, \cdots, \frac{d(d+1)}{2}$. We take the case that $\alpha = 1$, $\beta = d+1$ for instance. The other cases are the same. Let $\psi_{d+1} = (x_2, -x_1, 0, \cdots, 0)^T$. Then using (3.37) and the boundedness of $|\nabla u_\alpha|$ on $\partial D_1 \setminus B_R$, we have

$$
\begin{aligned}
a_{1(d+1)} &= -\int\limits_{\partial D_1} \frac{\partial u_1}{\partial \nu_0}\Big|_{+} \cdot \psi_{d+1} \\
&= -\int\limits_{\partial D_1 \cap B_R} \Big(\lambda(\nabla \cdot u_1)\vec{n} + \mu(\nabla u_\alpha + (\nabla u_1)^T)\vec{n}\Big) \cdot (x_2, -x_1, 0, \cdots, 0)^T \\
&= -\int\limits_{\partial D_1 \cap B_R} \left(\lambda(\sum_{k=1}^{d} \partial_{x_k} u_1^k) n_1 + \mu \sum_{l=1}^{d} \left(\partial_{x_1} u_1^l + \partial_{x_l} u_1^1\right) n_l\right) x_2 \\
&\quad + \int\limits_{\partial D_1 \cap B_R} \left(\lambda(\sum_{k=1}^{d} \partial_{x_k} u_1^k) n_2 + \mu \sum_{l=1}^{d} \left(\partial_{x_2} u_1^l + \partial_{x_l} u_1^2\right) n_l\right) x_1
\end{aligned}
$$

is bounded for $d \geq 2$, so $a_{1(d+1)}$.

Thus, estimates (4.6) and (4.7) are established.

STEP 3. Proof of (4.8) and (4.9). Firstly, we estimate $|a_{\alpha\beta}|$ for $\alpha, \beta = 1, \cdots, d$ with $\alpha \neq \beta$. By the definition,

$$
\begin{aligned}
a_{\alpha\beta} = a_{\beta\alpha} &= -\int\limits_{\partial D_1} \frac{\partial u_\alpha}{\partial \nu_0}\Big|_{+} \cdot \psi_\beta \\
&= -\int\limits_{\partial D_1} \lambda(\nabla \cdot u_\alpha) n_\beta + \mu \left((\nabla u_\alpha + (\nabla u_\alpha)^T)\vec{n}\right)_\beta \\
&= -\int\limits_{\partial D_1} \lambda \left(\sum_{k=1}^{d} \partial_{x_k} u_\alpha^k\right) n_\beta + \mu \sum_{l=1}^{d} \left(\partial_{x_\beta} u_\alpha^l + \partial_{x_l} u_\alpha^\beta\right) n_l.
\end{aligned}
$$

Denote

$$
\mathrm{I}_{\alpha\beta} := \int\limits_{\partial D_1 \cap B_R} \left(\sum_{k=1}^{d} \partial_{x_k} u_\alpha^k\right) n_\beta;
$$

and

$$
\begin{aligned}
\mathrm{II}_{\alpha\beta} &:= \int\limits_{\partial D_1 \cap B_R} \sum_{l=1}^{d} \left(\partial_{x_\beta} u_\alpha^l + \partial_{x_l} u_\alpha^\beta\right) n_l \\
&= \int\limits_{\partial D_1 \cap B_R} \sum_{l=1}^{d-1} \left(\partial_{x_\beta} u_\alpha^l + \partial_{x_l} u_\alpha^\beta\right) n_l + \int\limits_{\partial D_1 \cap B_R} \partial_{x_\beta} u_\alpha^d n_d + \int\limits_{\partial D_1 \cap B_R} \partial_{x_d} u_\alpha^\beta n_d \\
&=: \mathrm{II}_{\alpha\beta}^1 + \mathrm{II}_{\alpha\beta}^2 + \mathrm{II}_{\alpha\beta}^3,
\end{aligned}
$$

where

$$
\vec{n} = \frac{(-\nabla_{x'} h(x'), 1)}{\sqrt{1 + |\nabla_{x'} h(x')|^2}}.
$$

Due to (2.7), for $k = 1, 2, \cdots, d-1$,

$$|n_k| = \left| \frac{-\partial_{x_k} h(x')}{\sqrt{1 + |\nabla_{x'} h(x')|^2}} \right| \leq C|x'|, \quad \text{and } |n_d| = \frac{1}{\sqrt{1 + |\nabla_{x'} h(x')|^2}} \leq 1. \tag{4.11}$$

For $\alpha = 1, 2, \cdots, d$, $\beta = 1, 2, \cdots, d-1$, it follows from (3.37) and (4.11) that

$$\begin{aligned} |\mathrm{I}_{\alpha\beta}| &\leq \int\limits_{\partial D_1 \cap B_R} \left| \left(\sum_{k=1}^{d} \partial_{x_k} u_\alpha^k \right) n_\beta \right| \\ &\leq \int\limits_{\partial D_1 \cap B_R} \frac{C|x'|}{\varepsilon + |x'|^2} \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3, \end{cases} \end{aligned} \tag{4.12}$$

while,

$$|\mathrm{II}_{\alpha\beta}^1| \leq \int\limits_{\partial D_1 \cap B_R} \left| \sum_{l=1}^{d-1} \left(\partial_{x_\beta} u_\alpha^l + \partial_{x_l} u_\alpha^\beta \right) n_l \right| \leq \int\limits_{\partial D_1 \cap B_R} \frac{C|x'|}{\sqrt{\varepsilon + |x'|^2}} \leq C,$$

$$|\mathrm{II}_{\alpha\beta}^2| \leq \int\limits_{\partial D_1 \cap B_R} \left| \partial_{x_\beta} u_\alpha^d n_d \right| \leq \int\limits_{\partial D_1 \cap B_R} \frac{C}{\sqrt{\varepsilon + |x'|^2}} \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3, \end{cases}$$

and by the definition of $\tilde{u}_\alpha$ and (3.36),

$$\begin{aligned} |\mathrm{II}_{\alpha\beta}^3| \leq \int\limits_{\partial D_1 \cap B_R} \left| \partial_{x_d} u_\alpha^\beta n_d \right| &\leq \int\limits_{\partial D_1 \cap B_R} |(\partial_{x_d} \tilde{u}_\alpha^\beta) n_d| + \int\limits_{\partial D_1 \cap B_R} |(\partial_{x_d} (u_\alpha^\beta - \tilde{u}_\alpha^\beta) n_d| \\ &\leq \int\limits_{\partial D_1 \cap B_R} \frac{C}{\sqrt{\varepsilon + |x'|^2}} \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3. \end{cases} \end{aligned}$$

