NECESSARY AND SUFFICIENT CONDITIONS
ON SOLVABILITY FOR HESSIAN INEQUALITIES

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ABSTRACT. In this paper, we discuss the solvability of the Hessian inequality $\sigma_k^{\frac{1}{k}}(\lambda(D^2u)) \geq f(u)$ on the entire space $\mathbb{R}^n$ and provide a necessary and sufficient condition, which can be regarded as a generalized Keller-Osserman condition.

1. INTRODUCTION AND MAIN RESULTS

Many works have been done on the non-linear partial differential equation

$$(1.1) \qquad \Delta u = u^p, x \in \mathbb{R}^n,$$

where

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is the Laplacian of u and p is a positive constant (see [1], [15]).

These problems come from geometry. Briefly speaking, let (M, g) be a Riemannian manifold of dimension n, $n \geq 2$, and $K(\cdot)$ be a given function on M. We want to find a new metric G on M such that K is the scalar curvature of G and G is conformal to g. In the case $n \geq 3$, if we let $G = u^{\frac{4}{n-2}}g$, it is equivalent to the problem of finding positive solutions of the equation

$$(1.2) \qquad \frac{4(n-1)}{n-2}\Delta_g u - k_g u + K u^{\frac{n+2}{n-2}} = 0,$$

where Δ_g, k_g are the Laplacian and scalar curvature with respect to the metric g, respectively (see [4]). Especially, if we take $M = \mathbb{R}^n$, $g = (\delta_{ij})$ and $K(\cdot) \equiv -1$, then $k_g = 0$ and (1.2) reduces to (1.1) with $p = \frac{n+2}{n-2}$. In the case $n = 2$, if we let $G = e^{-2u}g$, it is equivalent to the problem of finding locally bounded solutions of the equation

$$(1.3) \qquad \Delta_g u - k_g + K e^{2u} = 0,$$

where Δ_g, k_g are the Laplacian and Gauss curvatures on M with respect to the metric g, respectively. Similarly, if we take $M = \mathbb{R}^n$, $g = (\delta_{ij})$ and $K(\cdot) \equiv -1$, then (1.3) reduces to the equation

$$(1.4) \qquad\qquad \Delta u = e^{2u}.$$

Details can be found in [12].

To state our results, we first need to fix some notation. For $k = 1, 2, \cdots, n$, let

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$$

denote the kth elementary symmetric function, and define

$$(1.5) \qquad\qquad \Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0, 1 \leq l \leq k\}.$$

For any $n \times n$ real symmetric matrix A, we let $\lambda(A)$ denote the eigenvalues of A.

Assume Ω is a domain in $\mathbb{R}^n$ and $D^2 u$ is the Hessian matrix of $u \in C^2(\Omega)$. It is easy to see that

$$\sigma_1(\lambda(D^2 u)) = \sum_{i=1}^n \lambda_i = \Delta u$$

and

$$\sigma_n(\lambda(D^2 u)) = \prod_{i=1}^n \lambda_i = \det(D^2 u).$$

We call a function $u \in C^2(\Omega)$ k-convex in Ω if $\lambda(D^2 u) \in \Gamma_k$ for all $x \in \Omega$, and let Φ^k denote the class of k-convex functions, i.e.

$$\Phi^k(\Omega) := \{u \in C^2(\Omega) : \lambda(D^2 u) \in \Gamma_k, \forall x \in \Omega\}.$$

There is a rich literature concerning the equation

$$(1.6) \qquad\qquad \sigma_k^{\frac{1}{k}}(\lambda(D^2 u)) = f,$$

for a positive function f. Caffarelli, Nirenberg and Spruck [2] established the regularity theory for equation (1.6) for Dirichlet boundary value problems and proved its existence. Krylov [14] and Evans [5] obtained the regularity for a more general class of fully non-linear elliptic equations not necessarily of divergence form. Trudinger and Xujia Wang ([19], [20], [21]) developed the theory of Hessian measures. Bo Guan [9] and John Urbas [22] have also made important contributions to the equation. Some of the techniques in these works can be modified to study equations in conformal geometry (see [3]).

A function $u \in \Phi^k(\mathbb{R}^n)$ is called a subsolution of the fully non-linear partial differential equation

$$(1.7) \qquad\qquad \sigma_k^{\frac{1}{k}}(\lambda(D^2 u)) = f(u), x \in \mathbb{R}^n$$

if u satisfies the inequality

$$\sigma_k^{\frac{1}{k}}(\lambda(D^2 u)) \geq f(u), x \in \mathbb{R}^n.$$

A famous result is that (1.1) has no positive subsolution if $p > 1$ (see [1]). This result can be led to by [17], where Osserman considered the necessary and sufficient condition under which the equation

$$(1.8) \qquad\qquad \Delta u = f(u), x \in \mathbb{R}^n,$$

has a subsolution, where f is a positive monotone increasing continuous function on $\mathbb{R}$. It has been proved that the equation (1.8) has a subsolution if and only if the function f satisfies the Keller-Osserman condition

$$(1.9) \qquad \int^{\infty} \left(\int_0^t f(s)ds \right)^{-\frac{1}{2}} dt = \infty,$$

where we omit the lower limit to admit an arbitrary positive number. Condition (1.9) is often used to study the so-called boundary blow-up (explosive, large) solutions (see [8], [7], [13], [16], [18], [23]).

