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On Jörgens, Calabi, and Pogorelov type theorem and isolated singularities of parabolic Monge–Ampère equations ☆

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ABSTRACT

In the paper, we extend Jörgens, Calabi, and Pogorelov's theorem on entire solutions of elliptic Monge–Ampère equations to parabolic equations associated with Gauss curvature flows. Our results include Gutiérrez and Huang's previous work as a special case. Besides, we also treat the isolated singularities for parabolic Monge–Ampère equations that was firstly studied by Jörgens for elliptic case in two dimensions.

1. Introduction

By suitably choosing Cartesian coordinate system $x_1, \ldots, x_n$, we say a complete surface $\Sigma = \{(x, u(x)):\ x \in \mathbb{R}^n\}$ is an *improper affine hypersurface* if $u(x)$ is a function satisfying the Monge–Ampère equation

$$\det D^2 u = \text{const} > 0 \quad \text{in } \mathbb{R}^n.$$

A celebrated theorem in affine geometry says that

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Theorem 1.1 *(Jörgens–Calabi–Pogorelov). A convex improper affine hypersurface is an elliptic paraboloid.*

The proof of this result is not trivial. It was given by Jörgens [21] for $n = 2$, then by Calabi [8] for $n \leqslant 5$ and eventually by Pogorelov [25] for arbitrary n. A simpler and more analytic proof, along the lines of affine geometry, of the theorem was later given by Cheng and Yau [11]. Recently, Caffarelli and Li [7] proved, by using the regular theory for Monge–Ampère equation developed in the fundamental papers [4] and [5], that this result holds for viscosity solutions. Please see also Chapter 4 of [17] for a proof. Note that in dimension two, Theorem 1.1 provides an elegant proof of Bernstein's theorem on minimal surfaces.

Theorem 1.1 was extended by Gutiérrez and Huang [18] to the solutions of following special parabolic Monge–Ampère equation

$$-u_t \det D^2 u = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{1.1}$$

This type differential operator was firstly introduced by Krylov [23] in 1976. It shares a lot of common features with elliptic Monge–Ampère operator, for instance it can be expressed as the Jacobian determinant of a mapping, see [9].

One purpose of this paper is to investigate this property for solutions of more general parabolic Monge–Ampère equations which may include other meaningful forms. Motivated by this, we would like to study the entire solutions to following parabolic Monge–Ampère equation

$$u_t = \rho\big(\log \det D^2 u\big) \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{1.2}$$

where $\rho(s) \in C^2(\mathbb{R})$, $u_t = D_t u$ and $D^2 u = D_x^2 u$ denote the first order derivative and Hessian of u with respect to t and x, respectively. Assume that $u = u(x,t)$ is convex in x for every $t \in (-\infty, 0]$ throughout this paper.

Eq. (1.2) appears in connection with the problem of the deformation of a surface by means of its nonhomogeneous Gauss curvature (speed is a function of Gauss curvature) which has drawn a great deal of attentions and undergone a rapid development. In particular, when $\rho(s) = e^{s/n}$ or s, then Eq. (1.2) gives appealing form

$$u_t = \big(\det D^2 u\big)^{\frac{1}{n}} \tag{1.3}$$

or

$$u_t = \log \det D^2 u. \tag{1.4}$$

The above two equations have been studied extensively in the geometric aspect, see [16,13,1,15,26] and references therein. Moreover, Eq. (1.4) has some applications in Minkowski problems, see [14]. Analytic aspect of Eqs. (1.3) and (1.4) has been investigated by some authors, see [24,20] for relevant results and a good survey. Finally, if $\rho(s) = -e^{-s}$, then we arrive at the interesting form (1.1).

As in standard parabolic equations theory, for integer $k \geqslant 0$ we say a function $u(x,t) \in C^{2k,k}(E)$ that means u is $2k$-th continuous differentiable with spatial variables x and k-th continuous differentiable with time variable t for $(x,t) \in E \subset \mathbb{R}^{n+1}$. The first result of this paper is the following theorem.

Theorem 1.2. *Suppose*

$$\rho'(s) > 0, \qquad \rho''(s) \leqslant \frac{1}{n}\rho'(s) \quad \textit{in } \mathbb{R}. \tag{1.5}$$

Let $u(x,t) \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ *be convex in* x *and satisfy* (1.2)*. Assume*

$$0 < m_1 = \inf_{(x,t)\in\mathbb{R}^n_-} \bigl(-u_t(x,t)\bigr) \leqslant \sup_{(x,t)\in\mathbb{R}^n_-} \bigl(-u_t(x,t)\bigr) = m_2 < \infty, \tag{1.6}$$

and

$$\bigl|\rho^{-1}(u_t)\bigr| \leqslant K < \infty. \tag{1.7}$$

Then $u(x,t) = P(x) + ct$, where c is a constant and $P(x)$ is a convex quadratic polynomial.

If we write $F(D^2u) = \rho(\log\det D^2u)$, condition (1.5) is necessary to ensure that $F(\cdot)$ is concave. The convexity or concavity of $F(\cdot)$ can guarantee the interior estimates of second order derivatives and thus is a vital ingredient in the theory of fully nonlinear elliptic and parabolic equations.

It is easy to see that Theorem 1.2 applies to Eqs. (1.1) and (1.4). Particularly, for Eq. (1.4) the condition (1.6) can be reduced to $|u_t| \leqslant C_0$ in $\mathbb{R}^n \times (-\infty, 0]$ for some $C_0 > 0$. This result for (1.1) has been obtained by Gutiérrez and Huang [18].

Corollary 1.1. *Let $u(x,t) \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be convex in x and a solution of Eq.* (1.3) *in $\mathbb{R}^n \times (-\infty, 0]$. Suppose that there exist positive constants m_1, m_2 such that*

$$m_1 \leqslant u_t(x,t) \leqslant m_2 \quad \text{for all } (x,t) \in \mathbb{R}^n \times (-\infty, 0]. \tag{1.8}$$

Then $u(x,t) = P(x) + ct$.

Proof. Replacing $u - (m_2+1)t$ to u, (1.3) implies

$$-u_t + \exp\left\{\frac{1}{n}\log\det D^2u\right\} - (m_2+1) = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]$$

and $1 \leqslant -u_t \leqslant m_2 - m_1 + 1$ in $\mathbb{R}^n \times (-\infty, 0]$. By the theorem above, we complete the proof. □

According to Evans–Krylov estimates and linear parabolic equations theory, we only need the solutions to be $C^{2,1}(\mathbb{R}^n \times (-\infty, 0])$ in Theorem 1.2. Nevertheless, we cannot reduce them to viscosity solutions, for a counterexample linked to Eq. (1.1) was constructed in [18].

