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Multi-valued solutions to fully nonlinear uniformly elliptic equations ☆

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ABSTRACT

In this paper, we discuss the existence and regularity of multi-valued viscosity solutions to fully nonlinear uniformly elliptic equations. We use the Perron method to prove the existence of bounded multi-valued viscosity solutions.

1. Introduction

In this paper, we study the multi-valued solutions of fully nonlinear uniformly elliptic equations

$$F\left(D^2u\right) = f(x),$$

where F is a map from $\mathbb{S}^{n\times n}$ to $\mathbb{R}$, $\mathbb{S}^{n\times n}$ denotes the space of $n\times n$ real symmetric matrix and $f(x)$ is a known function. Also, $F \in C^2(\mathbb{S}^{n\times n})$ satisfies the following conditions:

(i) F is uniformly elliptic, i.e., there exist positive constants λ, Λ such that for any $r \in \mathbb{S}^{n\times n}$, $\xi \in \mathbb{R}^n$,

$$\lambda|\xi|^2 \leqslant F_{ij}(r)\xi_i\xi_j \leqslant \Lambda|\xi|^2,$$

where $F_{ij} = \partial F/\partial r_{ij}$.

(ii) F is concave for r, i.e., F is a concave function in $\mathbb{S}^{n\times n}$.

Typical examples of fully nonlinear uniformly elliptic equations satisfying (i), (ii) are the following Pucci equation and Bellman equation.

1. Pucci equation:

$$\mathcal{M}^-\left(D^2u\right) = f(x), \qquad \mathcal{M}^+\left(D^2u\right) = f(x),$$

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where

$$\mathcal{M}^-(r,\lambda,\Lambda)=\mathcal{M}^-(r)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i,$$

$$\mathcal{M}^+(r,\lambda,\Lambda)=\mathcal{M}^+(r)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i,$$

$r\in\mathbb{S}^{n\times n}$ and $e_i=e_i(r)$ are the eigenvalues of r, $i=1,2,\dots,n$.

2. Bellman equation in stochastic control theory:

$$F\big(D^2u\big):=\inf_{\alpha\in\mathcal{A}}\big\{a_\alpha^{ij}D_{ij}u\big\}=f(x),$$

where $\mathcal{A}$ is any set, and $a_\alpha^{ij}\in\mathbb{R}$ satisfies

$$\lambda|\xi|^2\leqslant a_\alpha^{ij}\xi_i\xi_j\leqslant\Lambda|\xi|^2,\quad \xi\in\mathbb{R}^n,\ \alpha\in\mathcal{A}.$$

From the theory of analytic functions, we know the typical two-dimensional examples of multi-valued harmonic functions are

$$u_1(z)=Re\big(z^{\frac{1}{k}}\big),\quad z\in\mathbb{C}\backslash\{0\},$$
$$u_2(z)=Arg(z),\quad z\in\mathbb{C}\backslash\{0\},$$

and

$$u_3(z)=Re\big(\sqrt{(z-1)(z+1)}\big),\quad z\in\mathbb{C}\backslash\{\pm1\}.$$

By 1970s, Almgren [1] had realized that a minimal variety near a multiplicity-k disc could be well approximated by the graph of a multi-valued function minimizing a suitable analog of the ordinary Dirichlet integral. Many facts about harmonic functions are also true for these Dirichlet minimizing multi-valued functions. Evans [9–11], Levi [18] and Caffarelli [2,3] all studied the multi-valued harmonic functions. Evans [10] proved that the conductor potential of a surface with minimal capacity was a double-valued harmonic function. In [3], Caffarelli proved the Hölder continuity of the multi-valued harmonic functions.

At the beginning of this century, the multi-valued solutions of Eikonal equation were considered in [15,13], respectively. Later, Jin et al. provided a level set method for the computation of multi-valued geometric solutions to general quasilinear PDEs and multi-valued physical observables to semiclassical limit of Schrödinger equations, see [17,16].