In [11], Qinian Jin, Yanyan Li and Haoyuan Xu proved that a related equation,

$$(1.10) \qquad \sigma_k^{\frac{1}{k}}(\lambda(D^2 u)) = u^p, x \in \mathbb{R}^n,$$

has no k-convex positive subsolution for any $p > 1$. However the method they used cannot verify whether the condition $p > 1$ is optimal.

In order to answer this question, we pay attention to equation (1.7) and get a result comparable to that of Osserman ([17]).

Our main theorem is:

Theorem 1.1. *If $f(t)$ is a continuous function defined on $\mathbb{R}$ and satisfies*

$$(1.11) \qquad \begin{cases} f(t) > 0 \text{ is monotone non-decreasing in } (0, +\infty), \\ f(t) = 0 \text{ in } (-\infty, 0], \end{cases}$$

then equation (1.7) has a positive subsolution $u \in \Phi^k(\mathbb{R}^n)$ if and only if

$$(1.12) \qquad \int^{\infty} \left(\int_0^t f^k(s)ds \right)^{-\frac{1}{k+1}} dt = \infty.$$

We can see that when $k = 1$, equation (1.7) becomes equation (1.8), while condition (1.12) becomes the Keller-Osserman condition (1.9). In fact, equation (1.7) has a positive radial solution $u \in \Phi^k(\mathbb{R}^n)$ if and only if the function f in Theorem 1.1 satisfies condition (1.12).

By the main theorem, we can easily get the corollary below, which solves the problem of equation (1.10).

Corollary 1.2. *If the constant p is positive, (1.10) has a positive subsolution $u \in \Phi^k(\mathbb{R}^n)$ if and only if $p \leq 1$.*

If we strengthen the requirement of f from non-negative to positive, then the global subsolution of (1.7) we considered does not need to be positive. We have the following theorem:

Theorem 1.3. *If $f(t)$ is a positive, continuous and monotone non-decreasing function defined on $\mathbb{R}$, then equation (1.7) has a subsolution $u \in \Phi^k(\mathbb{R}^n)$ if and only if (1.12) holds.*

By Theorem 1.3, we get

Corollary 1.4. *The equation*

$$\sigma_k^{\frac{1}{k}}(\lambda(D^2 u)) = e^{2u}, x \in \mathbb{R}^n,$$

has no subsolution in $\Phi^k(\mathbb{R}^n)$.

In Section 2 we will introduce some results on radial solutions as preliminaries, and the proof of the main theorem will be given in Section 3.

2. Preliminary results on radial solutions

We need some properties of radial functions in the proof of the main theorem. For $R > 0$, let $B_R := \{x \in \mathbb{R}^n : |x| < R\}$.

Lemma 2.1. *Assume $\varphi(r) \in C^2[0, R)$, with $\varphi'(0) = 0$. Then for $v(x) = \varphi(r)$, where $r = |x| < R$, we have that $v(x) \in C^2(B_R)$,*

$$(2.1) \qquad \lambda(D^2 v) = \begin{cases} (\varphi''(r), \dfrac{\varphi'(r)}{r}, \cdots, \dfrac{\varphi'(r)}{r}), r \in (0, R), \\ (\varphi''(0), \varphi''(0), \cdots, \varphi''(0)), r = 0, \end{cases}$$

and then

$$(2.2) \quad \sigma_k(\lambda(D^2 v)) = \begin{cases} \dbinom{n-1}{k-1}\varphi''(r)(\dfrac{\varphi'(r)}{r})^{k-1} + \dbinom{n-1}{k}(\dfrac{\varphi'(r)}{r})^k, r \in (0, R), \\ \dbinom{n}{k}(\varphi''(0))^k, r = 0. \end{cases}$$

Proof. It is well-known that for $x \neq 0$, $1 \leq i, j \leq n$,

$$(2.3) \qquad\qquad \frac{\partial v}{\partial x_i}(x) = (\frac{\varphi'(r)}{r})x_i,$$

$$(2.4) \qquad \frac{\partial^2 v}{\partial x_i \partial x_j}(x) = (\frac{\varphi''(r)}{r^2})x_i x_j - (\frac{\varphi'(r)}{r^3})x_i x_j + (\frac{\varphi'(r)}{r})\delta_{ij}.$$

