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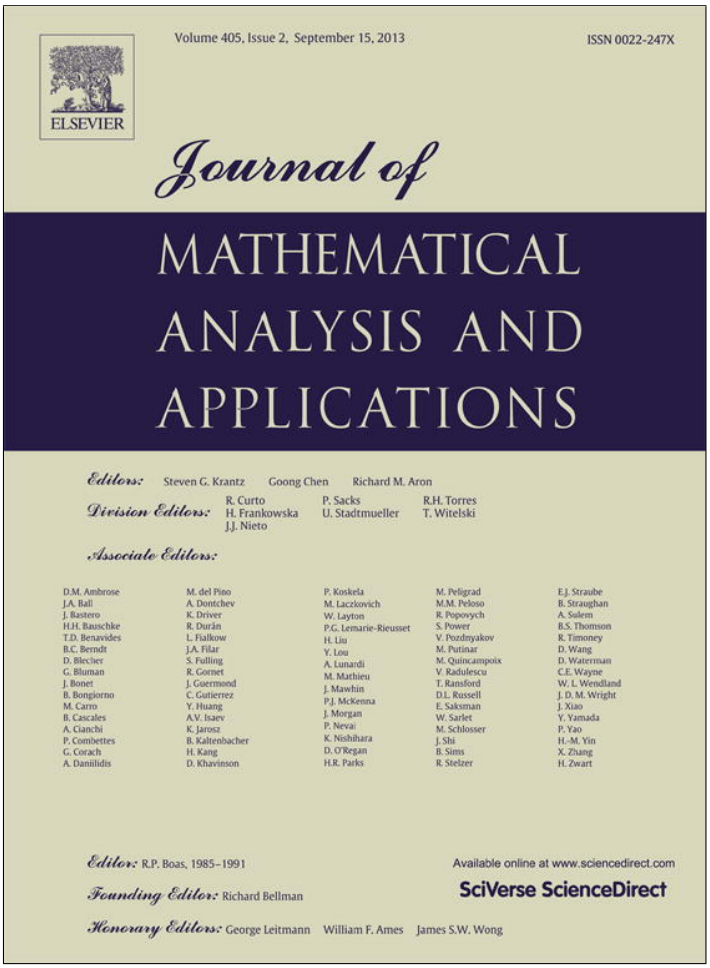

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On the exterior Dirichlet problem for Monge–Ampère equations

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ABSTRACT

In this paper, we use the Perron method to prove the existence of viscosity solutions to a class of Monge–Ampère equations on exterior domains in $\mathbb{R}^n (n \geq 2)$ with prescribed asymptotic behavior at infinity. This extends the Caffarelli–Li's result on $\det(D^2u) = 1$. We also obtain the existence of entire convex viscosity solutions to the Monge–Ampère equations with prescribed asymptotic behavior at infinity.

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1. Introduction and main results

In this paper, we will study the exterior Dirichlet problem for the Monge–Ampère equations

$$\det D^2u = g(x), \tag{1.1}$$

where $g \in C^0(\mathbb{R}^n)$ is a positive function.

For the special case $g \equiv 1$, (1.1) is reduced to the equation

$$\det D^2u = 1, \tag{1.2}$$

which is well understood. A classical theorem of Jörgens [13] for $n = 2$, Calabi [6] for $n \leq 5$ and Pogorelov [16] for all dimensions, respectively, states that any classical convex entire solution of (1.2) must be a quadratic polynomial. Cheng and Yau [7] gave a simpler and more analytical proof of the theorem. Another proof of this theorem was given by Jost and Xin in [14]. Caffarelli [4] extended the Jörgens–Calabi–Pogorelov theorem of classical solutions to viscosity solutions. It was proved by Trudinger and Wang in [17] that the only open convex subset Ω of $\mathbb{R}^n$ which admits a convex solution of $\det D^2u = 1$ in Ω with $\lim_{x\to\partial\Omega} u(x) = \infty$ is $\Omega = \mathbb{R}^n$.

In [5], Caffarelli and Li investigated the existence of solutions to (1.2) with prescribed asymptotic behavior at infinity in exterior domains of $\mathbb{R}^n$ $(n \geq 2)$. The similar exterior Dirichlet problem of (1.2) in $\mathbb{R}^2$ was studied by Ferrer, Martínez, and Milán in [8,9] using complex variable methods. Bao and Li [1] also considered the exterior problem of (1.2) for $n = 2$ with different asymptotic behaviors at infinity.

Recently, Bao, Li and Zhang [2] extended Caffarelli–Li's results in [5] to the case that $g(x) = 1 + O(|x|^{-\beta})$ at infinity for some $\beta > 2$.