In 2006, Caffarelli and Li investigated the multi-valued solutions of Monge–Ampère equations in [6] where they first introduced the geometric situation of multi-valued solutions and then obtained the existence, boundedness, regularity and the asymptotic behavior at infinity of multi-valued viscosity solutions. The multi-valued solutions for Dirichlet problem of Monge–Ampère equations on exterior planar domains were discussed by Ferrer, Martínez and Milán in [12] using the complex variable methods. Recently, the multi-valued solutions of Hessian equations have been studied in [8] and [7].

The geometric situation of the multi-valued functions was given in [6]. Let D be a bounded regular domain in $\mathbb{R}^n$, and let $\Sigma\subset D$ be homeomorphic in $\mathbb{R}^n$ to an $(n-1)$-dimensional closed disc, i.e., there exists a homeomorphism $\psi:\mathbb{R}^n\to\mathbb{R}^n$ such that $\psi(\Sigma)$ is an $(n-1)$-dimensional closed disc. Let $\Gamma=\partial\Sigma$, the boundary of Σ. Thus Γ is homeomorphic to an $(n-2)$-dimensional sphere for $n\geqslant3$.

Let $\mathbb{Z}$ be the set of integers and

$$M=(D\backslash\Gamma)\times\mathbb{Z}$$

denote a covering of $D\backslash\Gamma$ with the following standard parameterization: fixing an $x^*\in D\backslash\Gamma$, and connecting x^* by a smooth curve in $D\backslash\Gamma$ to a point x in $D\backslash\Gamma$. If the curve goes through Σ in the positive direction (fixing such a direction) $m\geqslant0$ times, then we arrive at (x,m) in M. If the curve goes through Σ in the negative direction $m\geqslant0$ times, then we arrive at $(x,-m)$ in M.

For $k=2,3,\dots$, we introduce an equivalence relation "$\sim k$" on M as follows: (x,m) and (y,j) in M are "$\sim k$" equivalent if $x=y$ and $m-j$ is an integer multiple of k. We let

$$M_k:=M/\sim k,$$

denote the k-sheet cover of $D\backslash\Gamma$, and let

$$\partial' M_k:=\bigcup_{m=1}^{k}\big(\partial D\times\{m\}\big).$$

For $n = 2$, we can understand the covering space M_k more clearly from the above example u_3. In this example, $\Gamma = \{1, -1\}$ and Σ is the interval $(-1, 1)$. Each time the point z goes around -1 or 1, it crosses the interval $(-1, 1)$ one time.

We define a distance in M_k as follows: For any $(x, m), (y, j) \in M_k$, let $l((x, m), (y, j))$ denote a smooth curve in M_k which connects (x, m) and (y, j), and let $|l((x, m), (y, j))|$ denote its length. Define

$$d\big((x, m), (y, j)\big) = \inf_l \big|l\big((x, m), (y, j)\big)\big|,$$

where the inf is taken over all smooth curves in M_k connecting (x, m) and (y, j). Then $d((x, m), (y, j))$ is a distance.

Definition 1.1. We call a function u is continuous at (x, m) in M_k if

$$\lim_{d((y,j),(x,m))\to 0} u(y, j) = u(x, m),$$

and $u \in C^0(M_k)$ if for any $(x, m) \in M_k$, u is continuous at (x, m).

Similarly we can define $u \in C^\alpha(M_k)$, $C^{0,1}(M_k)$ and $C^2(M_k)$, etc.

To the best of our knowledge, there isn't any result of multi-valued solutions to fully nonlinear uniformly elliptic equations. In this paper, we study the multi-valued solutions to Dirichlet problem:

$$F\big(D^2u\big) = f(x, m), \quad (x, m) \in M_k, \tag{1.1}$$

$$u = \varphi_m(x), \quad (x, m) \in \partial' M_k, \tag{1.2}$$

where

(H_1) $f \in C^0(M_k)$ and for some constants a and b, $a \leqslant f \leqslant b$,
(H_2) $\varphi_1, \ldots, \varphi_k \in C^0(\partial D)$.

To state our results, we recall the definition of viscosity solutions, see [4,5].