By (2.3) and $\varphi'(0) = 0$, we have

$$\lim_{x \to 0} \frac{\partial v}{\partial x_i}(x) = \lim_{x \to 0}(\frac{\varphi'(r) - \varphi'(0)}{r - 0})x_i = \varphi''(0) \cdot 0 = 0.$$

Similarly, using (2.4) we have

$$\lim_{x \to 0} \frac{\partial^2 v}{\partial x_i \partial x_j}(x) = \lim_{x \to 0}((\varphi''(r) - \frac{\varphi'(r)}{r})\frac{x_i x_j}{r^2} + (\frac{\varphi'(r)}{r})\delta_{ij}) = \varphi''(0)\delta_{ij}.$$

Define

$$\frac{\partial v}{\partial x_i}(0) = 0, \frac{\partial^2 v}{\partial x_i \partial x_j}(0) = \varphi''(0)\delta_{ij}.$$

Then it is straightforward to show that $v(x) \in C^2(B_R)$.

Let

$$a = \begin{cases} \dfrac{\varphi''(r)}{r^2} - \dfrac{\varphi'(r)}{r^3}, r \in (0, R), \\ 0, r = 0, \end{cases}$$

$$b = \begin{cases} \dfrac{\varphi'(r)}{r}, r \in (0, R), \\ \varphi''(0), r = 0, \end{cases}$$

and I denote the unit matrix; then

$$(2.5) \qquad\qquad D^2 v(x) = a x^T x + bI.$$

By calculations of linear algebra we know that the eigenvalues of a matrix such as (2.5) is $(ar^2 + b, b, \cdots, b)$. Hence (2.1) is proved. Equation (2.2) can thus be obtained easily from the definition of σ_k. $\qquad \square$

In order to make the presentation simpler, if $\varphi'(0) = 0$, since

$$\lim_{r \to 0} \frac{\varphi'(r)}{r} = \varphi''(0),$$

we will always think

$$\frac{\varphi'(r)}{r}|_{r=0} = \varphi''(0)$$

in the following passages. For example, we can exchange (2.1) for

$$\lambda(D^2 v) = (\varphi''(r), \frac{\varphi'(r)}{r}, \cdots, \frac{\varphi'(r)}{r}), r \in [0, R).$$

Hence $v(x) = \varphi(r)$ is a radial solution of equation (1.7) if and only if $\varphi(r)$ is a solution of the ODE equation

$$(2.6) \qquad \binom{n-1}{k-1}\varphi''(r)(\frac{\varphi'(r)}{r})^{k-1} + \binom{n-1}{k}(\frac{\varphi'(r)}{r})^k = f^k(\varphi(r)).$$

Furthermore, the following fact will be used, too.

Lemma 2.2. *Let $f(t)$ be a continuous function defined on $\mathbb{R}$ and satisfying (1.11). For any positive number a, assume $\varphi(r) \in C^0[0, R) \cap C^1(0, R)$ is a solution of the Cauchy problem*

$$(2.7) \qquad \begin{cases} \varphi'(r) = \left(\dfrac{r^{k-n}}{C_0} \displaystyle\int_0^r s^{n-1} f^k(\varphi) ds \right)^{\frac{1}{k}}, & r > 0, \\ \varphi(0) = a, \end{cases}$$

where $C_0 = \dfrac{(n-1)!}{k!(n-k)!}$. Then $\varphi(r) \in C^2[0, R)$, and it satisfies equation (2.6) with $\varphi'(0) = 0$ and

$$(2.8) \qquad \lambda_r := (\varphi''(r), \frac{\varphi'(r)}{r}, \cdots, \frac{\varphi'(r)}{r}) \in \Gamma_k$$

for $0 \le r < R$.

Proof. It is easy to see that $\varphi(r) \in C^2(0, R)$. Since

$$0 \le r^{k-n} \int_0^r s^{n-1} f^k(\varphi) ds \le r^{k-1} \int_0^r f^k(\varphi) ds \to 0, r \to 0,$$

we have

$$\lim_{r \to 0} \frac{\varphi(r) - \varphi(0)}{r - 0} = \lim_{r \to 0} \varphi'(\xi) = \lim_{\xi \to 0} \left(\frac{\xi^{k-n}}{C_0} \int_0^\xi s^{n-1} f^k(\varphi) ds \right)^{\frac{1}{k}} = 0,$$

where $\xi = \xi(r) \in (0, r)$. Hence $\varphi'(0) = 0$, and $\varphi(r) \in C^1[0, R)$. One can see that

$$\lim_{r \to 0} \frac{\varphi'(r) - \varphi'(0)}{r - 0} = \lim_{r \to 0} \left(\frac{\int_0^r s^{n-1} f^k(\varphi) ds}{C_0 r^n} \right)^{\frac{1}{k}}$$

$$= \lim_{r \to 0} \left(\frac{r^{n-1} f^k(\varphi(r))}{n C_0 r^{n-1}} \right)^{\frac{1}{k}}$$

$$= \left(\binom{n}{k}^{-1} f^k(a) \right)^{\frac{1}{k}}.$$

Consequently $\varphi(r) \in C^2[0, R)$.

