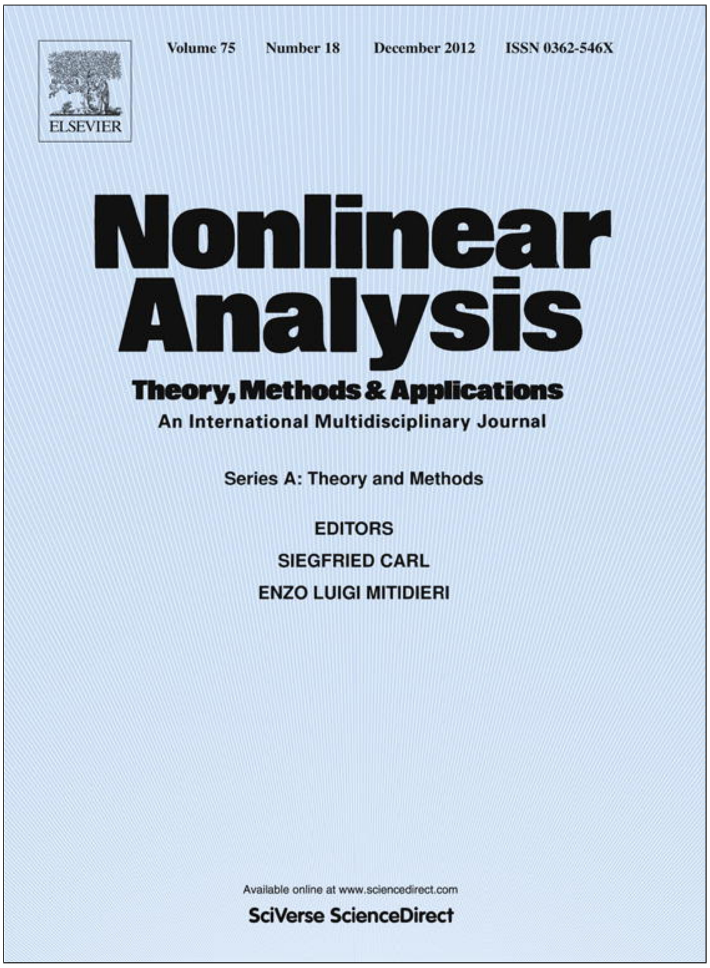
Volume 75 Number 18 December 2012 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
EDITORS
SIEGFRIED CARL
ENZO LUIGI MITIDIERI
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Nonlinear Analysis 75 (2012) 6448–6455

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the exterior Dirichlet problem for the Monge–Ampère equation in dimension two

Jiguang Bao, Haigang Li *

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

ARTICLE INFO

Article history:
Received 5 March 2012
Accepted 20 July 2012
Communicated by Enzo Mitidieri

Keywords:
Monge–Ampère equation
Exterior Dirichlet problem

ABSTRACT

In this paper, we consider the Dirichlet problem for the Monge–Ampère equation on exterior domains in dimension two and prove a theorem on the existence of solutions with prescribed asymptotic behavior at infinity.

1. Introduction

In this paper, we consider the solvability of the Dirichlet problem for the Monge–Ampère equation

$$\det(D^2u) = 1 \tag{1.1}$$

on an exterior domain $\mathbb{R}^2 \setminus \bar{D}$, where D is a bounded open convex subset of $\mathbb{R}^2$. This equation arises in the context of an affine differential geometry problem as the equation of a parabolic affine sphere in the unimodular affine real 3-space (see [1]). Contrary to studies of (1.1) in smooth bounded domains, less is known about the solutions of (1.1) when the domain is unbounded.

A classical theorem of Jörgens ($n = 2$ [2]), Calabi ($n \le 5$ [3]), and Pogorelov ($n \ge 2$ [4]) states that any classical convex solution of $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. A simpler and more analytic proof, along the lines of affine geometry, was later given by Cheng and Yau [5]. Caffarelli [6] extended the result for classical solutions to viscosity solutions. Another proof of this theorem was given by Jost and Xin [7]. Trudinger and Wang [8] proved that if Ω is an open convex subset of $\mathbb{R}^n$ and u is a convex C^2 solution of $\det(D^2u) = 1$ in Ω with $\lim_{x\to\partial\Omega} u(x) = \infty$, then $\Omega = \mathbb{R}^n$ and u is quadratic.

