Multi-valued solutions to fully nonlinear uniformly elliptic equations

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\textbf{ABSTRACT}
In this paper, we discuss the existence and regularity of multi-valued viscosity solutions to fully nonlinear uniformly elliptic equations. We use the Perron method to prove the existence of bounded multi-valued viscosity solutions.

1. Introduction
In this paper, we study the multi-valued solutions of fully nonlinear uniformly elliptic equations
\[ F(D^2u) = f(x), \]
where \( F \) is a map from \( S^{n\times n} \) to \( \mathbb{R} \), \( S^{n\times n} \) denotes the space of \( n \times n \) real symmetric matrix and \( f(x) \) is a known function. Also, \( F \in C^2(S^{n\times n}) \) satisfies the following conditions:
(i) \( F \) is uniformly elliptic, i.e., there exist positive constants \( \lambda, \Lambda \) such that for any \( r \in S^{n\times n} \), \( \xi \in \mathbb{R}^n \),
\[ \lambda|\xi|^2 \leq F_{ij}(r)\xi_i\xi_j \leq \Lambda|\xi|^2, \]
where \( F_{ij} = \partial F/\partial u_{ij} \).
(ii) \( F \) is concave for \( r \), i.e., \( F \) is a concave function in \( S^{n\times n} \).

Typical examples of fully nonlinear uniformly elliptic equations satisfying (i), (ii) are the following Pucci equation and Bellman equation.

1. Pucci equation:
\[ \mathcal{M}^{-}(D^2u) = f(x), \quad \mathcal{M}^{+}(D^2u) = f(x), \]

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where
\[ M^-(r, \lambda, \Lambda) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i, \]
\[ M^+(r, \lambda, \Lambda) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \]
\[ r \in \mathbb{S}^{n \times n} \text{ and } e_i = e_i(r) \text{ are the eigenvalues of } r, \quad i = 1, 2, \ldots, n. \]

2. Bellman equation in stochastic control theory:
\[ F(D^2 u) := \inf_{a \in \mathcal{A}} \left\{ a_{ij} D_{ij} u \right\} = f(x), \]
where \( \mathcal{A} \) is any set, and \( a_{ij} \in \mathbb{R} \) satisfies
\[ \lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \xi \in \mathbb{R}^n, \alpha \in \mathcal{A}. \]

From the theory of analytic functions, we know the typical two-dimensional examples of multi-valued harmonic functions are
\[ u_1(z) = \text{Re}(z^2), \quad z \in \mathbb{C}\setminus\{0\}, \]
\[ u_2(z) = \text{Arg}(z), \quad z \in \mathbb{C}\setminus\{0\}, \]
and
\[ u_3(z) = \text{Re}(\sqrt{(z-1)(z+1)}), \quad z \in \mathbb{C}\setminus\{\pm1\}. \]

By 1970s, Almgren [1] had realized that a minimal variety near a multiplicity-k disc could be well approximated by the graph of a multi-valued function minimizing a suitable analog of the ordinary Dirichlet integral. Many facts about harmonic functions are also true for these Dirichlet minimizing multi-valued functions. Evans [9–11], Levi [18] and Caffarelli [2,3] all studied the multi-valued harmonic functions. Evans [10] proved that the conductor potential of a surface with minimal capacity was a double-valued harmonic function. In [3], Caffarelli proved the Hölder continuity of the multi-valued harmonic functions.

At the beginning of this century, the multi-valued solutions of Eikonal equation were considered in [15,13], respectively. Later, Jin et al. provided a level set method for the computation of multi-valued geometric solutions to general quasilinear PDEs and multi-valued physical observables to semiclassical limit of Schrödinger equations, see [17,16].

In 2006, Caffarelli and Li investigated the multi-valued solutions of Monge–Ampère equations in [6] where they first introduced the geometric situation of multi-valued solutions and then obtained the existence, boundedness, regularity and the asymptotic behavior at infinity of multi-valued viscosity solutions. The multi-valued solutions for Dirichlet problem of Monge–Ampère equations on exterior planar domains were discussed by Ferrer, Martínez and Milán in [12] using the complex variable methods. Recently, Caffarelli proved the Hölder continuity of the multi-valued harmonic functions.

