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NEW MAXIMUM PRINCIPLES FOR FULLY NONLINEAR ODES OF SECOND ORDER

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Abstract. In the paper, we give the positive answer of an open problem of Li-Nirenberg under the weaker conditions, and we prove a new variation of the boundary point lemma for second order fully nonlinear ODEs by a new method. A simpler proof of Li-Nirenberg Theorem is also presented.

1. Introduction.

The maximum principle is one of the most useful and best known tools employed in the study of partial differential equations; see [1] and [2]. In many cases, the maximum principle enables us to obtain the existence, uniqueness, stability and regularity of the solutions without their explicit knowledge.

In 1958, A.D.Alexandrov [3] applied the boundary point lemma and the strong maximum principle to prove a famous theorem by the method of moving planes. Here it is:

Theorem 1. *Let M be a compact smooth hypersurface, embedded in $\mathbb{R}^{n+1}$ with the constant mean curvature. Then M is a sphere.*

Li and Nirenberg [4] extended Theorem 1 to the case of the monotonic mean curvature in 2005, and they proved for the case of one dimension.

Theorem 2. *Let M be a closed C^2 embedded curve in the plane, and stay on one side of any line parallel to the y-axis that is tangent to M. Assume the curvature $H(x,y)$ of M satisfies*

$$H(x,a) \geq H(x,b), \quad (x,a),(x,b) \in M, \quad a < b.$$

Then M is symmetric about some line $y = \lambda_0$.

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In another paper [5], Li and Nirenberg discussed the case of higher dimension.

To prove Theorem 2, a new version of the boundary point lemma is presented by Lemma 3.1 in [4].

Theorem 3. *Let $u, v \in C^2(0,b) \cap C^1[0,b]$ satisfying $u(0) = \frac{du}{dx}(0) = 0$, $u(x) \geq v(x) > 0$, and*

$$\frac{du}{dx} > 0,\ \frac{dv}{dx} \geq 0 \text{ or } \frac{du}{dx} \geq 0,\ \frac{dv}{dx} > 0,\ x \in (0,b].$$

Assume

$$\frac{\frac{d^2u}{dx^2}(t)}{\left(1+\left(\frac{du}{dx}\right)^2(t)\right)^{3/2}} \leq \frac{\frac{d^2v}{dx^2}(s)}{\left(1+\left(\frac{dv}{dx}\right)^2(s)\right)^{3/2}} \tag{1}$$

if $u(t) = v(s)$ for $0 < t \leq s < b$, then $u \equiv v$ on $[0,b]$.

Compared with the classical boundary point lemma, the difference of intrinsic quality is that the differential inequality (1) holds in the same functional values (maybe not the same independent variables). This is a shift idea, and it plays an important role to enrich theory of differential equations.

After that in [4], Li and Nirenberg extended the curvature operator in Theorem 3 to the divergence operator $(K(u'))'$, where $K \in C^1(\mathbb{R})$, and $K' > 0$. An open problem rises naturally; see Question 5.1 of [4].

Problem 1. *Can we replace $(K(u'))'$ by $K(u', u'')$ or $K(u, u', u'')$ in Theorem 3?*

Li and Nirenberg's proof of Theorem 3 is done by obtaining an explicit expression for u, and it depends strongly on the divergence form of (1). Their method is invalid in the case of fully nonlinear differential operator.

In this paper, we give the positive answer about the above open problem, and we obtain the new variation of the boundary points lemma for second order fully nonlinear ordinary differential equations, which is proved by a new method and under the weaker conditions.

We always assume that $K(p,q,r) \in C^0(\mathbb{R}^3)$, $K(p,q,r)$ is C^1 in (q,r), and

$$\frac{\partial K}{\partial r}(p,q,r) > 0,\ (p,q,r) \in \mathbb{R}^3, \tag{2}$$

$$K\left(u(t), \frac{du}{dx}(t), \frac{d^2u}{dx^2}(t)\right) \leq K\left(v(s), \frac{dv}{dx}(s), \frac{d^2v}{dx^2}(s)\right) \tag{3}$$

if $u(t) = v(s)$ for $t, s \in (a,b)$. Our main results are

Theorem 4. *Let $u, v \in C^2[a,b]$ satisfy (3), and*

$$u(a) = v(a),\ \frac{du}{dx}(a) = \frac{dv}{dx}(a), \tag{4}$$

$$\frac{du}{dx} > 0,\ \frac{dv}{dx} \geq 0,\ v(x) > v(a),\ \text{or } \frac{du}{dx} \geq 0,\ \frac{dv}{dx} > 0,\ u(x) > u(a),\ \text{in } (a,b). \tag{5}$$

Then $u \leq v$ on $[a,b]$.