Here we used the fact that $\tilde{u}_\alpha^\beta = 0$ if $\alpha \neq \beta$. Hence,

$$|\mathrm{II}_{\alpha\beta}| \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3. \end{cases}$$

This, together with (4.12), the boundedness of $|\nabla u_\alpha|$ on $\partial D_1 \setminus B_R$, and the symmetry of $a_{\alpha\beta} = a_{\beta\alpha}$, implies that for $\alpha, \beta = 1, \cdots, d$ with $\alpha \neq \beta$,

$$|a_{\alpha\beta}| = |a_{\beta\alpha}| \leq |\lambda||I_{\alpha\beta}| + |\mu||II_{\alpha\beta}| + C \leq \begin{cases} C|\log \varepsilon|, & d = 2, \\ C, & d \geq 3. \end{cases}$$

Therefore, (4.8) and (4.9) are proved. (4.10) is an immediate consequence of (4.5)–(4.9). The proof of the Lemma 4.2 is finished. □

Lemma 4.3.

$$|b_\beta| \leq C, \quad \beta = 1, \cdots, \frac{d(d+1)}{2}. \tag{4.13}$$

Consequently,

$$|P^i| \leq C, \quad i = 1, 2. \tag{4.14}$$

Proof. STEP 1. To estimate $|b_\beta|$ for $\beta = 1, \cdots, d$. We take $\beta = 1$ for instance. The other cases are the same. Denote

$$b_1 = \sum_{l=1}^{d} \int\limits_{\partial D_1} \frac{\partial u_{0l}}{\partial \nu_0}\Big|_+ \cdot \psi_1 := \sum_{l=1}^{d} b_{1l},$$

where u_{0l}, $l = 1, 2, \cdots, d$, is defined by (3.39). By definition,

$$\begin{aligned} b_{11} &= \int\limits_{\partial D_1} \frac{\partial u_{01}}{\partial \nu_0}\Big|_+ \cdot \psi_1 \\ &= \int\limits_{\partial D_1} \left[\lambda(\nabla \cdot u_{01}) n_1 + \mu \left((\nabla u_{01} + (\nabla u_{01})^T)\vec{n}\right)_1\right] \\ &= \int\limits_{\partial D_1} \left[\lambda \sum_{k=1}^{d} \partial_{x_k} u_{01}^k n_1 + \mu \sum_{i=1}^{d} (\partial_{x_1} u_{01}^i + \partial_{x_i} u_{01}^1) n_i\right]. \end{aligned}$$

Denote

$$\mathrm{I} := \int\limits_{\partial D_1} \sum_{k=1}^{d} \partial_{x_k} u_{01}^k n_1 = \int\limits_{\partial D_1} \sum_{k=1}^{d-1} \partial_{x_k} u_{01}^k n_1 + \int\limits_{\partial D_1} \partial_{x_d} u_{01}^d n_1 =: \mathrm{I}_1 + \mathrm{I}_2,$$

and

$$\begin{aligned} \mathrm{II} &:= \int\limits_{\partial D_1} \sum_{i=1}^{d} (\partial_{x_1} u_{01}^i + \partial_{x_i} u_{01}^1) n_i \\ &= \int\limits_{\partial D_1} \sum_{i=1}^{d-1} (\partial_{x_1} u_{01}^i + \partial_{x_i} u_{01}^1) n_i + \int\limits_{\partial D_1} \partial_{x_1} u_{01}^d n_d + \int\limits_{\partial D_1} \partial_{x_d} u_{01}^1 n_d \\ &=: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3. \end{aligned}$$

According to (3.42)–(3.43),

$$|\mathrm{I}_1| \leq \left| \int\limits_{\partial D_1} \sum_{k=1}^{d-1} \partial_{x_k} u_{01}^k n_1 \right| \leq \int\limits_{\partial D_1 \cap B_R} C|x'| + C \leq C;$$

$$|\mathrm{I}_2| \le \left| \int\limits_{\partial D_1} \partial_{x_d} u_{01}^d n_1 \right| \le \int\limits_{\partial D_1 \cap B_R} \frac{C|\nabla_{x'}\varphi^1(0)||x'|^2}{\varepsilon + |x'|^2} + C \le C.$$

So that,

$$|\mathrm{I}| \le |\mathrm{I}_1| + |\mathrm{I}_2| \le C. \tag{4.15}$$

By (3.42), (4.11) and the definition of $\tilde{u}_{01}$,

$$|\mathrm{II}_1| \le \left| \int\limits_{\partial D_1} \sum_{i=1}^{d-1} (\partial_{x_1} u_{01}^i + \partial_{x_i} u_{01}^1) n_i \right| \le \int\limits_{\partial D_1 \cap B_R} C|x'| + C \le C, \tag{4.16}$$

and

$$\begin{aligned} |\mathrm{II}_2| \le & \int\limits_{\partial D_1} |\partial_{x_1} u_{01}^d n_d| \\ \le & \int\limits_{\partial D_1 \cap B_R} |\partial_{x_1} \tilde{u}_{01}^d n_d| + \int\limits_{\partial D_1 \cap B_R} |\partial_{x_1} (u_{01}^d - \tilde{u}_{01}^d) n_d| + C \\ \le & C|\partial D_1 \cap B_R| + C \le C. \end{aligned} \tag{4.17}$$

