MIRROR SYMMETRY FOR A HESSIAN OVER-DETERMINED PROBLEM AND ITS GENERALIZATION

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ABSTRACT. In the paper, we apply the moving plane method to prove that if the right hand sides of equation and Neumann boundary condition are both independent of one variable, the domain and the solution to the Hessian over-determined problem are mirror symmetric. Our result generalizes the previous results on radial symmetry. In the end, we get the mirror symmetry of over-determined problems for more general equations, which include Weingarten curvature equation.

1. **Introduction.** In the theory of elasticity [15], by considering the torsion of a solid straight bar of cross section, we can get an over-determined problem for Poisson equation, that is

$$\begin{cases} \Delta u = n, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ \dfrac{\partial u}{\partial \gamma} = 1, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$ and $n \geq 2$), $\gamma(x)$ is the outer unit normal to $\partial\Omega$ at x.

This model can also be used to describe a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form and a liquid rising in a straight capillary tube of cross section.

In 1971, Serrin [13] applied the moving plane method and maximum principle to prove that: if $u \in C^2(\overline{\Omega})$ is a solution to problem (1), then up to a translation, Ω is a unit ball and $u(x) = \frac{|x|^2 - 1}{2}$. In the same year, Weinberger [16] proved the same conclusion by using Green formula. This conclusion states that, when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.

After Serrin's contribution to over-determined problem, lots of results have been obtained to extend this result. In 2008, Brandolini, Nitsch, Salani and Trombetti [1] considered over-determined problem for k-Hessian equation ($1 \leq k \leq n$),

$$\begin{cases} \sigma_k\left(\lambda(D^2u)\right) = \binom{n}{k}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ \dfrac{\partial u}{\partial\gamma} = 1, & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\sigma_k\left(\lambda(D^2u)\right)$ is the k-th elementary symmetric function of the eigenvalues of D^2u and $\binom{n}{k}$ denotes the combinatorial number. They proved that: if $u \in C^2(\overline{\Omega})$ is a solution to problem (2), then up to a translation, Ω is a unit ball and $u(x) = \frac{|x|^2-1}{2}$. In particular, when $k = 1$, problem (2) becomes problem (1); when $k = n$, problem (2) is the over-determined problem for Monge-Ampère equation. In the same year, they studied the stability of problem (2) for $k = 1$ in [2] and $k = n$ in [3].

Some results about over-determined problems in exterior domains can be found in [11] and [12]. And there are also many open problems about over-determined problems which have been proposed in [14].

In [7], B. Gidas, Weiming Ni and L. Nirenberg obtained a significant result about symmetry of solutions to Dirichlet problem of elliptic equations. For more results concerning symmetry properties of solutions of elliptic equations, we refer to [6] and [9].

In this paper, we apply the moving plane method to consider the over-determined problem for k-Hessian equation ($1 \leq k \leq n$) whose right hand sides of equation and Neumann condition are both independent of the n-th variable, that is

$$\begin{cases} \sigma_k\left(\lambda(D^2u)\right) = f(x'), & x \in \Omega, \\ u = c, & x \in \partial\Omega, \\ \dfrac{\partial u}{\partial\gamma} = \psi(x'), & x \in \partial\Omega, \end{cases} \tag{3}$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and c is a constant. For $k = 1$, problem (3) can be written as

$$\begin{cases} \Delta u = f(x'), & x \in \Omega, \\ u = c, & x \in \partial\Omega, \\ \dfrac{\partial u}{\partial\gamma} = \psi(x'), & x \in \partial\Omega. \end{cases} \tag{4}$$

Let $\Phi_k^{2,1}(\overline{\Omega}) = \{v \in C^{2,1}(\overline{\Omega}) : \sigma_i\left(\lambda\left(D^2v(x)\right)\right) > 0, x \in \overline{\Omega}, i = 1, 2, \cdots, k-1, k\}$ and Ω' be the projection of Ω in the x' direction. We can obtain the main theorem as follow.

Theorem 1.1. *Let Ω be a C^2 open, connected and bounded domain in $\mathbb{R}^n$, $f \in C^{0,1}(\overline{\Omega})$ be positive and $\psi \in C^{1,1}(\overline{\Omega})$. If $u \in \Phi_k^{2,1}(\overline{\Omega})$ is a solution to problem (3), then up to a translation in the x_n direction, Ω and u are symmetric about $x_n = 0$.*

In the case of Theorem 1.1, Ω must be $(k-1)$-convex (as level set of a k-convex function).

It would be interesting to see if Theorem remains valid under weaker regularity assumptions. For $k = 1$, we can reduce the regularity of u, f and ψ in Theorem 1.1 and then obtain the following corollary.