The story is quite different for elliptic case, see [7]. In fact, for elliptic Monge–Ampère equation, a result due to Cheng and Yau [12] says that for any convex domain $\Omega \subset \mathbb{R}^n$ there is a unique convex solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ to

$$\det D^2u = 1 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega,$$

which plays a crucial role in Caffarelli and Li's proof [7]. However, to our knowledge, there is no similar result for parabolic Monge–Ampère equation in bowl-shaped domains (see the definition in Section 2) so far. For the regularity of weak solutions in Aleksandroff generalized sense of Eq. (1.1), we refer to [9] and [19].

The other part of this paper is devoted to the removable singularities for Eq. (1.2). This problem for elliptic Monge–Ampère was also investigated by Jörgens [22] initially in two dimension. His result was extended to Monge–Ampère with general right hand side by Beyerstedt [2] in two dimensions as well. Eventually, Beyerstedt [3] and Schulz and Wang [27] established a similar result for higher dimensions independently. For parabolic Monge–Ampère equation, we have the following theorem.

Theorem 1.3. *Let $\mathbb{R}^{n+1}_- = \mathbb{R}^n \times (-\infty, 0)$ and $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1}_-$. Suppose that $u(x,t) \in C(\mathbb{R}^{n+1}_-) \cap C^{4,2}(\mathbb{R}^{n+1}_- \setminus X_0)$ is convex with respect to x and satisfies*

$$-u_t \det D^2 u = 1 \quad \text{in } \mathbb{R}^{n+1}_- \setminus X_0. \tag{1.9}$$

Then the isolated singularity at X_0 is removable if and only if there exists a smooth curve lying on the hyperplane $\{(x,t)\colon t = t_0\}$ and passing through the point X_0 such that u is $C^{1,0}$ along it.

Actually, the above result holds for general fully nonlinear parabolic equations with general isolated sets, particularly it is applicable to Eqs. (1.3) and (1.4), see Section 4 of this paper. Note that in our proof we only need $u \in C(\mathbb{R}^{n+1}_-)$ instead of being Lipschitz needed in [2,3,27] and [28].

The organization of the paper is as follows. In Section 2, Pogorelov type estimates are established. Then we prove Theorems 1.2 and 1.3 in Sections 3 and 4 respectively.

2. Pogorelov type estimates

Let $D \subset \mathbb{R}^{n+1}$ be bounded domain. For a fixed t we write

$$D(t) = \big\{x\colon (x,t) \in D\big\}, \tag{2.1}$$

and $t_0 = \inf\{t\colon D(t) \neq \emptyset\}$. The parabolic boundary of the bounded domain D is defined by

$$\partial_p D = \big(\overline{D(t_0)} \times \{t_0\}\big) \bigcup_{t \in \mathbb{R}} \big(\partial D(t) \times \{t\}\big),$$

where $\overline{D}$ denotes the closure of D and $\partial D(t)$ denotes the topological boundary of $D(t)$. We say that the set $D \subset \mathbb{R}^{n+1}$ is a bowl-shaped domain if $D(t)$ is strict convex for each t and $D(t_1) \subset D(t_2)$ for $t_1 \leqslant t_2$.

Definition 2.1. A function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $u = u(x,t)$, is called parabolically convex (or convex-monotone) if it is continuous, convex in x and non-increasing in t.

From the assumptions in Theorem 1.2, we see that u is parabolically convex.

Theorem 2.2. *Let $D \subset \mathbb{R}^n \times (-\infty, 0]$ be a bounded bowl-shaped domain. Assume that u is a smooth function satisfying* (1.2) *and* (1.6) *in D and $u = 0$ on $\partial_p D$. Then*

$$\big|D^2 u(x,t)\big| \leqslant \frac{C}{|u(x,t)|}, \quad x \in D,$$

where C depends on n, m_1, m_2, p, ρ, D and $\sup_D\{|Du| + |u|\}$.

Proof. Let

$$W = \sup_{(x,t)\in D,\, \xi \in \mathbb{S}^n} \big|u(x,t)\big| D_{\xi\xi} u(x,t) \exp\left\{\frac{\eta}{2}\big|Du(x,t)\big|^2\right\}$$

with

$$\eta = \frac{1}{4(1 + \sup_D |Du|^2)}.$$

Since $u = 0$ on $\partial_p D$ and u is strictly convex in $D \setminus \partial_p D$ with respect to x, it follows that the maximum W is attained at some point $X = (x_0, t_0) \in \overline{D} \setminus \partial_p D$ and some unit vector $\xi \in \mathbb{S}^n$. We may suppose

$\xi = e_1 = (1, 0, \dots, 0)$), then $D_{1j}u(X) = 0$ for $j > 1$. By rotating the coordinates $\{x_2, \dots, x_n\}$, we may assume $D^2u(X)$ is diagonal.

Set $F(D^2u) = \log\det D^2u$, we have

$$(F_{ij}) = \left(\frac{\partial F}{\partial u_{ij}}\right) = \left(D^2u\right)^{-1}, \qquad \frac{\partial^2 F}{\partial u_{ij}\partial u_{kl}} = F_{ij,kl} = -F_{ik}F_{jl}.$$

Let L be the linearized operator at X

$$L = -D_t + \rho'\big(F\big(D^2u(X)\big)\big)F_{ij}\big(D^2u(X)\big)D_{ij}.$$

Since W is achieved at (X, e_1), it follows that the function

$$h = \log|u| + \log D_{11}u + \frac{\eta}{2}|Du|^2$$

also attains its maximum at X, and consequently

$$Dh(X) = 0, \qquad h_t(X) \geqslant 0, \quad \text{and} \quad D^2h(X) \leqslant 0. \tag{2.2}$$

Since $(F_{ij}(D^2u(X)))$ is diagonal,

$$L(h)(X) = -D_t h(X) + \rho' F_{ii} D_{ii} h(X) \leqslant 0. \tag{2.3}$$

Now

$$D_i h = \frac{D_i u}{u} + \frac{D_{11i}u}{D_{11}u} + \eta\sum_{k=1}^n D_k u D_{ki} u, \tag{2.4}$$

$$\begin{aligned} D_{ij}h = {} & \frac{D_{ij}u}{u} - \frac{D_i u D_j u}{u^2} + \frac{D_{11ij}u}{D_{11}u} - \frac{D_{11i}u D_{11j}u}{(D_{11}u)^2} \\ & + \eta\sum_{k=1}^n D_{ki}u D_{kj}u + \eta\sum_{k=1}^n D_k u D_{kij}u, \end{aligned} \tag{2.5}$$

$$D_t h = \frac{D_t u}{u} + \frac{D_{11t}u}{D_{11}u} + \eta\sum_{k=1}^n D_k u D_{kt} u. \tag{2.6}$$

Substituting (2.4), (2.5) and (2.6) into (2.3), we have

$$\begin{aligned} & -\left(\frac{u_t}{u} + \frac{D_{11t}u}{D_{11}u} + \eta\sum_{k=1}^n D_k u D_{kt} u\right) \\ & \quad + \rho' F_{ii}\left(\frac{D_{ii}u}{u} - \frac{(D_i u)^2}{u^2} + \frac{D_{11ii}u}{D_{11}u} - \frac{(D_{11i}u)^2}{(D_{11}u)^2} + \eta\sum_{k=1}^n (D_{ki}u)^2 + \eta\sum_{k=1}^n D_k u D_{kii} u\right) \leqslant 0 \end{aligned}$$

valid at the point X. By collecting terms we obtain

$$\frac{-u_t}{u}+\frac{1}{D_{11}u}L(D_{11}u)+\eta\sum_{k=1}^{n}D_kuL(D_ku)$$
$$+\rho'F_{ii}\left(\frac{D_{ii}u}{u}-\frac{(D_iu)^2}{u^2}-\frac{(D_{11i}u)^2}{(D_{11}u)^2}+\eta(D_{ii}u)^2\right)\leqslant 0 \tag{2.7}$$

at X.

