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Local Maximum Principle of Semilinear Nonuniformly Elliptic Equations and Its Applications to Hessian Equations

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Abstract: We obtain a local maximum principle for the semilinear nonuniformly elliptic equations in divergence form, and then show the local $C^{1,1}$ estimate and a Bernstein type result for the solutions of the Hessian equations.

Key words: local maximum principle; nonuniformly elliptic equations; local $C^{1,1}$ estimate; Bernstein type result; Hessian equation

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1 Introduction

In this paper we show the local second order derivative bound for the solutions of the Hessian equations of the form

$$\left(S_k(D^2u)\right)^{\frac{1}{k}} = f(x) \quad \text{in } \Omega, \tag{1}$$

where Ω is a domain in $\mathbb{R}^n$, $S_k(D^2u)$ is the k-th elementary symmetric function of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the Hessian matrix D^2u, which is given by

$$S_k(D^2u) = \sum_{i_1<i_2<\cdots<i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},$$

with $k = 1, 2, \cdots, n$, and f is a positive function in Ω. We also deduce a Bernstein theorem for the Hessian equations

$$S_k(D^2u) = c \quad \text{in } \mathbb{R}^n, \tag{2}$$

here c is a positive constant.

The special case $k = 1$ of the equation (1) is the Poisson equations

$$\Delta u = f(x),$$

while for $k = n$, we have the Monge-Ampère equations

$$\det D^2u = f^n(x).$$

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The equations of this type were treated by Caffarelli, Nirenberg and Spruck[CNS] and Ivochkina[I], who demonstrated the existence of the classical solutions for the Dirichlet problem. Chou, Krylov, Trudinger, Urbas, Wang and others have also discussed the equations, see [CW], [K], [T2], [TW], [U].

We say $u \in C^2(\Omega)$ is k-convex if

$$S_j(D^2u(x)) > 0$$

for any $x \in \Omega$ and $j = 1, 2, \cdots, k$. When $k = 1$, the definition is equivalent to the definition of subharmonic functions, and a function is n-convex if and only if it is convex in Ω. If $u \in C^2(\Omega)$ is k-convex, then the equation (1) is elliptic and $\big(S_k(D^2u)\big)^{\frac{1}{k}}$ is a concave function of the second order derivative of u.

The main result of this paper is the following theorem.

Theorem 1 Let u be a C^3 k-convex solution of the equation (1), and $f \in W^{1,r}_{\mathrm{loc}}(\Omega)$ be positive with $r > n$. Then for any $p > 0$ and any concentric balls $B_R(y) \subset B_{2R}(y) \subset\subset \Omega$ we have

$$\sup_{B_R(y)} |D^2u| \le C\left(\big(R^{-n} + K(R)\big)\int_{B_{2R}(y)} |D^2u(x)|^{\frac{nk}{2}+p}dx\right)^{\frac{1}{p}}, \tag{3}$$

where C is a positive constant depending only on n, k, p and $\inf_{B_{2R}(y)} f$, and

$$K(R) = (R^{n(k-1)} + R^{n(k-2)})\|Df\|^{\frac{krn}{r-n}}_{L^r(B_{2R}(y))}.$$

A result similar to Theorem 1 is included in [U, Theorem 1.6 and 4.5], where the hypotheses of two theorems are satisfied respectively if

$$u \in W^{2,\frac{nk}{2}+p}_{\mathrm{loc}}(\Omega), \qquad f \in C^{1,1}(\Omega),$$

with $p > 0$, or

$$u \in C^4(\Omega), \qquad f \in C^{0,1}(\Omega).$$

We mention also for $f \in C^{1,1}(\overline{\Omega})$ Chou and Wang[CW, Theorem1.5] have proved an interior second derivative bound for the solutions which vanish on the boundary.