Corollary 1. *Let Ω be a C^2 open, connected and bounded domain in $\mathbb{R}^n$, $f \in C^0(\overline{\Omega})$ be positive and $\psi \in C^1(\overline{\Omega})$. If $u \in C^2(\overline{\Omega})$ is a solution to problem (4), then up to a translation in the x_n direction, Ω and u are symmetric about $x_n = 0$.*

Remark 1. If we take f and ψ as constants in Theorem 1.1 and Corollary 1, we can get that Ω and u are symmetric about any hyperplane passing through the origin. Therefore Ω is a ball and u is radial.

Our paper is organized as follows. In Section 2, we will give some notation and preliminaries. In Section 3 and 4, we will present the proof of Theorem 1.1 and Corollary 1 respectively. In Section 5, the moving plane method will be applied to the over-determined problem for a class of fully non-linear equations, which include Weingarten curvature equation.

2. Notation and preliminaries.

2.1. Hessian operator.

We first introduce the definition of k-th elementary symmetric function.

For $a = (a^1, a^2, \cdots, a^{n-1}, a^n) \in \mathbb{R}^n$ and $k \in \{1, 2, \cdots, n-1, n\}$, the k-th elementary symmetric function of a is defined as

$$\sigma_k(a) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k \leq n} a^{i_1} a^{i_2} \cdots a^{i_{k-1}} a^{i_k}.$$

Let $A = (a^{ij})$ be a real symmetric $n \times n$ matrix and $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \lambda_n$ be its eigenvalues. Then

$$\sigma_k(\lambda(A)) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{k-1}} \lambda_{i_k}.$$

It is easy to see that $\sigma_k(\lambda(A))$ is just the sum of all $k \times k$ principal minors of A (see[1]).

Denoting by

$$S_k(A) = \sigma_k(\lambda(A)), \tag{5}$$

and

$$S_k^{ij}(A) = \frac{\partial}{\partial a^{ij}} S_k(A), \quad i, j = 1, 2, \cdots, n-1, n, \tag{6}$$

then Euler identity for homogeneous functions gives us

$$S_k(A) = \frac{1}{k} S_k^{ij}(A) a^{ij},$$

here and throughout the paper, we adopt the Einstein summation convention for repeated indices.

Suppose M is a real symmetric $n \times n$ matrix,

$$M = \begin{pmatrix} m^{ij} & m^{in} \\ m^{ni} & m^{nn} \end{pmatrix},$$

$\widetilde{M}$ denotes

$$\widetilde{M} = \begin{pmatrix} m^{ij} & -m^{in} \\ -m^{ni} & m^{nn} \end{pmatrix}$$

and D denotes

$$D = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $D^{-1}MD = \widetilde{M}$, which means M and $\widetilde{M}$ are similar. So we have $\lambda(M) = \lambda(\widetilde{M})$. It follows that

$$S_k(M) = S_k(\widetilde{M}). \tag{7}$$

For $i = 1, 2, \cdots, n-2, n-1$, from differentiating (7) with respect to m^{in}, it follows that

$$S_k^{in}(M) = S_k^{in}(\widetilde{M}) \frac{\partial \widetilde{m}^{in}}{\partial m^{in}} = -S_k^{in}(\widetilde{M}).$$

So we have

$$S_k^{in}(M) + S_k^{in}(\widetilde{M}) = 0, \quad i = 1, 2, \cdots, n-2, n-1. \tag{8}$$

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in C^2(\overline{\Omega})$. We call $S_k(D^2u)$ the k-Hessian operator of u. It is obvious that

$$S_1(D^2u) = \Delta u \quad \text{and} \quad S_n(D^2u) = \det(D^2u).$$

The k-th Hessian operators are uniformly elliptic if restricted to the class of k-convex functions

$$\{v \in C^2(\overline{\Omega}) : S_i\left(D^2v\left(x\right)\right) > 0, x \in \overline{\Omega}, i = 1, 2, \cdots, k-1, k\}.$$

2.2. Moving plane method.
Before using the moving plane method to prove Theorem 1.1 and Corollary 1, we would like to introduce some notation.

$$
\begin{aligned}
T_\lambda &= \{x \in \mathbb{R}^n : x_n = \lambda\} && \text{the hyperplane,} \\
H_\lambda &= \{x : x_n > \lambda\} && \text{the above half-space,} \\
x^\lambda &= (x', 2\lambda - x_n) && \text{the reflection of } x \text{ about } T_\lambda, \\
\Sigma(T_\lambda) &= \{x : x^\lambda \in \Omega \cap H_\lambda\} && \text{the reflection domain of } \Omega \text{ about } T_\lambda, \\
\lambda_0 &= \sup\{x_n : x = (x', x_n) \in \Omega\} && \text{the } x_n\text{-extent of } \Omega.
\end{aligned}
$$