Definition 1.2. A function $u \in C^0(M_k)$ is called a viscosity subsolution of (1.1), if for any $(y, m) \in M_k$ and any function $\xi \in C^2(M_k)$ satisfying

$$u(x, m) \leqslant \xi(x, m), \quad (x, m) \in M_k \quad \text{and} \quad u(y, m) = \xi(y, m),$$

we have

$$F\big(D^2\xi(y, m)\big) \geqslant f(y, m).$$

A function $u \in C^0(M_k)$ is called a viscosity supersolution of (1.1), if for any $(y, m) \in M_k$ and any function $\xi \in C^2(M_k)$ satisfying

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we have

$$F\big(D^2\xi(y, m)\big) \leqslant f(y, m).$$

A function $u \in C^0(M_k)$ is called a viscosity solution of (1.1), if u is both a viscosity subsolution and a viscosity supersolution of (1.1).

A function $u \in C^0(M_k \cup \partial' M_k)$ is called a viscosity subsolution (supersolution, solution) of (1.1), (1.2), if u is a viscosity subsolution (supersolution, solution) of (1.1) and satisfies $u(x, m) \leqslant (\geqslant, =) \varphi_m(x)$ on $\partial' M_k$ for $m = 1, 2, \ldots, k$.

Our main result is as follows. Using the Perron method and the properties of Pucci operators, we obtain the existence of bounded multi-valued viscosity solutions of (1.1), (1.2).

Theorem 1.1. *If the conditions* (H_1) *and* (H_2) *hold, then the Dirichlet problem* (1.1), (1.2) *has at least a bounded viscosity solution* $u \in C^0(M_k \cup \partial' M_k)$. *Furthermore, if* $f \in C^\alpha(M_k)$, *we have* $u \in C^{2,\alpha}(M_k)$.

This paper is arranged as follows. In Section 2, we derive some lemmas for single-valued solutions to fully nonlinear uniformly elliptic equations. The existence of bounded multi-valued solutions is proved in Section 3.

2. Preliminaries

For the reader's convenience, we prove a few lemmas about single-valued solutions which will be used later. We shall always assume D is a bounded regular domain in $\mathbb{R}^n$.

Substituting $f(x)$ with $f(x) - F(0)$, we may assume $F(0) = 0$. From [4], we know that for any $M \in \mathbb{S}^{n\times n}$,

$$F(M) \leqslant Trace(AM),$$

where $A \in \mathbb{S}^{n\times n}$ and all the eigenvalues of A belong to $[\lambda, \Lambda]$. Hence we can make a linear change of space variables such that in the new variables,

$$Trace(AM) = Trace(M).$$

Therefore, for any $\varphi \in C^2$, we may suppose that

$$F\big(D^2\varphi\big) \leqslant \Delta\varphi. \tag{2.1}$$

Lemma 2.1. *Let $D' \Subset D$ be any open set, $V \in C^0(\overline{D})$, $\varphi \in C^0(\partial D)$ and c_0 be any constant. Then there exists a function $\underline{u} \in C^0(\overline{D}) \cap C^2(D)$ satisfying*

$$\begin{aligned} &F\big(D^2\underline{u}\big) \geqslant c_0, \quad x \in D,\\ &\underline{u} = \varphi, \quad x \in \partial D,\\ &\underline{u} \leqslant V, \quad x \in D'. \end{aligned}$$

Proof. From [4], let $\tilde{\varphi} \in C^{2,\alpha}(D) \cap C^0(\overline{D})$ satisfy

$$\begin{aligned} &\mathcal{M}^-\big(D^2\tilde{\varphi}\big) = 1, \quad x \in D,\\ &\tilde{\varphi} = \varphi(x), \quad x \in \partial D, \end{aligned}$$

and let $\rho \in C^{2,\alpha}(\overline{D})$ satisfy

$$\begin{aligned} &\mathcal{M}^-\big(D^2\rho\big) = 1, \quad x \in D,\\ &\rho = 0, \quad x \in \partial D. \end{aligned}$$