A direct calculation using (2.7) leads to

$$
\varphi''(r) = \frac{\varphi'(r)^{1-k}}{k} \left(\frac{(k-n)r^{k-n-1}}{C_0} \int_0^r s^{n-1} f^k(\varphi) ds + \frac{r^{k-1}}{C_0} f^k(\varphi(r)) \right)
$$

(2.9)

$$
= (1 - \frac{n}{k}) \frac{\varphi'(r)}{r} + \frac{1}{C_0 k} (\frac{\varphi'(r)}{r})^{1-k} f^k(\varphi(r)).
$$

By using (2.9), it is easy to verify that $\varphi(r)$ satisfies equation (2.6).

By (2.7), $\varphi'(r) \geq 0$. Since f and φ are both monotone non-decreasing, for $r \in (0, R)$,

$$
\varphi(r) \geq \varphi(0) = a > 0,
$$

and then

$$
f(\varphi(r)) \geq f(a) > 0.
$$

Hence, for $r \in [0, R)$, we know that $\dfrac{\varphi'(r)}{r} > 0$ and

$$
\sigma_k(\lambda_r) = \binom{n-1}{k-1} (\frac{\varphi'(r)}{r})^{k-1} \left(\varphi''(r) + \frac{n-k}{k} \frac{\varphi'(r)}{r} \right) = f^k(\varphi(r)) > 0,
$$

which means

$$
\varphi''(r) + \frac{n-k}{k} \frac{\varphi'(r)}{r} > 0,
$$

and then for $1 \leq l \leq k$,

$$
\begin{aligned}
\sigma_l(\lambda_r) &= \binom{n-1}{l-1} (\frac{\varphi'(r)}{r})^{l-1} \left(\varphi''(r) + \frac{n-l}{l} \frac{\varphi'(r)}{r} \right) \\
&\geq \binom{n-1}{l-1} (\frac{\varphi'(r)}{r})^{l-1} \left(\varphi''(r) + \frac{n-k}{k} \frac{\varphi'(r)}{r} \right) \\
&> 0.
\end{aligned}
$$

This implies that (2.8) is valid on $[0, R)$. $\qquad\square$

Finally we need to give a proof of the local existence of (2.7) near $r = 0$. The equipment we use is Euler's break line, and the process is similar to the proof of the classic ODE existence theorem (see [10]).

Lemma 2.3. *Let $f(t)$ be a continuous function defined on $\mathbb{R}$ and satisfying* (1.11). *For any positive number a, there exists a positive number R such that the Cauchy problem* (2.7) *has a solution in $[0, R]$.*

Proof. Define a functional $F(\cdot, \cdot)$ on

$$
\mathcal{R} := [0, l] \times \{\varphi \in C^2[0, l] : a - h < \varphi < a + h\}
$$

as

$$
F(r, \varphi) := \left(\frac{r^{k-n}}{C_0} \int_0^r s^{n-1} f^k(\varphi) ds \right)^{\frac{1}{k}},
$$

where l and h are small enough positive constants and C_0 is the same as in (2.7). Then (2.7) can be rewritten as

$$
\varphi'(r) = F(r, \varphi).
$$

It is easy to see that $F > 0$ for $r > 0$.

We defined a Euler's break line on $[0, l]$ as

$$(2.10) \qquad \begin{cases} \psi(0) = a, \\ \psi(r) = \psi(r_{i-1}) + F(r_{i-1}, \psi(r_{i-1}))(r - r_{i-1}), r_{i-1} < r \leq r_i, \end{cases}$$

where $0 = r_0 < r_1 < \cdots < r_m = l$.

Without loss of generality, we can assume that every point on Euler's break line that we defined above always lies in $\mathcal{R}$. Moreover, we can see from the following discussion that it lies below a straight line in $\mathcal{R}$. What we shall do is make sure that $\psi(r) < a + h$ for all $r \in [0, l]$, i.e. $(r, \psi) \in \mathcal{R}$.