The condition (5) is optimal in some sense.

Remark 1. *If we replace*

$$\frac{du}{dx} > 0,\ \frac{dv}{dx} \geq 0,\ \ or\ \frac{du}{dx} \geq 0,\ \frac{dv}{dx} > 0$$

in (5) by both $\frac{du}{dx} \geq 0$ *and* $\frac{dv}{dx} \geq 0$ *in* (a,b), *then the conclusion of Theorem 4 may fail. See Example 1. The similar counterexample can be found in* [4].

Example 1. See Figure 1. Let $a = 0, b = 4\pi$,

$$u(x) = \begin{cases} 0 & , \quad x \in [0,\pi], \\ x - \pi - sin(x-\pi), & x \in (\pi, 4\pi], \end{cases}$$

$$v(x) = \begin{cases} x - sinx, & x \in [0, 2\pi), \\ 2\pi & , \quad x \in [2\pi, 4\pi]. \end{cases}$$

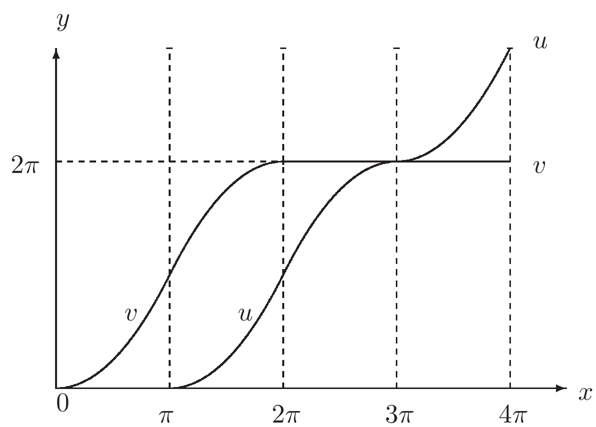

Figure 1

Remark 2. *If there is not the condition* $v(x) > v(a)$ *or* $u(x) > u(a)$ *in (5), then the conclusion of Theorem 4 may also fail. See Example 2.*

Example 2. See Figure 2. Let $a = 0, b = 2\pi$,

$$u(x) = x - sinx, \quad x \in [0, 2\pi),$$

$$v(x) = \begin{cases} 0 & , \quad x \in [0,\pi], \\ x - \pi - sin(x-\pi), & x \in (\pi, 2\pi]. \end{cases}$$

If we assume $u \geq v$ and K is the mean curvature operator in Theorem 4, then we can get Theorem 3 at once, i.e., Lemma 3.1 of [4].

Theorem 4 extends Li-Nirenberg's result in [4] from the mean curvature operator and the divergence operator to the most general form of the fully nonlinear ordinary differential operator of second order. So, we solve completely the open Problem 1.

The rest part of the paper is organized as follows. Theorem 4 is proved in the next section. In Section 3, we present the new strong and weak maximum principles of the fully nonlinear second order ordinary differential equations and some remarks. In the last section, we give a simpler proof of Theorem 2.

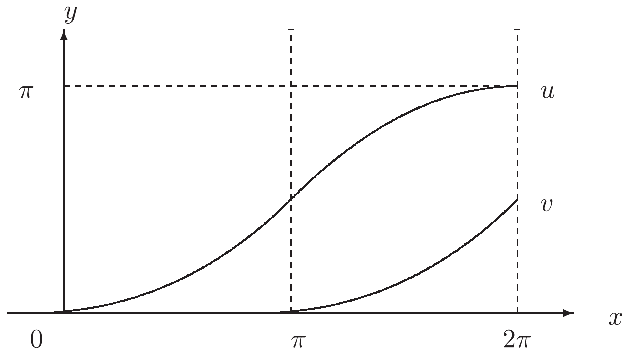

Figure 2

2. **Proof of Main Result.** In this section, we prove Theorem 4.

