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RESEARCH ARTICLE

On uniqueness and existence of viscosity solutions to Hessian equations in exterior domains

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Abstract In this paper, we obtain the uniqueness and existence of viscosity solutions with prescribed asymptotic behavior at infinity to Hessian equations in exterior domains.

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1 Introduction

In this paper, we study the Dirichlet problem of Hessian equation in exterior domains:

$$\sigma_k(\lambda(\mathrm{D}^2 u)) = 1, \quad x \in \mathbb{R}^n \backslash \overline{\Omega}, \tag{1}$$

$$u = \varphi(x), \quad x \in \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $0 \in \Omega$, $\varphi \in C^2(\partial\Omega)$. Here,

$$\sigma_k(\lambda(\mathrm{D}^2 u)) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad (k = 1, \ldots, n)$$

is the kth elementary symmetric function of $\lambda(\mathrm{D}^2 u) = (\lambda_1, \ldots, \lambda_n)$, the eigenvalues of the Hessian matrix $\mathrm{D}^2 u$.

The Hessian equation (1) is an important class of fully nonlinear elliptic equations. For $k = 1$, (1) is the Poisson equation $\Delta u = 1$, and for $k = n$, (1) is the Monge-Ampère equation $\det(\mathrm{D}^2 u) = 1$. There exist many excellent

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results in interior domains of Hessian equations, see [3–10] and the references therein. For instance, Caffarelli et al. [3] established the classical solvability of the Dirichlet problem of Hessian equations. Trudinger [8] demonstrated the existence and uniqueness of weak solutions, and Urbas [10] proved the existence of viscosity solutions.

To work in the realm of elliptic equations, we have to restrict the solutions to a class of functions. Let

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0,\ j = 1, 2, \ldots, k\}.$$

Γ_k is a symmetric cone, that is, any permutation of λ is in Γ_k if $\lambda \in \Gamma_k$. When $k = 1$, Γ_k is the half space $\{\lambda \in \mathbb{R}^n \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n > 0\}$. When $k = n$, Γ_k is the positive cone

$$\Gamma^+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0,\ i = 1, \ldots, n\}.$$

Following [3], we give the definition of k-convex function.

Definition 1 A function $u \in C^2(\mathbb{R}^n \backslash \overline{\Omega})$ is called k-convex if $\lambda(\mathrm{D}^2 u) \in \overline{\Gamma}_k$ for all $x \in \mathbb{R}^n \backslash \overline{\Omega}$.

It follows from [3] that (1) is degenerate elliptic for the k-convex functions, and $\sigma_j^{1/j}(\lambda(r)),\ j = 1, 2, \ldots, k$, is concave for r with $\lambda(r) \in \Gamma_k$.

For the readers' convenience, we recall the definition of viscosity solutions.

Definition 2 A function $u \in C^0(\mathbb{R}^n \backslash \overline{\Omega})$ is called a viscosity subsolution to (1), if for any $y \in \mathbb{R}^n \backslash \overline{\Omega}$, $\xi \in C^2(\mathbb{R}^n \backslash \overline{\Omega})$ satisfying

$$u(x) \leqslant \xi(x),\ x \in \mathbb{R}^n \backslash \overline{\Omega}; \quad u(y) = \xi(y),$$

we have

$$\sigma_k(\lambda(\mathrm{D}^2 \xi(y))) \geqslant 1.$$

A function $u \in C^0(\mathbb{R}^n \backslash \overline{\Omega})$ is called a viscosity supersolution to (1), if for any $y \in \mathbb{R}^n \backslash \overline{\Omega}$, any k-convex function $\xi \in C^2(\mathbb{R}^n \backslash \overline{\Omega})$ satisfying

$$u(x) \geqslant \xi(x),\ x \in \mathbb{R}^n \backslash \overline{\Omega}; \quad u(y) = \xi(y),$$

we have

$$\sigma_k(\lambda(\mathrm{D}^2 \xi(y))) \leqslant 1.$$

A function $u \in C^0(\mathbb{R}^n \backslash \overline{\Omega})$ is called a viscosity solution to (1), if u is both a viscosity subsolution and a viscosity supersolution to (1).