The geometric situation of the multi-valued functions was given in [6]. Let \( D \) be a bounded regular domain in \( \mathbb{R}^n \), and let \( \Sigma \subset D \) be homeomorphic in \( \mathbb{R}^n \) to an \((n-1)\)-dimensional closed disc, i.e., there exists a homeomorphism \( \psi : \mathbb{R}^n \to \mathbb{R}^n \) such that \( \psi(\Sigma) \) is an \((n-1)\)-dimensional closed disc. Let \( \Gamma = \partial \Sigma \), the boundary of \( \Sigma \). Thus \( \Gamma \) is homeomorphic to an \((n-2)\)-dimensional sphere for \( n \geq 3 \).

Let \( \mathbb{Z} \) be the set of integers and
\[ M = (D \setminus \Gamma) \times \mathbb{Z} \]
denote a covering of \( D \setminus \Gamma \) with the following standard parameterization: fixing an \( x^* \in D \setminus \Gamma \), and connecting \( x^* \) by a smooth curve in \( D \setminus \Gamma \) to a point \( x \in D \setminus \Gamma \). If the curve goes through \( \Sigma \) in the positive direction (fixing such a direction) \( m \geq 0 \) times, then we arrive at \((x, m)\) in \( M \). If the curve goes through \( \Sigma \) in the negative direction \( m \geq 0 \) times, then we arrive at \((x, -m)\) in \( M \).

For \( k = 2, 3, \ldots \), we introduce an equivalence relation “\( \sim k \)” on \( M \) as follows: \((x, m)\) and \((y, j)\) in \( M \) are “\( \sim k \)” equivalent if \( x = y \) and \( m - j \) is an integer multiple of \( k \). We let
\[ M_k := M/ \sim k, \]
denote the \( k \)-sheet cover of \( D \setminus \Gamma \), and let
\[ \partial' M_k := \bigcup_{m=1}^{k} (\partial D \times \{m\}). \]
For $n = 2$, we can understand the covering space $M_k$ more clearly from the above example $u_3$. In this example, $\Gamma = \{1, -1\}$ and $\Sigma$ is the interval $(-1, 1)$. Each time the point $z$ goes around $-1$ or $1$, it crosses the interval $(-1, 1)$ one time.

We define a distance in $M_k$ as follows: For any $(x, m), (y, j) \in M_k$, let $l((x, m), (y, j))$ denote a smooth curve in $M_k$ which connects $(x, m)$ and $(y, j)$, and let $l((x, m), (y, j))$ denote its length. Define

$$
    d((x, m), (y, j)) = \inf_{l} l((x, m), (y, j)),
$$

where the inf is taken over all smooth curves in $M_k$ connecting $(x, m)$ and $(y, j)$. Then $d((x, m), (y, j))$ is a distance.

**Definition 1.1.** We call a function $u$ is continuous at $(x, m)$ in $M_k$ if

$$
    \lim_{d((y, j), (x, m)) \to 0} u(y, j) = u(x, m),
$$

and $u \in C^0(M_k)$ if for any $(x, m) \in M_k$, $u$ is continuous at $(x, m)$.

Similarly, we can define $u \in C^a(M_k), C^{0,1}(M_k)$ and $C^2(M_k)$, etc.

To the best of our knowledge, there isn’t any result of multi-valued solutions to fully nonlinear uniformly elliptic equations. In this paper, we study the multi-valued solutions to Dirichlet problem:

$$
    \begin{align*}
    F(D^2 u) &= f(x, m), & (x, m) \in M_k, \\
    u &= \varphi_m(x), & (x, m) \in \partial' M_k,
    \end{align*}
$$

(1.1) (1.2)

where

$(H_1)$ $f \in C^0(M_k)$ and for some constants $a$ and $b$, $a \leq f \leq b$.

$(H_2)$ $\varphi_1, \ldots, \varphi_k \in C^0(\partial D)$.