Proof of Theorem 4. Let $\aleph = \{\alpha \in [a,b] \mid u(x) \le v(x),\ x \in [\alpha, b]\}$. If $\aleph = \phi$, then $u(b) > v(b)$. If $\aleph \neq \phi$, then set $\beta = inf \aleph$. We have $u(\beta) = v(\beta)$, and $u(x) \le v(x)$, $x \in [\beta, b]$. So, we may assume

$$u(b) \ge v(b) \tag{6}$$

without loss of generality.

Case 1. $\dfrac{du}{dx} > 0$, $\dfrac{dv}{dx} \ge 0$, and $v > v(a)$ in (a,b).

From the implicit function theorem, we can get $t = t(s) \in C^0[a,b] \cap C^2(a,b)$, such that

$$u(t(s)) = v(s), \quad s \in [a,b], \tag{7}$$

and

$$t(a) = a, \quad t(s) \in [a,b].$$

Here (6) guarantees the function $t(s)$ well defined on whole interval $[a,b]$. Differentiating (7) with respect to s, we find

$$\frac{du}{dt}(t(s)) \cdot \frac{dt}{ds}(s) = \frac{dv}{ds}(s). \tag{8}$$

By (3), (7) and the mean value theorem, we have

$$\begin{aligned} 0 &\ge K\left(u(t(s)), \frac{du}{dt}(t(s)), \frac{d^2u}{dt^2}(t(s))\right) - K\left(v(s), \frac{dv}{ds}(s), \frac{d^2v}{ds^2}(s)\right) \\ &= a(s) \cdot \left(\frac{d^2u}{dt^2}(t(s)) - \frac{d^2v}{ds^2}(s)\right) + b(s) \cdot \left(\frac{du}{dt}(t(s)) - \frac{dv}{ds}(s)\right), \end{aligned} \tag{9}$$

where

$$a(s) = \int_0^1 \frac{\partial K}{\partial r}\left(v(s), \theta \cdot \frac{du}{dt}(t(s)) + (1-\theta) \cdot \frac{dv}{ds}(s), \theta \cdot \frac{d^2u}{dt^2}(t(s)) + (1-\theta) \cdot \frac{d^2v}{ds^2}(s)\right) d\theta,$$

$$b(s) = \int_0^1 \frac{\partial K}{\partial q}\left(v(s), \theta \cdot \frac{du}{dt}(t(s)) + (1-\theta) \cdot \frac{dv}{ds}(s), \theta \cdot \frac{d^2u}{dt^2}(t(s)) + (1-\theta) \cdot \frac{d^2v}{ds^2}(s)\right) d\theta.$$

Since (2) and $u, v \in C^2[a, b]$, $a(s)$ and $b(s)$ are bounded functions on $[a, b]$, and $a(s)$ has a positive lower bound. In view of (8) and (9),

$$\begin{aligned} & \frac{d}{ds}\left(\left(\frac{du}{dt}(t(s))\right)^2 - \left(\frac{dv}{ds}(s)\right)^2\right) \\ = \ & 2\frac{dv}{ds}(s) \cdot \left(\frac{d^2u}{dt^2}(t(s)) - \frac{d^2v}{ds^2}(s)\right) \\ \leq \ & 2\frac{dv}{ds}(s) \cdot \left(-\frac{b(s)}{a(s)} \cdot \left(\frac{du}{dt}(t(s)) - \frac{dv}{ds}(s)\right)\right) \\ = \ & -A(s) \cdot \left(\left(\frac{du}{dt}(t(s))\right)^2 - \left(\frac{dv}{ds}(s)\right)^2\right), \end{aligned} \tag{10}$$

where

$$A(s) = \begin{cases} \dfrac{2b(s)}{a(s)} \cdot \dfrac{\frac{dv}{ds}(s)}{\frac{du}{dt}(t(s)) + \frac{dv}{ds}(s)}, & s \in (a, b], \\ 0, & s = a, \end{cases}$$

and $A(s)$ is bounded function on $[a, b]$. By (10), we have

$$\frac{d}{ds}\left(\left(\left(\frac{du}{dt}(t(s))\right)^2 - \left(\frac{dv}{ds}(s)\right)^2\right) e^{\int_a^s A(\tau)d\tau}\right) \leq 0.$$