A function $u \in C^0(\mathbb{R}^n \backslash \Omega)$ is called a viscosity subsolution (resp. supersolution, solution) to (1)-(2) if u is a viscosity subsolution (resp. supersolution, solution) to (1) and $u \leqslant$ (resp. $\geqslant, =$) $\varphi(x)$ on $\partial\Omega$.

Definition 3 A function $u \in C^0(\mathbb{R}^n \backslash \overline{\Omega})$ is called k-convex if in the viscosity sense $\sigma_j(\lambda(\mathrm{D}^2 u)) \geqslant 0$ in $\mathbb{R}^n \backslash \overline{\Omega}$, $j = 1, 2, \ldots, k$.

Motivated by the work of [1], in this paper, we investigate the Drichlet problem of Hessian equations in exterior domains. Using the Perron method, we get the uniqueness and existence of viscosity solutions with prescribed asymptotic behavior at infinity to Hessian equations.

Theorem 1 *For* $n \geqslant 3$, *let* $\Omega \subset \mathbb{R}^n$ *be a* C^2, *bounded, and strictly convex domain; and let* $0 \in \Omega$, $\varphi \in C^2(\partial\Omega)$. *Then there exists* c_0 *such that for any* $c > c_0$, *there exists a unique* k*-convex function* $u \in C^0(\mathbb{R}^n \backslash \Omega)$ *satisfying* (1)-(2) *in the viscosity sense and*

$$\limsup_{x\to\infty}\left(|x|^{n-2}\left|u(x)-\left(\frac{c_*}{2}|x|^2+c\right)\right|\right)<\infty, \tag{3}$$

where $c_* = (1/\mathrm{C}_n^k)^{1/k}$, $\mathrm{C}_n^k = n!/(k!\,(n-k)!)$.

This paper is arranged as follows. In Section 2, we give some lemmas which will be used later. In Section 3, we prove Theorem 1.

2 Preliminaries

The following lemma can be found in [2].

Lemma 1 *Let* Ω *be a bounded strictly convex domain in* $\mathbb{R}^n$, $\partial\Omega \in C^2$, $\varphi \in C^2(\overline{\Omega})$. *Then there exists a constant* c *only depending on* n, φ *and* Ω *such that for any* $\xi \in \partial\Omega$, *there exists* $\overline{x}(\xi) \in \mathbb{R}^n$ *satisfying*

$$|\overline{x}(\xi)| \leqslant c, \quad w_\xi < \varphi, \quad x \in \overline{\Omega}\backslash\{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\left(|x-\overline{x}(\xi)|^2 - |\xi-\overline{x}(\xi)|^2\right), \quad x \in \mathbb{R}^n.$$

Lemma 2 *Let* Ω *be a domain in* $\mathbb{R}^n$, *and let* $f \in C^0(\mathbb{R}^n)$ *be nonnegative. Assume that* k*-convex functions* $v \in C^0(\overline{\Omega})$, $u \in C^0(\mathbb{R}^n)$ *satisfy, respectively,*

$$\sigma_k(\lambda(\mathrm{D}^2 v)) \geqslant f(x), \quad x \in \Omega,$$
$$\sigma_k(\lambda(\mathrm{D}^2 u)) \geqslant f(x), \quad x \in \mathbb{R}^n.$$

Moreover,

$$u \leqslant v, \quad x \in \overline{\Omega}, \tag{4}$$
$$u = v, \quad x \in \partial\Omega.$$

Set

$$w(x) = \begin{cases} v(x), & x \in \Omega,\\ u(x), & x \in \mathbb{R}^n\backslash\Omega.\end{cases}$$

Then $w \in C^0(\mathbb{R}^n)$ *is a* k*-convex function and satisfies, in the viscosity sense,*

$$\sigma_k(\lambda(\mathrm{D}^2 w)) \geqslant f(x), \quad x \in \mathbb{R}^n.$$

Proof Let $y \in \mathbb{R}^n,\ \xi \in C^2(\mathbb{R}^n)$ satisfying $w(y) = \xi(y)$,

$$w(x) \leqslant \xi(x), \quad x \in \mathbb{R}^n. \tag{5}$$

If $y \in \Omega$, then we have

$$v(y) = w(y) = \xi(y), \quad v(x) = w(x) \leqslant \xi(x), \quad x \in \Omega.$$

Therefore,

$$\sigma_j(\lambda(\mathrm{D}^2\xi(y))) \geqslant 0, \quad 1 \leqslant j \leqslant k,$$
$$\sigma_k(\lambda(\mathrm{D}^2\xi(y))) \geqslant f(y).$$