Now, we need only to estimate II_3. Note that

$$\mathrm{II}_3 = \int\limits_{\partial D_1 \cap B_R} \partial_{x_d} \tilde{u}_{01}^1 n_d + \int\limits_{\partial D_1 \cap B_R} \partial_{x_d} (u_{01}^1 - \tilde{u}_{01}^1) n_d =: \mathrm{II}_3^1 + \mathrm{II}_3^2.$$

By the definitions of $\tilde{u}_{01}^1$ and $\bar{v}$,

$$\partial_{x_d} \tilde{u}_{01}^1 = -(\varphi^1(x', h(x')) - \varphi^1(0)) \partial_{x_d} \bar{v} = -\frac{\varphi^1(x', h(x')) - \varphi^1(0)}{\varepsilon + h_1(x') - h(x')}.$$

From the expression of $\partial D_1 \cap B_R : x_d = \varepsilon + h_1(x'), \ |x'| < R$, we have $dS = \sqrt{1 + |\nabla_{x'} h_1(x')|^2} dx'$. Then, by the Taylor expansion (3.44), we have

$$\begin{aligned} \mathrm{II}_3^1 = & \int\limits_{\partial D_1 \cap B_R} \partial_{x_d} \tilde{u}_{01}^1 n_d \\ = & \int\limits_{|x'| < R} \frac{-(\varphi^1(x', h(x')) - \varphi^1(0))}{\varepsilon + h_1(x') - h(x')} dx' \\ = & -\int\limits_{|x'| < R} \frac{\nabla_{x'} \varphi^1(0) x' + O(|x'|^2)}{\varepsilon + h_1(x') - h(x')} dx'. \end{aligned}$$

Since

$$\frac{1}{C(\varepsilon+|x'|^2)} \leq \frac{1}{\varepsilon+h_1(x')-h(x')} \leq \frac{C}{\varepsilon+|x'|^2}, \quad |x'| \leq R, \tag{4.18}$$

it follows that

$$\left| \int\limits_{|x'|<R} \frac{O(|x'|^2)}{\varepsilon+h_1(x')-h(x')} dx' \right| \leq C.$$

While, according to (4.18), we have

$$\begin{aligned} &\int\limits_{|x'|<R} \frac{\nabla_{x'}\varphi^1(0)x'}{\varepsilon+h_1(x')-h(x')} dx' \\ &= \int\limits_{|x'|<R} \frac{\nabla_{x'}\varphi^1(0)x'}{\varepsilon+\frac{1}{2}x'^T(\nabla^2_{x'}(h_1-h)(0'))x'} dx' \\ &+ \int\limits_{|x'|<R} \frac{O(|x'|^{3+\gamma})}{(\varepsilon+\frac{1}{2}x'^T(\nabla^2_{x'}(h_1-h)(0'))x')(\varepsilon+h_1(x')-h(x'))} dx'. \end{aligned}$$

For the positive matrix $(\nabla^2_{x'}(h_1-h)(0'))$, there exists orthogonal matrix O, such that

$$O^T(\nabla^2_{x'}(h_1-h)(0'))O = diag(\lambda_1, \cdots, \lambda_{d-1}),$$

where $\lambda_i \geq \kappa_1,\ i=1,\cdots,d-1$. Under the orthogonal transform $y'=Ox'$, we obtain

$$\int\limits_{|x'|<R} \frac{\nabla_{x'}\varphi^1(0)x'}{\varepsilon+\frac{1}{2}x'^T(\nabla^2_{x'}(h_1-h)(0'))x'} dx' = \int\limits_{|y'|<R} \frac{\nabla_{x'}\varphi^1(0)O^Ty'}{\varepsilon+\Sigma_{i=1}^{d-1}\lambda_i y_i^2} dy' = 0,$$

and

$$\left| \int\limits_{|x'|<R} \frac{O(|x'|^{3+\gamma})}{(\varepsilon+\frac{1}{2}x'^T(\nabla^2_{x'}(h_1-h)(0'))x')(\varepsilon+h_1(x')-h(x'))} dx' \right| \leq C.$$

Therefore,

$$|\mathrm{II}_3^1| = \left| \int\limits_{\partial D_1 \cap B_R} \partial_{x_d}\tilde{u}_{01}^1 n_d \right| \leq C. \tag{4.19}$$

On the other hand, in view of (3.41),

$$|\mathrm{II}_3^2| \le \left| \int\limits_{\partial D_1 \cap B_R} \partial_{x_d}(u_{01}^1 - \tilde{u}_{01}^1) n_d \right| \le C.$$

This, together with (4.19), implies that

$$|\mathrm{II}_3| \le C. \tag{4.20}$$

Combining (4.15)–(4.17) and (4.20), we have

$$|b_{11}| \le C.$$

Next, for $l = 2, \cdots, d$,

$$\begin{aligned} b_{1l} &= \int\limits_{\partial D_1} \frac{\partial u_{0l}}{\partial \nu_0}\Big|_+ \cdot \psi_1 \\ &= \int\limits_{\partial D_1} \left[\lambda(\nabla \cdot u_{0l}) n_1 + \mu \left((\nabla u_{0l} + (\nabla u_{0l})^T)\vec{n}\right)_1\right] \\ &= \int\limits_{\partial D_1} \left[\lambda \sum_{k=1}^d \partial_{x_k} u_{0l}^k n_1 + \mu \sum_{i=1}^d (\partial_{x_1} u_{0l}^i + \partial_{x_i} u_{0l}^1) n_i\right]. \end{aligned}$$