To state our results, we recall the definition of viscosity solutions, see [4,5].

**Definition 1.2.** A function $u \in C^0(M_k)$ is called a viscosity subsolution of (1.1), if for any $(y, m) \in M_k$ and any function $\xi \in C^2(M_k)$ satisfying

$$
    u(x, m) \leq \xi(x, m), \quad (x, m) \in M_k \quad \text{and} \quad u(y, m) = \xi(y, m),
$$

we have

$$
    F(D^2 \xi(y, m)) \geq f(y, m).
$$

A function $u \in C^0(M_k)$ is called a viscosity supersolution of (1.1), if for any $(y, m) \in M_k$ and any function $\xi \in C^2(M_k)$ satisfying

$$
    u(x, m) \geq \xi(x, m), \quad (x, m) \in M_k \quad \text{and} \quad u(y, m) = \xi(y, m),
$$

we have

$$
    F(D^2 \xi(y, m)) \leq f(y, m).
$$

A function $u \in C^0(M_k)$ is called a viscosity solution of (1.1), if $u$ is both a viscosity subsolution and a viscosity supersolution of (1.1).

A function $u \in C^0(M_k \cup \partial' M_k)$ is called a viscosity subsolution (supersolution, solution) of (1.1), (1.2), if $u$ is a viscosity subsolution (supersolution, solution) of (1.1) and satisfies $u(x, m) \leq (\geq, =) \varphi_m(x)$ on $\partial' M_k$ for $m = 1, 2, \ldots, k$.

Our main result is as follows. Using the Perron method and the properties of Pucci operators, we obtain the existence of bounded multi-valued viscosity solutions of (1.1), (1.2).

**Theorem 1.1.** If the conditions $(H_1)$ and $(H_2)$ hold, then the Dirichlet problem (1.1), (1.2) has at least a bounded viscosity solution $u \in C^0(M_k \cup \partial' M_k)$. Furthermore, if $f \in C^a(M_k)$, we have $u \in C^{2,a}(M_k)$.

This paper is arranged as follows. In Section 2, we derive some lemmas for single-valued solutions to fully nonlinear uniformly elliptic equations. The existence of bounded multi-valued solutions is proved in Section 3.
2. Preliminaries

For the reader’s convenience, we prove a few lemmas about single-valued solutions which will be used later. We shall always assume \( D \) is a bounded regular domain in \( \mathbb{R}^n \).

Substituting \( f(x) \) with \( f(x) - F(0) \), we may assume \( F(0) = 0 \). From [4], we know that for any \( M \in S^{n \times n} \),

\[
F(M) \leq \operatorname{Trace}(AM),
\]

where \( A \in S^{n \times n} \) and all the eigenvalues of \( A \) belong to \( [\lambda, \Lambda] \). Hence we can make a linear change of space variables such that in the new variables,

\[
\operatorname{Trace}(AM) = \operatorname{Trace}(M).
\]

Therefore, for any \( \phi \in C^2 \), we may suppose that

\[
F(D^2 \phi) \leq \Delta \phi. \tag{2.1}
\]

Lemma 2.1. Let \( D' \subset D \) be any open set, \( V \in C^0(\overline{D}) \), \( \phi \in C^0(\partial D) \) and \( c_0 \) be any constant. Then there exists a function \( u \in C^0(\overline{D}) \cap C^2(D) \) satisfying

\[
F(D^2 u) \geq c_0, \quad x \in D,
\]

\[
u = \phi, \quad x \in \partial D,
\]

\[
\underline{u} \leq V, \quad x \in D'.
\]

Proof. From [4], let \( \tilde{\phi} \in C^{2,\alpha}(D) \cap C^0(\overline{D}) \) satisfy

\[
\mathcal{M}^-(D^2 \tilde{\phi}) = 1, \quad x \in D,
\]

\[
\tilde{\phi} = \phi(x), \quad x \in \partial D,
\]

and let \( \rho \in C^{2,\alpha}(\overline{D}) \) satisfy

\[
\mathcal{M}^-(D^2 \rho) = 1, \quad x \in D,
\]

\[
\rho = 0, \quad x \in \partial D.
\]