So,

$$\left(\left(\frac{du}{dt}(t(s))\right)^2 - \left(\frac{dv}{ds}(s)\right)^2\right) e^{\int_a^s A(\tau)d\tau}$$

is a decreasing function of s on $[a, b]$. Thus from (4),

$$\left(\left(\frac{du}{dt}(t(s))\right)^2 - \left(\frac{dv}{ds}(s)\right)^2\right) e^{\int_a^s A(\tau)d\tau} \leq 0.$$

Thereby

$$\frac{du}{dt}(t(s)) \leq \frac{dv}{ds}(s) = \frac{du}{dt}(t(s)) \cdot \frac{dt}{ds}(s),$$
$$\frac{dt}{ds}(s) \geq 1,\ t(s) \geq s,\ s \in (a, b).$$

Here we have used the fact that $\frac{du}{dt}(t(s)) > 0$ in (a, b), since $v > v(a)$ and $t(s) > a$ in (a, b). From the monotone of u again, we know

$$u(s) \leq u(t(s)) = v(s), \qquad s \in [a, b].$$

We have proved Case 1.

Case 2. $\frac{du}{dx} \geq 0$, $\frac{dv}{dx} > 0$ and $u > u(a)$ in (a, b).

Since $\frac{dv}{dx} > 0$ for $x \in (a, b)$, and $u(a) = v(a) < v(b) \leq u(b)$, there must be $c_0 \in (a, b]$ with $u(c_0) = v(b)$ by the intermediate value theorem of continuous functions. Let

$$c = sup\ \{c_0 \in (a, b] \mid u(x) \leq v(b),\ x \in (a, c_0)\},$$

then $u(c) = v(b)$. By the implicit function theorem, we have $s = s(t) \in C^0[a,c] \cap C^2(a,c)$, such that

$$u(t) = v(s(t)), \qquad t \in [a,c], \tag{11}$$

and

$$s(a) = a, \quad s(t) \in [a,b].$$

Here $u(c) = v(b)$ guarantees the function $s(t)$ well defined on whole interval $[a,c]$. Differentiating (11) with respect to t, we find

$$\frac{du}{dt}(t) = \frac{dv}{ds}(s(t)) \cdot \frac{ds}{dt}(t). \tag{12}$$

By (3), (11) and the mean value theorem, we have

$$\begin{aligned} 0 &\geq K\left(u(t), \frac{du}{dt}(t), \frac{d^2u}{dt^2}(t)\right) - K\left(v(s(t)), \frac{dv}{ds}(s(t)), \frac{d^2v}{ds^2}(s(t))\right) \\ &= c(t) \cdot \left(\frac{d^2u}{dt^2}(t) - \frac{d^2v}{ds^2}(s(t))\right) + d(t) \cdot \left(\frac{du}{dt}(t) - \frac{dv}{ds}(s(t))\right), \end{aligned} \tag{13}$$

where

$$c(t) = \int_0^1 \frac{\partial K}{\partial r}\left(u(t), \theta\cdot\frac{du}{dt}(t) + (1-\theta)\cdot\frac{dv}{ds}(s(t)), \theta\cdot\frac{d^2u}{dt^2}(t) + (1-\theta)\cdot\frac{d^2v}{ds^2}(s(t))\right)d\theta,$$

$$d(t) = \int_0^1 \frac{\partial K}{\partial q}\left(u(t), \theta\cdot\frac{du}{dt}(t) + (1-\theta)\cdot\frac{dv}{ds}(s(t)), \theta\cdot\frac{d^2u}{dt^2}(t) + (1-\theta)\cdot\frac{d^2v}{ds^2}(s(t))\right)d\theta.$$

Since (2) and $u, v \in C^2[a,b]$, so $c(t)$ and $d(t)$ are bounded functions on $[a,c]$, and $c(t)$ has a positive lower bound. In view of (12) and (13),

$$\frac{d}{dt}\left(\left(\frac{du}{dt}(t)\right)^2 - \left(\frac{dv}{ds}(s(t))\right)^2\right) \tag{14}$$

$$\begin{aligned} &= 2\frac{du}{dt}(t) \cdot \left(\frac{d^2u}{dt^2}(t) - \frac{d^2v}{ds^2}(s(t))\right) \\ &\leq 2\frac{du}{dt}(t) \cdot \left(-\frac{d(t)}{c(t)} \cdot \left(\frac{du}{dt}(t) - \frac{dv}{ds}(s(t))\right)\right) \\ &= -B(t)\left(\left(\frac{du}{dt}(t)\right)^2 - \left(\frac{dv}{ds}(s(t))\right)^2\right), \end{aligned} \tag{15}$$