Similarly, making use of (3.42), (3.43) and (4.11), we have

$$\left| \int\limits_{\partial D_1} \sum_{k=1}^d \partial_{x_k} u_{0l}^k n_1 \right| \le \left| \int\limits_{\partial D_1} \sum_{k=1}^{d-1} \partial_{x_k} u_{0l}^k n_1 \right| + \left| \int\limits_{\partial D_1} \partial_{x_d} u_{0l}^d n_1 \right| \le C, \tag{4.21}$$

and recalling the definition of $\tilde{u}_{0l}$, and $\tilde{u}_{0l}^1 = 0$,

$$\begin{aligned} &\left| \int\limits_{\partial D_1} \sum_{i=1}^d (\partial_{x_1} u_{0l}^i + \partial_{x_i} u_{0l}^1) n_i \right| \\ \le &\left| \int\limits_{\partial D_1} \sum_{i=1}^{d-1} (\partial_{x_1} u_{0l}^i + \partial_{x_i} u_{0l}^1) n_i \right| + \left| \int\limits_{\partial D_1} \partial_{x_1} (u_{0l}^d - \tilde{u}_{0l}^d) n_d \right| \\ &+ \left| \int\limits_{\partial D_1} \partial_{x_1} \tilde{u}_{0l}^d n_d \right| + \left| \int\limits_{\partial D_1} \partial_{x_d} (u_{0l}^1 - \tilde{u}_{0l}^1) n_d \right| \\ \le &\, C. \end{aligned}$$

This implies that

$$|b_{1l}| \le C, \quad l = 2, \cdots, d.$$

Hence,

$$|b_1| \leq C.$$

STEP 2. To estimate $|b_\beta|$ for $\beta = d+1, \cdots, \frac{d(d+1)}{2}$. By using (2.13) and (2.14), we have

$$\begin{aligned}
|b_\beta| &= \left| \int\limits_\Omega \left(\mathbb{C}^0 e(u_0), e(u_\beta)\right) dx \right| \\
&\leq C \int\limits_\Omega |\nabla u_0||\nabla u_\beta| dx \\
&\leq \int\limits_{\Omega_R} \frac{C|\nabla_{x'}\varphi(0)||x'|(\varepsilon + |x'|)}{(\varepsilon + |x'|^2)^2} dx + C \\
&\leq C.
\end{aligned}$$

The proof of Lemma 4.3 is completed. □

Proof of Proposition 2.4. Step 1. Proof of (2.15).

Let u^ε be the solution of (2.1). By Theorem 6.6 in the appendix in [15], u^ε is the minimizer of

$$I_\infty[u] := \frac{1}{2} \int\limits_\Omega \left(\mathbb{C}^0 e(u), e(u)\right) dx$$

on $\mathcal{A}$ defined by (1.4). It follows that

$$\|u^\varepsilon\|^2_{H^1(\Omega)} \leq C\|e(u^\varepsilon)\|^2_{L^2(\Omega)} \leq C I_\infty[u^\varepsilon] \leq C.$$

By the Sobolev trace embedding theorem,

$$\|u^\varepsilon\|_{L^2(\partial D_1 \cap B_R)} \leq C.$$

Recalling that

$$u^\varepsilon = \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C^\alpha \psi_\alpha, \quad \text{on } \partial D_1.$$

If $\mathcal{C} := (C^1, C^2, \cdots, C^{\frac{d(d+1)}{2}}) = 0$, there is nothing to prove. Otherwise,

$$C \geq |\mathcal{C}| \left\| \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \widehat{C}_\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \cap B_R)}, \tag{4.22}$$

where $\widehat{C}_\alpha = \frac{C^\alpha}{|C|}$ and $|\widehat{C}| = 1$. It is easy to see that

$$\left\| \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \widehat{C}_\alpha \psi_\alpha \right\|_{L^2(\partial D_1 \cap B_R)} \geq \frac{1}{C}. \tag{4.23}$$

Indeed, if not, along a subsequence $\varepsilon \to 0$, $\widehat{C}_\alpha \to \overline{C}_\alpha$, and

$$\left\| \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \overline{C}_\alpha \psi_\alpha \right\|_{L^2(\partial D_1^* \cap B_R)} = 0,$$

where ∂D_1^* is the limit of ∂D_1 as $\varepsilon \to 0$ and $|\overline{C}| = 1$. This implies

$$\sum_{\alpha=1}^{\frac{d(d+1)}{2}} \overline{C}_\alpha \psi_\alpha = 0 \quad \text{on } \partial D_1^* \cap B_R.$$

But $\{\psi_\alpha|_{\partial D_1^* \cap B_R}\}$ is easily seen to be linear independent, according to Lemma 6.1 in the appendix of [16], we must have $\overline{C} = 0$. This is a contradiction. (2.15) follows from (4.22) and (4.23).

Step 2. Proof of (2.16). According to Lemma 4.1, the matrix

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

is positive definite, so invertible. Moreover,

$$A \geq \frac{1}{C} I_{d\times d}, \qquad D \geq \frac{1}{C} I_{\frac{d(d-1)}{2} \times \frac{d(d-1)}{2}}. \tag{4.24}$$

Therefore, from (4.2), we have

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} \begin{pmatrix} P^1 \\ P^2 \end{pmatrix}.$$

For $d \geq 4$, it is easy to see from Lemma 4.1 and Lemma 4.3 that

$$|X^1| \leq C.$$

Next, we prove (2.16) for $d = 2, 3$. By Lemma 6.2 in Appendix of [16] and Lemma 4.2,

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} + (Errors),$$

where

$$|(Errors)| \sim o(\rho_d(\varepsilon)). \tag{4.25}$$

Then,

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} + (Errors) \begin{pmatrix} P^1 \\ P^2 \end{pmatrix} = \begin{pmatrix} A^{-1}P^1 \\ D^{-1}P^2 \end{pmatrix} + (Errors).$$

Therefore,

$$X^1 = A^{-1}P^1 + Errors = \frac{1}{\det A} A^* P^1 + Errors, \tag{4.26}$$

where $A^* = \left(a^*_{\alpha\beta}\right)$ is the adjoint matrix of A. Following Lemma 4.2, it is clear that

$$A^* \sim \begin{pmatrix} \frac{\tilde{c}_1}{(\rho_d(\varepsilon))^{d-1}} & \cdots & o\left(\frac{1}{(\rho_d(\varepsilon))^{d-1}}\right) \\ \vdots & \ddots & \vdots \\ o\left(\frac{1}{(\rho_d(\varepsilon))^{d-1}}\right) & \cdots & \frac{\tilde{c}_d}{(\rho_d(\varepsilon))^{d-1}} \end{pmatrix},$$

for some constants $\tilde{c}_\alpha \neq 0, \alpha = 1, \cdots, d$, independent of ε. In view of (4.10) and (4.14), we obtain

$$|X^1| \leq C\rho_d(\varepsilon).$$

Therefore,

$$|C^\alpha - \varphi^\alpha(0)| \leq C\rho_d(\varepsilon), \quad \alpha = 1, \cdots, d.$$

Proposition 2.4 is established. □

5. Proof of Theorem 1.3 (Lower bound)

In order to prove Theorem 1.3, we first prove $b_\beta \to b^*_\beta$, as $\varepsilon \to 0$, $\beta = 1, \cdots, d$.