Define in \( D \), \( u(x) = \tilde{\phi}(x) + \mu \rho(x) \), where \( \mu \) is a positive constant to be determined. Then \( u \in C^2(D) \cap C^0(\overline{D}) \) and

\[
u(x) = \phi(x), \quad x \in \partial D. \tag{2.2}
\]

On the other hand, for any function \( v \in C^2(D) \), we have

\[
F(D^2 v) = F(D^2 v) - F(0)
\]

\[
= F_{ij}(\theta D^2 v)D_{ij}v
\]

\[
\geq \lambda \sum_{e_i > 0} e_i + A \sum_{e_i < 0} e_i
\]

\[
= \mathcal{M}^-(D^2 v),
\]

where \( 0 < \theta < 1 \) and \( e_i \) are the eigenvalues of \( D^2 v \). Therefore, by the concavity of \( \mathcal{M}^- \)

\[
F(D^2 u) = F(D^2 \tilde{\phi} + \mu D^2 \rho)
\]

\[
\geq \mathcal{M}^-(D^2 \tilde{\phi} + \mu D^2 \rho)
\]

\[
\geq \mathcal{M}^-(D^2 \tilde{\phi}) + \mathcal{M}^-(\mu D^2 \rho)
\]

\[
= 1 + \mu.
\]

Then we can choose \( \mu = \mu(c_0) \) such that

\[
F(D^2 u) \geq c_0, \quad x \in D. \tag{2.3}
\]
By the strong maximum principle, there exists a constant $\rho_0 > 0$ such that $\rho \leq -\rho_0$ in $D'$. Thus we can again choose $\mu = \mu(\rho, \psi, V, D')$ large such that
\[
\mu \leq \psi + \mu \rho \leq \sup_{D'} \psi - \mu \rho_0 \leq \inf_{D'} V \leq V, \quad x \in D'.
\] (2.4)

From (2.2), (2.3), (2.4), we know that the lemma holds.

Lemma 2.2. Let $\Omega \subset D$ be an open set and $f \in C^0(D)$. Suppose that $v \in C^0(\Omega)$ and $u \in C^0(D)$ satisfy respectively
\[
F (D^2 v) \geq f(x), \quad x \in \Omega,
\]
\[
F (D^2 u) \geq f(x), \quad x \in D.
\]

Moreover,
\[
u \leq v, \quad x \in \Omega,
u = v, \quad x \in \partial \Omega.
\] (2.5)

Set
\[
w(x) = \begin{cases} v(x), & x \in \Omega, \\
u(x), & x \in D \setminus \Omega. \end{cases}
\]

Then $w \in C^0(D)$ satisfies in the viscosity sense
\[
F (D^2 w) \geq f(x).
\]

Proof. For any $x_0 \in D$, $\xi \in C^2(D)$ satisfying $w(x_0) = \xi(x_0)$,
\[
w(x) \leq \xi(x), \quad x \in D,
\] (2.6)

by the definition of $w$, if $x_0 \in \Omega$, we have
\[
v(x_0) = \xi(x_0), \quad v(x) \leq \xi(x), \quad x \in \Omega.
\]

Then
\[
F (D^2 \xi(x_0)) \geq f(x_0).
\]

If $x_0 \in D \setminus \Omega$, we have
\[
u(x_0) = \xi(x_0), \quad u(x) \leq \xi(x), \quad x \in D \setminus \Omega.
\]

From (2.5), (2.6),
\[
u(x) \leq \xi(x), \quad x \in D.
\]

Hence
\[
F (D^2 \xi(x_0)) \geq f(x_0).
\]

Lemma 2.2 is proved.

The following lemma may be a known result. But for the readers convenience, we give the lemma and its proof.