where

$$B(t) = \begin{cases} \dfrac{2d(t)}{c(t)} \cdot \dfrac{\dfrac{du}{dt}(t)}{\dfrac{du}{dt}(t) + \dfrac{dv}{ds}(s(t))}, & t \in (a,c], \\ 0, & t = a, \end{cases}$$

and $B(t)$ is bounded on $[a,c]$. By (15)

$$\frac{d}{dt}\left(\left(\left(\frac{du}{dt}(t)\right)^2 - \left(\frac{dv}{ds}(s(t))\right)^2\right) e^{\int_a^t B(\tau)d\tau}\right) \leq 0.$$

So,

$$\left(\left(\frac{du}{dt}(t)\right)^2 - \left(\frac{dv}{ds}(s(t))\right)^2\right) e^{\int_a^t B(\tau)d\tau}$$

is a decreasing function of t on $[a, c]$. Thus from (1.4)

$$\left(\left(\frac{du}{dt}(t)\right)^2 - \left(\frac{dv}{ds}(s(t))\right)^2\right) e^{\int_a^t B(\tau)d\tau} \leq 0.$$

Thereby

$$\frac{dv}{ds}(s(t)) \cdot \frac{ds}{dt}(t) = \frac{du}{dt}(t) \leq \frac{dv}{ds}(s(t)),$$
$$\frac{ds}{dt} \leq 1, \ s(t) \leq t, \ t \in (a, c).$$

Here we have used the fact that $\frac{dv}{ds}(s(t)) > 0$ in (a, c), since $u > u(a)$ and $s(t) > a$ in (a, c). Because of monotone of v again, we have

$$u(t) = v(s(t)) \leq v(t), \qquad t \in [a, c].$$

In particular, $v(b) = u(c) \leq v(c)$. Since v is strictly monotone, $c = b$. Now we gain

$$u(t) \leq v(t), \quad t \in [a, b].$$

We have proved Case 2.

So far we have completed the proof of Theorem 4. □

3. **New Strong and Weak Maximum Principles.** We can use the similar method as in Section 2 to prove the new variations of strong maximum principle (Lemma 5.1 of [4]) and weak maximum principle (Lemma 5.2 of [4]) for (3). Their proofs will be omitted.

Theorem 5. *(New Strong Maximum Principle) Let $u, v \in C^2(a, b)$ satisfying (3), $u(x) \geq v(x)$ in (a, b), and*

$$\max\{\frac{du}{dx}, \frac{dv}{dx}\} > 0, \quad in\ (a, b). \tag{16}$$

Then $u > v$ or $u \equiv v$ in (a, b).

Remark 3. *If we replace (16) by both $\frac{du}{dx} \geq 0$ and $\frac{dv}{dx} \geq 0$ in (a, b), then the conclusion of Theorem 5 may fail. See Example 3.*

Example 3. See Figure 3. Let $a = 0, b = 3\pi$,

$$u(x) = x - sinx, \quad x \in [0, 3\pi],$$
$$v(x) = \begin{cases} x - sinx, & x \in [0, 2\pi), \\ 2\pi \quad , & x \in [2\pi, 3\pi]. \end{cases}$$

Theorem 6. *(New Weak Maximum Principle) Let $u, v \in C^2(a, b) \cap C^0[a, b]$ satisfying (3), $u(a) \geq v(a)$, $u(b) > v(x)$, $x \in (a, b)$, and*

$$\frac{du}{dx} > 0 \ or\ \frac{dv}{dx} > 0 \ in\ (a, b). \tag{17}$$

Then $u \geq v$ on $[a, b]$.