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Especially, for $g(x) = (1+|x|^2)^{-\frac{n+2-\frac{1}{\alpha}}{2}}$, the Monge–Ampère equation

$$\det D^2u = (1+|x|^2)^{-\frac{n+2-\frac{1}{\alpha}}{2}} \tag{1.3}$$

comes from the study of translating solutions to the K^α-flow, i.e., the flow by α powers of Gauss curvature. A more detailed background of Eq. (1.3) can be found in [15], in which the authors and Jian studied the exterior Dirichlet problem of (1.3).

Motivated by [5,2,15], we study the exterior Dirichlet problem of the Monge–Ampère equation (1.1) with more general prescribed asymptotic behavior at infinity, where g satisfies the following assumption:

(A) *$g \in C^0(\mathbb{R}^n)$ is a positive function with*

$$g(x) = g_0(|x|) + O(|x|^{-\beta}) \quad \text{as } |x| \to \infty,$$

where $g_0 \in C^0([0,+\infty))$ and $\beta > 0$.

One of the main results in this paper is the following theorem.

Theorem 1.1. *Let D be a bounded, strictly convex open subset of $\mathbb{R}^n (n \geq 2)$ with $\partial D \in C^2$ and $\phi \in C^2(\partial D)$. Assume that g satisfies* (A) *with*

$$b_1 r^\alpha \leq g_0(r) \leq b_2 r^\alpha, \quad r \geq r_0$$

for some positive constants b_1, b_2, r_0. If

$$-\frac{n(\min\{\beta, n\} - 2)}{n-1} < \alpha < \infty, \tag{1.4}$$

then for any given $b \in \mathbb{R}^n$, there exists some constant c^*, depending only on $n, b, b_1, b_2, \alpha, \beta, D$ and ϕ such that for every $c > c^*$ there exists a locally convex viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ to the Dirichlet problem

$$\begin{cases} \det D^2u = g(x), & \text{in } \mathbb{R}^n \setminus \bar{D}, \\ u = \phi, & \text{on } \partial D. \end{cases} \tag{1.5}$$

Moreover, u satisfies

$$\limsup_{|x|\to\infty} |x|^{\min\{\beta,n\}-2+\alpha-\frac{\alpha}{n}} |u(x) - f_0(|x|) - b\cdot x - c| < \infty, \tag{1.6}$$

where $f_0(|x|)$ is the radially symmetric solution of $\det D^2u = g_0(|x|)$ in $\mathbb{R}^n$ with $f_0(0) = f_0'(0) = 0$, given explicitly by (2.2).

Remark 1.1. We can obtain $u \in C^0(\mathbb{R}^n \setminus D) \cap C^\infty(\mathbb{R}^n \setminus \bar{D})$ by the regularity theory of the Monge–Ampère equation, if $g \in C^\infty$; see [11,3].

Remark 1.2. It is necessary for the condition (1.4) by counterexample in the last section. We can see that $c > c^*$ is also necessary from Wang and Bao's results in [18], in which they obtained the necessary and sufficient conditions on existence and convexity of radial solutions for (1.2) on exterior domains.

Remark 1.3. Our result is compatible with Theorem 1.5 in [5] for $g = 1$, and Theorem 1.1 in [2] for $g_0 = 1$, respectively.

Remark 1.4. There are many issues to be solved, one of which is to weaken the boundary condition $\phi \in C^2(\partial D)$ to $\phi \in C^0(\partial D)$. Besides, the Dirichlet problems on the general unbounded domains or half space are also interesting problems.

In this paper, we also obtain the existence of entire convex viscosity solution to the Monge–Ampère equation (1.1) with given asymptotic behavior at infinity.

Theorem 1.2. *Assume g, g_0 and f_0 are as in Theorem 1.1. For any given $b \in \mathbb{R}^n$, there exists some constant c^*, depending only on n, b, b_1, b_2, α and β, such that for every $c > c^*$ there exists an entire convex viscosity solution $u \in C^0(\mathbb{R}^n)$ to Eq. (1.1). Moreover, u satisfies (1.6).*

The paper is organized as follows. In Section 2, we find the radially symmetric solution to $\det D^2u = g_0$ by the ODE method and show the asymptotic behavior of the solution at infinity for $\alpha \in (-n, \infty)$. Theorems 1.1 and 1.2 will be proved by the Perron method in Sections 3 and 4, respectively. Finally, we give a counterexample to show the necessity of the condition (1.4) in Section 5.

2. Radially symmetric solutions of $\det D^2u = g_0$

Let $u(x) = f(|x|)$. Then

$$\det(D^2u(x)) = \left(\frac{f'(r)}{r}\right)^{n-1} f''(r), \quad r = |x|.$$

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The equation $\det D^2u(x) = g_0(|x|)$ can be rewritten as

$$(f'(r))^{n-1}f''(r) = r^{n-1}g_0(r). \tag{2.1}$$

By the similar method as in [15], we can find a convex radially symmetric solution $u(x) := f(|x|) \in C^2(\mathbb{R}^n)$ if we define $f''(0) = g_0^{\frac{1}{n}}(0)$.