Caffarelli and Li [9] extended the Jörgens–Calabi–Pogorelov theorem to exterior domains. Let u be a convex viscosity solution of $\det(D^2u) = 1$ in $\mathbb{R}^n \setminus \bar{D}$, then for $n \ge 3$, there exist an $n \times n$ real symmetric positive definite matrix A, a vector $b \in \mathbb{R}^n$, and a constant $c \in \mathbb{R}$ such that

$$\limsup_{|x|\to\infty} \left(|x|^{n-2} \left| u(x) - \left[\frac{1}{2}x^TAx + b\cdot x + c \right] \right| \right) < \infty; \tag{1.2}$$

for $n = 2$, there exist a 2×2 real symmetric positive definite matrix A, a vector $b \in \mathbb{R}^2$, and constants $c, d \in \mathbb{R}$ such that

$$\limsup_{|x|\to\infty} \left(|x| \left| u(x) - \left[\frac{1}{2}x^TAx + b\cdot x + d\ln\sqrt{x^TAx} + c \right] \right| \right) < \infty; \tag{1.3}$$

* Corresponding author.
E-mail addresses: jgbao@bnu.edu.cn (J. Bao), hgli@bnu.edu.cn (H. Li).

0362-546X/$ – see front matter
doi:10.1016/j.na.2012.07.017

moreover, if $D = \varnothing$, then $d = 0$. The similar asymptotic problems in $\mathbb{R}^2$ are studied by Ferrer et al. in [10,11] using the complex variable method. It is shown that any solution behaves like a quadratic polynomial plus a logarithmic term at infinity, i.e.

$$\limsup_{|x|\to\infty} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + d \ln \sqrt{x^T A x} + c \right) \right| < \infty. \tag{1.4}$$

See also Delanoë [12]. We remark that the theorem of Jörgens, Calabi, Pogorelov is an easy consequence of the above results. Indeed, let $u \in C^2$ be a convex solution of

$$\det(D^2 u) = 1 \quad \text{in } \mathbb{R}^n, \ n \geq 3,$$

then for some c, b, and $n \times n$ real symmetric positive definite matrix A with $\det(A) = 1$,

$$E(x) := u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \to 0 \quad \text{as } |x| \to \infty.$$

Since

$$\det(A + D^2 E) - \det(A) = \det(D^2 u) - 1 = 0$$

and $(A + D^2 E) = (D^2 u)$ is positive definite, it follows from the mean value theorem that for some positive definite matrix function $(a_{ij}(x))$,

$$a_{ij} D_{ij} E = 0 \quad \text{in } \mathbb{R}^n, \ n \geq 3.$$

By the maximum principle, $E(x) \equiv 0$, i.e.,

$$u(x) \equiv \frac{1}{2} x^T A x + b \cdot x + c.$$

The above results on asymptotic behavior at infinity of the solutions of (1.1) enable us to study the existence of solutions for the exterior Dirichlet problem of the Monge–Ampère equation with these prescribed asymptotic behaviors at infinity. Whereas in dimensions $n \geq 3$ the existence theory for the exterior Dirichlet problem of Monge–Ampère equations has been established in [9] and extended to k-Hessian equations in [13,14], the analogue existence problem for Monge–Ampère equations in dimension two has not been studied until very recently. In this paper, we will investigate the existence theorem to the Dirichlet problem for (1.1) on exterior domains in $\mathbb{R}^2$, with an appropriate asymptotic behavior at infinity.

In order to impose some restriction on the behavior of the solution at infinity, it will be worthwhile to recall some classical results concerning Laplace's equation $\Delta u = 0$ in $\mathbb{R}^n$, $n \geq 2$. First of all, in dimensions $n \geq 3$ Laplace's equation has a radial symmetric solution $|x|^{2-n}$, which tends to zero as $|x| \to \infty$, while in dimension two the radial symmetric solution is $\ln |x|$, tending to $+\infty$ as $|x| \to \infty$. Second, in treating the exterior Dirichlet problem for $\Delta u = 0$ it is clear that some restriction on the behavior of the solution at infinity is also necessary in order to insure uniqueness. Indeed, for the case $n = 2$, it is well known that the boundedness of solutions alone suffices for uniqueness. So that for some d, c, and $\varphi \in C^2(\partial B_1)$, where $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$, the following exterior Dirichlet problem

$$\begin{cases} \Delta u = 1, & \text{in } \mathbb{R}^2 \setminus \overline{B_1}, \\ u = \varphi, & \text{on } \partial B_1, \\ u - \left(\frac{1}{4}|x|^2 + d \ln |x| + c \right) = O(1), & \text{as } |x| \to \infty \end{cases}$$

has a unique solution; see [15].