If $y \in \mathbb{R}^n \backslash \Omega$, then we have

$$u(y) = w(y) = \xi(y), \quad u(x) = w(x) \leqslant \xi(x), \quad x \in \mathbb{R}^n \backslash \Omega.$$

By (4) and (5),

$$u(x) \leqslant \xi(x), \quad x \in \mathbb{R}^n.$$

Therefore,

$$\sigma_j(\lambda(\mathrm{D}^2\xi(y))) \geqslant 0, \quad 1 \leqslant j \leqslant k,$$
$$\sigma_k(\lambda(\mathrm{D}^2\xi(y))) \geqslant f(y).$$

This completes the proof. □

The following existence result for viscosity solutions on a ball can be found in [5].

Lemma 3 *Let B be a ball in $\mathbb{R}^n$, and let $f \in C^0(\overline{B})$ be nonnegative. Suppose that $\underline{u}, \overline{u} \in C^0(\overline{B})$ are, respectively, viscosity subsolution and supersolution of*

$$\sigma_k(\lambda(\mathrm{D}^2 u)) = f(x), \quad x \in B,$$

and satisfy

$$\underline{u}|_{\partial B} = \overline{u}|_{\partial B} = \varphi \in C^0(\partial B).$$

Then there exists a unique k-convex function $u \in C^0(\overline{B})$ satisfying

$$\sigma_k(\lambda(\mathrm{D}^2 u)) = f(x), \quad x \in B,$$
$$u = \varphi(x), \quad x \in \partial B.$$

Lemma 4 *Let B be a ball in $\mathbb{R}^n$, and let $f \in C^0(\overline{B})$ be nonnegative. Suppose that $\underline{u} \in C^0(\overline{B})$ satisfies, in the viscosity sense,*

$$\sigma_k(\lambda(\mathrm{D}^2 \underline{u})) \geqslant f(x), \quad x \in B.$$

Then the Dirichlet problem

$$\sigma_k(\lambda(\mathrm{D}^2 u)) = f(x), \quad x \in B, \tag{6}$$

$$u = \underline{u}(x), \quad x \in \partial B \tag{7}$$

has a unique k-convex viscosity solution $u \in C^0(\overline{B})$.

Proof Clearly, $\underline{u}$ is a viscosity subsolution of (6)-(7). From Lemma 3, we only need to prove (6)-(7) has a viscosity supersolution $\overline{u} \in C^0(\overline{B})$ satisfying $\overline{u} = \underline{u}$ on ∂B.

Let $v \in C^2(B) \cap C^0(\overline{B})$ satisfy

$$\Delta v = 0, \quad x \in B,$$
$$v = \underline{u}, \quad x \in \partial B.$$

We claim that v is a viscosity supersolution of (6). Indeed, suppose that v is not a viscosity supersolution of (6). Then there exist $y \in B$ and some k-convex function $\xi \in C^2(B)$ such that

$$v(x) \geqslant \xi(x), \quad x \in B, \quad v(y) = \xi(y), \tag{8}$$

but

$$\sigma_k(\lambda(\mathrm{D}^2\xi(y))) > f(y).$$

By the k-convexity of ξ and the Newton-Maclaurin inequality

$$\sigma_1(\lambda) \geqslant n(\sigma_k(\lambda)/\mathrm{C}_n^k)^{1/k}, \quad \lambda \in \overline{\Gamma}_k,$$

we know

$$\Delta\xi(y) \geqslant n(f(y)/\mathrm{C}_n^k)^{1/k} > 0.$$

But from (8), we get

$$\mathrm{D}^2 v(y) \geqslant \mathrm{D}^2\xi(y).$$

Hence,

$$\Delta\xi(y) \leqslant \Delta v(y) = 0.$$

This is a contradiction. The lemma is proved. □

3 Proof of Theorem 1

In this section, we prove Theorem 1. We divide the proof into six steps.

Step 1 We construct a viscosity subsolution w_a of (1)-(2).