Remark 4. *If we replace (17) by both $\frac{du}{dx} \geq 0$ and $\frac{dv}{dx} \geq 0$ in (a, b), then the conclusion of Theorem 6 may fail. See Example 1.*

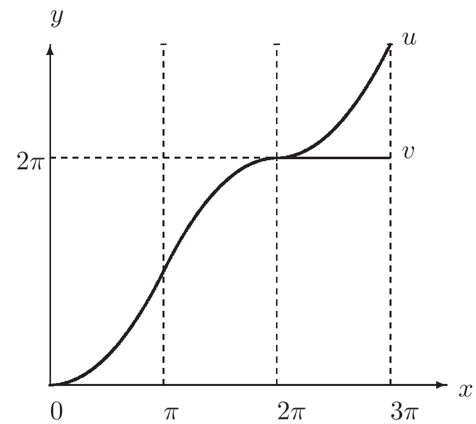

Figure 3

Remark 5. *If we have not the condition $u(b) > v(x)$, $x \in (a,b)$, then the conclusion of Theorem 6 may fail. See Example 4.*

Example 4. See Figure 4. Let $a = 0, b = \frac{7\pi}{8}$,

$$u(x) = x, \quad x \in [0, \frac{7\pi}{8}],$$

$$v(x) = \frac{3\sqrt{2}\pi}{4} \cdot sinx, \quad x \in [0, \frac{7\pi}{8}].$$

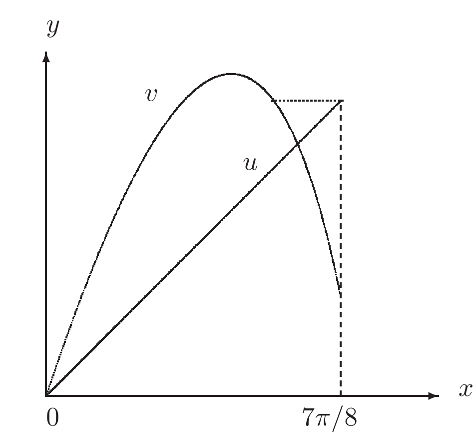

Figure 4

4. **A New Proof of Theorem 2.** We first give the main lemma for the proof of Theorem 2.

Lemma 1. *Let $u, v \in C^2(a,b)$ satisfying*

$$\frac{u''}{(1+u'^2)^{3/2}} \le \frac{v''}{(1+v'^2)^{3/2}}, \quad x \in (a,b). \tag{18}$$

Suppose that there exists $\xi \in (a,b)$, such that

$$u(\xi) = v(\xi), \qquad u'(\xi) = v'(\xi). \tag{19}$$

Then $u \le v$ in (a,b).

Proof. Set

$$f(q) = \frac{q}{\sqrt{1+q^2}}, \quad q \in \mathbb{R}.$$

Then

$$f'(q) = \left(\frac{q}{\sqrt{1+q^2}}\right)' = \frac{1}{(1+q^2)^{3/2}} > 0, \quad q \in \mathbb{R}, \tag{20}$$

which implies that $f(q)$ is strictly increasing in $\mathbb{R}$. Let

$$F(x) = f(u'(x)) - f(v'(x)) = \frac{u'(x)}{\sqrt{1+u'(x)^2}} - \frac{v'(x)}{\sqrt{1+v'(x)^2}}, \quad x \in (a,b).$$

So $F(\xi) = 0$, and by (18)

$$F'(x) = f'(u')u'' - f'(v')v'' = \frac{u''(x)}{(1+u'^2(x))^{3/2}} - \frac{v''(x)}{(1+v'^2(x))^{3/2}} \le 0, \quad x \in (a,b).$$

It follows that $F(x) \ge 0$ in (ξ, b), that is,

$$\frac{u'}{\sqrt{1+u'^2}} \le \frac{v'}{\sqrt{1+v'^2}} \text{ in } (\xi, b).$$

Therefore we have by (20) $u' \le v'$ and then $u \le v$ in (ξ, b), since $u(\xi) = v(\xi)$. It can be proved in similar fashion that $u \le v$ in (a, ξ). ☐

Remark 6. *The condition (19) in Lemma 1 may be replaced by*

$$\lim_{x\to a+0} u(x) = \lim_{x\to a+0} v(x), \text{ and}$$
$$\lim_{x\to a+0} u'(x) = \lim_{x\to a+0} v'(x) = -\infty, \tag{21}$$

or

$$\lim_{x\to b-0} u(x) = \lim_{x\to b-0} v(x), \text{ and}$$
$$\lim_{x\to b-0} u'(x) = \lim_{x\to b-0} v'(x) = +\infty. \tag{22}$$

Now we present a simpler proof of Theorem 2 than Li-Nirenberg's, without turn in the picture as in [3] or [4].