Define in D, $\underline{u}(x) = \tilde{\varphi}(x) + \mu\rho(x)$, where μ is a positive constant to be determined. Then $\underline{u} \in C^2(D) \cap C^0(\overline{D})$ and

$$\underline{u}(x) = \varphi(x), \quad x \in \partial D. \tag{2.2}$$

On the other hand, for any function $v \in C^2(D)$, we have

$$\begin{aligned} F\big(D^2 v\big) &= F\big(D^2 v\big) - F(0)\\ &= F_{ij}\big(\theta D^2 v\big) D_{ij} v\\ &\geqslant \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i\\ &= \mathcal{M}^-\big(D^2 v\big), \end{aligned}$$

where $0 < \theta < 1$ and e_i are the eigenvalues of $D^2 v$. Therefore, by the concavity of $\mathcal{M}^-$

$$\begin{aligned} F\big(D^2\underline{u}\big) &= F\big(D^2\tilde{\varphi} + \mu D^2\rho\big)\\ &\geqslant \mathcal{M}^-\big(D^2\tilde{\varphi} + \mu D^2\rho\big)\\ &\geqslant \mathcal{M}^-\big(D^2\tilde{\varphi}\big) + \mathcal{M}^-\big(\mu D^2\rho\big)\\ &= 1 + \mu. \end{aligned}$$

Then we can choose $\mu = \mu(c_0)$ such that

$$F\big(D^2\underline{u}\big) \geqslant c_0, \quad x \in D. \tag{2.3}$$

By the strong maximum principle, there exists a constant $\rho_0 > 0$ such that $\rho \leqslant -\rho_0$ in D'. Thus we can again choose $\mu = \mu(c_0, \rho_0, \tilde{\varphi}, V, D')$ large such that

$$\underline{u} = \tilde{\varphi} + \mu\rho \leqslant \sup_{D'} \tilde{\varphi} - \mu\rho_0 \leqslant \inf_{D'} V \leqslant V, \quad x \in D'. \tag{2.4}$$

From (2.2), (2.3), (2.4), we know that the lemma holds. □

Lemma 2.2. *Let $\Omega \Subset D$ be an open set and $f \in C^0(D)$. Suppose that $v \in C^0(\overline{\Omega})$ and $u \in C^0(D)$ satisfy respectively*

$$F\left(D^2 v\right) \geqslant f(x), \quad x \in \Omega,$$
$$F\left(D^2 u\right) \geqslant f(x), \quad x \in D.$$

Moreover,

$$u \leqslant v, \quad x \in \overline{\Omega},$$
$$u = v, x \quad \in \partial\Omega. \tag{2.5}$$

Set

$$w(x) = \begin{cases} v(x), & x \in \Omega, \\ u(x), & x \in \overline{D} \setminus \Omega. \end{cases}$$

Then $w \in C^0(D)$ satisfies in the viscosity sense

$$F\left(D^2 w\right) \geqslant f(x).$$

Proof. For any $x_0 \in D$, $\xi \in C^2(D)$ satisfying $w(x_0) = \xi(x_0)$,

$$w(x) \leqslant \xi(x), \quad x \in D, \tag{2.6}$$

by the definition of w, if $x_0 \in \Omega$, we have

$$v(x_0) = \xi(x_0), \qquad v(x) \leqslant \xi(x), \quad x \in \Omega.$$

Then

$$F\left(D^2 \xi(x_0)\right) \geqslant f(x_0).$$

If $x_0 \in D \setminus \Omega$, we have

$$u(x_0) = \xi(x_0), \qquad u(x) \leqslant \xi(x), \quad x \in D \setminus \Omega.$$

From (2.5), (2.6),

$$u(x) \leqslant \xi(x), \quad x \in D.$$

Hence

$$F\left(D^2 \xi(x_0)\right) \geqslant f(x_0).$$

Lemma 2.2 is proved. □

The following lemma may be a known result. But for the readers convenience, we give the lemma and its proof.

Lemma 2.3. *Let $f \in C^0(\overline{D})$. Suppose that $\underline{u} \in C^0(\overline{D})$ satisfies in the viscosity sense*

$$F\left(D^2 \underline{u}\right) \geqslant f(x), \quad x \in D,$$

then the Dirichlet problem

$$F\left(D^2 u\right) = f(x), \quad x \in D, \tag{2.7}$$
$$u = \underline{u}(x), \quad x \in \partial D \tag{2.8}$$

has a viscosity solution $u \in C^0(\overline{D})$.