Lemma 5.1. *For $d \geq 3$, $\beta = 1, 2, \cdots, d$,*

$$|b_\beta - b^*_\beta| \leq C\left(|\nabla_{x'}\varphi(0)| + \|\nabla^2\varphi\|_{L^\infty(\partial D)}\right)\varepsilon^{\gamma_d};$$

for $d = 2$, if $\nabla_{x'}\varphi^\beta(0) = 0$ for $\beta = 1$ or 2, then, for $\alpha \neq \beta$,

$$|b_\beta - b^*_\beta| \leq C\left(|\nabla_{x'}\varphi^\alpha(0)| + \|\nabla^2\varphi\|_{L^\infty(\partial D)}\right)\varepsilon^{\gamma_2},$$

where

$$\gamma_d = \begin{cases} \frac{d-2}{2(d-1)}, & d \geq 3, \\ \frac{1}{6}, & d = 2. \end{cases}$$

Consequently,

$$b_\beta \to b_\beta^*, \quad \textit{as } \varepsilon \to 0, \quad \beta = 1, 2, \cdots, d.$$

Proof. We here prove the case $\beta = 1$ for instance. The other cases are the same. It follows from the definitions of u_0 and u_1 and the integration by parts formula (1.5) that

$$b_1 = \int\limits_{\partial D_1} \frac{\partial u_0}{\partial \nu_0}\Big|_+ \cdot u_1 = \int\limits_{\Omega} \left(\mathbb{C}^0 e(u_1), e(u_0)\right) = \int\limits_{\partial D} \frac{\partial u_1}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0)).$$

Similarly,

$$b_1^* = \int\limits_{\partial D_1^*} \frac{\partial u_0^*}{\partial \nu_0}\Big|_+ \cdot \psi_1 = \int\limits_{\partial D} \frac{\partial u_1^*}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0)),$$

where u_1^* satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u_1^* = 0, & \text{in } \Omega^*, \\ u_1^* = \psi_1, & \text{on } \partial D_1^* \setminus \{0\}, \\ u_1^* = 0, & \text{on } \partial D. \end{cases} \tag{5.1}$$

Thus,

$$b_1 - b_1^* = \int\limits_{\partial D} \frac{\partial (u_1 - u_1^*)}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0)).$$

Similarly as before, in order to estimate the difference $u_1 - u_1^*$, we introduce two auxiliary functions

$$\tilde{u}_1 = \begin{pmatrix} \bar{v} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad \tilde{u}_1^* = \begin{pmatrix} \bar{v}^* \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\bar{v}$ is defined in Section 3, and $\bar{v}^*$ satisfies $\bar{v}^* = 1$ on $\partial D_1^* \setminus \{0\}$, $\bar{v}^* = 0$ on ∂D, and

$$\bar{v}^* = \frac{x_d - h(x')}{h_1(x') - h(x')}, \quad \text{on } \Omega_R^*, \qquad \|\bar{v}^*\|_{C^2(\overline{\Omega^* \setminus \Omega_{R/2}^*})} \leq C,$$

where $\Omega^*_r := \left\{ x \in \Omega^* \;\middle|\; |x'| < r \right\}$, for $r < R$. By (2.7) and (2.8), we have, for $x \in \Omega^*_R$,

$$|\nabla_{x'}(\tilde{u}_1^1 - \tilde{u}_1^{*1})| \leq \frac{C}{|x'|}, \tag{5.2}$$

and

$$|\partial_{x_d}(\tilde{u}_1^1 - \tilde{u}_1^{*1})| = \left| \frac{1}{h_1(x') - h(x')} - \frac{1}{\varepsilon + h_1(x') - h(x')} \right| \leq \frac{C\varepsilon}{|x'|^2(\varepsilon + |x'|^2)}. \tag{5.3}$$

Applying Corollary 3.4 to (5.1), we obtain

$$|\nabla(u_1^* - \tilde{u}_1^*)(x)| \leq \frac{C}{|x'|}, \quad x \in \Omega^*_R; \tag{5.4}$$

and

$$|\nabla_{x'} u_1^*(x)| \leq \frac{C}{|x'|}, \quad |\partial_{x_d} u_1^*(x)| \leq \frac{C}{|x'|^2}, \quad x \in \Omega^*_R. \tag{5.5}$$

Define a cylinder

$$\mathcal{C}_r := \left\{ x \in \mathbb{R}^d \;\middle|\; |x'| < r, 0 \leq x_n \leq \varepsilon + 2 \max_{|x'|=r} h_1(x') \right\},$$

for $r < R_0$. Next, we divide into two steps to estimate the difference $u_1 - u_1^*$.

STEP 1. Notice that $u_1 - u_1^*$ satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu}(u_1 - u_1^*) = 0, & \text{in } D \setminus (\overline{D_1 \cup D_1^*}), \\ u_1 - u_1^* = \psi_1 - u_1^*, & \text{on } \partial D_1 \setminus D_1^*, \\ u_1 - u_1^* = u_1 - \psi_1, & \text{on } \partial D_1^* \setminus (D_1 \cup \{0\}), \\ u_1 - u_1^* = 0, & \text{on } \partial D. \end{cases}$$

We first estimate $|u_1 - u_1^*|$ on $\partial(D_1 \cup D_1^*) \setminus \mathcal{C}_{\varepsilon^\gamma}$, where $0 < \gamma < 1/2$ to be determined later. For ε sufficiently small, in view of the definition of u_1^*,

$$|\partial_{x_d} u_1^*(x)| \leq C, \quad x \in \Omega^* \setminus \Omega^*_R,$$

we have, for $x \in \partial D_1 \setminus D_1^*$,

$$|(u_1 - u_1^*)(x', x_d)| = |u_1^*(x', x_d - \varepsilon) - u_1^*(x', x_d)| \leq C\varepsilon. \tag{5.6}$$