Differentiate Eq. (1.2) to obtain at X,

$$-D_{kt}u+\rho'F_{ii}D_{iik}u=0,\quad k=1,\dots,n.$$

That is $L(D_ku)=0$. Next, let us compute $L(D_{11}u)(X)$. Differentiating Eq. (1.2) twice with respect to x_1 yields

$$-D_{11t}u+\rho''F_{ij}D_{1ij}uF_{kl}D_{1kl}u+\rho'F_{ij,kl}D_{1ij}uD_{1kl}u+\rho'F_{ij}D_{11ij}u=0.$$

Therefore, at X we have

$$L(D_{11}u)=-\rho''\left(\sum_{i=1}^{n}F_{ii}D_{1ii}u\right)^2+\rho'F_{ik}F_{jl}D_{ij1}uD_{kl1}u.$$

Since $\rho''\leqslant\frac{1}{n}\rho'$, we obtain

$$L(D_{11}u)\geqslant\rho'\left(-\frac{1}{n}\left(\sum_{i=1}^{n}F_{ii}D_{1ii}u\right)^2+F_{ik}F_{jl}D_{ij1}uD_{kl1}u\right)$$

at X.

Noting again that $F_{ij}(D^2u(X))=(D^2u)^{-1}(X)$ is diagonal again and $\rho'>0$, in view of (2.7), we have the inequality

$$\frac{n\rho'-u_t}{u}+\rho'\left(-\frac{(D_{ii1}u)^2}{D_{11}u(D_{ii}u)^2}+\frac{(D_{ij1}u)^2}{D_{11}uD_{ii}uD_{jj}u}\right.$$
$$\left.+\frac{1}{D_{ii}u}\left(-\frac{(D_iu)^2}{u^2}-\frac{(D_{11i}u)^2}{(D_{11}u)^2}+\eta(D_{ii}u)^2\right)\right)\leqslant 0,$$

where we have used the inequality

$$\frac{1}{n}\left(\sum_{i=1}^{n}\frac{D_{ii1}u}{D_{ii}u}\right)^2\leqslant\sum_{i=1}^{n}\frac{(D_{ii1}u)^2}{(D_{ii}u)^2}.$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(D_{ij1}u)^2}{D_{11}uD_{ii}uD_{jj}u}=2\sum_{i=1}^{n}\frac{(D_{11i}u)^2}{(D_{11}u)^2D_{ii}u}-\frac{(D_{111}u)^2}{(D_{11}u)^3}$$
$$+\sum_{i=2}^{n}\frac{(D_{ii1}u)^2}{D_{11}u(D_{ii}u)^2}+\sum_{i=2}^{n}\sum_{j=2,j\neq i}^{n}\frac{(D_{ij1}u)^2}{D_{11}uD_{ii}uD_{jj}u},$$

we have

$$\frac{C}{u}-\frac{(D_{111}u)^2}{(D_{11}u)^3}+\sum_{i=2}^{n}\frac{(D_{11i}u)^2}{(D_{11}u)^2D_{ii}u}-\sum_{i=1}^{n}\frac{(D_iu)^2}{(D_{ii}u)u^2}+\eta\Delta u\leqslant 0. \tag{2.8}$$

Since $D_ih(X)=0$, and $D^2u(X)$ is diagonal, it follows from (2.4) that

$$\frac{D_{111}u}{D_{11}u}=-\frac{D_1u}{u}-\eta D_1uD_{11}u,$$
$$\frac{D_iu}{u}=-\frac{D_{11i}u}{D_{11}u}-\eta D_iuD_{ii}u,\quad i=2,\dots,n,$$

at X. Therefore by (2.8) we get

$$\frac{C}{u}-\frac{2(D_1u)^2}{u^2D_{11}u}-2\eta\sum_{i=2}^{n}\frac{D_iuD_{11i}u}{D_{11}u}-2\eta^2|Du|^2\Delta u+\eta\Delta u\leqslant 0.$$

Using $Dh(X)=0$ again,

$$-\eta\sum_{i=2}^{n}\frac{D_iuD_{11i}u}{D_{11}u}=\sum_{i=2}^{n}\left(\frac{\eta(D_iu)^2}{u}+\eta^2(D_iu)^2D_{ii}u\right).$$

Hence

$$\frac{C}{u}-\frac{2(D_1u)^2}{u^2D_{11}u}-2\eta^2|Du|^2\Delta u+\eta\Delta u\leqslant 0.$$

By the choice of η,

$$\frac{C}{u}-\frac{2(D_1u)^2}{u^2D_{11}u}+\frac{D_{11}u}{8(1+\sup_D|Du|^2)}\leqslant 0.$$

Multiply the inequality above by $8u^2D_{11}u\exp\{\eta|Du|^2\}(1+\sup_D|Du|^2)$, we obtain (for a different C)

$$W\leqslant C$$

valid at the point X. Hence

$$\left|D^2u(x,t)\right|\leqslant\frac{C}{|u(x,t)|}, \tag{2.9}$$

where C depends on n, m_1, m_2, p, ρ, D and $\sup_D\{|Du|+|u|\}$. This completes the proof of the theorem. □

The proof of Theorem 2.2 is essentially due to Pogorelov [25].

3. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2, the essential idea of our proof follows closely from [7] and [18]. However, different from (1.1) and standard elliptic Monge–Ampère operator, our differential operator cannot be expressed as Jacobian determinant of a mapping and does not enjoy convenient scaling form. We find a new normalization approach of the solutions and their level sets. Applying the Pogorelov type estimates to the normalized solution in the small domains, due to the assumption (1.6) and then Evans–Krylov estimates, we shall get the $C^{2+\alpha,1+\alpha/2}$ estimates for the normalized solutions, where $C^{2+\alpha,1+\alpha/2}$ is the standard parabolic Hölder space. By rescaling, we show that the Hölder norms of the first order derivatives in t and second derivatives in x of the solutions must be zero, then Theorem 1.2 follows.