Denote by $f_0(r)$ the solution of (2.1) with $f_0(0) = 0, f_0'(0) = 0$. Then

$$f_0(r) = \int_0^r \left[\int_0^\tau ns^{n-1}g_0(s)ds\right]^{\frac{1}{n}} d\tau. \tag{2.2}$$

Assume that

$$b_1r^\alpha \le g_0(r) \le b_2r^\alpha, \quad \text{for } r > r_0 \gg 1 \tag{2.3}$$

for some fixed constant r_0. Since

$$\int_0^\tau s^{n-1}g_0(s)ds = \int_0^{r_0} s^{n-1}g_0(s)ds + \int_{r_0}^\tau s^{n-1}g_0(s)ds,$$

then, we obtain from (2.3) that

$$\frac{nb_1}{n+\alpha}\tau^{n+\alpha} + m_1 \le n\int_0^\tau s^{n-1}g_0(s)ds \le \frac{nb_2}{n+\alpha}\tau^{n+\alpha} + m_2, \quad \text{for } \tau > 0, \tag{2.4}$$

where m_i $(i = 1, 2)$ depend on r_0, n, α and b_i $(i = 1, 2)$. Therefore,

$$f_0(r) = O\left(r^{2+\frac{\alpha}{n}}\right), \quad \text{as } r \to \infty.$$

3. Proof of Theorem 1.1

By subtracting a linear function from u, we need only to prove the theorem for $b = 0$. Under the assumption (A), we can find two positive continuous functions $\underline{g}$ and $\bar{g}$ satisfying

$$0 < \underline{g}(|x|) \le g(x) \le \bar{g}(|x|), \quad \forall x \in \mathbb{R}^n \tag{3.1}$$

with

$$\underline{g}(|x|) = g_0(|x|) - c_1|x|^{-\beta}, \tag{3.2}$$

$$\bar{g}(|x|) = g_0(|x|) + c_2|x|^{-\beta}, \tag{3.3}$$

for some positive constants c_1 and c_2 and $|x|$ sufficiently large.

By the similar technique as in [5, Lemma A.3], we obtain the following existence theorem in bounded convex domains, which is needed for us to use the Perron method.

Lemma 3.1. *Let Ω be a smooth, bounded, strictly convex subset in $\mathbb{R}^n$ $(n \ge 2)$, and let g be a positive continuous function on $\bar{\Omega}$ and $\varphi \in C^0(\partial\Omega)$. Assume that $\underline{u} \in C^0(\bar{\Omega})$ is a convex viscosity subsolution to $\det D^2u = g(x)$. Then the Dirichlet problem*

$$\begin{cases} \det D^2u = g(x), & \text{in } \Omega, \\ u = \underline{u}, & \text{on } \partial\Omega \end{cases}$$

has a unique convex viscosity solution $u \in C^0(\bar{\Omega})$.

Lemma 3.2. *Let $\Omega_1 \subset \Omega_2$ be two bounded domains in $\mathbb{R}^n$ and $g \in C^0(\mathbb{R}^n)$ an nonnegative function. Suppose that convex functions $v \in C^0(\bar{\Omega}_2)$, $u \in C^0(\mathbb{R}^n \setminus \Omega_1)$ satisfy*

$$\det D^2v \ge g(x), \quad x \in \Omega_2,$$
$$\det D^2u \ge g(x), \quad x \in \mathbb{R}^n \setminus \bar{\Omega}_1$$

in the viscosity sense, respectively, and

$$u < v, \quad x \in \partial\Omega_1,$$
$$u > v, \quad x \in \partial\Omega_2.$$

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Set

$$w(x) := \begin{cases} v(x), & x \in \Omega_1, \\ \max\{v(x), u(x)\}, & x \in \Omega_2 \setminus \bar{\Omega}_1, \\ u(x), & x \in \mathbb{R}^n \setminus \bar{\Omega}_2. \end{cases}$$

Then, $w \in C^0(\mathbb{R}^n)$ is a convex function and satisfies

$$\det D^2 w \geq g(x), \quad x \in \mathbb{R}^n,$$

in the viscosity sense.

Proof. Let $\bar{x} \in \mathbb{R}^n$, $\varphi \in C^2(\mathbb{R}^n)$ satisfy

$$w(\bar{x}) = \varphi(\bar{x}), \qquad w(x) \leq \varphi(x), \quad x \in \mathbb{R}^n. \tag{3.4}$$

If $\bar{x} \in \Omega_1$, we have

$$v(\bar{x}) = w(\bar{x}) = \varphi(\bar{x}), \qquad v(x) \leq w(x) \leq \varphi(x), \quad x \in \Omega_2. \tag{3.5}$$

By the definition of viscosity subsolution, we obtain

$$\det D^2 \varphi(\bar{x}) \geq g(\bar{x}). \tag{3.6}$$

If $\bar{x} \in \mathbb{R}^n \setminus \bar{\Omega}_2$, we have

$$u(\bar{x}) = w(\bar{x}) = \varphi(\bar{x}), \qquad u(x) \leq w(x) \leq \varphi(x), \quad x \in \mathbb{R}^n \setminus \bar{\Omega}_1. \tag{3.7}$$

Then (3.6) holds, since u is a viscosity subsolution of $\det D^2 u = g(x)$ in $\mathbb{R}^n \setminus \bar{\Omega}_1$.