To prove Theorem 1 we use a local maximum principle for the semilinear nonuniformly elliptic equations in divergence form

$$D_j(a^{ij}(x,u)D_iu - f_j(x)) - g(x) = 0 \ \text{ in } \Omega, \tag{4}$$

that has independent interest and generalizes a local pointwise estimate first proved by De Giorgi[G], also see [GT, Theorem 8.17]. We will assume that the equations (4) is nonuniformly elliptic in Ω, that is, there exist positive constants λ, Λ and nonnegative constants α, β such that

$$\lambda|u|^{-\alpha}|\xi|^2 \le a^{ij}(x,u)\xi_i\xi_j \le \Lambda|u|^{\beta}|\xi|^2 \tag{5}$$

for all $x \in \Omega$, $u \in \mathbb{R}^1$ and $\xi \in \mathbb{R}^n$. If $\alpha = \beta = 0$, the equation (4) becomes uniformly elliptic in Ω. A function $u \in W^{1,s}_{\mathrm{loc}}(\Omega)$ with

$$s = \frac{n(\beta+2)}{n+\beta}$$

is said a weak solution (subsolution, supersolution) of the equations (4) respectively in Ω if

$$\int_\Omega \left((a^{ij}(x,u)D_iu - f_j(x))D_jv + vg(x)\right)dx = 0 \ (\le 0,\ \ge 0) \tag{6}$$

for all nonnegative functions $v \in W_0^{1,s}(\Omega)$. v is a legitimate test function in (6) will be seen in the beginning of Section 2. The following local maximum principle is proved by using the Moser technique of iteration from [Mo].

Theorem 2 Let $a^{ij}(x,u)$ satisfy the condition (5), and u be a locally bounded subsolution of the equations (4) in Ω, such that

$$u^{\alpha-1}f_j \in L^r_{\mathrm{loc}}(\Omega), \qquad u^{\alpha-1}g \in L^{\frac{r}{2}}_{\mathrm{loc}}(\Omega)$$

for $r > n$ and $j = 1, 2, \cdots, n$. Then we have for any ball $B_{2R}(y) \subset\subset \Omega$ and

$$p > \max\left\{0, 1+\beta - \frac{n}{2}(\alpha+\beta)\right\},$$

$$\sup_{B_R(y)} u \le C\left(\left(R^{-n} + K^{\frac{rn}{r-n}}\right)\int_{B_{2R}(y)} u^{p+\frac{n}{2}(\alpha+\beta)}(x)dx\right)^{\frac{1}{p}}, \tag{7}$$

where C is a positive constant depending only on $n, p, \lambda, \Lambda, \alpha, \beta$ and r, and

$$K = \sum_{j=1}^n \|u^{\alpha-1}f_j\|_{L^r(B_{2R}(y))} + \|u^{\alpha-1}g\|^{\frac{1}{2}}_{L^{\frac{r}{2}}(B_{2R}(y))}.$$

The ordinary local maximum principle has been known for many years and its value in the theory of PDEs is well known. The local maximum principle for the uniformly elliptic equations of the special form

$$D_i(a^{ij}(x)D_ju) = 0$$

was established in the pioneering work of de Giorgi[G]. The proof was extended to the linear equations

$$D_i(a^{ij}(x)D_ju + b^i(x)u - f^i(x)) + c^i(x)D_iu + d(x)u - g(x) = 0$$

by Morrey[M], Stampacchia[St] and to the quasilinear equations

$$D_iA^i(x,u,Du) + B(x,u,Du) = 0$$

by Ladyzhenskaya and Ural'tseva[LU]. We mention also Trudinger's work on nonuniformly elliptic equations, see [T1] and its references.

A celebrated result of Calabi[Ca], that generalizes a two dimensional theorem by Jörgens[J], asserts that if u is a C^5 convex solution of the Monge-Ampère equation

$$\det D^2u = 1 \quad \text{in } \mathbb{R}^n \tag{8}$$

and $n \le 5$, then u is a quadratic polynomial. This statement was extended to all dimensions by Pogorelov[P] and Cheng and Yau[CY]. We also investigate the validity of the results for the Hessian equations (2). A Bernstein type result is derived by means of Theorem 1 and Evans-Krylov Theorem[CC].