Proof. By Theorem 3.4 in [14], if (2.7), (2.8) has a viscosity subsolution g and a viscosity supersolution h satisfying $g \leqslant h$ in D, we know that (2.7), (2.8) has a viscosity solution.

Clearly, $\underline{u}$ is a viscosity subsolution of (2.7), (2.8). So we only need to prove that (2.7), (2.8) has a viscosity supersolution $\bar{u}$ satisfying $\bar{u} \geqslant \underline{u}$ in D.

Let $f_0 = \inf_D f(x)$ and $\bar{u} \in C^2(D) \cap C^0(\overline{D})$ satisfy

$$\Delta\bar{u} = f_0, \quad x \in D,$$
$$\bar{u} = \underline{u}, \quad x \in \partial D.$$

Then $\bar{u}$ satisfies in the viscosity sense

$$F\big(D^2\bar{u}\big) \leqslant f(x), \quad x \in D. \tag{2.9}$$

Indeed, for any $x_0 \in D$, $\xi \in C^2(D)$ satisfying

$$\bar{u}(x_0) = \xi(x_0), \qquad \bar{u}(x) \geqslant \xi(x), \quad x \in D,$$

we have

$$D^2(\bar{u} - \xi)(x_0) \geqslant 0.$$

Therefore

$$F\big(D^2\bar{u}(x_0)\big) \geqslant F\big(D^2\xi(x_0)\big).$$

By (2.1), we have $F(D^2\bar{u}(x_0)) \leqslant \Delta\bar{u}(x_0)$. Hence

$$F\big(D^2\xi(x_0)\big) \leqslant F\big(D^2\bar{u}(x_0)\big) \leqslant f_0 \leqslant f(x_0),$$

and then (2.9) holds. Consequently $\bar{u}$ is a viscosity supersolution of (2.7), (2.8). From the comparison principle, $\bar{u} \geqslant \underline{u}$ in D. The lemma is proved. □

3. Existence of bounded multi-valued solutions

In this section, we prove Theorem 1.1. We first introduce a comparison principle in M_k, see [6].

Lemma 3.1. *Let $u, \upsilon \in C^0(M_k) \cap L^\infty(M_k)$ satisfy $\Delta u \geqslant 0 \geqslant \Delta\upsilon$ in M_k and*

$$\liminf_{dist((x,m),\partial' M_k)\to 0} \big(u(x,m) - \upsilon(x,m)\big) \leqslant 0,$$

then $u \leqslant \upsilon$ in M_k.

Proof of Theorem 1.1. We divide the proof into three steps.

Step 1. We construct a viscosity subsolution of (1.1), (1.2).

Let $P \in C^2(D) \cap C^0(\overline{D})$ satisfy

$$F\big(D^2P\big) = b, \quad x \in D,$$
$$P = c, \quad x \in \partial D, \tag{3.1}$$

where $a \leqslant f \leqslant b$ and

$$c = \inf_{x\in\partial D} \min_{1\leqslant m\leqslant k} \big\{\varphi_m(x)\big\}.$$

By Lemma 2.1, we know that there exist functions $\underline{u}_1, \ldots, \underline{u}_k \in C^0(\overline{D}) \cap C^2(D)$ satisfying

$$F\big(D^2\underline{u}_m\big) \geqslant b, \quad x \in D,$$
$$\underline{u}_m = \varphi_m, \quad x \in \partial D,$$
$$\underline{u}_m < P, \quad x \in D',$$

where D' is an open set satisfying $\Sigma \Subset D' \Subset D$.

Define

$$\underline{u}(x,m) = \max\{\underline{u}_m(x), P(x)\}, \quad x \in D.$$

Then $\underline{u}(x,m) = \varphi_m(x)$ on ∂D,

$$\underline{u}(x,m) = P(x), \quad x \in D', \tag{3.2}$$

and $\underline{u} \in C^0(M_k \cup \partial' M_k)$ is a viscosity subsolution of (1.1), (1.2).

Step 2. We define the Perron solution of (1.1).