In the case of $\bar{x} \in \Omega_2 \setminus \bar{\Omega}_1$, if $w(\bar{x}) = v(\bar{x})$, then we have (3.5), which implies (3.6). If $w(\bar{x}) = u(\bar{x})$, then we obtain (3.7), which implies (3.6). The lemma is completed. □

Definition 3.3. The subfunction class S_c for some constant c is defined as follows: a function $v \in S_c$ if and only if

(1) $v \in C^0(\mathbb{R}^n \setminus D)$ and $v \leq \phi$ on ∂D;
(2) v is a locally convex viscosity subsolution of (1.1) in $\mathbb{R}^n \setminus \bar{D}$;
(3) $v(x) \leq \bar{w}(x) + c$, $\forall\, x \in \mathbb{R}^n \setminus D$, where

$$\bar{w}(x) := \int_1^{|x|} \left[n \int_0^{\tau} s^{n-1} \underline{g}(s) ds \right]^{\frac{1}{n}} d\tau.$$

Lemma 3.4. *There exists some constant $\tilde{c}$, depending only on n, α, β, D and ϕ, such that, for any $c > \tilde{c}$, we can find a viscosity subsolution $\underline{u} \in S_c$.*

Proof. Fix $R_2 > R_1 > 1$ such that $D \subset\subset B_{R_1}(0)$ and $R_2 > 3R_1$. Let

$$C := \max_{x \in \overline{B_{R_2}}} g(x) > 0.$$

According to Lemma 5.1 in [5], for any boundary point $\xi \in \partial D$, we can construct a convex smooth solution $v_\xi(x)$ to the equation

$$\det D^2 u = C \quad \text{in } \mathbb{R}^n$$

with

$$v_\xi(\xi) = \phi(\xi), \quad v_\xi < \phi \text{ on } \partial D \setminus \{\xi\}.$$

Define

$$V(x) := \sup_{\xi \in \partial D} v_\xi(x), \quad x \in \overline{B_{R_2}(0)}.$$

We can see that $V(x)$ is a convex viscosity subsolution of (1.1) in $B_{R_2}(0)$ and satisfies

$$V(\xi) \leq \phi(\xi), \quad \xi \in \partial D.$$

On the other hand, by the definition of V, for any $\xi \in \partial D$,

$$V(\xi) \geq v_\xi(\xi) = \phi(\xi).$$

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Therefore,

$$V = \phi \quad \text{on } \partial D.$$

For $a \geq 0$, we define a function

$$\underline{w}_a(x) := \inf_{x \in B_{R_1}} V(x) + \int_{2R_1}^{|x|} [\bar{h}(\tau) + a]^{\frac{1}{n}} d\tau,$$

where

$$\bar{h}(\tau) = n \int_0^\tau s^{n-1} \bar{g}(s) ds. \tag{3.8}$$

It is easy to check that $\underline{w}_a$ is a locally convex subsolution of (1.1) in $\mathbb{R}^n \setminus B_1(0)$ and

$$\underline{w}_a(x) \leq V(x), \quad \text{for } 1 \leq |x| \leq R_1.$$

In view of $R_2 > 3R_1$, we choose $a_1 > 0$ large enough such that for $a \geq a_1$,

$$\underline{w}_a(x) \geq \inf_{x \in B_{R_1}} V(x) + \int_{2R_1}^{3R_1} [\bar{h}(\tau) + a]^{\frac{1}{n}} d\tau \geq 1 + V(x), \quad |x| = R_2. \tag{3.9}$$

Let $h_0(\tau) = n \int_0^\tau s^{n-1} g_0(s) ds$. Then $f_0(r) = \int_0^r [h_0(\tau)]^{\frac{1}{n}} d\tau$. Now, we rewrite $\underline{w}_a$ as

$$\begin{aligned} \underline{w}_a(x) &= f_0(|x|) + \inf_{x \in B_{R_1}} V(x) + \int_{2R_1}^{|x|} [\bar{h}(\tau) + a]^{\frac{1}{n}} d\tau - \int_0^{|x|} [h_0(\tau)]^{\frac{1}{n}} d\tau \\ &= f_0(|x|) + \underline{\mu}(a) - \int_{|x|}^\infty (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\bar{h}(\tau) - h_0(\tau) + a}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau, \end{aligned} \tag{3.10}$$

where

$$\underline{\mu}(a) := \inf_{x \in B_{R_1}} V(x) - f_0(2R_1) + \int_{2R_1}^\infty (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\bar{h}(\tau) - h_0(\tau) + a}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau.$$