Lemma 2.3. Let $f \in C^0(D)$. Suppose that $u \in C^0(D)$ satisfies in the viscosity sense
\[
F (D^2 u) \geq f(x), \quad x \in D,
\]
then the Dirichlet problem
\[
F (D^2 u) = f(x), \quad x \in D,
\] (2.7)
\[
u = u(x), \quad x \in \partial D
\] (2.8)

has a viscosity solution $u \in C^0(D)$. 
**Proof.** By Theorem 3.4 in [14], if (2.7), (2.8) has a viscosity subsolution $g$ and a viscosity supersolution $h$ satisfying $g \leq h$ in $D$, we know that (2.7), (2.8) has a viscosity solution.

Clearly, $u$ is a viscosity subsolution of (2.7), (2.8). So we only need to prove that (2.7), (2.8) has a viscosity supersolution $u$ satisfying $u \geq u$ in $D$.

Let $f_0 = \inf_D f(x)$ and $u \in C^2(D) \cap C^0(\overline{D})$ satisfy

$$\Delta u = f_0, \quad x \in D,$$

$$u = u, \quad x \in \partial D.$$ 

Then $u$ satisfies in the viscosity sense

$$F(D^2 u) \leq f(x), \quad x \in D. \quad (2.9)$$

Indeed, for any $x_0 \in D, \xi \in C^2(D)$ satisfying

$$u(x_0) = \xi(x_0), \quad u(x) \geq \xi(x), \quad x \in D,$$

we have

$$D^2(u - \xi)(x_0) \geq 0.$$ 

Therefore

$$F(D^2 u(x_0)) \geq F(D^2 \xi(x_0)).$$

By (2.1), we have $F(D^2 u(x_0)) \leq \Delta u(x_0)$. Hence

$$F(D^2 \xi(x_0)) \leq F(D^2 u(x_0)) \leq f_0 \leq f(x_0),$$

and then (2.9) holds. Consequently $u$ is a viscosity supersolution of (2.7), (2.8). From the comparison principle, $u \geq u$ in $D$. The lemma is proved. $\square$

### 3. Existence of bounded multi-valued solutions

In this section, we prove Theorem 1.1. We first introduce a comparison principle in $M_k$, see [6].

**Lemma 3.1.** Let $u, \nu \in C^0(\overline{M_k}) \cap L^\infty(\overline{M_k})$ satisfy $\Delta u \geq 0 \geq \Delta \nu$ in $M_k$ and

$$\liminf_{\text{dist}(x,m), \partial M_k} \mu(x,m) - \mu(x,m) \leq 0,$$

then $u \leq \nu$ in $M_k$.

**Proof of Theorem 1.1.** We divide the proof into three steps.

**Step 1.** We construct a viscosity subsolution of (1.1), (1.2).

Let $P \in C^2(D) \cap C^0(\overline{D})$ satisfy

$$F(D^2 P) = b, \quad x \in D,$$

$$P = c, \quad x \in \partial D,$$

where $a \leq f \leq b$ and $c = \inf_{x \in \partial D} \min_{1 \leq m \leq k} \{\psi_m(x)\}$.

By Lemma 2.1, we know that there exist functions $u_1, \ldots, u_k \in C^0(\overline{D}) \cap C^2(D)$ satisfying

$$F(D^2 u_m) \geq b, \quad x \in D,$$

$$u_m = \psi_m, \quad x \in \partial D,$$

$$u_m < P, \quad x \in D',$$

where $D'$ is an open set satisfying $\Sigma \subseteq D' \subseteq D$. 

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Define

\[ u(x, m) = \max \{ u_m(x), P(x) \}, \quad x \in D. \]

Then \( u(x, m) = \phi_m(x) \) on \( \partial D \),

\[ u(x, m) = P(x), \quad x \in D', \tag{3.2} \]

and \( u \in C^0(M_k \cup \partial' M_k) \) is a viscosity subsolution of (1.1), (1.2).