Theorem 3 Suppose u is a C^3 k-convex solution of the equation (2). If there exist positive constants p and C, such that

$$\int_{B_R(0)} |D^2u(x)|^{\frac{nk}{2}+p}dx \le CR^n \tag{9}$$

for any $R>0$. Then u is a quadratic polynomial.

Theorem 3 extends the Bernstein theorem [S] for the Monge-Ampère equation (8) under the assumption that D^2u is bounded to the Hessian equations (2).

The paper is organized as follows. The local maximum principle for the nonuniformly elliptic equations (4) is proved in Section 2. The proof of Theorem 1 and 3 are then carried out in Section 3.

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2 Local Maximum Principle

We begin with explaining the validity of the relation (6). If

$$u \in W^{1,s}_{\rm loc}(\Omega), \quad f_j \in L^{\frac{s}{s-1}}_{\rm loc}(\Omega), \quad g \in L^{\frac{ns}{ns-n+s}}_{\rm loc}(\Omega), \quad v \in W^{1,s}_0(\Omega) \tag{10}$$

with

$$s=\frac{n(\beta+2)}{n+\beta} \quad \text{and} \quad j=1,2,\cdots,n,$$

we have by (5) and Hölder inequality[GT,(7.11)]

$$\begin{aligned}\|a^{ij}D_iuD_jv\|_{L^1_{\rm loc}(\Omega)} &\le \Lambda\||u|^\beta|Du||Dv|\|_{L^1_{\rm loc}(\Omega)}\\ &\le \Lambda\||u|^\beta\|_{L^{\frac{s}{s-2}}_{\rm loc}(\Omega)}\|Du\|_{L^s_{\rm loc}(\Omega)}\|Dv\|_{L^s(\Omega)}.\end{aligned}$$

Noting

$$\frac{\beta s}{s-2}=\frac{ns}{n-s}, \quad s<n$$

and using Sobolev inequality[GT,(7.26)], we get that

$$\||u|^\beta\|_{L^{\frac{s}{s-2}}_{\rm loc}(\Omega)} = \|u\|^\beta_{L^{\frac{\beta s}{s-2}}_{\rm loc}(\Omega)} = \|u\|^\beta_{L^{\frac{ns}{n-s}}_{\rm loc}(\Omega)} \le C\|Du\|^\beta_{L^s_{\rm loc}(\Omega)}.$$

Thus

$$\|a^{ij}D_iuD_jv\|_{L^1_{\rm loc}(\Omega)} \le C\|Du\|^{\beta+1}_{L^s_{\rm loc}(\Omega)}\|Dv\|_{L^s(\Omega)},$$

where $C=C(n,\beta,\Lambda)$. Similarly, we also have

$$\|f_jD_jv\|_{L^1_{\rm loc}(\Omega)} \le \|f_j\|_{L^{\frac{s}{s-1}}_{\rm loc}(\Omega)}\|Dv\|_{L^s(\Omega)},$$

$$\|gv\|_{L^1_{\rm loc}(\Omega)} \le \|g\|_{L^{\frac{ns}{ns-n+s}}_{\rm loc}(\Omega)}\|v\|_{L^{\frac{ns}{n-s}}(\Omega)} \le C\|g\|_{L^{\frac{ns}{ns-n+s}}_{\rm loc}(\Omega)}\|Dv\|_{L^s(\Omega)}.$$

Consequently the definition (6) is reasonable under the assumption (10). Now we prove Theorem 2 by using a Moser type iteration.

Proof of Theorem 2 Replacing $u^+, \Omega^+ = \{x \in \Omega \mid u(x) > 0\}$ by u, Ω, we see that there is no loss of generality in assuming $u > 0$. For constant $q > 1$ and nonnegative cutoff function $\eta \in C_0^1(\Omega)$, the test function $v = \eta^2 u^{q-1}$ is valid in (6). By the definition of the weak subsolutions, we obtain

$$\begin{aligned} 0 \ge & \int_\Omega (a^{ij} D_i u - f_j) D_j(\eta^2 u^{q-1}) dx + \int_\Omega \eta^2 u^{q-1} g dx \\ = & (q-1) \int_\Omega \eta^2 u^{q-2} a^{ij} D_i u D_j u dx + 2 \int_\Omega \eta u^{q-1} a^{ij} D_i u D_j \eta dx \\ & - (q-1) \int_\Omega \eta^2 u^{q-2} f_j D_j u dx - 2 \int_\Omega \eta u^{q-1} f_j D_j \eta dx + \int_\Omega \eta^2 u^{q-1} g dx. \end{aligned}$$