It follows from (3.3) that

$$\begin{aligned} \bar{h}(\tau) &= h_0(\tau) + n \int_0^1 s^{n-1} (\bar{g}(s) - g_0(s)) ds + n \int_1^\tau s^{n-1} (\bar{g}(s) - g_0(s)) ds \\ &= h_0(\tau) + \hat{c}_1 + \hat{c}_2 \tau^{n-\beta}, \end{aligned} \tag{3.11}$$

as τ large enough, where $\hat{c}_i$ $(i = 1, 2)$ depend on n, c_1, c_2 and β. It is easy to compute that $\alpha + \beta > 0$ since $\alpha > -\frac{n(\min\{\beta, n\} - 2)}{n-1}$, which together with (2.4) and (3.11) implies that for τ sufficiently large,

$$\hat{c}_3 \tau^{1 - \min\{\beta, n\} - \alpha + \frac{\alpha}{n}} \leq (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\bar{h}(\tau) - h_0(\tau) + a}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} \leq \hat{c}_4 \tau^{1 - \min\{\beta, n\} - \alpha + \frac{\alpha}{n}},$$

where $\hat{c}_i$ $(i = 3, 4)$ depends on a, α, β, n, c_i $(i = 1, 2)$. Then, in view of $-\frac{n(\min\{\beta, n\} - 2)}{n-1} < \alpha < \infty$, i.e., $1 - \min\{\beta, n\} - \alpha + \frac{\alpha}{n} < -1$, we have

$$|\underline{\mu}(a)| < \infty \tag{3.12}$$

and

$$\underline{w}_a(x) = f_0(|x|) + \underline{\mu}(a) + O\left(|x|^{2 - \min\{\beta, n\} - \alpha + \frac{\alpha}{n}} \right), \quad \text{as } |x| \to \infty. \tag{3.13}$$

As above, let

$$\underline{h}(\tau) = n \int_0^\tau s^{n-1} \underline{g}(s) ds,$$

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then $\bar{w}(x)$ can be rewritten as

$$\begin{aligned}\bar{w}(x) &= f_0(|x|) + \int_1^{|x|} [\underline{h}(\tau)]^{\frac{1}{n}} d\tau - \int_0^{|x|} [h_0(\tau)]^{\frac{1}{n}} d\tau \\ &= f_0(|x|) + \bar{\mu}_0 - \int_{|x|}^{\infty} (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\underline{h}(\tau) - h_0(\tau)}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau, \end{aligned} \tag{3.14}$$

where

$$\bar{\mu}_0 := -f_0(1) + \int_1^{\infty} (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\underline{h}(\tau) - h_0(\tau)}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau.$$

By the similar calculating as (3.11)–(3.13), we obtain $|\bar{\mu}_0| < \infty$ and

$$\bar{w}(x) = f_0(|x|) + \bar{\mu}_0 + O\left(|x|^{2-\min\{\beta,n\}-\alpha+\frac{\alpha}{n}} \right), \quad \text{as } |x| \to \infty. \tag{3.15}$$

In view of (3.1), for $a > 0$,

$$(h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\bar{h}(\tau) - h_0(\tau) + a}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} \geq (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\underline{h}(\tau) - h_0(\tau)}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\}.$$

Then,

$$\underline{w}_a(x) \leq \bar{w}(x) + \underline{\mu}(a) - \bar{\mu}_0, \quad \forall\, x \in \mathbb{R}^n \setminus B_1(0). \tag{3.16}$$

It is clear that $\underline{\mu}(a)$ is continuous, monotonic increasing for a, and $\underline{\mu}(a) \to \infty$ as $a \to \infty$. We can choose $a_2 > 0$ large enough such that, for $a > a_2$,

$$V(x) \leq \bar{w}(x) + \underline{\mu}(a) - \bar{\mu}_0, \quad |x| \leq R_2. \tag{3.17}$$

Set $a^* = \max\{a_1, a_2\}$. Then for any $a > a^*$, (3.9), (3.16) and (3.17) hold.

Define

$$\underline{u}_a(x) := \begin{cases} V(x), & |x| < R_1, \\ \max\{V(x), \underline{w}_a(x)\}, & R_1 \leq |x| < R_2, \\ \underline{w}_a(x), & |x| \geq R_2. \end{cases} \tag{3.18}$$

Obviously,

$$\underline{u}_a(x) = V(x) = \phi(x), \quad x \in \partial D. \tag{3.19}$$

By Lemma 3.2, $\underline{u}_a$ is a convex viscosity subsolution of (1.1) in $\mathbb{R}^n$. For $c > c^* := \underline{\mu}(a^*)$, there is a number $a > a^*$, such that $c = \underline{\mu}(a)$.