Let $a > -1$. Set

$$w_a(x) = \min_{\partial\Omega} \varphi - \int_{|\sqrt{c_*}x|}^{\overline{r}} (s^n + a)^{1/n} \mathrm{d}s, \quad x \in \mathbb{R}^n,$$

where $\overline{r} = 2\sqrt{c_*}\,\mathrm{diam}\,\Omega$. Then $w_a \in C^0(\mathbb{R}^n)$, and

$$w_a \leqslant \varphi, \quad x \in \partial\Omega, \tag{9}$$

and

$$w_a(x) = \frac{c_*}{2}|x|^2 + \mu(a) - \int_{|\sqrt{c_*}x|}^{\infty} s\Big[\Big(1+\frac{a}{s^n}\Big)^{1/n} - 1\Big]\mathrm{d}s, \quad x \in \mathbb{R}^n, \tag{10}$$

where

$$\mu(a) := \min_{\partial\Omega}\varphi + \int_{\overline{r}}^{\infty} s\Big[\Big(1+\frac{a}{s^n}\Big)^{1/n} - 1\Big]\mathrm{d}s - \frac{1}{2}\overline{r}^2.$$

A direct calculation gives

$$D_{ij}w_a = (|y|^n + a)^{\frac{1}{n}-1}c_*\Big[\Big(|y|^{n-1} + \frac{a}{|y|}\Big)\delta_{ij} - \frac{ay_iy_j}{|y|^3}\Big], \quad |x| > 0,$$

where $y = \sqrt{c_*}\,x$. By rotating the coordinates, we may set $y = (R, 0, \ldots, 0)^{\mathrm{T}}$, and therefore,

$$\mathrm{D}^2w_a = (R^n + a)^{\frac{1}{n}-1}c_*\mathrm{diag}\Big(R^{n-1}, R^{n-1} + \frac{a}{R}, \ldots, R^{n-1} + \frac{a}{R}\Big),$$

where $R = |y|$. Consequently, $\lambda(\mathrm{D}^2w_a) \in \Gamma_k$ for $|x| > 0$. By Newton-Maclaurin inequality,

$$\sigma_k(\lambda(\mathrm{D}^2w_a)) \geqslant C_n^k(\sigma_n(\lambda(\mathrm{D}^2w_a)))^{k/n} = C_n^k(c_*^n)^{k/n} = 1, \quad |x| > 0. \tag{11}$$

Step 2 We define the Perron solution u_c of (1).

Fix $a_0 > -1$ such that $c_0 := \mu(a_0) \geqslant c_1$. For any $c > c_0$ and $x \in \mathbb{R}^n\backslash\overline{\Omega}$, let $S_{c,x}$ denote the set of k-convex functions $w \in C^0(\mathbb{R}^n\backslash\Omega)$ satisfying, in the viscosity sense,

$$\sigma_k(\lambda(\mathrm{D}^2w)) \geqslant 1, \quad y \in \mathbb{R}^n\backslash\overline{\Omega},$$
$$w \leqslant \varphi, \quad y \in \partial\Omega,$$

and for any $y \in \mathbb{R}^n\backslash\Omega,\ |y - x| \leqslant 2\,\mathrm{diam}\,\Omega$,

$$w(y) \leqslant \frac{c_*}{2}|y|^2 + c.$$

Then, for all $\mu^{-1}(c_0) < a < \mu^{-1}(c)$, by (11), (9), and (10), $w_a \in S_{c,x}$. Consequently, $S_{c,x} \neq \emptyset$. Define

$$u_c(x) = \sup\{w(x)\colon\ w \in S_{c,x}\}, \quad x \in \mathbb{R}^n\backslash\overline{\Omega}.$$

Step 3 We prove u_c can be extended to a continuous function in $\mathbb{R}^n\backslash\Omega$ and $u_c = \varphi$ on $\partial\Omega$.

By the expression of $w_\xi(x)$ in Lemma 1, we can fix some constant c_1 such that for any $\xi \in \partial\Omega$,

$$w_\xi(x) \leqslant \frac{c_*}{2}|x|^2 + c_1, \quad \mathrm{dist}(x, \partial\Omega) \leqslant 1, \quad x \in \mathbb{R}^n\backslash\Omega. \tag{12}$$