In fact, for any $(r, \varphi) \in \mathcal{R}$, we have

$$(2.11) \qquad \begin{aligned} F(r, \varphi(r)) &\leq \left(\frac{r^{k-n}}{C_0} \int_0^r s^{n-1} ds \right)^{\frac{1}{k}} f(a+h) \\ &= \frac{r}{C} f(a+h) \\ &\leq \frac{l}{C} f(a+h). \end{aligned}$$

It implies

$$M := \max_{\mathcal{R}} F(r, \varphi) \leq \frac{l}{C} f(a+h).$$

Hence, for the break line (r, ψ), we have

$$0 \leq \psi(r) \leq a + Mr \leq a + \frac{l^2}{C} f(a+h).$$

Therefore, once h is fixed, we can choose l sufficiently small to make sure that

$$0 \leq \psi(r) < a + h.$$

In the next step, we will prove that Euler's break line ψ is an ε-approximation solution of (2.7). To do this, we only need to prove that for any small $\epsilon > 0$, we can choose appropriate points $\{r_i\}_{i=1,\ldots,m}$ to make the break line satisfy

$$(2.12) \qquad |\frac{d\psi(r)}{dr} - F(r, \psi(r))| < \varepsilon, r \in [0, l].$$

As a matter of fact, by (2.11) it is easy to see that

$$\lim_{r \to 0} F(r, \varphi) = 0$$

is valid uniformly for any $\varphi \in C^2[0, l], a - h < \varphi < a + h$. Then for each $\varepsilon > 0$, there exists $\overline{r} \in (0, l)$ such that for $0 \leq r < \overline{r}$, we have

$$F(r, \psi) < \frac{\varepsilon}{2}.$$

Then

$$|\frac{d\psi(r)}{dr} - F(r, \psi(r))| = |F(r_{i-1}, \psi(r_{i-1})) - F(r, \psi(r))| < \varepsilon.$$

For $\overline{r} \leq r \leq l$,

$$
\begin{aligned}
&|\frac{d\psi(r)}{dr} - F(r, \psi(r))| \\
&= |F(r_{i-1}, \psi(r_{i-1})) - F(r, \psi(r))| \\
&= C| \left(r_{i-1}^{k-n} \int_0^{r_{i-1}} s^{n-1} f^k(\psi) ds \right)^{\frac{1}{k}} - \left(r^{k-n} \int_0^r s^{n-1} f^k(\psi) ds \right)^{\frac{1}{k}} | \\
\text{(2.13)} \quad &\leq C|r_{i-1}^{k-n} \int_0^{r_{i-1}} s^{n-1} f^k(\psi) ds - r^{k-n} \int_0^r s^{n-1} f^k(\psi) ds|^{\frac{1}{k}} \\
&\leq C \left(|r_{i-1}^{k-n} - r^{k-n}| \int_0^{r_{i-1}} s^{n-1} f^k(\psi) ds + r^{k-n} \int_{r_{i-1}}^r s^{n-1} f^k(\psi) ds \right)^{\frac{1}{k}} \\
&\leq C \left(|r_{i-1}^{k-n} - r^{k-n}| l^n f^k(a+h) + \overline{r}^{k-n} (r^n - r_{i-1}^n) f^k(a+h) \right)^{\frac{1}{k}}.
\end{aligned}
$$

Since functions r^{k-n} and r^n are both Liptchitz continuous on $[\overline{r}, l]$, for the above ε there exists $\delta(\varepsilon) > 0$ such that

$$
\text{(2.14)} \quad
\begin{aligned}
&|r'^{k-n} - r''^{k-n}| < \left(l^n f^k(a+h) \right)^{-1} (\frac{\varepsilon}{2C})^k, \\
&|r'^n - r''^n| < \left(\overline{r}^{k-n} f^k(a+h) \right)^{-1} (\frac{\varepsilon}{2C})^k,
\end{aligned}
$$

where $r', r'' \in [\overline{r}, l]$ and $|r' - r''| < \delta(\varepsilon)$.

Noting that $\delta(\varepsilon)$ is independent of the definition of ψ, we can assume $r_1 = \overline{r}$ and

$$
\max_{2 \leq i \leq m} |r_{i-1} - r_i| < \min\{\overline{r}, \delta(\varepsilon)\};
$$

then we get (2.12).

Thus, Euler's break line ψ is an ε-approximation solution of (2.7).

The next step is to find a solution of (2.7) by the Euler break line we defined. Assume $\{\varepsilon_j\}_{j=1}^\infty$ is a positive constant sequence converging to 0. For ε_j, there is an ε_j-approximation solution ψ_j on $[0, l]$, defined as above. It is easy to know that

$$
|\psi_j(r') - \psi_j(r'')| \leq M|r' - r''|,
$$

where $r', r'' \in [0, l]$. That is to say, $\{\psi_j\}$ is equicontinuous and uniformly bounded. Therefore by the Ascoli-Arzela Lemma, we can find a uniformly convergent subsequence, still denoted as $\{\psi_j\}$, without loss of generality.