Let T_{λ_0} be the starting position and we move T_λ along the negative direction of x_n axis. From [10] we can show that during the motion, Ω contains $\Sigma(T_\lambda)$ until one of the following two events (critical positions) occurs:

(1) $\partial\Sigma(T_\lambda)$ becomes internally tangent to $\partial\Omega$ at $P \notin T_\lambda$;

(2) T_λ reaches a position where it is orthogonal to $\partial\Omega$ at some point $Q \in T_\lambda$.

During the proof below, we will discuss the two critical cases and a key ingredient in the proof is a corner lemma. For readers' convenience, we will state it below and the proof can be found in [13].

Let D^* be a bounded domain with C^2 boundary in $\mathbb{R}^n$ and let T be a hyperplane containing the unit outer normal to ∂D^* at some point Q. And D denotes the portion of D^* lying on some particular side of T. We assume the coefficient b^i's are uniformly bounded in D and there exist three positive constants K_1, K_2 and K such that

$$K_1|\xi|^2 \le \hat{a}^{ij}(x)\xi_i\xi_j \le K_2|\xi|^2, \tag{9}$$

and

$$|\hat{a}^{ij}\left(x\right)\xi_i\eta_j| \le K\left(|\xi \cdot \eta| + |\xi|d\left(x\right)\right), \tag{10}$$

where $x \in D$, $\xi = (\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n)$ is an arbitrary real vector, $\eta = (\eta_1, \eta_2, \cdots, \eta_{n-1}, \eta_n)$ is the unit normal to the hyperplane T and $d(x)$ is the distance from x to T.

Lemma 2.1. *Suppose $w \in C^2(\overline{D})$ satisfies*

$$\hat{a}^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + b^i(x)\frac{\partial w}{\partial x_i} \geq 0, \quad x \in D,$$

and $w \leq 0$ in D and $w(Q) = 0$. Let s be any direction at Q which enters D non-tangentially. Then at Q, we have either

$$\frac{\partial w}{\partial s} < 0, \quad or \quad \frac{\partial^2 w}{\partial s^2} < 0,$$

unless $w \equiv 0$ in D.

Remark 2. From the above lemma, we can see that if w doesn't vanish in D, then either $Dw(Q) \neq 0$ or $D^2 w(Q) \neq 0$.

3. **Proof of Theorem 1.1.**

Proof of Theorem 1.1. We will divide our proof into four steps.

Step 1. First of all, we define a new function,

$$v(x) = u(x', 2\lambda - x_n), \quad x \in \overline{\Sigma(T_\lambda)}.$$

Clearly, we can get that $v \in C^{2,1}\left(\overline{\Sigma(T_\lambda)}\right)$. And for each $x \in \overline{\Sigma(T_\lambda)}$, we have

$$Dv(x) = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_{n-1}}, -\frac{\partial u}{\partial x_n})(x', 2\lambda - x_n), \tag{11}$$

$$D^2 v(x) = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & \dfrac{\partial^2 u}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 u}{\partial x_1 \partial x_{n-1}} & -\dfrac{\partial^2 u}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 u}{\partial x_2 \partial x_1} & \dfrac{\partial^2 u}{\partial x_2^2} & \cdots & \dfrac{\partial^2 u}{\partial x_2 \partial x_{n-1}} & -\dfrac{\partial^2 u}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2 u}{\partial x_{n-1} \partial x_1} & \dfrac{\partial^2 u}{\partial x_{n-1} \partial x_2} & \cdots & \dfrac{\partial^2 u}{\partial x_{n-1}^2} & -\dfrac{\partial^2 u}{\partial x_{n-1} \partial x_n} \\ -\dfrac{\partial^2 u}{\partial x_n \partial x_1} & -\dfrac{\partial^2 u}{\partial x_n \partial x_2} & \cdots & -\dfrac{\partial^2 u}{\partial x_n \partial x_{n-1}} & \dfrac{\partial^2 u}{\partial x_n^2} \end{pmatrix}$$
$$\cdot (x', 2\lambda - x_n), \tag{12}$$

and

$$\frac{\partial v}{\partial \gamma}(x) = \frac{\partial u}{\partial \gamma}(x', 2\lambda - x_n) = \psi(x'), \quad x \in \partial\Sigma(T_\lambda) \setminus T_\lambda.$$

Since u is the solution to problem (3), v satisfies

$$\begin{cases} S_k(D^2 v) = f(x'), & x \in \Sigma(T_\lambda), \\ v = u, & x \in \partial\Sigma(T_\lambda) \cap T_\lambda, \\ v = c, & x \in \partial\Sigma(T_\lambda) \setminus T_\lambda. \end{cases}$$

where S_k is defined in (5).