For $x \in \partial D_1^* \setminus (D_1 \cup \mathcal{C}_{\varepsilon^\gamma})$, by (2.12),

$$\begin{aligned} |(u_1 - u_1^*)(x', x_d)| &= |u_1(x', x_d) - u_1(x', x_d + \varepsilon)| \\ &\leq \frac{C\varepsilon}{\varepsilon + |x'|^2} \leq C\varepsilon^{1-2\gamma}. \end{aligned} \tag{5.7}$$

By using (5.3), (3.36) and (5.4), we have, for $x \in \Omega_R^*$ with $|x'| = \varepsilon^\gamma$,

$$\begin{aligned}|\partial_{x_d}(u_1 - u_1^*)(x', x_d)| &= |\partial_{x_d}(\tilde{u}_1 - \tilde{u}_1^*) + \partial_{x_d}(u_1 - \tilde{u}_1) + \partial_{x_d}(u_1^* - \tilde{u}_1^*)|\,(x', x_d)\\ &\le \frac{C\varepsilon}{|x'|^2(\varepsilon + |x'|^2)} + \frac{C}{|x'|}\\ &\le \frac{C}{\varepsilon^{4\gamma-1}} + \frac{C}{\varepsilon^\gamma}.\end{aligned}$$

Thus, for $x \in \Omega_R^*$ with $|x'| = \varepsilon^\gamma$, recalling $u_1 - u_1^* = 0$ on ∂D, we have

$$\begin{aligned}|(u_1 - u_1^*)(x', x_d)| &= |(u_1 - u_1^*)(x', x_d) - (u_1 - u_1^*)(x', h(x'))|\\ &\le \sup_{h(x') \le x_d \le h_1(x')} \left|\partial_{x_d}(u_1 - u_1^*)(x', x_d)\right|_{|x'|=\varepsilon^\gamma} \cdot (h_1(x') - h(x'))\\ &\le (\frac{C}{\varepsilon^{4\gamma-1}} + \frac{C}{\varepsilon^\gamma}) \cdot \varepsilon^{2\gamma} \le C(\varepsilon^{1-2\gamma} + \varepsilon^\gamma).\end{aligned} \tag{5.8}$$

Letting $1 - 2\gamma = \gamma$, we take $\gamma = 1/3$. Combining (5.6), (5.7) and (5.8), and recalling $u_1 - u_1^* = 0$ on ∂D, we obtain

$$|(u_1 - u_1^*)(x)| \le C\varepsilon^{1/3}, \quad x \in \partial(D \setminus (\overline{D_1 \cup D_1^* \cup \mathcal{C}_{\sqrt[3]{\varepsilon}}})).$$

Applying the maximum principle for Lamé systems, see [35],

$$|(u_1 - u_1^*)(x)| \le C\varepsilon^{1/3}, \quad \text{in} \ \ D \setminus \overline{(D_1 \cup D_1^* \cup \mathcal{C}_{\sqrt[3]{\varepsilon}})}.$$

Then using the standard interior and boundary estimates for Lamé system, we have, for any $0 < \tilde{\gamma} < 1/3$,

$$|\nabla(u_1 - u_1^*)(x)| \le C\varepsilon^{\tilde{\gamma}}, \quad \text{in} \ \ D \setminus \overline{(D_1 \cup D_1^* \cup \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}})}.$$

This implies that

$$|\mathcal{B}^{out}| := \left| \int_{\partial D \setminus \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{\partial(u_1 - u_1^*)}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0)) \right| \le C\varepsilon^{\tilde{\gamma}}, \tag{5.9}$$

where $0 < \tilde{\gamma} < 1/3$ will be determined later.

STEP 2. In the following, we estimate

$$\mathcal{B}^{in} := \int_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{\partial(u_1 - u_1^*)}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0))$$

$$
\begin{aligned}
&= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{\partial(\tilde{u}_1 - \tilde{u}_1^*)}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0)) \\
&\quad + \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{\partial(w_1 - w_1^*)}{\partial \nu_0}\Big|_+ \cdot (\varphi(x) - \varphi(0)) \\
&=: \mathcal{B}_{\tilde{u}} + \mathcal{B}_w
\end{aligned}
$$

where $w_1 = u_1 - \tilde{u}_1,\ w_1^* = u_1^* - \tilde{u}_1^*$. By definition,

$$
\begin{aligned}
\mathcal{B}_{\tilde{u}} = &\int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \Bigg\{ \lambda \sum_{k=1}^{d} \partial_{x_1}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_k (\varphi^k(x) - \varphi^k(0)) \\
&\quad + \mu \sum_{k=1}^{d} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) \Big[n_1(\varphi^k(x) - \varphi^k(0)) + n_k(\varphi^1(x) - \varphi^1(0)) \Big] \Bigg\} \\
&=: \lambda \left(\mathcal{B}_{\tilde{u}}^1 + \mathcal{B}_{\tilde{u}}^2 \right) + \mu \left(\mathcal{B}_{\tilde{u}}^3 + \mathcal{B}_{\tilde{u}}^4 + \mathcal{B}_{\tilde{u}}^5 + \mathcal{B}_{\tilde{u}}^6 \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{B}_{\tilde{u}}^1 &:= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \sum_{k=1}^{d-1} \partial_{x_1}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_k (\varphi^k(x) - \varphi^k(0)), \\
\mathcal{B}_{\tilde{u}}^2 &:= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \partial_{x_1}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_d (\varphi^d(x) - \varphi^d(0)), \\
\mathcal{B}_{\tilde{u}}^3 &:= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \sum_{k=1}^{d-1} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_1 (\varphi^k(x) - \varphi^k(0)), \\
\mathcal{B}_{\tilde{u}}^4 &:= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \partial_{x_d}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_1 (\varphi^d(x) - \varphi^d(0)), \\
\mathcal{B}_{\tilde{u}}^5 &:= \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \sum_{k=1}^{d-1} \partial_{x_k}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_k (\varphi^1(x) - \varphi^1(0)),
\end{aligned}
$$

and

$$
\mathcal{B}_{\tilde{u}}^6 := \int\limits_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \partial_{x_d}(\tilde{u}_1^1 - \tilde{u}_1^{*1}) n_d (\varphi^1(x) - \varphi^1(0)).
$$