Let $\mathbb{S}$ denote the set of viscosity subsolutions $v \in C^0(M_k \cup \partial' M_k)$ of (1.1), (1.2) satisfying

$$\limsup_{x \to \bar{x}} \max_{1 \leqslant m \leqslant k} \left[v(x,m) - \tilde{h}(x,m)\right] \leqslant 0, \quad \bar{x} \in \Gamma,$$

where $\tilde{h} \in C^0(M_k) \cap L^\infty(M_k)$ [6] satisfies

$$\Delta \tilde{h} = a, \quad (x,m) \in M_k,$$
$$\tilde{h} = \varphi_m(x), \quad x \in \partial D.$$

Therefore by (2.1),

$$\Delta \tilde{h} = a \leqslant \Delta P, \quad x \in D \backslash \Gamma,$$
$$P = c \leqslant \varphi_m = \tilde{h}, \quad x \in \partial D.$$

Then from Lemma 3.1,

$$P \leqslant \tilde{h}, \quad (x,m) \in M_k.$$

Thus in view of (3.2), $\underline{u} \in \mathbb{S}$ and $\mathbb{S} \neq \emptyset$.

Define in M_k,

$$u(x,m) = \sup\{v(x,m) \mid v \in \mathbb{S}\}.$$

Then from [14], $u \in C^0(M_k \cup \partial' M_k)$ and u is a viscosity subsolution of (1.1). Because $\underline{u} \leqslant u$ in M_k and $\underline{u} = \varphi_m$ on $\partial' M_k$ for $m = 1, 2, \ldots, k$, then

$$u(x,m) = \varphi_m(x), \quad (x,m) \in \partial' M_k.$$

Step 3. We prove that u is a viscosity solution of (1.1).

For any $x_0 \in D \backslash \Gamma$, fix $\varepsilon > 0$ such that $\overline{B} = \overline{B_\varepsilon(x_0)} \subset D \backslash \Gamma$. The lifting of B into M_k is the union of k disjoint balls denoted as $\{B^{(i)}\}_{i=1}^k$. Then, in each $B^{(i)}$, the Dirichlet problem

$$F(D^2 \tilde{u}) = f, \quad (x,m) \in B^{(i)},$$
$$\tilde{u} = u, \quad (x,m) \in \partial B^{(i)} \tag{3.3}$$

has a viscosity solution $\tilde{u} \in C^0(\overline{B^{(i)}})$. From the comparison principle,

$$u \leqslant \tilde{u}, \quad (x,m) \in B^{(i)}. \tag{3.4}$$

Define in M_k,

$$w(x,m) = \begin{cases} \tilde{u}(x,m), & (x,m) \in B^{(i)}, \\ u(x,m), & (x,m) \in M_k \backslash \{B^{(i)}\}_{i=1}^k. \end{cases}$$

Then $w \in C^0(M_k \cup \partial' M_k)$.

From (3.4) and Lemma 2.2, we know in the viscosity sense,

$$F(D^2 w) \geqslant f, \quad (x,m) \in M_k.$$

Moreover,

$$w(x,m) = u(x,m) = \varphi_m(x), \quad x \in \partial D.$$

Consequently w is a viscosity subsolution of (1.1), (1.2). By (2.1), we have

$$\Delta\tilde{h} \leqslant \Delta u, \quad (x,m) \in M_k,$$
$$u \leqslant \tilde{h}, \quad (x,m) \in \partial' M_k.$$

Then Lemma 3.1 leads to

$$u \leqslant \tilde{h}, \quad (x,m) \in M_k.$$

As a result, $w \in \mathbb{S}$.

According to the definition of u, we have $u \geqslant w$ in M_k, and then $\tilde{u} \leqslant u$ in $B^{(i)}$. By (3.4),

$$u = \tilde{u}, \quad (x,m) \in B^{(i)}.$$

Because $\tilde{u} \in C^0(\overline{B^{(i)}})$ is a viscosity solution of (3.3) and x_0 is arbitrary, then $u \in C^0(M_k \cup \partial' M_k)$ is a viscosity solution to (1.1), (1.2).

Furthermore, if $f \in C^\alpha(M_k)$, we have $\tilde{u} \in C^{2,\alpha}(B^{(i)})$, therefore $u \in C^{2,\alpha}(M_k)$. Theorem 1.1 is completed. □

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