Assume $\lim_{j \to \infty} \psi_j = \varphi$. Then $\varphi(0) = a$, and $\varphi'(0) = 0$.

Since ψ_j is an ε_j-approximation solution, we have

$$
\text{(2.15)} \quad \frac{d\psi_j(r)}{dr} = F(r, \psi_j(r)) + \triangle_j(r),
$$

where $|\triangle_j(r)| < \varepsilon_j$, for $r \in [0, l]$. Integrating (2.15) from 0 to $r(\leq l)$, we have

$$
\psi_j(r) = a + \int_0^r F(s, \psi_j(s)) ds + \int_0^r \triangle_j(s) ds.
$$

Let $j \to \infty$,

$$
\text{(2.16)} \quad
\begin{aligned}
\varphi(r) &= a + \lim_{j \to \infty} \left(\int_0^r F(s, \psi_j(s)) ds + \int_0^r \triangle_j(s) ds \right) \\
&= a + \int_0^r F(s, \varphi(s)) ds.
\end{aligned}
$$

Since ψ_j is continuous, we know that φ is continuous. By (2.16), φ is continuously differentiable. Differentiating (2.16), we can see that φ satisfies equation (2.7) in $[0, l]$. $\square$

In fact, a local solution also exists for any real number a if we do not consider only the positive ones. Once a is positive, it is easy to know the solution φ is positive, too.

3. Proof of the main theorem

We will prove the main theorem by three lemmas:

Lemma 3.1. *Let $f(t)$ be a monotone non-decreasing continuous function defined on $\mathbb{R}$. Suppose there exists a function $\varphi(r) \in C^2[0, R)$ satisfying (2.6) and (2.8) for $r \in [0, R)$, with $\varphi'(0) = 0$ and $\varphi(r) \to \infty$ as $r \to R$. Then if $u(x) \in \Phi^k(\mathbb{R}^n)$ is a positive subsolution of (1.7), we have $u(x) \le \varphi(|x|)$ at each point in B_R.*

Proof. Let $v(x) = \varphi(r)$, and then by Lemma 2.1, we know $\lambda(D^2 v(x)) \in \Gamma_k$ for $x \in B_R$. Therefore $v(x) \in \Phi^k(B_R)$.

Let $L[w] = \sigma_k^{\frac{1}{k}}(\lambda(D^2 w)) - f(w)$. If $u > v$ somewhere, then there is some constant $a > 0$ such that $u - a$ touches v from below, which means $u - a - v \le 0$ in B_R. Suppose $u - a$ touches v at some interior point x_0 in B_R. Then there is $R' \in (0, R)$ such that $x_0 \in B_{R'}$. Since $v(x) = \varphi(|x|) \to \infty$ as $x \to \partial B_R$ and u is bounded in B_R, we can assume $\sup_{\partial B_{R'}}(u - a - v) < 0$.

It follows from (1.11) that in $B_{R'}$

$$
\begin{aligned}
L[u - a] &= \sigma_k^{\frac{1}{k}}(\lambda(D^2(u - a))) - f(u - a) \\
&= (\sigma_k^{\frac{1}{k}}(\lambda(D^2 u)) - f(u)) + (f(u) - f(u - a)) \\
&\ge 0 = L[v].
\end{aligned}
$$

Now $u - a$ is a subsolution and v is a solution (with respect to L). By the maximum principle,

$$
0 = \sup_{B_{R'}}(u - a - v) = \sup_{\partial B_{R'}}(u - a - v) < 0,
$$

which is impossible. $\square$

Lemma 3.2. *Let the continuous function $f(t)$ satisfy (1.11) on $\mathbb{R}$. Then equation (1.7) has a positive subsolution $u \in \Phi^k(\mathbb{R}^n)$ if and only if the Cauchy problem (2.7) has a positive solution $\varphi(r) \in C^2[0, \infty)$ for some positive number a.*

Proof. First, the sufficient condition is obvious. If there exists such a solution $\varphi(r)$ of (2.7) for $R = +\infty$, let $v(x) = \varphi(|x|)$. By Lemma 2.1 and Lemma 2.2, $\sigma_k(\lambda(D^2 v(x))) = f^k(v(x))$ and $\lambda(D^2 v(x)) \in \Gamma_k$ for $x \in \mathbb{R}^n$. Thus $v(x) \in \Phi^k(\mathbb{R}^n)$ is a required solution of (1.7).