Next, we define

$$w(x) = u(x) - v(x), \quad x \in \overline{\Sigma(T_\lambda)},$$

then w satisfies

$$
\begin{cases}
\hat{a}^{ij} \dfrac{\partial^2 w}{\partial x_i \partial x_j} = 0, & x \in \Sigma(T_\lambda), \\
w = 0, & x \in \partial\Sigma(T_\lambda) \cap T_\lambda, \\
w \leq 0, & x \in \partial\Sigma(T_\lambda) \setminus T_\lambda,
\end{cases}
\tag{13}
$$

where

$$
\hat{a}^{ij}(x) = \int_0^1 S_k^{ij}\left(sD^2 u(x) + (1-s)D^2 v(x)\right) ds, \quad i,j = 1,2,\cdots,n-1,n.
$$

The last condition in (13) is obtained by the fact that $u < c$ in Ω which is deduced via strong maximum principle. It follows from $u, v \in \Phi_k^{2,1}\left(\overline{\Sigma(T_\lambda)}\right)$ that $(\hat{a}^{ij}(x))$ is positive definite.

By strong maximum principle, we can get from problem (13) that either

$$
w < 0, \quad x \in \Sigma(T_\lambda),
\tag{14}
$$

or $w \equiv 0$ in $\Sigma(T_\lambda)$. It is obvious that the latter case means that u and Ω are symmetric about $x_n = \lambda$. Therefore, we only need to show that for the two critical cases mentioned in (1) and (2) of Section 2, (14) is impossible. Assume that (14) is true, we will consider the two critical cases respectively.

Step 2. Let us consider the first critical case, that is, $\partial\Sigma(T_\lambda)$ becomes internally tangent to $\partial\Omega$ at $P \notin T_\lambda$. Since $P \in \partial\Omega \cap \partial\Sigma(T_\lambda)$, we have $u(P) = v(P)$. Thus $w(P) = u(P) - v(P) = 0$ and

$$
\frac{\partial w}{\partial \gamma}(P) = \frac{\partial u}{\partial \gamma}(P) - \frac{\partial v}{\partial \gamma}(P) = \psi(P') - \psi(P') = 0.
$$

However, by applying Hopf's Lemma (see [8] Lemma 3.4) to the linearized problem (13) and (14), we have

$$
\frac{\partial w}{\partial \gamma}(P) > 0,
$$

which is a contradiction. Hence (14) is impossible for the first critical case.

Step 3. Let us consider the second critical case, that is, T_λ reaches a position where it is orthogonal to $\partial\Omega$ at some point $Q \in T_\lambda$.

We shall use Lemma 2.1 to make contradiction. Since $u, v \in \Phi_k^{2,1}\left(\overline{\Sigma(T_\lambda)}\right)$, it is obvious that (9) in Lemma 2.1 holds in our case. Now let us verify the condition (10) in Lemma 2.1.

If $x \in \partial\Sigma(T_\lambda) \cap T_\lambda$, by the definition of v, we can get

$$
sD^2 v(x) + (1-s)D^2 u(x) = \widetilde{sD^2 u(x) + (1-s)D^2 v(x)}, \quad s \in [0,1].
\tag{15}
$$

And by taking $t = 1 - s$, we have

$$
\int_0^1 S_k^{in}\left(sD^2 u(x) + (1-s)D^2 v(x)\right) ds = \int_0^1 S_k^{in}\left(tD^2 v(x) + (1-t)D^2 u(x)\right) dt,
$$

i.e.

$$
\int_0^1 S_k^{in}\left(sD^2 u(x) + (1-s)D^2 v(x)\right) ds = \int_0^1 S_k^{in}\left(sD^2 v(x) + (1-s)D^2 u(x)\right) ds.
\tag{16}
$$