Let u be a solution to (1.2) satisfying the assumptions in Theorem 1.2. For convenience, we rewrite (1.6) below

$$m_1 \leqslant -u_t(x,t) \leqslant m_2 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{3.1}$$

Owing to (1.7), there exist two positive constants λ_1, λ_2 (depending only on m_1, m_2 and ρ) such that

$$0 < \lambda_1 \leqslant \det D^2 u \leqslant \lambda_2 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{3.2}$$

By subtracting a linear function on x, we may also assume that

$$u(0,0) = 0, \qquad Du(0,0) = 0 \tag{3.3}$$

is valid in the following.

We state a normalization theorem of John–Cordoba and Gallegos and refer to [17] for a proof.

Lemma 3.1. *If $\Omega \subset \mathbb{R}^n$ is a bounded convex set with nonempty interior and E is the ellipsoid of minimum volume containing Ω centered at the center of mass of Ω, then*

$$\alpha_n E \subset \Omega \subset E,$$

where $\alpha_n = n^{-\frac{3}{2}}$ and αE denotes the α-dilation of E with respect to its center.

Given $H > 0$, let

$$Q_H = \big\{(x,t)\colon\ u(x,t) < H\big\} \quad \text{and} \quad Q_H(t_0) = \big\{x\colon\ (x,t_0) \in Q_H\big\}. \tag{3.4}$$

Let x_H be the mass center of $Q_H(0)$, E the ellipsoid of minimum volume containing $Q_H(0)$ with center x_H, and T_H an affine transform that normalizes the $Q_H(0)$, that is $T_H(E) = B_1(0)$ and

$$B_{\alpha_n}(0) \subset T_H Q_H(0) \subset B_1(0). \tag{3.5}$$

The following lemma gives an estimate for the shape of Q_H.

The following results about elementary properties of level sets of Monge–Ampère equations are not new, particularly Lemmas 3.2, 3.3, 3.6 and Corollary 3.1 are essentially contained in [19], for completeness we give proofs of them.

Lemma 3.2. *Let u be parabolically convex and satisfy* (3.1), (3.2) *and* (3.3). *Then there exist constants $\varepsilon_0, \varepsilon_1$, and ε_2 such that for all $H > 0$*

$$\varepsilon_0 E \times [-\varepsilon_1 H, 0] \subset Q_H \subset E \times [-\varepsilon_2 H, 0], \tag{3.6}$$

where ε_i $(i = 0, 1, 2)$ depend only on ρ, p, n, m_j $(j = 1, 2)$.

Proof. Let $(x,t) \in Q_H$. Since $u(0,0) = 0$, $u \geqslant 0$, we have $u(x,t) - u(x,0) \leqslant H$. It follows from (3.1) that $t \geqslant -H/m_1$. Hence, $(x,t) \in E \times [-H/m_1, 0]$. Then the second inclusion follows with $\varepsilon_2 = 1/m_1$.

On the other hand, by elliptic Monge–Ampère equation theory (see Lemma 3.3.1 of [17]), we have

$$\gamma Q_H(0) \subset Q_{(1-(1-\gamma)\alpha_n/2)H}(0),$$

where $0 < \gamma < 1$, α_n as in Lemma 3.1. Setting $\gamma = 1/2$ and noting that $\alpha_n E \subset Q_H(0)$, then we have

$$u(x,t) \leqslant u(x,0) - m_2 t \leqslant (1 - \alpha_n/4)H - m_2 t < H,$$

if $(x,t) \in \frac{1}{2}\alpha_n E \times [-\varepsilon_1 H, 0]$ and $\varepsilon_1 = \alpha_n/8m_2$. Thus the first inclusion follows with $\varepsilon_0 = \alpha_n/2$, $\varepsilon_1 = \alpha_n/8m_2$. $\square$

For the convenience, throughout the paper, we use the symbol $a \approx b$ to denote that the quality a/b is bounded by two positive universal constants from above and below.

Lemma 3.3. *Let u be parabolically convex and satisfy* (3.1), (3.2) *and* (3.3). *Let H, T_H be the same as in* (3.5), *then*

$$|\det T_H|^{-\frac{2}{n}} \approx H.$$

Proof. For $y = T_H x \in T_H Q_H(0)$, let

$$v(y) = |\det T_H|^{\frac{2}{n}} \big(u\big(T_H^{-1}(y,0)\big) - H\big),$$

then $v(y)$ is convex and $v(y) = 0$ for $y \in \partial(T_H(Q_H(0)))$. We have

$$\det D^2 v(y) = \det D^2 u\big(T_H^{-1} y, 0\big).$$

So

$$\lambda_1 \leqslant \det D^2 v \leqslant \lambda_2. \tag{3.7}$$

Hence, the Monge–Ampère measure $\mathcal{M}$ with density $\det D^2 v(y)$ has the doubling property

$$\mathcal{M}\big(T_H\big(Q_H(0)\big)\big) \leqslant \frac{2^n \lambda_2}{\lambda_1} \mathcal{M}\bigg(\frac{1}{2} T_H\big(Q_H(0)\big)\bigg).$$

Indeed,

$$\begin{aligned}\mathcal{M}\big(T_H\big(Q_H(0)\big)\big) &= \int\limits_{T_H(Q_H(0))} \det D^2 v(y)\,\mathrm{d}y \\ &\leqslant \int\limits_{T_H(Q_H(0))} \lambda_2\,\mathrm{d}y \\ &= \frac{2^n\lambda_2}{\lambda_1} \int\limits_{\frac{1}{2}T_H(Q_H(0))} \lambda_1\,\mathrm{d}y \\ &\leqslant \frac{2^n\lambda_2}{\lambda_1}\mathcal{M}\bigg(\frac{1}{2}T_H\big(Q_H(0)\big)\bigg).\end{aligned}$$

We may then apply Proposition 3.2.3 of [17] to obtain

$$\mathcal{M}\big(T_H\big(Q_H(0)\big)\big) \approx \Big|\min_{T_H(Q_H(0))} v(y)\Big|^n,$$

with comparison constants depending only on the dimension n and $\frac{\lambda_2}{\lambda_1}$. Since $u(0,0)=0$ and $u \geqslant 0$, we have that

$$\min_{T_H(Q_H(0))} v(y) = -|\det T_H|^{\frac{2}{n}} H.$$

On the other hand, by (3.7) and the normalization of $Q_H(0)$ we get

$$\mathcal{M}\big(T_H\big(Q_H(0)\big)\big) = \int\limits_{T_H(Q_H(0))} \det D^2 v(y)\,\mathrm{d}y \approx 1.$$

Therefore

$$H \approx |\det T_H|^{-\frac{2}{n}}. \qquad \Box$$

Set

$$\mathcal{T}_H(x,t) = \bigg(T_H x, \frac{t}{|\det T_H|^{-2/n}}\bigg), \qquad \mathcal{T}_H(Q_H) = Q_H^*.$$

Then Lemmas 3.2 and 3.3 imply that

$$B_{\varepsilon_0} \times [-\varepsilon_1, 0] \subset Q_H^* \subset B_1 \times [-\varepsilon_2, 0], \tag{3.8}$$

where ε_i $(i=0,1,2)$ depend only on ρ, p, n, m_j $(j=1,2)$. Let

$$u^*(y,s) = |\det T_H|^{\frac{2}{n}}\big(u\big(\mathcal{T}_H^{-1}(y,s)\big) - H\big),$$

then for $(y,s) \in Q_H^*$

$$\frac{\partial u^*(y,s)}{\partial s} = \frac{\partial u}{\partial t}\big(\mathcal{T}_H^{-1}(y,s)\big), \qquad \det D^2 u^*(y,s) = \det D^2 u\big(\mathcal{T}_H^{-1}(y,s)\big).$$

So

$$m_1 \leqslant -u_s^* \leqslant m_2, \qquad \lambda_1 \leqslant \det D^2 u^* \leqslant \lambda_2 \quad \text{in } Q_H^* \tag{3.9}$$

and

$$u^* = 0 \quad \text{on } \partial_p Q_H^*.$$

The following lemma and its proof can be found in [7].