By (12), for $\overline{\xi} \in \partial\Omega$ and $x \in \mathbb{R}^n\backslash\overline{\Omega}$, x sufficiently close to $\overline{\xi}$, we have $w_{\overline{\xi}} \in S_{c,x}$. Consequently, $u_c(x) \geqslant w_{\overline{\xi}}(x)$ for x sufficiently close to $\overline{\xi}$. Thus,

$$\liminf_{x\to\overline{\xi}} u_c(x) \geqslant \liminf_{x\to\overline{\xi}} w_{\overline{\xi}}(x) = \varphi(\overline{\xi}).$$

On the other hand,

$$\limsup_{x\to\overline{\xi}} u_c(x) \leqslant \varphi(\overline{\xi}).$$

Indeed, if along a sequence $x_i \to \overline{\xi}$,

$$\lim_{i\to\infty} u_c(x_i) \geqslant \varphi(\overline{\xi}) + 3\delta$$

for some $\delta > 0$, then by the definition of u_c, there exists $w_i \in S_{c,x_i}$ such that

$$w_i(x_i) \geqslant \varphi(\overline{\xi}) + 2\delta$$

for large i. But $w_i \in C^0(\mathbb{R}^n\backslash\Omega)$, then for any ξ close to $\overline{\xi}$,

$$w_i(\xi) \leqslant \varphi(\overline{\xi}) + \delta.$$

This is a contradiction.

Step 4 We prove u_c satisfies (1).

By the definition of u_c, u_c is a viscosity subsolution of (1). We only need to prove that u_c is a viscosity supersolution of (1).

For any $x \in \mathbb{R}^n\backslash\overline{\Omega}$, fix $0 < \varepsilon < 2\,\mathrm{diam}\,\Omega$ such that

$$B = B_\varepsilon(x) \subset \mathbb{R}^n\backslash\overline{\Omega}.$$

From Lemma 4, the Dirichlet problem

$$\begin{aligned} \sigma_k(\lambda(\mathrm{D}^2\widetilde{u})) &= 1, \quad y \in B, \\ \widetilde{u} &= u_c, \quad y \in \partial B, \end{aligned} \tag{13}$$

has a unique k-convex viscosity solution $\widetilde{u} \in C^0(\overline{B})$. By the comparison principle, $u_c \leqslant \tilde{u}$ in B. Define

$$\widetilde{w}(y) = \begin{cases} \widetilde{u}(y), & y \in B, \\ u_c(y), & y \in (\mathbb{R}^n\backslash\Omega)\backslash B. \end{cases}$$

Then $\widetilde{w} \in S_{c,x}$. Indeed, by the definition of u_c,

$$u_c(y) \leqslant \frac{c_*}{2}\,|y|^2 + c, \quad y \in B.$$

Let

$$\widetilde{v}(y) = \frac{c_*}{2}\,|y|^2 + c.$$

Then

$$\sigma_k(\lambda(\mathrm{D}^2\widetilde{u})) = 1 = \sigma_k(\lambda(\mathrm{D}^2\widetilde{v})), \quad y \in B,$$
$$\widetilde{u} = u_c \leqslant \widetilde{v}, \quad y \in \partial B.$$

From the comparison principle, for any $y \in B$,

$$\widetilde{u} \leqslant \widetilde{v},$$

i.e.,

$$\widetilde{u}(y) \leqslant \frac{c_*}{2}|y|^2 + c.$$

By Lemma 2,

$$\sigma_k(\lambda(\mathrm{D}^2\widetilde{w})) \geqslant 1, \quad y \in \mathbb{R}^n\backslash\overline{\Omega}.$$

Therefore, $\widetilde{w} \in S_{c,x}$. And thus, by the definition of u_c, $u_c \geqslant \widetilde{w}$ in $\mathbb{R}^n\backslash\Omega$ and $u_c \geqslant \widetilde{u}$ in B. Hence,

$$u_c \equiv \widetilde{u}, \quad y \in B. \tag{14}$$

However, $\widetilde{u}$ satisfies (13), we have, in the viscosity sense,

$$\sigma_k(\lambda(\mathrm{D}^2 u_c)) = 1, \quad y \in B.$$

Because x is arbitrary, we know that u_c is a viscosity supersolution of (1).

Step 5 We prove u_c satisfies (3).