New Proof of Theorem 2. Our proof relies only on Lemma 1 and Remark 6. The condition (M stays on one side of any line parallel to the y-axis that is tangent to M) of Theorem 2 implies that there are just two lines, say $x = a$ and $x = b$ with $a < b$, parallel to the y-axis which are tangent to M. We carry out the moving plane method in strip-shaped domain $\{(x,y) \mid x \in [a,b], y \in (-\infty, +\infty)\}$, which is similar to those in the proof of Theorem 1.4 in [4].

Let M be the boundary of an open set U in the plane. For λ less than, but close to, $\max_M y$, take

$$S_\lambda = \{(x,y) \in M \mid y > \lambda\},$$

and reflect it in the line $y = \lambda$. The reflected piece of curve, S'_λ, lies in $\overline{U}$. Decrease λ and continue to reflect S_λ so that S'_λ continues to lie in $\overline{U}$. There will be a first value λ_0 of λ, such that for any $\lambda < \lambda_0$, S'_λ does not lie in $\overline{U}$. Note that it is allowed for M to have some flat segment. We then obtain the following.

Case 1. See Figure 5. S'_{λ_0} touches M at some point (ξ, η) with $a < \xi < b$ and $\eta < \lambda_0$.

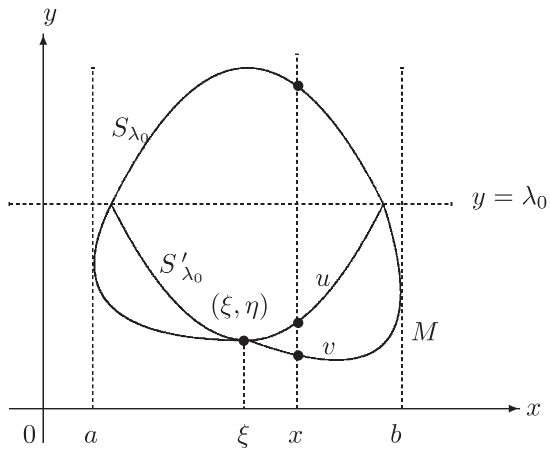

Figure 5

Case 2. See Figure 6. At the point (a, λ_0) or (b, λ_0), S'_{λ_0} and M are tangent to each other.

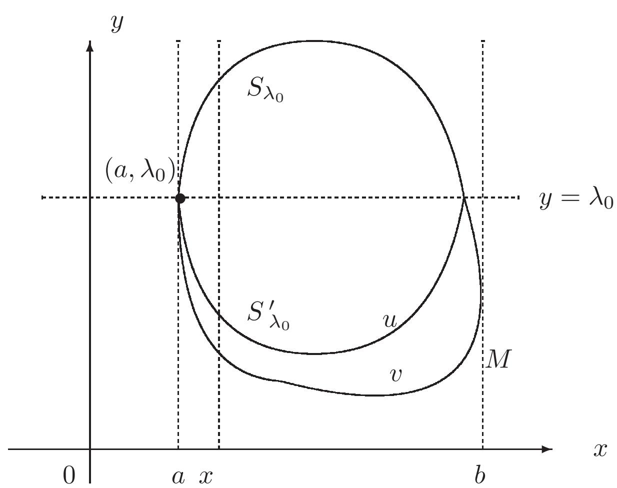

Figure 6

By the condition of Theorem 2, we may describe S'_{λ_0} and M on $[a, b]$ as graphs of functions $u, v \in C^2(a, b) \cap C^0[a, b]$ with

$$u(x) \geq v(x) \quad \text{and} \quad u(\xi) = v(\xi) = \eta \quad \text{for some } \xi \in [a, b],$$

and

$$\frac{u''}{(1 + u'^2)^{3/2}} \leq \frac{v''}{(1 + v'^2)^{3/2}}, \quad x \in (a, b).$$

In Case 1, $\xi \in (a, b)$, $u(\xi) = v(\xi)$, $u'(\xi) = v'(\xi)$. In Case 2,

$$\xi = a, \; u(a) = v(a), \; u'(a + 0) = v'(a + 0) = -\infty, \quad \text{or}$$

$$\xi = b,\ u(b) = v(b),\ u'(b-0) = v'(b-0) = +\infty.$$

Therefore we have that the conditions (19), (21) or (22) hold. By Lemma 1 and Remark 6 it follows that $u \leq v$, and then $u \equiv v$ in (a, b). This is the desired symmetry. □

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