Recently, Wang and Bao in [16] have studied radial solutions of the exterior Dirichlet problem of (1.1). They showed that for any $\rho \in \mathbb{R}$, the Dirichlet problem

$$\begin{cases} \det(D^2 u) = 1, \quad (D^2 u) > 0, & \text{in } \mathbb{R}^2 \setminus \overline{B_1}, \\ u = \rho, & \text{on } \partial B_1 \end{cases} \tag{1.5}$$

has a unique radial solution

$$u(x) = \rho + \frac{1}{2}\left(|x|\sqrt{|x|^2 + d} + d \ln \left(|x| + \sqrt{|x|^2 + d} \right) \right) - \frac{1}{2}\left(\sqrt{1+d} + d \ln \left(1 + \sqrt{1+d} \right) \right)$$

in $C^1(\mathbb{R}^2 \setminus B_1(0)) \cap C^2(\mathbb{R}^2 \setminus \overline{B_1(0)})$, satisfying

$$u(x) = \frac{1}{2}|x|^2 + \frac{d}{2} \ln |x| + c + O\left(|x|^{-2}\right), \quad x \to \infty, \tag{1.6}$$

if and only if $d \geq -1$ and

$$c = \rho + \frac{d}{4}(1 + 2\ln 2) - \frac{1}{2}\left[\sqrt{1+d} + d \ln \left(1 + \sqrt{1+d} \right) \right]. \tag{1.7}$$

In this paper, we consider the following exterior Dirichlet problem

$$\begin{cases} \det(D^2 u) = 1, & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\ u = \varphi, & \text{on } \partial D, \end{cases} \tag{1.8}$$

where D is a bounded open convex subset of $\mathbb{R}^2$ with $0 \in D$, and $\varphi \in C^2(\partial D)$. Let

$$\mathcal{A} = \{A : A \text{ is a real } 2 \times 2 \text{ symmetric positive definite matrix, with } \det(A) = 1\},$$

and

$$E_r = \{x \in \mathbb{R}^2 : x^T A x < r^2\}.$$

Then the main result is the following.

Theorem 1.1. *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^2$, and $\varphi \in C^2(\partial D)$. Then for any given $b \in \mathbb{R}^2$ and any given $A \in \mathcal{A}$, there exists some constant d_*, depending only on D, φ, b, and A, such that for every $d > d_*$ there exists a unique local convex function $u \in C^0(\mathbb{R}^2 \setminus D) \cap C^\infty(\mathbb{R}^2 \setminus \bar{D})$ that satisfies (1.8) and*

$$O\left(|x|^{-2}\right) \le V(x) \le M_d + O\left(|x|^{-2}\right) \quad \text{as } x \to +\infty, \tag{1.9}$$

where

$$V(x) = u(x) - \left(\frac{1}{2}x^T A x + b \cdot x + d \ln \sqrt{x^T A x} + c(d)\right),$$

$$M_d = \text{osc}_{\partial D}\varphi + \max_{\partial D} \int_{\sqrt{x^T A x}}^{\bar{r}} \sqrt{s^2 + d}\, ds,$$

$\bar{r} = \min\{r > 0 : D \subset E_r\}$, and $c(d)$ is a function of d.

Remark 1.1. In fact, we expect by (1.6) that the solution has the following asymptotic behavior at infinity

$$\limsup_{|x| \to \infty} \left(|x|^2 \left| u(x) - \left(\frac{1}{2}x^T A x + b \cdot x + d \ln \sqrt{x^T A x} + c\right)\right|\right) < \infty, \tag{1.10}$$

where c is a function of d. Especially, if D is an ellipse with respect to A, i.e., $D = E_{r_0} = \{x \in \mathbb{R}^2 : x^T A x \le {r_0}^2\}$, and $\varphi = \rho$, a constant, then $\bar{r} = r_0$ and $M_d = 0$. At this moment, (1.9) actually is (1.10).

Remark 1.2. In dimensions $n \ge 3$, the corresponding existence theorem could be found in [9], where the asymptotic formula at infinity is (1.2).