Next, we will prove the necessary condition. On the contrary, suppose that no such function $\varphi(r)$ exists globally. By Lemma 2.3, for any positive number a, (2.7) has a positive solution $\varphi(r)$ on some interval which cannot be a global solution. Hence we assume $[0, R)$ is the maximal interval in which the solution exists. Since $\varphi'(r) > 0$ for $r > 0$, we know $\varphi(r) \to \infty$ as $r \to R$. Then $\varphi(|x|)$ satisfies (2.6) and (2.8) in B_R by Lemma 2.2. By Lemma 3.1, any positive solution $u(x) \in \Phi^k(\mathbb{R}^n)$ of (1.7) would satisfy $u(x) \le \varphi(|x|)$ for $x \in B_R$. In particular we

have $u(0) \le \varphi(0) = a$. However, since a is arbitrary, we take $a = \dfrac{u(0)}{2}$ for granted and obtain a contradiction, which means the necessary condition holds. $\qquad\square$

Lemma 3.3. *If $f(t)$ is a continuous function defined on $\mathbb{R}$ and satisfies* (1.11), *then the Cauchy problem* (2.7) *has a positive solution $\varphi(r) \in C^2[0,\infty)$ for some positive number a if and only if* (1.12) *holds.*

Proof. First, we will prove the sufficient condition. Suppose no such solution of (2.7) exists. As in the proof of Lemma 3.2, the problem (2.7) has a solution $\varphi(r)$ with $a = 1$, valid on the maximal existence interval $[0, R)$, and $\varphi(r) \to \infty$ as $r \to R$. By Lemma 2.2, we know that φ satisfies equation (2.6).

For $0 < r < R$, since $\varphi'(r) > 0$, by (2.6), we have

$$\binom{n-1}{k-1}\varphi''(r)(\varphi'(r))^{k-1} < r^{k-1}f^k(\varphi(r)) \le R^{k-1}f^k(\varphi(r)).$$

Multiplying by $\varphi'(r)$ on both sides, we get

$$(3.1) \qquad \varphi''(r)(\varphi'(r))^k < CR^{k-1}f^k(\varphi(r))\varphi'(r).$$

Here the constant C depends only on n and k. Moreover, to simplify the presentation, in the sequel we will use C to denote some constant that depends only on n and k unless we mention its value specifically. Integrating (3.1) from 0 to r and using $\varphi'(0) = 0$ and $\varphi(0) = 1$, we get

$$(\varphi'(r))^{k+1} < CR^{k-1}\int_1^{\varphi(r)} f^k(s)ds,$$

i.e.

$$(3.2) \qquad \left(\int_1^\varphi f^k(s)ds\right)^{-\frac{1}{k+1}} d\varphi < CR^{\frac{k-1}{k+1}}dr.$$

Integrating (3.2) on r from 0 to R and using the fact that $\varphi(0) = 1$ and $\varphi(R) = \infty$, we have

$$\int_1^\infty \left(\int_1^\varphi f^k(s)ds\right)^{-\frac{1}{k+1}} d\varphi < CR^{\frac{2k}{k+1}} < \infty.$$

This contradicts (1.12).

Second, we will prove the necessary condition. On the contrary, suppose

$$(3.3) \qquad \int^\infty \left(\int_0^t f^k(s)ds\right)^{-\frac{1}{k+1}} dt < +\infty.$$

Set

$$g(t) = \left(\int_0^t f^k(s)ds\right)^{-\frac{1}{k+1}}.$$

Since $\displaystyle\int^\infty g(t)dt < \infty$, we have $\displaystyle\int_s^\infty g(t)dt \to 0$ as $s \to \infty$. Observe furthermore that g is decreasing in $(0,\infty)$. Therefore

$$tg(t) \le 2\int_{\frac{t}{2}}^t g(s)ds < 2\int_{\frac{t}{2}}^\infty g(s)ds \to 0,$$

as $t \to \infty$. Thus as $t \to \infty$, we have

$$(3.4) \qquad t^{-(k+1)} \int_0^t f^k(s)ds \to \infty.$$

Since f is non-decreasing,

$$(3.5) \qquad t^{-(k+1)} \int_0^t f^k(s)ds \le t^{-(k+1)}(t-0)f^k(t) = (\frac{f(t)}{t})^k.$$

By (3.4) and (3.5), we have

$$(3.6) \qquad \frac{f(t)}{t} \to \infty,$$

as $t \to \infty$. Hence there exists $t_1 > 0$ such that $f(t) > t - \varphi(0)$ for $t > t_1$. Since $\varphi(r) > \varphi(0) = a$, f is positive and non-decreasing in $(0, \infty)$, and φ satisfies equation (2.6), we have

$$C_0 \left((\varphi'(r))^k r^{n-k} \right)' = r^{n-1} f^k(\varphi(r)) \ge r^{n-1} f^k(a).$$

After integrating, we have

$$(\varphi'(r))^k r^{n-k} \ge C r^n,$$

i.e.