Then by (16), (15) and (8), for $i = 1, 2, \cdots, n-2, n-1$ and $x \in \partial\Sigma(T_\lambda) \cap T_\lambda$, we have

$$2\hat{a}^{in}(x)$$

$$= 2\int_0^1 S_k^{in}\left(sD^2u(x) + (1-s)D^2v(x)\right)ds$$

$$= \int_0^1 S_k^{in}\left(sD^2u(x) + (1-s)D^2v(x)\right)ds + \int_0^1 S_k^{in}\left(sD^2v(x) + (1-s)D^2u(x)\right)ds$$

$$= \int_0^1 S_k^{in}\left(sD^2u(x) + (1-s)D^2v(x)\right)ds + \int_0^1 S_k^{in}\left(\widetilde{sD^2u(x) + (1-s)D^2v(x)}\right)ds$$

$$= 0.$$

By $u, v \in C^{2,1}\left(\overline{\Sigma(T_\lambda)}\right)$ and the definition of $\hat{a}^{in}$, we can get $\hat{a}^{in}(x)$ is Lipschitz continuous on $\overline{\Sigma(T_\lambda)}$. So there exists $L > 0$, for $x = (x', x_n) \in \Sigma(T_\lambda)$ and $x_0 = (x', \lambda) \in \partial\Sigma(T_\lambda) \cap T_\lambda$, such that

$$|\hat{a}^{in}(x)| = |\hat{a}^{in}(x) - \hat{a}^{in}(x_0)| \leq L|x - x_0| = Ld(x),$$

where $d(x)$ is the distance from x to T_λ. And now the unit outer normal of T_λ is $\eta = (0, 0, \cdots, 0, 1)$, so for arbitrary vector $\xi = (\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n)$, we have

$$|\sum_{i,j=1}^n \hat{a}^{ij}(x)\xi_i\eta_j| = |\sum_{i=1}^n \hat{a}^{in}(x)\xi_i|,$$

$$\leq \sum_{i=1}^{n-1}|\hat{a}^{in}(x)||\xi_i| + |\hat{a}^{nn}(x)||\xi_n|$$

$$\leq (n-1)Ld(x)|\xi| + K_2|\xi\cdot\eta|$$

$$\leq K\left(|\xi|d(x) + |\xi\cdot\eta|\right).$$

This completes the proof of condition (10).

Since $w < 0$ in $\Sigma(T_\lambda)$ and $w(Q) = 0$, from Lemma 2.1, at Q we obtain that

$$\frac{\partial w}{\partial s} < 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2} < 0,$$

which contradicts with the fact $Dw(Q) = 0$ and $D^2w(Q) = 0$ that will be obtained in Step 4. Hence (14) is also impossible for the second critical case.

Step 4. We shall show that $Dw(Q) = 0$ and $D^2w(Q) = 0$. By (11) and (12), it is obvious that at Q

$$\frac{\partial w}{\partial x_l} = 0,$$

$$\frac{\partial w}{\partial x_n} = 2\frac{\partial u}{\partial x_n},$$

$$\frac{\partial^2 w}{\partial x_k\partial x_l} = \frac{\partial^2 w}{\partial x_n^2} = \frac{\partial^2 w}{\partial x_n\partial x_1} = 0,$$

$$\frac{\partial^2 w}{\partial x_n\partial x_l} = 2\frac{\partial^2 u}{\partial x_n\partial x_l},$$

for $k, l = 1, 2, \cdots, n-2, n-1$. So we only need to prove that at Q

$$\frac{\partial w}{\partial x_n} = 0,$$

$$\frac{\partial^2 w}{\partial x_n \partial x_l} = 0, \quad l = 1, 2, \cdots, n-2, n-1.$$

Since $\partial\Omega \in C^{2,1}$, we consider a rectangular coordinate frame with origin at Q, the x_1 axis being directed along the inward normal to $\partial\Omega$ at Q and the x_n axis being normal to T_λ. In this frame we can represent $\partial\Omega$ locally by the equation

$$x_1 = \phi(x_2, x_3, \cdots, x_n), \quad \phi \in C^{2,1}, \quad \phi(0') = 0, \quad D'\phi(0') = 0,$$

where D' denotes $(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \cdots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_n})$. Since $u \in C^{2,1}(\overline{\Omega})$, the Dirichlet boundary condition $u = c$ on $\partial\Omega$ can be expressed as a twice differentiable identity

$$u(\phi, x_2, x_3, \cdots, x_n) \equiv c. \tag{17}$$

Differentiating (17) with respect to x_k, $k = 2, 3 \cdots, n-1, n$, we obtain that

$$\frac{\partial u}{\partial x_k} + \frac{\partial u}{\partial x_1} \frac{\partial \phi}{\partial x_k} = 0. \tag{18}$$

By $D'\phi(0') = 0$, we find that for $k = 2, 3, \cdots, n-1, n$,

$$\frac{\partial u}{\partial x_k}(0) = 0,$$

in particular,

$$\frac{\partial w}{\partial x_n}(0) = 2\frac{\partial u}{\partial x_n}(0) = 0.$$