2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^2$, $g \in C^0(\Omega)$ a positive function, and $u \in C^0(\Omega)$ a locally convex function. We say that u is a viscosity subsolution (resp. supersolution) of

$$\det(D^2 u) = g \quad \text{in } \Omega \tag{2.1}$$

or a viscosity solution of

$$\det(D^2 u) \ge (\text{resp.} \le)\, g \quad \text{in } \Omega$$

if for every $\bar{x} \in \Omega$ and every convex $\psi \in C^2(\Omega)$ satisfying

$$\psi \ge (\text{resp.} \le)\, u \quad \text{on } \Omega \quad \text{and} \quad \psi(\bar{x}) = u(\bar{x})$$

we have

$$\det(D^2 \psi(\bar{x})) \ge (\text{resp.} \le)\, g(\bar{x}).$$

u is a viscosity solution of (2.1) if u is both a viscosity subsolution and a viscosity supersolution of (2.1). (See, e.g., [17,18].)

Let $\varphi \in C^0(\partial D)$. A function $u \in C^0(\mathbb{R}^2 \setminus D)$ is a viscosity subsolution (resp. supersolution, solution) of the Dirichlet problem (1.8), if u is a viscosity subsolution (resp. supersolution, solution) of (1.1) and $u \le (\text{resp.} \ge, =)\, \varphi(x)$ on ∂D.

The proof of Theorem 1.1 can be reduced to the establishment of the existence of viscosity solutions of (1.8), based on the following lemma. (See, e.g., Lemma 3.1 on p. 561 in [9].)

Lemma 2.1. *Let u be a locally convex viscosity solution of (1.8), then u is C^∞ in $\mathbb{R}^2 \setminus \bar{D}$.*

Recall that any real symmetric matrix A has an eigen-decomposition $A = U^T \Lambda U$ where U is an orthogonal matrix, and Λ is a diagonal matrix. That is, A may be regarded as a real diagonal matrix Λ that has been re-expressed in some new coordinate system, and the eigenvalues $\lambda(A) = \lambda(\Lambda)$. Let $y = Ux$ and $v(y) = u(U^{-1}y)$, then (1.8) and (1.9) become

$$\begin{cases} \det(D_y^2 v) = 1, & \text{in } \mathbb{R}^2 \setminus \widetilde{\bar{D}}, \\ v = \varphi(U^{-1}y), & \text{on } \partial\widetilde{D} \end{cases}$$

and

$$O\left(|U^{-1}y|^{-2}\right) \leq v(y) - \left(\frac{1}{2}y^T \Lambda y + bU^{-1} \cdot y + d \ln \sqrt{y^T \Lambda y} + c\right) \leq \widetilde{M}_d + O\left(|U^{-1}y|^{-2}\right),$$

where $\widetilde{D}$ is transformed from D under $y = Ux$. So, without loss of generality, we always assume that A is diagonal. Further, if A is diagonal and $A \in \mathcal{A}$, then $\det(A) = 1$, and we can find a diagonal matrix Q with $\det Q = 1$ such that $QAQ = I \in \mathcal{A}$. Clearly, $\lambda(I)$ is not necessarily the same as $\lambda(A)$, but under transformation $y = Qx$, we still have

$$\det\left(D_x^2 u\right) = \det\left(QD_y^2 uQ\right) = \det\left(D_y^2 u\right).$$

Therefore, in the following we assume without loss of generality that $A = I$. Further, by subtracting a linear function from u, we only need to prove Theorem 1.1 for the case that $A = I$, $b = 0$, and $B_2(0) \subset D$. These will be assumed below.

The following lemma holds for $n \geq 2$, and its proof can be found in [9].

Lemma 2.2. *Let $\varphi \in C^2(\partial D)$. There exists some constant C, depending only on n, the convexity of D, $\|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D, such that, for every $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi)| \leq C \quad \text{and} \quad w_\xi < \varphi \quad \text{on } \bar{D} \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\left(|x - \bar{x}(\xi)|^2 - |\xi - \bar{x}(\xi)|^2\right), \quad x \in \mathbb{R}^n. \tag{2.2}$$

We remark that this lemma holds for any invertible and symmetric matrix A, where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}\left((x - \bar{x}(\xi))^T A(x - \bar{x}(\xi)) - (\xi - \bar{x}(\xi))^T A(\xi - \bar{x}(\xi))\right), \quad x \in \mathbb{R}^n$$

and C depends also on an upper bound of A. The interested readers can refer to [14].