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a convex open domain with* $\operatorname{diam}(\Omega) \leqslant 1$*, and let v be a convex solution of*

$$\det D^2 v \leqslant 1 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

Then

$$v(x) \geqslant \begin{cases} -C(n)\operatorname{dist}(x, \partial\Omega)^{2/n} & \text{for any } x \in \Omega,\ n \geqslant 3, \\ -C(\alpha)\operatorname{dist}(x, \partial\Omega)^{\alpha} & \text{for any } x \in \Omega,\ n = 2,\ 0 < \alpha < 1. \end{cases}$$

Lemma 3.5. *Given $\varepsilon > 0$, let $\Omega_\varepsilon = \{(x,t) \in Q_H^*\colon u^*(x,t) < -\varepsilon\}$. Assume that u satisfies the assumptions in Theorem* 1.2*. Then*

$$\left|Du^*(x,t)\right| \leqslant C \quad \text{for } (x,t) \in \Omega_\varepsilon, \tag{3.10}$$

$$\left|D^2 u^*(x,t)\right| \leqslant C \quad \text{for } (x,t) \in \Omega_{3\varepsilon}, \tag{3.11}$$

where $C > 0$ depends only on ρ, p, n, ε, m_j $(j = 1, 2)$.

Proof. Let $v(x,t) = \lambda_2^{-\frac{1}{n}} u^*(x,t)$. By (3.2), we have $\det D^2 v \leqslant 1$. Since Q_H^* is a bowl-shaped domain, it follows from Lemma 3.4 and (3.5) that for $(x_0, t_0) \in \Omega_\varepsilon$

$$\operatorname{dist}\big(x_0, \partial Q_H^*(t_0)\big)^{2/n} \geqslant -\frac{v(x_0,t_0)}{C(n)} \geqslant \frac{\varepsilon}{C(n)\lambda_2^{\frac{1}{n}}}, \quad n \geqslant 3,$$

and

$$\operatorname{dist}\big(x_0, \partial Q_H^*(t_0)\big)^{\alpha} \geqslant -\frac{v(x_0,t_0)}{C(\alpha)} \geqslant \frac{\varepsilon}{C(\alpha)\lambda_2^{\frac{1}{n}}}, \quad n = 2.$$

Hence, $\operatorname{dist}(x_0, \partial Q_H^*(t_0)) > C(\varepsilon)$. The function $u^*(x, t_0)$ is convex in $Q_H^*(t_0)$ and $u^*(x, t_0) = 0$ on $\partial Q_H^*(t_0)$. Hence by Lemma 3.2.1 of [17] we obtain

$$\left|Du^*(x_0,t_0)\right| \leqslant \frac{-u^*(x_0,t_0)}{\operatorname{dist}(x_0, \partial Q_H^*(t_0))} \leqslant C,$$

where we have used the fact

$$-u^*(x_0,t_0) = -|\det T_H|^{\frac{2}{n}}\big(u\big(\mathcal{T}_H^{-1}(y,s)\big) - H\big) \leqslant |\det T_H|^{\frac{2}{n}} H \approx 1.$$

Thus (3.10) follows.

Next, consider the function $\omega(x,t) = u^*(x,t) + 2\varepsilon$. From (3.10), we have $|D\omega(x,t)| \leqslant C$ for $(x,t) \in \Omega_{2\varepsilon}$. Note that $\omega(x,t) = u^* + 2\varepsilon < -3\varepsilon + 2\varepsilon = -\varepsilon$, i.e., $|\omega(x,t)| > \varepsilon$ for $(x,t) \in \Omega_{3\varepsilon}$. Applying Theorem 2.2 to ω on the set $\Omega_{2\varepsilon}$, then we obtain (3.11). □

Corollary 3.1. *There exist constants C_1, C_2 depending on ρ, p, n, ε, m_j $(j=1,2)$ such that*

$$C_1 I \leqslant D^2 u^*(x,t) \leqslant C_2 I \quad \text{for all } (x,t) \in \Omega_\varepsilon. \tag{3.12}$$

Proof. Since $u^* = 0$ on $\partial_p Q_H^*$, then $\Omega_{\varepsilon/3} \subset Q_H^*$. Applying (3.11) of Lemma 3.5 to u^* on $\Omega_{\varepsilon/3}$, we have $D^2 u^* \leqslant C_2 I$. Since $\det D^2 u^* \geqslant \lambda_1$, we obtain

$$\lambda_{\min}\big(D^2 u^*\big) \geqslant \frac{\lambda_1}{C_2^{n-1}} =: C_1$$

where $\lambda_{\min}(D^2 u^*)$ is the minimum eigenvalue of $D^2 u^*$. □

Recall that E is the ellipsoid of minimum volume containing $Q_H(0)$ center at x_H the mass center of $Q_H(0)$. By rotating the coordinate system, we may suppose that the axes of the ellipsoid E coincide with the coordinate axes. If $T = T_H$ is an affine transformation that normalizes $Q_H(0)$, then $T(E) = B_1(0)$, $T(x_H) = 0$, and $Tx = A(x - x_H)$, $A = A_H = \operatorname{diag}\{\mu_1, \dots, \mu_n\}$.

Lemma 3.6. *Let A and μ_i, $i = 1, \dots, n$ be as above, then*

$$\frac{\lambda_{\min}}{C_2} \leqslant H\mu_i^2 \leqslant \frac{\lambda_{\max}}{C_1}, \quad i = 1, \dots, n, \tag{3.13}$$

where C_1, C_2 are the same as in Corollary 3.1 and $\lambda_{\max}, \lambda_{\min} > 0$ denotes the maximum and the minimum eigenvalue of $D^2 u(0)$, respectively.