Similarly, the Neumann boundary condition $\frac{\partial u}{\partial \gamma} = \psi(x')$ on $\partial\Omega$ can also be written locally as an identity

$$\frac{\partial u}{\partial x_1} - \sum_{k=2}^{n} \frac{\partial u}{\partial x_k} \frac{\partial \phi}{\partial x_k} \equiv -\psi(x')\{1 + \sum_{k=2}^{n}(\frac{\partial \phi}{\partial x_k})^2\}^{\frac{1}{2}}. \tag{19}$$

Next differentiating (19) with respect to x_n and evaluating at 0, we can get

$$\frac{\partial^2 u}{\partial x_1 \partial x_n}(0) - \sum_{k=2}^{n} \frac{\partial u}{\partial x_k}(0)\frac{\partial^2 \phi}{\partial x_k \partial x_n}(0') = -\frac{\partial \psi}{\partial x_n}(0'),$$

i.e.

$$\frac{\partial^2 w}{\partial x_1 \partial x_n}(0) = 2\frac{\partial^2 u}{\partial x_1 \partial x_n}(0) = 0.$$

Now it remains to prove that

$$\frac{\partial^2 w}{\partial x_n \partial x_l}(0) = 0, \quad l = 2, 3, \cdots, n-2, n-1.$$

We do this by using the following Taylor expansion of w at 0. For $\xi \in \Sigma(T_\lambda)$, and $\xi \to 0$, we have

$$w(\xi) = w(0) + \sum_{l=1}^{n} \frac{\partial w}{\partial x_l}(0)\xi_l + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial^2 w}{\partial x_k \partial x_l}(0)\xi_l\xi_k + o(|\xi|^2),$$

$$= \sum_{l=2}^{n-1} \frac{\partial^2 w}{\partial x_n \partial x_l}(0)\xi_l\xi_n + o(|\xi|^2).$$

Fix $l \in \{2, 3, \cdots, n-2, n-1\}$, define $\xi(\delta) = \delta(1, 0, \cdots, 0, \pm 1, 0, \cdots, 0, -1)$ for $\delta > 0$, where ± 1 is the l-th component of ξ and when $\frac{\partial^2 w}{\partial x_n \partial x_l}(0) \leq 0$ plus sign is taken, when $\frac{\partial^2 w}{\partial x_n \partial x_l}(0) > 0$ minus sign is taken. Now we choose δ sufficiently small, such that $\xi(\delta) \in \Sigma(T_\lambda)$. It follows from the Taylor expansion that

$$w(\xi(\delta)) = \delta^2 |\frac{\partial^2 w}{\partial x_n \partial x_l}(0)| + o(\delta^2), \quad \delta \to 0 + .$$

Since $w < 0$ in $\Sigma(T_\lambda)$, we see that it forces $\frac{\partial^2 w}{\partial x_n \partial x_l}(0) = 0$. So far we have proved that $Dw(0) = 0$ and $D^2 w(0) = 0$. $\qquad \square$

4. Proof of Corollary 1.

Proof of Corollary 1. We can define v and w the same as Step 1 in the proof of Theorem 1.1. And we can also find that w satisfies

$$\begin{cases} \Delta w = 0, & x \in \Sigma(T_\lambda), \\ w = 0, & x \in \partial\Sigma(T_\lambda) \cap T_\lambda, \\ w \leq 0, & x \in \partial\Sigma(T_\lambda) \setminus T_\lambda. \end{cases} \tag{20}$$

By strong maximum principle, we can get from problem (20) that either

$$w < 0, \quad x \in \Sigma(T_\lambda), \tag{21}$$

or $w \equiv 0$ in $\Sigma(T_\lambda)$. It is obvious that the latter case means that u and Ω are symmetric about $x_n = \lambda$. Therefore, we only need to show that for the two critical cases, (21) is impossible. Assume that (21) is true, we will consider the two critical cases respectively.

The argument of the first critical case is completely the same as Step 2 in the proof of Theorem 1.1. Now we consider the second critical case. A same argument can turn out that $Dw(Q) = 0$ and $D^2 w(Q) = 0$. Since w satisfies problem (20), we can take $\hat{a}^{ij}(x) = \delta_{ij}$, $i, j = 1, 2, \cdots, n-1, n$ in Lemma 2.1. Then it is obvious that conditions (9) and (10) hold. So by Lemma 2.1, at Q we have

$$\frac{\partial w}{\partial s} < 0 \quad \text{or} \quad \frac{\partial^2 w}{\partial s^2} < 0,$$

which contradicts with $Dw(Q) = 0$ and $D^2 w(Q) = 0$. Hence (21) is also impossible for the second critical position. $\qquad \square$

5. Over-determined problems for more general equations.
In this section, we want to show that our method can also be applied to the over-determined problem for a class of fully non-linear elliptic equations, which can be regarded as a generalization of Theorem 1.1.