3. Proof of Theorem 1.1

Proof of Theorem 1.1. For $r > 0$, let $B_r = \left\{x \in \mathbb{R}^2 : |x| < r\right\}$. Fix

$$\bar{r} = \min\{r > 0 : D \subset B_r\}.$$

For $d > 0$, $\alpha \in \mathbb{R}$, set

$$\omega_d(x) = \alpha + \int_{\bar{r}}^{|x|} \sqrt{s^2 + d}\, ds.$$

Then ω_d is a locally convex smooth solution of (1.1) in $\mathbb{R}^2 \setminus \{0\}$,

$$\omega_d(x) = \alpha + \frac{1}{2}\left(|x|\sqrt{|x|^2 + d} + d \ln\left(|x| + \sqrt{|x|^2 + d}\right)\right) - \frac{1}{2}\left(\bar{r}\sqrt{\bar{r}^2 + d} + d \ln\left(\bar{r} + \sqrt{\bar{r}^2 + d}\right)\right). \tag{3.1}$$

As $s \to +\infty$, by Taylor expansion,

$$s\sqrt{s^2 + d} = s^2\sqrt{1 + \frac{d}{s^2}} = s^2 + \frac{d}{2} - \frac{1}{8}\frac{d^2}{s^2} + O\left(\frac{d^3}{s^4}\right),$$

and

$$\ln\left(s + \sqrt{s^2 + d}\right) = \ln s + \ln\left(1 + \sqrt{1 + \frac{d}{s^2}}\right) = \ln s + \ln 2 + \frac{1}{4}\frac{d}{s^2} + O\left(\frac{d^2}{s^4}\right).$$

Substituting these expansions into (3.1), we have

$$\omega_d(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\ln|x| + \nu(d) + O\left(\frac{d^2}{|x|^2}\right), \quad \text{as } x \to \infty. \tag{3.2}$$

Here

$$\nu(d) = \alpha + \frac{d}{4}(1 + 2\ln 2) - \frac{1}{2}\left[\bar{r}\sqrt{\bar{r}^2 + d} + d\ln\left(\bar{r} + \sqrt{\bar{r}^2 + d}\right)\right], \tag{3.3}$$

with

$$\frac{\partial \nu}{\partial d} = \frac{1}{2}\ln\left[\frac{2}{\bar{r} + \sqrt{\bar{r}^2 + d}}\right] < 0, \quad \text{for } \bar{r} > 1,$$

and

$$\lim_{d\to\infty} \nu(d) = -\infty.$$

Denote

$$\mu(d, r) = \alpha + \int_{\bar{r}}^{r} \sqrt{s^2 + d}\, ds - \frac{r^2}{2}, \tag{3.4}$$

then

$$\omega_d(x) = \frac{1}{2}|x|^2 + \mu(d, |x|). \tag{3.5}$$

From

$$\frac{\partial \mu(d, r)}{\partial r} = \sqrt{r^2 + d} - r > 0, \tag{3.6}$$

it follows that for fixed $d > 0$, $\mu(d, r)$ is strictly increasing in r, and

$$\lim_{r\to\infty} \mu(d, r) = +\infty.$$

On the other hand, by calculus,

$$\frac{\partial \mu(d, r)}{\partial d} = \frac{1}{2}\int_{\bar{r}}^{r} \frac{ds}{\sqrt{s^2 + d}} = \frac{1}{2}\ln\left[\frac{r + \sqrt{r^2 + d}}{\bar{r} + \sqrt{\bar{r}^2 + d}}\right] > 0, \quad \text{for } r > \bar{r}, \tag{3.7}$$

which implies that for fixed $r > \bar{r}$, $\mu(d, r)$ is strictly increasing in d. And in view of the definition of $\mu(d, r)$, (3.4), we have for $r > \bar{r}$

$$\mu(d, r) \sim O\left(\sqrt{d}\right), \quad \text{as } d \to \infty.$$

Hence

$$\lim_{d\to\infty} \mu(d, r) = +\infty. \tag{3.8}$$

It is clear from (3.4) that

$$\mu(0, r) = \alpha - \frac{1}{2}\bar{r}^2 < \alpha.$$

Thus, in view of (3.7) and (3.8), for every $c > \alpha$ and $r > \bar{r}$, there exists a unique $d = d(c, r)$ such that