$$(\varphi'(r))^k \ge C r^k.$$

Hence

$$\varphi(r) \ge C r^2 + a.$$

There exists $r_1 > 0$ such that $\varphi(r) > t_1$ for $r > r_1$. Therefore, for $r > r_1$, we have

$$(3.7) \qquad f(\varphi(r)) > \varphi(r) - a.$$

Then by (2.6),

$$\binom{n-1}{k-1} \varphi''(r)(\varphi'(r))^k < r^{k-1} f^k(\varphi(r))\varphi'(r),$$

which comes to

$$\binom{n-1}{k-1}(\varphi'(r))^{k+1} < (k+1)\int_0^r s^{k-1} f^k(\varphi(s))\varphi'(s)ds$$

$$< (k+1)r^{k-1}\int_0^r f^k(\varphi(s))\varphi'(s)ds.$$

This, together with (3.7), implies that

$$\binom{n-1}{k-1}(\frac{\varphi'(r)}{r})^{k+1} < \frac{k+1}{r^2}\int_a^{\varphi(r)} f^k(t)dt$$

$$< \frac{k+1}{r^2}(\varphi(r) - a)f^k(\varphi(r))$$

$$< \frac{k+1}{r^2} f^{k+1}(\varphi(r)).$$

Therefore, there exists $r_2 > \max\{r_1, 1\}$ such that for $r > r_2$ we have

$$(3.8) \qquad \binom{n-1}{k}(\frac{\varphi'(r)}{r})^k < \frac{1}{2} f^k(\varphi(r)).$$

By (2.6) and (3.8), we have

$$\binom{n-1}{k-1} \varphi''(r)(\frac{\varphi'(r)}{r})^{k-1} > \frac{1}{2} f^k(\varphi(r)).$$

Then for $r > r_2 > 1$,

$$\varphi''(r)(\varphi'(r))^{k-1} > \frac{1}{2}\binom{n-1}{k-1}^{-1} r^{k-1} f^k(\varphi(r)) > \frac{1}{2}\binom{n-1}{k-1}^{-1} f^k(\varphi(r)).$$

Similar to the discussion above, we will use C to denote some constant which depends only on n, k in the sequel. Since

$$\varphi''(r)(\varphi'(r))^k > C f^k(\varphi(r))\varphi'(r),$$

it is easy to know

$$(\varphi'(r))^{k+1} > C \int_a^{\varphi(r)} f^k(t)dt,$$

i.e.

$$\left(\int_a^\varphi f^k(t)dt\right)^{-\frac{1}{k+1}} d\varphi > Cdr.$$

Hence, we have

$$(3.9) \quad \int_{\varphi(r_2)}^\infty \left(\int_a^t f^k(s)ds\right)^{-\frac{1}{k+1}} dt \geq \int_{\varphi(r_2)}^\varphi \left(\int_a^t f^k(s)ds\right)^{-\frac{1}{k+1}} dt > C(r - r_2).$$

For $t > 2a$, we have

$$\int_0^a f^k(s)ds \leq af^k(a) < \frac{t}{2}f^k\left(\frac{t}{2}\right) \leq \int_{\frac{t}{2}}^t f^k(s)ds.$$

Then

$$(3.10) \quad \int_0^{\frac{t}{2}} f^k(s)ds < \int_a^t f^k(s)ds.$$

By (3.9) and (3.10), we have

$$
\begin{aligned}
(3.11) \qquad C(r - r_2) &< \int_{\varphi(r_2)}^\infty \left(\int_a^t f^k(s)ds\right)^{-\frac{1}{k+1}} dt \\
&< \int_{\varphi(r_2)}^\infty \left(\int_0^{\frac{t}{2}} f^k(s)ds\right)^{-\frac{1}{k+1}} dt \\
&= 2\int_{\frac{\varphi(r_2)}{2}}^\infty \left(\int_0^t f^k(s)ds\right)^{-\frac{1}{k+1}} dt.
\end{aligned}
$$

Since r can be arbitrarily large, we get a contradiction between (3.3) and (3.11), which completes the proof. $\qquad \square$

We give a brief proof of Theorem 1.1.

Proof of Theorem 1.1. If $f(t)$ is a continuous function defined on $\mathbb{R}$ and satisfies (1.11), then by Lemma 3.2, equation (1.7) has a positive subsolution $u \in \Phi^k(\mathbb{R}^n)$ if and only if the Cauchy problem (2.7) has a positive solution $\varphi(r) \in C^2[0, \infty)$ for some positive number a, which is equivalent to condition (1.12) by Lemma 3.3.

The proof of Theorem 1.3 is quite similar to that of Theorem 1.1. Most of the properties we need are almost the same as we have discussed. Since f is now a positive, continuous and monotone non-decreasing function defined on $\mathbb{R}$, we don't need a to be positive in Lemma 2.2, Lemma 2.3 and Lemma 3.2 to get similar conclusions. $\qquad \square$

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