**Step 2. We define the Perron solution of (1.1).**

Let \( S \) denote the set of viscosity subsolutions \( v \in C^0(M_k \cup \partial' M_k) \) of (1.1), (1.2) satisfying

\[ \limsup_{x \to x_i} \max_{1 \leq m \leq k} [v(x, m) - \tilde{h}(x, m)] \leq 0, \quad \tilde{x} \in \Gamma, \]

where \( \tilde{h} \in C^0(M_k) \cap L^\infty(M_k) \) [6] satisfies

\[ \Delta \tilde{h} = a, \quad (x, m) \in M_k, \]
\[ \tilde{h} = \phi_m(x), \quad x \in \partial D. \]

Therefore by (2.1),

\[ \Delta \tilde{h} = a \leq \Delta P, \quad x \in D \setminus \Gamma, \]
\[ P = c \leq \phi_m = \tilde{h}, \quad x \in \partial D. \]

Then from Lemma 3.1,

\[ P \leq \tilde{h}, \quad (x, m) \in M_k. \]

Thus in view of (3.2), \( u \in S \) and \( S \neq \emptyset \).

Define in \( M_k \),

\[ u(x, m) = \sup \{ v(x, m) \mid v \in S \}. \]

Then from [14], \( u \in C^0(M_k \cup \partial' M_k) \) and \( u \) is a viscosity subsolution of (1.1). Because \( u \leq u \) in \( M_k \) and \( u = \phi_m \) on \( \partial' M_k \) for \( m = 1, 2, \ldots, k \), then

\[ u(x, m) = \phi_m(x), \quad (x, m) \in \partial' M_k. \]

**Step 3. We prove that \( u \) is a viscosity solution of (1.1).**

For any \( x_0 \in D \setminus \Gamma \), fix \( \varepsilon > 0 \) such that \( B = B_\varepsilon(x_0) \subset D \setminus \Gamma \). The lifting of \( B \) into \( M_k \) is the union of \( k \) disjoint balls denoted as \( \{ B^{(i)} \}_{i=1}^k \). Then, in each \( B^{(i)} \), the Dirichlet problem

\[ F(D^2 \tilde{u}) = f, \quad (x, m) \in B^{(i)}, \]
\[ \tilde{u} = u, \quad (x, m) \in \partial B^{(i)} \]

has a viscosity solution \( \tilde{u} \in C^0(\overline{B^{(i)}}) \). From the comparison principle,

\[ u \leq \tilde{u}, \quad (x, m) \in B^{(i)}. \tag{3.4} \]

Define in \( M_k \),

\[ w(x, m) = \begin{cases} \tilde{u}(x, m), & (x, m) \in B^{(i)}, \\ u(x, m), & (x, m) \in M_k \setminus \{ B^{(i)} \}_{i=1}^k. \end{cases} \]

Then \( w \in C^0(M_k \cup \partial' M_k) \).

From (3.4) and Lemma 2.2, we know in the viscosity sense,

\[ F(D^2 w) \geq f, \quad (x, m) \in M_k. \]

Moreover,

\[ w(x, m) = u(x, m) = \phi_m(x), \quad x \in \partial D. \]

Consequently \( w \) is a viscosity subsolution of (1.1), (1.2). By (2.1), we have
\[ \Delta \tilde{h} \leq \Delta u, \quad (x, m) \in M_k, \]
\[ u \leq \tilde{h}, \quad (x, m) \in \partial' M_k. \]

Then Lemma 3.1 leads to
\[ u \leq \tilde{h}, \quad (x, m) \in M_k. \]

As a result, \( w \in S \).

According to the definition of \( u \), we have \( u \geq w \) in \( M_k \), and then \( \tilde{u} \leq u \) in \( B^{(i)} \). By (3.4),
\[ u = \tilde{u}, \quad (x, m) \in B^{(i)}. \]

Because \( \tilde{u} \in C^0(B^{(i)}) \) is a viscosity solution of (3.3) and \( x_0 \) is arbitrary, then \( u \in C^0(M_k \cup \partial' M_k) \) is a viscosity solution to (1.1), (1.2).

Furthermore, if \( f \in C^\alpha(M_k) \), we have \( \tilde{u} \in C^{2,\alpha}(B^{(i)}) \), therefore \( u \in C^{2,\alpha}(M_k) \). Theorem 1.1 is completed. \( \Box \)

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