$$\mu(d(c, r), r) = c. \tag{3.9}$$

Recalling (3.6), $\mu(d, r)$ is strictly increasing in r, we know that for $d > 0$,

$$\mu(d, r) \le \mu(d, 2\bar{r}) \quad \text{for } r \le 2\bar{r}.$$

Combining with (3.5), we have, for $c > \alpha$ (or $d > d_2 = d(\alpha, 2\bar{r})$),

$$\omega_{d(c,2\bar{r})}(x) = \frac{1}{2}|x|^2 + \mu\left(d(c, 2\bar{r}), |x|\right) \le \frac{1}{2}|x|^2 + \mu(d(c, 2\bar{r}), 2\bar{r}) = \frac{1}{2}|x|^2 + c, \quad \text{in } B_{2\bar{r}}.$$

Clearly,

$$\omega_d \le \alpha, \quad \text{in } B_{\bar{r}} \setminus \bar{D}, \ \forall\, d > 0. \tag{3.10}$$

Let

$$\alpha := \min\left\{w_\xi(x) \mid \xi \in \partial D,\ x \in \overline{B_{\bar{r}}} \setminus D\right\},$$
$$\gamma := \max\left\{w_\xi(x) \mid \xi \in \partial D,\ x \in \overline{B_{\bar{r}}} \setminus D\right\},$$

where $w_\xi(x)$ is given by (2.2). Then by Lemma 2.2, we have

$$\gamma \geq \alpha = \min_{\partial D} \varphi.$$

Set

$$\underline{w}(x) = \max\left\{w_\xi(x) \mid \xi \in \partial D\right\}.$$

It is clear by Lemma 2.2 that $\underline{w}$ is a locally Lipschitz function in $\mathbb{R}^2 \setminus D$, and $\underline{w} = \varphi$ on ∂D. Since w_ξ is a smooth convex solution of (1.1), $\underline{w}$ is a viscosity subsolution of (1.1) in $\mathbb{R}^2 \setminus \bar{D}$. We now fix a number $\hat{r} > 2\bar{r}$, by (3.5),

$$\omega_d(x) = \frac{\hat{r}^2}{2} + \mu(d, \hat{r}) \quad \text{on } \partial B_{\hat{r}}$$

and choose another number $d_3 > 0$, depending only on $\bar{r}, \hat{r}, D, \max_{\partial D}|\varphi|$, by (3.8), such that

$$\omega_{d_3}(\hat{r}) = \min_{\partial B_{\hat{r}}} \omega_{d_3} > \max_{\partial B_{\hat{r}}} \underline{w}. \tag{3.11}$$

We now fix the value of

$$d_* \geq \max\{d_2, d_3\}.$$

Then for $d > d_*$, denoting $c' = \nu(d)$, we have by (3.2)

$$\omega_d(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\ln|x| + c' + O\left(|x|^{-2}\right), \quad \text{as } x \to \infty. \tag{3.12}$$

For $d > d_*$, from (3.5) and (3.7), it follows that

$$\frac{\partial \omega_d}{\partial d} > 0 \quad \text{on } \partial B_{\hat{r}},$$

therefore, by (3.11)

$$\omega_{d(c,\hat{r})} \geq \omega_{d_3} > \underline{w}, \quad \text{on } \partial B_{\hat{r}}. \tag{3.13}$$

By (3.10), we have

$$\omega_{d(c,\hat{r})} \leq 0 \leq \underline{w}, \quad \text{in } B_{\bar{r}} \setminus \bar{D}. \tag{3.14}$$

Now we define, for $d > d_*$,

$$\underline{u}(x) = \begin{cases} \max\left\{\omega_d(x), \underline{w}(x)\right\}, & x \in B_{\hat{r}} \setminus D, \\ \omega_d(x), & x \in \mathbb{R}^2 \setminus B_{\hat{r}}. \end{cases}$$

We know from (3.14) that

$$\underline{u} = \underline{w}, \quad \text{in } B_{\bar{r}} \setminus \bar{D}, \tag{3.15}$$

and in particular

$$\underline{u} = \underline{w} = 0, \quad \text{on } \partial D. \tag{3.16}$$

We know from (3.13) that $\underline{u} = \omega_d$ in a neighborhood of $\partial B_{\hat{r}}$. Therefore $\underline{u}$ is locally Lipschitz in $\mathbb{R}^2 \setminus D$. Since both ω_d and $\underline{w}$ are viscosity subsolutions of (1.1) in $\mathbb{R}^2 \setminus \bar{D}$, so is $\underline{u}$.