From (3.16) and (3.17), we have, for $c > c^*$,

$$\underline{u}_a(x) \leq \bar{w}(x) + c - \bar{\mu}_0, \quad \forall\, x \in \mathbb{R}^n.$$

Then $\underline{u}_a \in S_{c-\bar{\mu}_0}$. Moreover, by (3.13),

$$\underline{u}_a(x) = f_0(|x|) + \underline{\mu}(a) + O\left(|x|^{2-\min\{\beta,n\}-\alpha+\frac{\alpha}{n}} \right), \quad \text{as } |x| \to \infty. \tag{3.20}$$

Taking $\tilde{c} = c^* - \bar{\mu}_0$, the lemma is proved. □

Define for $c > c^*$,

$$u_c(x) := \sup\{v(x) : v \in S_{c-\bar{\mu}_0}\}, \quad x \in \mathbb{R}^n \setminus \bar{D}.$$

Lemma 3.5. *The function $u_c(x) \in C^0(\mathbb{R}^n \setminus D)$ is a locally convex viscosity solution to the exterior Dirichlet problem (1.5) and $u_c \leq \bar{w}(x) + c - \bar{\mu}_0$ in $\mathbb{R}^n \setminus D$.*

Proof. We will divide the proof into two steps.

The first step is to prove that $u_c(x)$ is a locally convex viscosity subsolution of (1.1) with $u_c = \phi$ on ∂D and $u_c \leq \bar{w}(x) + c - \bar{\mu}_0$ in $\mathbb{R}^n \setminus D$.

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As in [15], from the definition of u_c and $S_{c-\bar{\mu}_0}$, we know that u_c is a locally convex viscosity subsolution and $u_c \le \bar{w}(x) + c - \bar{\mu}_0$ in $\mathbb{R}^n \setminus \bar{D}$. Since

$$u_c(x) \ge \underline{u}_a(x) \quad \text{in } \mathbb{R}^n \setminus \bar{D}$$

with $c = \underline{\mu}(a)$ and $\underline{u}_a$ is continuous on the boundary of D, so, for any $\xi_0 \in \partial D$,

$$\liminf_{x\to\xi_0} u_c(x) \ge \underline{u}_a(\xi_0) = \phi(\xi_0). \tag{3.21}$$

On the other hand, for any $v \in S_{c-\bar{\mu}_0}$, v is a viscosity subsolution of (1.1) in $\mathbb{R}^n \setminus \bar{D}$, i.e., for every $\bar{x} \in \mathbb{R}^n \setminus \bar{D}$ and every function $\varphi \in C^2(\mathbb{R}^n \setminus \bar{D})$ satisfying

$$\varphi(\bar{x}) = v(\bar{x}), \quad \varphi \ge v \text{ on } \mathbb{R}^n \setminus \bar{D},$$

we have $\det D^2\varphi(\bar{x}) \ge g(\bar{x}) > 0$. By Remark 1.3.2 in [11], we know $D^2\varphi(\bar{x}) \ge 0$. Then,

$$\triangle\varphi(\bar{x}) \ge n[\det D^2\varphi(\bar{x})]^{\frac{1}{n}} > 0,$$

which means that v is a viscosity subsolution of $\triangle v = 0$ in $\mathbb{R}^n \setminus \bar{D}$ and $v \le \phi$ on ∂D.

Choosing a ball $B_R(0) \supset D$. It is well-known that the Dirichlet problem

$$\begin{cases} \triangle v^+ = 0, & \text{in } B_R(0) \setminus \bar{D}, \\ v^+ = \phi, & \text{on } \partial D, \\ v^+ = u_c, & \text{on } \partial B_R(0) \end{cases} \tag{3.22}$$

has a unique classical solution $v^+ \in C^2(B_R(0) \setminus \bar{D}) \cap C^0(\overline{B_R(0) \setminus D})$; see [10]. Applying a comparison principle to v^+ and $v \in S_{c-\bar{\mu}_0}$,

$$v \le v^+ \quad \text{in } \overline{B_R(0) \setminus D}.$$

Therefore, $u_c \le v^+$ in $B_R(0) \setminus \bar{D}$ and

$$\limsup_{x\to\xi_0} u_c(x) \le v^+(\xi_0) = \phi(\xi_0).$$

The second step is to show that $u_c(x)$ is a viscosity solution of (1.1) in $\mathbb{R}^n \setminus \bar{D}$.