According to (5.2), (5.3) and the Taylor expansion of $\varphi^k(x)$,

$$\begin{aligned}
&\left|\mathcal{B}_{\tilde{u}}^{1}\right|+\left|\mathcal{B}_{\tilde{u}}^{3}\right|+\left|\mathcal{B}_{\tilde{u}}^{5}\right| \\
&\leq \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{C}{|x'|}\cdot|x'|\cdot|\nabla_{x'}\varphi(0)||x'| + \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}|x'|^{2} \\
&\leq C|\nabla_{x'}\varphi(0)|\varepsilon^{(\frac{1}{3}-\tilde{\gamma})d} + C\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d+1)}\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}; \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
\left|\mathcal{B}_{\tilde{u}}^{2}\right| &\leq \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{C}{|x'|}\cdot|\nabla_{x'}\varphi(0)||x'| + \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}|x'| \\
&\leq C|\nabla_{x'}\varphi(0)|\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-1)} + C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}\varepsilon^{(\frac{1}{3}-\tilde{\gamma})d}; \qquad (5.11)
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathcal{B}_{\tilde{u}}^{4}\right| &\leq \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{C}{|x'|^{2}}\cdot|x'|\cdot|\nabla_{x'}\varphi(0)||x'| + \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}|x'| \\
&\leq C|\nabla_{x'}\varphi(0)|\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-1)} + C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}\varepsilon^{(\frac{1}{3}-\tilde{\gamma})d}. \qquad (5.12)
\end{aligned}$$

For $d = 2$ if $\nabla_{x'}\varphi^{1}(0) = 0$, we have

$$\left|\mathcal{B}_{\tilde{u}}^{6}\right| \leq \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} C\|\nabla^{2}\varphi^{1}\|_{L^{\infty}(\partial D)} \leq C\|\nabla^{2}\varphi^{1}\|_{L^{\infty}(\partial D)}\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-1)}. \qquad (5.13)$$

If $d \geq 3$, we have

$$\begin{aligned}
\left|\mathcal{B}_{\tilde{u}}^{6}\right| &\leq \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{C}{|x'|^{2}}\cdot|\nabla_{x'}\varphi(0)||x'| + \int_{\partial D\cap\mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)} \\
&\leq C|\nabla_{x'}\varphi(0)|\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-2)} + C\|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-1)}. \qquad (5.14)
\end{aligned}$$

Hence, combining (5.10)–(5.14) yields that for $\varepsilon > 0$ sufficiently small, if $d = 2$ and $\nabla_{x'}\varphi^{1}(0) = 0$,

$$|\mathcal{B}_{\tilde{u}}| \leq C\left(|\nabla_{x'}\varphi^{2}(0)| + \|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}\right)\varepsilon^{\frac{1}{3}-\tilde{\gamma}}; \qquad (5.15)$$

and if $d \geq 3$,

$$|\mathcal{B}_{\tilde{u}}| \leq C\left(|\nabla_{x'}\varphi(0)| + \|\nabla^{2}\varphi\|_{L^{\infty}(\partial D)}\right)\varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-2)}. \qquad (5.16)$$

We now estimate $\mathcal{B}_{w}$. It follows from Corollary 3.4 that

$$|\nabla w_{1}(x)| \leq \frac{C}{\sqrt{\delta(x)}}, \quad 0 < |x'| \leq R, \qquad (5.17)$$

and

$$|\nabla w_1^*(x)| \leq \frac{C}{|x'|}, \quad 0 < |x'| \leq R. \tag{5.18}$$

By definition,

$$\mathcal{B}_w = \int_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \left\{ \lambda \sum_{k,l=1}^{d} \partial_{x_k}(w_1^k - w_1^{*k}) n_l (\varphi^l(x) - \varphi^l(0)) \right.$$
$$\left. + \mu \sum_{k,l=1}^{d} [\partial_{x_l}(w_1^k - w_1^{*k}) + \partial_{x_k}(w_1^l - w_1^{*l})] n_l (\varphi^k(x) - \varphi^k(0)) \right\}.$$

By (5.17), (5.18) and the Taylor expansion of $\varphi^l(x)$,

$$\begin{aligned}
|\mathcal{B}_w| &\leq \int_{\partial D \cap \mathcal{C}_{\varepsilon^{\frac{1}{3}-\tilde{\gamma}}}} \frac{C}{|x'|} \cdot \left(|\nabla_{x'} \varphi^l(0)||x'| + \|\nabla^2 \varphi^l\|_{L^\infty(\partial D)} |x'|^2 \right) \\
&\leq C |\nabla_{x'} \varphi^l(0)| \varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-1)} + \|\nabla^2 \varphi^l\|_{L^\infty(\partial D)} \varepsilon^{(\frac{1}{3}-\tilde{\gamma})d} \\
&\leq C \left(|\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-1)}.
\end{aligned} \tag{5.19}$$

This, together with (5.15), implies that, for $d = 2$, if $\nabla_{x'} \varphi^1(0) = 0$,

$$\left| \mathcal{B}^{in} \right| \leq |\mathcal{B}_{\tilde{u}}| + |\mathcal{B}_w| \leq C \left(|\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{\frac{1}{3}-\tilde{\gamma}}.$$

Combining with (5.9), we now simply choose $\tilde{\gamma} = \gamma_2 = 1/6$, such that $\frac{1}{3} - \gamma_2 = \gamma_2$. Thus, we have, for $d = 2$,

$$|b_1 - b_1^*| \leq |\mathcal{B}^{in}| + |\mathcal{B}^{out}| \leq C \left(|\nabla_{x'} \varphi^2(0)| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{1/6}.$$