For $d > d_*$,

$$\bar{u}(x) := \omega_d(x) + M_d \tag{3.17}$$

is a locally convex solution of (1.1), where

$$M_d = \text{osc}_{\partial D}\varphi + \max_{\partial D}\int_{|x|}^{\bar{r}} \sqrt{s^2 + d}\, ds.$$

Then we have

$$\omega_d \leq \bar{u}, \quad \text{on } \mathbb{R}^2 \setminus D. \tag{3.18}$$

By (3.13) and the above, we have, for $d > d_*$,

$$w_\xi \leq \bar{u}, \quad \text{on } \partial(B_{\hat{r}} \setminus D), \forall\, \xi \in \partial D.$$

By the comparison principle for smooth convex solutions of (1.1), we have

$$w_\xi \le \bar{u}, \quad \text{in } B_{\hat{r}} \setminus \bar{D},\ \forall\, \xi \in \partial D.$$

Thus

$$\underline{w} \le \bar{u}, \quad \text{in } B_{\hat{r}} \setminus \bar{D}.$$

This, combining with (3.18), implies that

$$\underline{u} \le \bar{u}, \quad \text{in } \mathbb{R}^2 \setminus D.$$

For any $d > d_*$, let $\mathcal{S}_d$ denote the set of $v \in C^0(\mathbb{R}^2 \setminus D)$ which are viscosity subsolutions of (1.1) in $\mathbb{R}^2 \setminus \bar{D}$ satisfying

$$v = \varphi, \quad \text{on } \partial D, \tag{3.19}$$

and

$$\underline{u} \le v \le \bar{u}, \quad \text{in } \mathbb{R}^2 \setminus D. \tag{3.20}$$

We know that $\underline{u} \in \mathcal{S}_d$. Let

$$u(x) := \sup\{v(x) \mid v \in \mathcal{S}_d\}, \quad x \in \mathbb{R}^2 \setminus D.$$

Then u is a subsolution. By (3.12), and the definitions of $\underline{u}$ and $\bar{u}$,

$$u(x) \ge \underline{u} = \omega_d(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\ln|x| + c' + O\left(|x|^{-2}\right), \quad \text{as } x \to \infty, \tag{3.21}$$

and

$$u(x) \le \bar{u} = \frac{1}{2}|x|^2 + \frac{d}{2}\ln|x| + c' + M_d + O\left(|x|^{-2}\right), \quad \text{as } x \to \infty,$$

where $c' = \nu(d)$. The estimate (1.9) follows.

Next, we prove that u satisfies the boundary condition. It is obvious from (3.16) that

$$\liminf_{x\to\xi} u(x) \ge \lim_{x\to\xi} \underline{u}(x) = \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

So we only need to prove that

$$\limsup_{x\to\xi} u(x) \le \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

Let $\omega_c^+ \in C^2(\overline{B_{\bar{r}} \setminus D})$ be defined by

$$\begin{cases} \Delta\omega_c^+ = 0, & \text{in } B_{\bar{r}} \setminus \bar{D}, \\ \omega_c^+ = \varphi, & \text{on } \partial D, \\ \omega_c^+ = \max\limits_{\partial B_{\bar{r}}} \bar{u}, & \text{on } \partial B_{\bar{r}}. \end{cases}$$

It is easy to see that a viscosity subsolution v of (1.1) satisfies $\Delta v \ge 0$ in viscosity sense. Therefore, for every $v \in \mathcal{S}_c$, by $v \le \omega_c^+$ on $\partial(B_{\bar{r}} \setminus D)$, we have

$$v \le \omega_c^+ \quad \text{in } B_{\bar{r}} \setminus \bar{D}.$$

It follows that

$$u \le \omega_c^+ \quad \text{in } B_{\bar{r}} \setminus \bar{D},$$

and then

$$\limsup_{x\to\xi} u(x) \le \lim_{x\to\xi} \omega_c^+(x) = \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