For any $x_0 \in \mathbb{R}^n \setminus \bar{D}$, choose a ball $B_\varepsilon(x_0) \subset \mathbb{R}^n \setminus \bar{D}$. By Lemma 3.1, there is a unique convex viscosity solution $\tilde{u} \in C^0(\overline{B_\varepsilon(x_0)})$ to the Dirichlet problem

$$\begin{cases} \det D^2\tilde{u} = g(x) & \text{in } B_\varepsilon(x_0), \\ \tilde{u} = u_c & \text{on } \partial B_\varepsilon(x_0). \end{cases}$$

According to the definition of $\bar{w}(x)$ and u_c, we can see that

$$\begin{cases} \det D^2(\bar{w} + c - \bar{\mu}_0) \le g(x), & \text{in } B_\varepsilon(x_0), \\ \bar{w} + c - \bar{\mu}_0 \ge u_c & \text{on } \partial B_\varepsilon(x_0). \end{cases}$$

Using the comparison principle of viscosity solutions, see the main theorem in [12], we have $\tilde{u} \ge u_c$ and $\tilde{u} \le \bar{w} + c - \bar{\mu}_0$ on $\overline{B_\varepsilon(x_0)}$.

Define

$$\tilde{w}(x) = \begin{cases} \tilde{u}(x), & x \in B_\varepsilon(x_0), \\ u_c(x), & x \in \mathbb{R}^n \setminus (D \cup B_\varepsilon(x_0)). \end{cases}$$

Obviously, $\tilde{w}$ is a locally convex viscosity subsolution with $\tilde{w} = \phi$ on ∂D and $u_c \le \bar{w} + c - \bar{\mu}_0$ in $\mathbb{R}^n \setminus D$; therefore, $\tilde{w} \in S_{c-\bar{\mu}_0}$. Then, $u_c \ge \tilde{w}$ on $B_\varepsilon(x_0)$ by the definition of u_c. It follows that $u_c \equiv \tilde{u}$ on $B_\varepsilon(x_0)$, and $u_c \in C^0(\mathbb{R}^n \setminus D)$ is a viscosity solution to (1.5). □

Proof of Theorem 1.1. From Lemma 3.5, we need only to prove that u_c satisfies (1.6). In fact, by the definition of u_c and Lemma 3.5, we have

$$\underline{u}_a \le u_c \le \bar{w}(x) + c - \bar{\mu}_0 \quad \text{in } \mathbb{R}^n \setminus D,$$

where $c = \underline{\mu}(a)$, which, together with (3.15) and (3.20), implies

$$\limsup_{|x|\to\infty} |x|^{\min\{\beta,n\}-2+\alpha-\frac{\alpha}{n}} |u_c(x) - f_0(|x|) - c| < \infty.$$

The theorem is completed. □

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4. Proof of Theorem 1.2

As above, we define a class of subfunctions as follows.

Definition 4.1. The subfunction class $\hat{S}_c$ for some constant c is defined as follows: a function v is in $\hat{S}_c$ if and only if
(1) $v \in C^0(\mathbb{R}^n)$ is a convex viscosity subsolution of (1.1) in $\mathbb{R}^n$;
(2) $v(x) \le \bar{w}(x) + c,\ \forall x \in \mathbb{R}^n$.

From the proof of Lemma 3.4, we can see that $\underline{u}_a$ is a convex viscosity subsolution of (1.1) in $\mathbb{R}^n$, and $\underline{u}_a(x) \le \bar{w}(x) + c - \bar{\mu}_0$ with $c = \underline{\mu}(a)$ and $a > a^*$. Therefore, $\underline{u}_a \in \hat{S}_{c-\bar{\mu}_0}$ for $c > c^*$. Moreover,

$$\underline{u}_a(x) = f_0(|x|) + \underline{\mu}(a) + O\left(|x|^{2-\min\{\beta,n\}-\alpha+\frac{\alpha}{n}}\right), \quad \text{as } |x| \to \infty. \tag{4.1}$$

Lemma 4.2. *If we define*

$$\hat{u}_c(x) := \sup\{v(x) : v \in \hat{S}_{c-\bar{\mu}_0}\}, \quad x \in \mathbb{R}^n,\ c > c^*.$$

Then $\hat{u}_c$ is an entire convex viscosity solution of (1.1) with $\hat{u}_c \le \bar{w}(x) + c - \bar{\mu}_0$ in $\mathbb{R}^n$.

Proof. The proof is similar to the proof of Lemma 3.5. □

Proof of Theorem 1.2. It follows from Lemma 4.2 that for any $c > c^*$, there exists an entire convex viscosity solution $\hat{u}_c \in C^0(\mathbb{R}^n)$ to (1.1). We need only to prove (1.6). By the definition of $\hat{u}_c$ and Lemma 4.2, we have

$$\underline{u}_a \le \hat{u}_c \le \bar{w} + c - \bar{\mu}_0, \quad \text{in } \mathbb{R}^n,$$

where $c = \underline{\mu}(a)$. Then the asymptotic behavior (1.6) follows from (3.15) and (4.1). The theorem is completed. □

5. Example

In this part, we will give a counterexample to show the necessity of $\alpha > -\frac{n(\min\{\beta,n\}-2)}{n-1}$ in Theorem 1.1. From Theorems 4.2 and 4.3 in [15] where $\beta = 0$, we can see that it is necessary for $\alpha > -\frac{n(n-2)}{n-1}$ in Theorem 1.1.