Now we consider the over-determined problem

$$\begin{cases} F(D^2 u, Du) = f(x'), & x \in \Omega, \\ u < c, & x \in \Omega, \\ u = c, & x \in \partial\Omega, \\ \frac{\partial u}{\partial \gamma} = \psi(x'), & x \in \partial\Omega, \end{cases} \tag{22}$$

where $F \in C^2(\mathbb{R}^{n \times n} \times \mathbb{R}^n$ satisfying the ellipticity condition

$$\left(F^{ij}(M, p) \right) > 0, \tag{23}$$

where $F^{ij}(M,p) = \frac{\partial F}{\partial m^{ij}}(M,p)$ and

$$F(M,p) = F(\widetilde{M}, \widetilde{p}), \tag{24}$$

for any real symmetric $n \times n$ matrix $M = (m^{ij})$, $p = (p^1, p^2, \cdots, p^{n-1}, p^n) \in \mathbb{R}^n$ and $\widetilde{p} = (p^1, p^2, \cdots, p^{n-1}, -p^n)$. For M and p bounded, F is uniformly elliptic.

Then we can obtain the following theorem:

Theorem 5.1. *Let Ω be a C^2 open, connected and bounded domain in $\mathbb{R}^n$, $f \in C^{0,1}(\overline{\Omega})$ and $\psi \in C^{1,1}(\overline{\Omega})$. If $u \in C^{2,1}(\overline{\Omega})$ is a solution to problem (22), then up to a translation in the x_n direction, Ω and u are symmetric about $x_n = 0$.*

Remark 3. If we take f and ψ as constants in Theorem 5.1, we can get that Ω and u are symmetric about any hyperplane passing through the origin. Therefore Ω is a ball and u is radial.

Remark 4. If we take $F(M,p) = \sigma_k(\lambda(M))$, $k = 1, 2, \cdots, n-1, n$, then it is obvious that Theorem 5.1 reduces to Theorem 1.1.

Remark 5. This $F(M,p)$ in Theorem 5.1 also includes so called Weingarten curvature equation, see Subsection 5.2.

5.1. **Proof of Theorem 5.1.** The proof is similar with the proof of Theorem 1.1, so we will only give a sketch.

We can define v and w the same as Step 1 in the proof of Theorem 1.1. Thus w satisfies

$$\begin{cases} \overline{a}^{ij} \dfrac{\partial^2 w}{\partial x_i \partial x_j} + b^i \dfrac{\partial w}{\partial x_i} = 0, & x \in \Sigma(T_\lambda), \\ w = 0, & x \in \partial\Sigma(T_\lambda) \cap T_\lambda, \\ w \leq 0, & x \in \partial\Sigma(T_\lambda) \setminus T_\lambda, \end{cases} \tag{25}$$

where

$$\overline{a}^{ij}(x) = \int_0^1 F^{ij}\left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x)\right) ds,$$

and

$$b^i(x) = \int_0^1 \frac{\partial F}{\partial p^i}\left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x)\right) ds,$$

for $i, j = 1, 2, \cdots, n-1, n$. Then we can make the same argument as the proof of Theorem 1.1 except using Lemma 2.1. It remains to verify that $\left(\overline{a}^{ij}(x)\right)$ satisfies the condition (10). In order to show this, we only need to prove that

$$\overline{a}^{in}(x) = 0, \quad x \in \partial\Sigma(T_\lambda) \cap T_\lambda.$$

In fact, by differentiating (24) with respect to m^{in}, we have

$$F^{in}(M,p) + F^{in}(\widetilde{M}, \widetilde{p}) = 0, \quad i = 1, 2, \cdots, n-2, n-1. \tag{26}$$

And by the definition of v, we can get that for $x \in \partial\Sigma(T_\lambda) \cap T_\lambda$,

$$sD^2v(x) + (1-s)D^2u(x) = \widetilde{sD^2u(x) + (1-s)D^2v(x)}, \quad s \in [0,1], \tag{27}$$

and

$$sDv(x) + (1-s)Du(x) = \widetilde{sDu(x) + (1-s)Dv(x)}, \quad s \in [0,1]. \tag{28}$$