Proof. Since $T = T_H$ normalizes $Q_H(0)$ and by (3.2) the Monge–Ampère measure with density $\det D^2 u(x,0)$ is doubling, by Theorem 3.3.8 of [17] applied to the sections $Q_H(0)$, $Q_{\tau H}(0)$ with $0 < \tau < 1$, we get that

$$B\big(T(0), K\tau\big) \subset T Q_{\tau H}(0),$$

where K is a constant depending on n, λ_1, λ_2. Let $\eta > 0$, then as in the proof of Lemma 3.2, we obtain

$$Q_{\tau H}(0) \times \left[-\frac{\eta H}{m_2}, 0\right] \subset Q_{(\tau+\eta)H}.$$

By applying $\mathcal{T}_H$ we have for some $\varepsilon_1 > 0$ depending on ρ, p, n, ε, m_j $(j=1,2)$

$$B\big(T(0), K\tau\big) \times [-\varepsilon_1 \eta, 0] \subset \mathcal{T}_H Q_{(\tau+\eta)H},$$

where we have used the fact $|\det T_H|^{-2/n} \approx H$. If we pick η such that $\tau + \eta < 1$ then

$$\mathcal{T}_H Q_{(\tau+\eta)H} \subset \big\{(x,t)\colon u^*(x,t) < -(1-\tau-\eta)H|\det T_H|^{2/n}\big\}.$$

Setting $\tau = 1/2$ and $\eta = 1/4$, we obtain

$$B\big(T(0), c_0\big) \times [-c_1, 0] \subset \Omega_\varepsilon = \big\{(x,t)\colon u^*(x,t) < -\varepsilon\big\}, \tag{3.14}$$

provided $c_0 = K/2$, $c_1 = \varepsilon_1/4$ and $\varepsilon \leqslant \delta_0 := \frac{1}{4}H|\det T_H|^{2/n}$. On the other hand,

$$D^2u^*\big(T(0), 0\big) = |\det T|^{2/n}\big(A^{-1}\big)^t D^2u(0,0)A^{-1}. \tag{3.15}$$

Combining (3.12) and (3.15), we obtain

$$C_1 I \leqslant |\det T|^{2/n}\big(A^{-1}\big)^t D^2u(0,0)A^{-1} \leqslant C_2 I.$$

Note that $A^{-1} = \mathrm{diag}\{1/\mu_1, \ldots, 1/\mu_n\}$ and (3.2), therefore

$$\frac{C_1}{\lambda_{\max}} \leqslant \frac{|\det T|^{2/n}}{\mu_i^2} \leqslant \frac{C_2}{\lambda_{\min}}, \quad i = 1, \ldots, n.$$

Thus (3.13) follows. □

Proof of Theorem 1.2. Given $\varepsilon > 0$, from (3.9) and (3.11), we have

$$\big\|u^*\big\|_{C^{2,1}(\Omega_\varepsilon)} \leqslant C(\varepsilon).$$

From (1.5), we see that $G(M) := \rho(\log\det M)$ is concave for symmetric positive definite matrix M. By (3.14) and Evans–Krylov estimates (see [24] or [18]), we have

$$\big[D_{ij}u^*\big]_{C^\alpha(B(T(0),c_0)\times[-c_1,0])} \leqslant C(\varepsilon), \tag{3.16}$$

$$\big[u_s^*\big]_{C^{\alpha/2}(B(T(0),c_0)\times[-c_1,0])} \leqslant C(\varepsilon), \tag{3.17}$$

where $\alpha \in (0,1)$. Since

$$u^*(y,s) = |\det T_H|^{2/n}\left[u\left(\left(\frac{y_1}{\mu_1}, \ldots, \frac{y_n}{\mu_n}\right) + x_H, |\det T_H|^{-2/n}s\right) - H\right],$$

then

$$D_{ij}u^*(y,s) = \frac{|\det T_H|^{2/n}}{\mu_i\mu_j} D_{ij}u\left(\left(\frac{y_1}{\mu_1}, \ldots, \frac{y_n}{\mu_n}\right) + x_H, |\det T_H|^{-2/n}s\right)$$

and

$$u_s^*(y,s) = u_t\left(\left(\frac{y_1}{\mu_1}, \ldots, \frac{y_n}{\mu_n}\right) + x_H, |\det T_H|^{-2/n}s\right).$$

From (3.16) and (3.17), we obtain

$$[D_{ij}u]_{C^\alpha(\mathcal{T}_H)^{-1}(B(T(0),c_0)\times[-c_1,0])} \leqslant C\frac{\mu_i\mu_j}{|\det T_H|^{2/n}}\Big(\max_i \mu_i\Big)^\alpha,$$

$$[u_t]_{C^{\alpha/2}(\mathcal{T}_H)^{-1}(B(T(0),c_0)\times[-c_1,0])} \leqslant C\big(|\det T_H|^{2/n}\big)^{\alpha/2}.$$

By Lemma 3.6, together with $T(0) = -Ax_H$, it follows that

$$B\big(0, c_2H^{1/2}\big) \times [-c_3H, 0] \subset (\mathcal{T}_H)^{-1}\big(B(T(0), c_0) \times [-c_1, 0]\big),$$

where

$$c_2 = c_0 \left(\frac{C_2}{\lambda_{\min}} \right)^{-1/2}, \qquad c_3 = \frac{c_1 |\det T_H|^{-2/n}}{H}.$$

Recalling the fact $|\det T_H|^{-2/n} \approx H$ again, consequently we obtain

$$[D_{ij}u]_{C^\alpha(B(0,\, c_2 H^{1/2}) \times [-c_3 H, 0])} \leqslant C H^{-\alpha/2},$$

and

$$[D_t u]_{C^{\alpha/2}(B(0,\, c_2 H^{1/2}) \times [-c_3 H, 0])} \leqslant C H^{-\alpha/2}.$$

By letting $H \to \infty$ we obtain that $D_{ij}u$ and u_t are constants on each bounded set and the proof is complete. □

4. Isolated singularities of parabolic Hessian equation

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $D^2 u$, then

$$\rho\left(\log \det D^2 u\right) = \rho\left(\sum_{i=1}^{n} \log \lambda_i\right)$$

if $\lambda_i > 0$. In view of this, we consider more general equation

$$-u_t + f\left(\lambda\left(D^2 u\right)\right) = 0. \tag{4.1}$$

The Dirichlet problem of (4.1) of elliptic type was studied by Caffarelli, Nirenberg and Spruck [6]. We say Eq. (4.1) is parabolic if $f(\lambda(M_1)) > f(\lambda(M_2))$ for any $M_1, M_2 \in \Gamma, M_1 > M_2$, where Γ is a convex cone of symmetric matrices $\mathbb{S}^{n\times n}$. We call u an admissible solution to (4.1), if $D^2 u(x,t) \in \Gamma$.

There are several interesting particular forms of f in our setting, for instance,

$$f\left(\lambda\left(D^2 u\right)\right) = S_k\left(\lambda\left(D^2 u\right)\right) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k-th elementary symmetric polynomial. The parabolic k-Hessian equation includes the heat equations ($k = 1$)

$$-u_t + \Delta u = 0$$

and parabolic Monge–Ampère equation ($k = n$)

$$-u_t + \det D^2 u = 0.$$

See [24] for a complete description and related results of parabolic Hessian equations.