Fix a ball $B_1(0) \subseteq B_R(0) \subset \mathbb{R}^n (n \ge 2)$ and a constant d. We consider the locally convex radially symmetric solution of the exterior Dirichlet problem

$$\begin{cases} \det D^2 u = g(|x|) & \text{in } \mathbb{R}^n \setminus \overline{B_R(0)}, \\ u = d, & \text{on } \partial B_R(0), \end{cases} \tag{5.1}$$

where

$$g(|x|) = g_0(|x|) + |x|^{-\beta} \quad \text{for } |x| \ge R$$

with

$$g_0(|x|) = \begin{cases} 1, & |x| \le 1, \\ |x|^{\alpha}, & |x| > 1. \end{cases}$$

Theorem 5.1. *Assume $\alpha = -\frac{n(\beta-2)}{n-1}$ for $0 < \beta < n$. Then any radially symmetric locally convex solution $u(x) := f(|x|) \in C^0(\mathbb{R}^n \setminus B_R(0)) \cap C^2(\mathbb{R}^n \setminus \overline{B_R(0)})$ to the exterior problem (5.1) has the following asymptotic behavior at infinity:*

$$\limsup_{|x|\to\infty} |x|^{n-\beta} \left|u(x) - f_0(|x|) - c_1 - c_2 \ln |x|\right| < \infty, \tag{5.2}$$

where $c_2 = \frac{1}{n-\beta}\left(\frac{n}{n+\alpha}\right)^{\frac{1}{n}-1}$, and c_1 depends on d, R, n, β and $f'(R)$.

Proof. Assume that $u(x) := f(|x|) \in C^0(\mathbb{R}^n \setminus B_R(0)) \cap C^2(\mathbb{R}^n \setminus \overline{B_R(0)})$ is a locally convex radially symmetric solution of (5.1). Then $f''(r) > 0$, $\frac{f'(r)}{r} > 0$ for $r > R$, $r = |x|$ and

$$((f'(r))^n)' = nr^{n-1}g(r).$$

Integrating the above equation on $[R, r]$ for $r > R$, we obtain

$$f'(r) = \left[n\int_R^r s^{n-1}g(s)ds + b\right]^{\frac{1}{n}},$$

where $b = (f'(R))^n \ge 0$.

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Let

$$h(r) = n\int_R^r s^{n-1}g(s)ds, \qquad h_0(r) = n\int_0^r s^{n-1}g_0(s)ds.$$

Under the assumption that $g(r) = g_0(r) + r^{-\beta}$ for $r \geq R$, it is easy to see that

$$\begin{aligned} h(r) &= h_0(r) - h_0(R) + n\int_R^r s^{n-\beta-1}ds \\ &= h_0(r) - c_0 + \frac{n}{n-\beta}r^{n-\beta} \end{aligned}$$

for $r \geq R$, where $c_0 = h_0(R) + \frac{n}{n-\beta}R^{n-\beta}$. Then, by recalling the definition of f_0,

$$\begin{aligned} f(|x|) &= \int_R^{|x|} [h(\tau) + b]^{\frac{1}{n}} d\tau + f(R) \\ &= f_0(|x|) - f_0(R) + d + \int_R^{|x|} (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\frac{n}{n-\beta}\tau^{n-\beta} - c_0 + b}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau. \end{aligned}$$

In view of $\alpha = -\frac{n(\beta-2)}{n-1}$ for $0 < \beta < n$, by Taylor's expansion, we obtain

$$\begin{aligned} (h_0(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{\frac{n}{n-\beta}\tau^{n-\beta} - c_0 + b}{h_0(\tau)} \right]^{\frac{1}{n}} - 1 \right\} &= \frac{1}{n-\beta}\left(\frac{n}{n+\alpha}\right)^{\frac{1}{n}-1}\tau^{-1} \\ &\quad + \frac{b-c_0}{n}\left(\frac{n}{n+\alpha}\right)^{\frac{1}{n}-1}\tau^{\beta-n-1} + o(\tau^{\beta-n-1}), \end{aligned}$$

as $\tau \to \infty$. Therefore,

$$f(|x|) = f_0(|x|) + c_1 + c_2 \ln|x| + O(|x|^{\beta-n}), \quad |x| \to \infty,$$

where $c_2 = \frac{1}{n-\beta}\left(\frac{n}{n+\alpha}\right)^{\frac{1}{n}-1}$, and c_1 depends only on n, β, b, d and R. □

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