By the definition of u_c,

$$u_c(x) \leqslant \frac{c_*}{2}|x|^2 + c, \quad x \in \mathbb{R}^n\backslash\overline{\Omega}.$$

Then,

$$u_c(x) - \frac{c_*}{2}|x|^2 - c \leqslant 0 \leqslant \frac{1}{|x|^{n-2}}, \quad x \in \mathbb{R}^n\backslash\overline{\Omega}. \tag{15}$$

On the other hand, from (10), as $|x| \to \infty$,

$$w_a(x) = \frac{c_*}{2}|x|^2 + \mu(a) - O(|x|^{2-n}).$$

Because $w_a \in S_{c,x}$, as $|x| \to \infty$,

$$u_c(x) - \frac{c_*}{2}|x|^2 - \mu(a) \geqslant -O(|x|^{2-n}).$$

Let $a \to \mu^{-1}(c)$. Then

$$u_c(x) - \frac{c_*}{2}|x|^2 - c \geqslant -O(|x|^{2-n}). \tag{16}$$

And thus, from (15) and (16), for some constant C, we have

$$\Big|u_c(x) - \Big(\frac{c_*}{2}|x|^2 + c\Big)\Big| \leqslant \frac{C}{|x|^{n-2}}.$$

Then

$$\limsup_{x\to\infty}\Big(|x|^{n-2}\Big|u_c(x)-\Big(\frac{c_*}{2}|x|^2+c\Big)\Big|\Big)<\infty.$$

Step 6 We prove the uniqueness.

Suppose that u and v satisfy (1)–(3). By the comparison principle of viscosity solutions to Hessian equations and

$$\lim_{x\to\infty}(u-v)=0,$$

we know $u\equiv v$ in $\mathbb{R}^n\backslash\Omega$.

The proof is completed. □

We conclude this paper with a brief examination of radially symmetric solutions to (1)–(3).

Example 1 *Let $\Omega=B_1(0)$ be the unit ball in $\mathbb{R}^n$. Then for any constant C, there exists a constant c_0, such that for any $c>c_0$, the Dirichlet problem*

$$\sigma_k(\lambda(\mathrm{D}^2u))=1,\quad x\in\mathbb{R}^n\backslash\overline{B}_1(0), \tag{17}$$

$$u=C,\quad x\in\partial B_1(0), \tag{18}$$

has a radial solution satisfying (3).

Proof Assume that

$$u=u(r)=u(|x|)$$

is the radial solution of (17)-(18). A direct calculation gives that

$$\sigma_k(\lambda(\mathrm{D}^2u))=\mathrm{C}_{n-1}^{k-1}\frac{r^{1-n}}{k}(r^{n-k}(u')^k)'=1.$$

Then, we have

$$u=c_*\int_1^{|x|}(s^k+as^{k-n})^{1/k}\mathrm{d}s+C,$$

where $c_*=(1/\mathrm{C}_n^k)^{1/k}$, and $a\geqslant -1$ is a constant to be determined. Consequently,

$$\begin{aligned}
u&=c_*\int_1^{|x|}\Big\{s\Big[\Big(1+\frac{a}{s^n}\Big)^{1/k}-1\Big]+s\Big\}\mathrm{d}s+C\\
&=\frac{c_*}{2}|x|^2+C-\frac{c_*}{2}+c_*\int_1^{\infty}s\Big[\Big(1+\frac{a}{s^n}\Big)^{1/k}-1\Big]\mathrm{d}s\\
&\quad-c_*\int_{|x|}^{\infty}s\Big[\Big(1+\frac{a}{s^n}\Big)^{1/k}-1\Big]\mathrm{d}s\\
&=\frac{c_*}{2}|x|^2+C-\frac{c_*}{2}+c_*\int_1^{\infty}s\Big[\Big(1+\frac{a}{s^n}\Big)^{1/k}-1\Big]\mathrm{d}s+O(|x|^{2-n})\\
&=\frac{c_*}{2}|x|^2+\mu_0(a)+O(|x|^{2-n}),
\end{aligned}$$

where

$$\mu_0(a) = C - \frac{c_*}{2} + c_* \int_1^{\infty} s\Big[\Big(1 + \frac{a}{s^n}\Big)^{1/k} - 1\Big]\mathrm{d}s, \quad a \geqslant -1.$$

Clearly, $\mu_0(a)$ is strictly increasing on $[-1, +\infty)$ and $\mu_0(+\infty) = +\infty$. For $c_0 = \mu_0(-1)$ and $c > c_0$, we have a constant $a > -1$ such that $c = \mu_0(a)$, and then (3) holds. □

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