Let $E \subset \mathbb{R}^{n+1}_-$ be a bounded closed measurable set and $u \in C^{4,2}(\mathbb{R}^{n+1}_- \setminus E)$ be an admissible solution to

$$-u_t + f\left(\lambda\left(D^2 u\right)\right) = 0 \quad \text{in } \mathbb{R}^{n+1}_- \setminus E. \tag{4.2}$$

In the following, we consider the problem of what assumptions imposed on u and E are enough to ensure that u is a smooth solution in entire $\mathbb{R}^{n+1}_-$. For this, we have

Theorem 4.1. *Assume u and E are as above. Let Q be a bowl-shaped domain satisfying $E \Subset Q$. Let $v \in C^{4,2}(\overline{Q})$ be an admissible solution to*

$$\begin{cases} -v_t + f\big(\lambda\big(D^2 v\big)\big) = 0 & \text{in } Q, \\ v = u & \text{on } \partial_p Q. \end{cases} \tag{4.3}$$

Suppose there exists a nonnegative integer $l \leqslant n-2$ such that $\dim E(t) \leqslant l$ for any $t<0$, where $\dim E(t)$ is the Hausdorff dimension of $E(t)$ in $\mathbb{R}^{n+1}$. Suppose further that for any $(x,t) \in E$, there are $l+2$ independent C^2 curves $(c_i(s), t)$ lying on $\mathbb{R}^n \times \{t\}$ and passing through (x,t) such that $u(c_i(s), t) \in C^1$. Then $u \equiv v$ on $\overline{Q}$.

Under some assumptions of f, the Dirichlet problem (4.3) is well studied, see [24]. Particularly, when $f = S_k$ and Q is a cylinder with a strict convex bottom, then there exists a unique solution of (4.3).

Recall that $E(t) = \{x\colon\ (x,t) \in E\}$. Similar result for elliptic equations was obtained by [28], but it further needed u is locally Lipschitz continuous. To prove the theorem, we need a special version of Aleksandroff maximum principle (see Lemma 4.3).

Let Q be a bowl-shaped domain in $\mathbb{R}^{n+1}$ and $u \in C(Q)$. For $(x_0, t_0), (x,t) \in Q$, the parabolic normal mapping of u is the set value function defined by

$$\begin{aligned} \Phi_{x_0}(x,t) = \big\{&(p,h) \in \mathbb{R}^{n+1}\colon\ u(y,s) \leqslant u(x,t) + p(y-x), \\ &h = u(x,t) - p\cdot(x-x_0), \text{ for any } y \in Q(s) \text{ with } s \leqslant t\big\}. \end{aligned}$$

We call the set

$$\Gamma_u = \big\{(x,t) \in Q\colon\ \Phi_{x_0}(x,t) \neq \emptyset\big\}$$

contact set of u. It is not difficult to see that the contact set of u is independent of the choice of (x_0, t_0). Denote

$$\Phi_{x_0}(Q) = \Phi_{x_0}(\Gamma_u) = \bigcup_{(x,t)\in\Gamma_u} \Phi_{x_0}(x,t).$$

Lemma 4.1. *Assume $u \in C^{2,1}(Q) \cap C(\overline{Q})$, then we have for $(x,t) \in \Gamma_u$*

$$\begin{gathered} p = D_x u(x,t), \qquad h = u(x,t) - D_x u(x,t)(x-x_0), \\ D_t u(x,t) \geqslant 0, \qquad -D_x^2 u(x,t) \geqslant 0, \end{gathered}$$

where $(p,h) \in \Phi_{x_0}(x,t)$.

Proof. By the definition of $\Phi_{x_0}(y,t)$, it is easy to see that this lemma holds. □

Lemma 4.2. *Let $E \Subset Q$ be a closed measurable set, $u \in C^{4,2}(\overline{Q} \setminus E) \cap C(\overline{Q})$, Γ_u be the contact set of u, and $0 \leqslant g \in C(\mathbb{R}^n)$. If $|\Phi_{x_0}(E \cap \Gamma_u)|_{n+1} = 0$, where $|\cdot|_{n+1}$ is the $(n+1)$-dimension Lebesgue measure, then*

$$\int_{\Phi_{x_0}(\Gamma_u)} g(p)\,\mathrm{d}p\,\mathrm{d}h \leqslant \int_{\Gamma_u \setminus E} g(Du) u_t \det\big(-D^2 u\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.4}$$

Proof. Denote the Jacobian determinant of the mapping Φ_{x_0} by $J(x,t) = |\det D\Phi_{x_0}| = u_t \det(-D^2u)$. Let $A = \{(x,t) \in Q_T \setminus E\colon\ J(x,t) = 0\}$. According to Sard Theorem, $|\Phi_{x_0}(A)|_{n+1} = 0$. Therefore, in view of Lemma 4.1, $J(x,t) > 0$ in $B := \Gamma_u \setminus (A \cup E)$.

At the first step, we assume B is open. Thus there exists a sequence of cubes $\{C_i\}_{i=1}^{\infty}$, $C_i \cap C_j = \emptyset$ if $i \neq j$ such that $B = \bigcup_{i=1}^{\infty} C_i$, and $\Phi_{x_0} : C_i \to \Phi_{x_0}(C_i)$ is a diffeomorphism. Hence,

$$\int\limits_{\Phi_{x_0}(C_i)} g(p)\,\mathrm{d}p\,\mathrm{d}h = \int\limits_{C_i} g(Du)u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\begin{aligned}\int\limits_{\Phi_{x_0}(B)} g(p)\,\mathrm{d}p\,\mathrm{d}h &\leqslant \sum_i \int\limits_{\Phi_{x_0}(C_i)} g(p)\,\mathrm{d}p\,\mathrm{d}h \\ &= \sum_i \int\limits_{C_i} g(Du)u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int\limits_{B} g(Du)u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Next, if B is only a measurable set, there exists an open set $G \subset Q$ such that $G \supset B$ and $J(x,t) > 0$ in G. Since B is measurable, one can choose an open set sequence $\{O_i\}_{i=1}^{\infty}$ such that $B \subset O_i$ and $|O_i \setminus B|_{n+1} \to 0$ when $i \to \infty$. For the open set $G \cap O_i$, due to the proof above, we obtain

$$\int\limits_{\Phi_{x_0}(G\cap O_i)} g(p)\,\mathrm{d}p\,\mathrm{d}h \leqslant \int\limits_{G\cap O_i} g(Du)u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t.$$

Let $i \to \infty$, it follows that

$$\begin{aligned}\int\limits_{\Phi_{x_0}(B)} g(p)\,\mathrm{d}p\,\mathrm{d}h &= \int\limits_{B} g(Du)u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int\limits_{\Gamma_u\setminus E} g(Du)u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Taking into account that $|\Phi_{x_0}(E \cap \Gamma_u)|_{n+1} = |\Phi_{x_0}(A)|_{n+1} = 0$, we complete the proof. □

Lemma 4.3. *Assume $u \in C^{4,2}(\overline{Q} \setminus E) \cap C(\overline{Q})$ and $u|_{\partial_p Q_T} \leqslant 0$. If $|\Phi_{x_0}(E \cap \Gamma_u)|_{n+1} = 0$, then*

$$\sup_{Q_T} u \leqslant \left(\frac{n+1}{\omega_n}\right)^{\frac{1}{n+1}} d^{\frac{n}{n+1}} \Biggl(\int\limits_{\Gamma_u\setminus E} u_t \det\bigl(-D^2u\bigr)\,\mathrm{d}x\,\mathrm{d}t\Biggr)^{\frac{1}{n+1}}, \tag{4.5}$$

where ω_n is the volume of n-dimension unite ball, $d = \operatorname{diam}\Omega$.