Finally, we prove that u is a supersolution of (1.1). For $\bar{x} \in \mathbb{R}^2 \setminus \bar{D}$, fix some ϵ satisfying $0 < \epsilon < 2\mathrm{diam}(D)$ and $B_\epsilon(\bar{x}) \subset \mathbb{R}^2 \setminus \bar{D}$. By the definition of u, $u \le \bar{u}$. It is well-known that there is a unique convex viscosity solution $\widetilde{u} \in C^0(\overline{B_\epsilon(\bar{x})})$ to

$$\begin{cases} \det(D^2\widetilde{u}) = 1, & x \in B_\epsilon, \\ \widetilde{u} = u, & x \in \partial B_\epsilon. \end{cases}$$

By the maximum principle (see, e.g., [9]), $\widetilde{u} \geq u$ on B_ϵ. Define

$$w(y) = \begin{cases} \widetilde{u}(y), & \text{if } y \in B_\epsilon, \\ u(y), & \text{if } y \in \mathbb{R}^2 \setminus (D \cup B_\epsilon(\bar{x})). \end{cases}$$

Clearly, $w \in \mathcal{S}_d$. So, by the definition of u, $u \geq w$ on $B_\epsilon(\bar{x})$. It follows that $u \equiv \widetilde{u}$ on $B_\epsilon(\bar{x})$. Therefore u is a viscosity solution of (1.1). By Lemma 2.1, Theorem 1.1 is established. □

Acknowledgments

The first author was partially supported by SRFDPHE (2010 0003110003). The second author was partially supported by NSFC (11126038) and SRFDPHE (20100003120005). They were both partially supported by NSFC (11071020) and Program for Changjiang Scholars and Innovative Research Team in University in China (IRT0908). The authors would like to thank the referee for valuable comments and suggestions.

References

[1] A.M. Li, U. Simon, Z. Zhao, Global Affine Differential Geometry of Hypersurfaces, Walter de Gruyter, Berlin, New York, 1993.
[2] K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, Math. Ann. 127 (1954) 130–134.
[3] E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, Michigan Math. J. 5 (1958) 105–126.
[4] A.V. Pogorelov, On the improper convex affine hyperspheres, Geom. Dedicata 1 (1972) 33–46.
[5] S.Y. Cheng, S.T. Yau, Complete affine hypersurfaces, I. The completeness of affine metrics, Comm. Pure Appl. Math. 39 (1986) 839–866.
[6] L.A. Caffarelli, Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation, Ann. of Math. (2) 131 (2) (1990) 135–150.
[7] J. Jost, Y.L. Xin, Some aspects of the global geometry of entire space-like submanifolds, Results Math. 40 (2001) 233–245.
[8] N.S. Trudinger, X.-J. Wang, The Bernstein problem for affine maximal hypersurface, Invent. Math. 140 (2000) 399–422.
[9] L. Caffarelli, Y.Y. Li, An extension to a theorem of Jörgens, Calabi, and Pogorelov, Comm. Pure Appl. Math. 56 (2003) 549–583.
[10] L. Ferrer, A. Martínez, F. Milán, An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres, Math. Z. 230 (1999) 471–486.
[11] L. Ferrer, A. Martínez, F. Milán, The space of parabolic affine spheres with fixed compact boundary, Monatsh. Math. 130 (1) (2000) 19–27.
[12] P. Delanoë, Partial decay on simple manifolds, Ann. Global Anal. Geom. 10 (1992) 3–61.
[13] L.M. Dai, J.G. Bao, On uniqueness and existence of viscosity solutions to Hessian equations in exterior domains, Front. Math. China 6 (2011) 221–230.
[14] J.G. Bao, H.G. Li, Y.Y. Li, On the exterior Dirichlet problem for Hessian equations, Trans. Amer. Math. Soc. (2011). arXiv:1112.4665v1 (in press).
[15] N. Meyers, J. Serrin, The exterior Dirichlet problem for second order elliptic partial differential equations, J. Math. Mech. 9 (1960) 513–538.
[16] C. Wang, J.G. Bao, Necessary and sufficient conditions on existence and convexity of solutions for Dirichlet problems of Hessian equations on exterior domains, Proc. Amer. Math. Soc (in press).
[17] L.A. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, in: American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.
[18] C.E. Gutiérrez, The Monge–Ampère Equation, in: Progress in Nonlinear Differential Equations and their Applications, vol. 44, Birkhäuser Boston, Inc., Boston, MA, 2001.