And by taking $t = 1 - s$, we have

$$\int_0^1 F^{in} \left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x) \right) ds$$

$$= \int_0^1 F^{in} \left(tD^2v(x) + (1-t)D^2u(x), tDv(x) + (1-t)Du(x) \right) dt,$$

i.e.

$$\int_0^1 F^{in} \left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x) \right) ds \qquad (29)$$

$$= \int_0^1 F^{in} \left(sD^2v(x) + (1-s)D^2u(x), sDv(x) + (1-s)Du(x) \right) ds.$$

Then by (29), (27), (28) and (26), for $i = 1, 2, \cdots, n-2, n-1$ and $x \in \partial \Sigma(T_\lambda) \cap T_\lambda$, we have

$$2\overline{a}^{in}(x) = 2 \int_0^1 F^{in} \left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x) \right) ds$$

$$= \int_0^1 F^{in} \left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x) \right) ds$$

$$+ \int_0^1 F^{in} \left(sD^2v(x) + (1-s)D^2u(x), sDv(x) + (1-s)Du(x) \right) ds$$

$$= \int_0^1 F^{in} \left(sD^2u(x) + (1-s)D^2v(x), sDu(x) + (1-s)Dv(x) \right) ds$$

$$+ \int_0^1 F^{in} \left(\widetilde{sD^2u(x) + (1-s)D^2v(x)}, \widetilde{sDu(x) + (1-s)Dv(x)} \right) ds$$

$$= 0.$$

And then we can follow the proof of Theorem 1.1 to complete our proof.

5.2. An application: Weingarten curvature equation. In Theorem 5.1, if we take $F(M, p) = \sigma_k \left(\lambda \left(A(M, p) \right) \right)$, $k = 1, 2, \cdots, n-1, n$, where $A(M, p) = \left(a^{ij}(M, p) \right)$,

$$a^{ij}(M, p) = \frac{1}{h} \{ m^{ij} - \frac{p^i p^l m^{lj}}{h(1+h)} - \frac{p^j p^k m^{ki}}{h(1+h)} + \frac{p^i p^j p^l p^k m^{lk}}{h^2(1+h)^2} \}, \quad i, j = 1, 2, \cdots, n-1, n,$$

and $h = (1 + |p|^2)^{\frac{1}{2}}$. Then the equation in (22) becomes

$$\sigma_k \left(\lambda \left(A \left(D^2u, Du \right) \right) \right) = f(x'), \quad x \in \Omega,$$

which is called as Weingarten curvature equation. This equation was studied by L.Caffarelli, L.Nirenberg and J.Spruck in [5].

By the computations from Section 1 in [4], the eigenvalues of the real symmetric matrix $A \left(D^2u(x), Du(x) \right) = \left(a^{ij}(x) \right)$ denote the principal curvatures of the C^2 graph $(x, u(x))$. For $k = 1$,

$$\sigma_1 \left(\lambda \left(A \left(D^2u, Du \right) \right) \right) = \mathrm{div} \left(\frac{Du}{(1 + |Du|^2)^{\frac{1}{2}}} \right),$$

which denotes the mean curvature of the hypersurface, see [8]. And for $k = n$,

$$\sigma_n \left(\lambda \left(A \left(D^2u, Du \right) \right) \right) = \frac{\det D^2u}{(1 + |Du|^2)^{\frac{n+2}{2}}},$$

which denotes the Gauss curvature of the hypersurface, also see [8].

If $u \in C^{2,1}(\overline{\Omega})$ and for $i = 1, 2, \cdots, k-1, k,$

$$\sigma_i \left(\lambda \left(A \left(D^2 u(x), Du(x) \right) \right) \right) > 0, \quad x \in \overline{\Omega},$$

then the Weingarten curvature equation satisfies the ellipticity condition (23) and the condition $u < c$ in Ω can be deduced by the equation and Dirichlet boundary condition in problem (22) via strong maximum principle. Since

$$a^{ij}(M, p) = a^{ij}(\widetilde{M}, \widetilde{p}), \quad i, j < n \quad \text{and} \quad i = j = n,$$
$$a^{ij}(M, p) = - a^{ij}(\widetilde{M}, \widetilde{p}), \quad i < n, j = n \quad \text{and} \quad j < n, i = n,$$

we can get that

$$\sigma_k \left(\lambda \left(A(M, p) \right) \right) = \sigma_k \left(\lambda \left(A \left(\widetilde{M}, \widetilde{p} \right) \right) \right).$$

Thus, by applying Theorem 5.1, the mirror symmetry of Ω and the solution to the over-determined problem for Weingarten curvature equation can be obtained.

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