Proof. Assume $M := \sup_Q u > 0$, otherwise, there is nothing to prove. Since $u|_{\partial_p Q} \leqslant 0$, there is a point $(x_0, t_0) \in Q$ such that $u(x_0, t_0) = M$. At this point, consider the parabolic normal mapping Φ_{x_0}. We claim

$$\mathcal{N} = \left\{(p, h) \in \mathbb{R}^{n+1}\colon |p| < \frac{M}{d},\ d|p| < h < M\right\} \subset \Phi_{x_0}(\Gamma_u). \tag{4.6}$$

Indeed, for any point $(p, h) \in \mathcal{N}$, in the $(n+2)$-dimensional Euclidean space $\mathbb{R}^{n+2}$ with coordinates (x, t, z) we move the n-dimensional hyperplane $z = p(x - x_0) + h$ in positive direction of t. Note that the hyperplane lies above the surface $z = u(x, t_{\min})$ on $Q(t_{\min}) \times \{t_{\min}\}$ and $h < u(x_0, t_0)$, where $t_{\min} = \inf\{t\colon\ Q(t) \neq \emptyset\}$. In the process of moving, let t_1 be the first time when the hyperplane touches the surface $z = u(x, t)$ and (x_1, t_1) be one of the touching points. Since $u|_{\partial_p Q} \leqslant 0$ and $|p(x - x_0) + h| > 0$, we have $(x_1, t_1) \in Q$, $t_{\min} < t_1 \leqslant t_0$. Note that

$$u(x, t) \leqslant p(x - x_0) + h \quad \text{for } t \leqslant t_1, \tag{4.7}$$

$$u(x_1, t_1) = p(x_1 - x_0) + h. \tag{4.8}$$

Substituting (4.8) into (4.7), we have

$$\begin{aligned} u(x, t) &\leqslant p(x - x_0) + u(x_1, t_1) - p(x_1 - x_0) \\ &= u(x_1, t_1) - p(x - x_1) \quad \text{for } t \leqslant t_1. \end{aligned} \tag{4.9}$$

Combining (4.8) and (4.9), it follows that $(x_1, t_1) \in \Gamma_u$ and $(p, h) \in \Phi_{x_0}(x_1, t_1)$. Thus we proved the claim.

According to Lemma 4.2, we have

$$\begin{aligned} \int\limits_{\Gamma_u \setminus E} u_t \det\big(-D^2 u\big)\, \mathrm{d}x\, \mathrm{d}t &\geqslant \big|\Phi_{x_0}(\Gamma_u)\big|_{n+1} \geqslant |\mathcal{N}|_{n+1} \\ &= n\omega_n \int\limits_0^{Md^{-1}} r^{n-1}\, \mathrm{d}r \int\limits_{rd}^{M} \mathrm{d}h \\ &= \frac{\omega_n M^{n+1}}{(n+1)d^n}. \end{aligned}$$

This completes the proof. □

The use of moving hyperplane in the above proof follows from Chen [10]. As remarked in [10], the original proof of Tso (see [24]) making use of moving paraboloid may fail to find the contact point (x_1, t_1). Therefore, here the parabolic normal mapping Φ is a little bit different from standard definition.

Proof of Theorem 4.1. Suppose $w(x, t) := u - v$. First of all, we verify $|\Phi_{x_0}(E \cap \Gamma_w)|_{n+1} = 0$. For any point $(y_0, t_0) \in E \cap \Gamma_w$, let $(c_i(s), t_0)$ with $1 \leqslant i \leqslant l + 2$ be the independent curves passing through (y_0, t_0) and lying on $\Omega \times \{t_0\}$ such that $w(c_i(s), t_0) \in C^1$. Without loss of generality, we may assume $c_i(0) = y_0$. Let $(p, h) \in \Phi_{x_0}(y_0, t_0)$, then

$$w\big(c_i(s), t_0\big) \leqslant p\big(c_i(s) - y_0\big) + w(y_0, t_0)$$

which implies

$$\left.\frac{\mathrm{d}w(c_i(s), t_0)}{\mathrm{d}s}\right|_{s=0} = p\left.\frac{\mathrm{d}c_i(s)}{\mathrm{d}s}\right|_{s=0}.$$

Since $c_i(s)$ are independent, by the knowledge of linear algebra $\Phi_{x_0}(y_0, t_0)$ is a subset in a subspace of dimension $n+1-(l+2) = n-l-1$. It follows that

$$\dim \Phi_{x_0}(E \cap \Gamma_w) \leqslant 1 + l + n - l - 1 = n < n+1.$$

Consequently, $|\Phi_{x_0}(E \cap \Gamma_w)|_{n+1} = 0$.

On the other hand, for any point $(x, t) \in \Gamma_w \setminus E$, owing to Lemma 4.1 we obtain

$$w_t(x, t) \geqslant 0, \qquad -D^2 w(x, t) \geqslant 0.$$

If $w_t(x, t)\det(-D^2 w(x, t)) > 0$, then

$$u_t(x, t) > v_t(x, t), \qquad D^2 u(x, t) < D^2 v(x, t).$$

It follows that

$$0 = -u_t + f\big(\lambda\big(D^2 u\big)\big) < -v_t + f\big(\lambda\big(D^2 v\big)\big) = 0.$$

This contradiction leads to

$$w_t \det\big(-D^2 w\big) = 0 \quad \text{in } \Gamma_w \setminus E.$$

Now, applying Lemma 4.3 to w, we have

$$u - v = w \leqslant 0.$$

By the same procedure, one can prove $v - u \leqslant 0$. In combination, we complete the proof. □

Proof of Theorem 1.3. In view of Theorem 4.1, we only need to show that there exists a bowl-shaped domain $Q \subset \mathbb{R}^{n+1-}$ such that $X_0 \in Q$ and the Dirichlet problem

$$\begin{cases} -v_t \det D^2 v = 1 & \text{in } Q, \\ v = u & \text{on } \partial_p Q, \end{cases}$$

is solvable. This is a well-known result, see [24] or [19]. □

It is easy to see that isolated point X_0 in Theorem 1.3 can be replaced by a closed set E as in Theorem 4.1. For Eqs. (1.3) and (1.4), their first boundary value problem has been well established. Therefore, Theorem 1.3 applies to them.

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