For $d \geq 3$, combining (5.19) together with (5.16) yields that

$$\left| \mathcal{B}^{in} \right| \leq |\mathcal{B}_{\tilde{u}}| + |\mathcal{B}_w| \leq C \left(|\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{(\frac{1}{3}-\tilde{\gamma})(d-2)}.$$

Therefore, using (5.9) again and picking $\tilde{\gamma} = \gamma_d = \frac{d-2}{3(d-1)}$ (such that $(\frac{1}{3} - \gamma_d)(d-2) = \gamma_d$), we have, for $d \geq 3$,

$$|b_1 - b_1^*| \leq |\mathcal{B}^{in}| + |\mathcal{B}^{out}| \leq C \left(|\nabla_{x'} \varphi(0)| + \|\nabla^2 \varphi\|_{L^\infty(\partial D)} \right) \varepsilon^{\gamma_d}.$$

The proof of Lemma 5.1 is completed. □

Proof of Theorem 1.3. Under the assumptions of Theorem 1.3 that $b_{k_0}^* \neq 0$ for some integer $1 \leq k_0 \leq d$, it follows from Lemma 5.1 that there exists a universal constant

$C_0 > 0$ and a sufficiently small number $\varepsilon_0 > 0$, such that, for $0 < \varepsilon < \varepsilon_0$,

$$|b_{k_0}| > \frac{C_0}{2} > 0. \tag{5.20}$$

By the definition of $A^* = (a^*_{\alpha\beta})_{d\times d}$, where $a^*_{\alpha\beta}$ is the cofactor of $a_{\alpha\beta}$, and Lemma 4.2, we have

$$a^*_{\alpha\alpha} \sim \frac{1}{(\rho_d(\varepsilon))^{d-1}}, \ \alpha = 1, \cdots, d; \quad a^*_{\alpha\beta} \sim \frac{1}{(\rho_d(\varepsilon))^{d-2}}, \quad \alpha \neq \beta. \tag{5.21}$$

According to (4.13), (4.25), (4.26), (5.20) and (5.21), for sufficiently small ε,

$$\begin{aligned} |C^{k_0} - \varphi^{k_0}(0)| &= \left| \frac{1}{\det A}\Big[a^*_{k_0k_0}b_{k_0} + \sum_{\beta \neq k_0} a^*_{k_0\beta}b_\beta\Big] + Errors \right| \\ &\geq \frac{1}{2}\frac{1}{\det A} a^*_{k_0k_0}|b_{k_0}| \geq \frac{\rho_d(\varepsilon)}{C}. \end{aligned} \tag{5.22}$$

On the other hand, in view of Corollary 3.4, we obtain

$$\begin{aligned} |\partial_{x_d}u^{k_0}_{k_0}| &= |\partial_{x_d}\tilde{u}^{k_0}_{k_0} + \partial_{x_d}(u^{k_0}_{k_0} - \tilde{u}^{k_0}_{k_0})| \\ &\geq |\partial_{x_d}\tilde{u}^{k_0}_{k_0}| - |\partial_{x_d}(u^{k_0}_{k_0} - \tilde{u}^{k_0}_{k_0})| \geq \frac{1}{C(\varepsilon + |x'|^2)}, \quad x \in \Omega_R. \end{aligned} \tag{5.23}$$

At the same time, since $\tilde{u}^{k_0}_\alpha = 0$ if $\alpha \neq k_0$, it is easy to see from Corollary 3.4 that

$$\begin{aligned} |\partial_{x_d}u^{k_0}_{\alpha}| &= |\partial_{x_d}\tilde{u}^{k_0}_{\alpha} + \partial_{x_d}(u^{k_0}_{\alpha} - \tilde{u}^{k_0}_{\alpha})| \\ &= |\partial_{x_d}(u^{k_0}_{\alpha} - \tilde{u}^{k_0}_{\alpha})| \leq \frac{C}{\sqrt{\varepsilon + |x'|^2}}, \quad \alpha \neq k_0, \quad x \in \Omega_R. \end{aligned} \tag{5.24}$$

Therefore, by a combination of the estimates (5.22), (5.23), (5.24), we get, for $(0', x_d) \in \overline{P_1P}$,

$$\begin{aligned} \left|\sum_{\alpha=1}^{d}(C^\alpha - \varphi^\alpha(0))\nabla u_\alpha\right| &\geq \left|\sum_{\alpha=1}^{d}(C^\alpha - \varphi^\alpha(0))\partial_{x_d}u^{k_0}_\alpha\right| \\ &\geq \left|(C^{k_0} - \varphi^{k_0}(0))\partial_{x_d}u^{k_0}_{k_0}\right| - \left|\sum_{\alpha\neq k_0}^{d}(C^\alpha - \varphi^\alpha(0))\partial_{x_d}u^{k_0}_\alpha\right| \\ &\geq \frac{\rho_d(\varepsilon)}{C\varepsilon}. \end{aligned} \tag{5.25}$$

Here we used the assumption that $b^*_\alpha = 0$ for $\alpha \neq k_0$, $1 \leq \alpha \leq d$ when $d \geq 4$. By means of Corollary 2.3, (2.15), Lemma 4.2 and Lemma 4.3, especially for $x = (0', x_d) \in \overline{P_1P}$,

$$\left|\sum_{\alpha=d+1}^{\frac{d(d+1)}{2}} C^\alpha \nabla u_\alpha\right| \leq \frac{C(\varepsilon+|x'|)}{\varepsilon+|x'|^2} \leq C, \tag{5.26}$$

$$|\nabla u_0| \leq \frac{C|\nabla\varphi(0)||x'|}{\varepsilon+|x'|^2} + C \leq C. \tag{5.27}$$

Combining (5.25), (5.26), (5.27) and (2.5) immediately yields that for $x = (0', x_d) \in \overline{P_1P}$,

$$|\nabla u(0', x_d)| \geq \frac{\rho_d(\varepsilon)}{C\varepsilon}, \quad 0 < x_d < \varepsilon.$$

Theorem 1.3 is thus established. □

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