Using the Cauchy inequality and the ellipticity condition (5), we can estimate

$$\begin{aligned} 0 \ge & \frac{q-1}{2} \int_\Omega \eta^2 u^{q-2} a^{ij} D_i u D_j u dx - \frac{2}{q-1} \int_\Omega u^q a^{ij} D_i \eta D_j \eta dx \\ & - (q-1) \int_\Omega \eta^2 u^{q-2} f_j D_j u dx - 2 \int_\Omega \eta u^{q-1} f_j D_j \eta dx + \int_\Omega \eta^2 u^{q-1} g dx \\ \ge & \frac{\lambda(q-1)}{2} \int_\Omega \eta^2 u^{q-\alpha-2} |Du|^2 dx - \frac{2\Lambda}{q-1} \int_\Omega u^{q+\beta} |D\eta|^2 dx \\ & - (q-1) \int_\Omega \eta u^{\frac{q-\alpha-2}{2}} D_j u \cdot \eta u^{\frac{q+\alpha-2}{2}} f_j dx - 2 \int_\Omega \eta u^{\frac{q+\alpha-2}{2}} f_j \cdot u^{\frac{q-\alpha}{2}} D_j \eta dx \\ & + \int_\Omega \eta^2 u^{q-1} g dx \\ \ge & \frac{\lambda(q-1)}{4} \int_\Omega \eta^2 u^{q-\alpha-2} |Du|^2 dx - \frac{2\Lambda}{q-1} \int_\Omega u^{q+\beta} |D\eta|^2 dx \\ & - \left(\frac{2(q-1)}{\lambda} + 1 \right) \int_\Omega \eta^2 u^{q+\alpha-2} |f|^2 dx - \int_\Omega u^{q-\alpha} |D\eta|^2 dx + \int_\Omega \eta^2 u^{q-1} g dx. \end{aligned}$$

It is clearly from (5) that

$$\lambda u^{q-\alpha} \le \Lambda u^{q+\beta}, \tag{11}$$

and thus we get

$$\int_\Omega \eta^2 \left| D u^{\frac{q-\alpha}{2}} \right|^2 dx \le C \int_\Omega \left(u^{q+\beta} |D\eta|^2 + \eta^2 u^{q-\alpha} \left((u^{\alpha-1}|f|)^2 + u^{\alpha-1}|g| \right) \right) dx, \tag{12}$$

where C, a constant depending only on λ, Λ and q, is bounded when q is bounded away from 1.

Now we have from the Sobolev inequality[GT, Theorem7.10] and (12), (11)

$$\begin{aligned} \left(\int_\Omega \eta^{\frac{2n}{n-2}} u^{\frac{n(q-\alpha)}{n-2}} dx \right)^{\frac{n-2}{n}} & = \left(\int_\Omega \left(\eta u^{\frac{q-\alpha}{2}} \right)^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\ & \le C \int_\Omega \left| D\left(\eta u^{\frac{q-\alpha}{2}} \right) \right|^2 dx \\ & \le C \int_\Omega \left(u^{q-\alpha} |D\eta|^2 + \eta^2 \left| D u^{\frac{q-\alpha}{2}} \right|^2 \right) dx \\ & \le C \int_\Omega \left(u^{q+\beta} |D\eta|^2 + \eta^2 u^{q-\alpha} \left((u^{\alpha-1}|f|)^2 + u^{\alpha-1}|g| \right) \right) dx. \end{aligned} \tag{13}$$

By the Hölder inequality and the interpolation inequality[GT, (7.10)], we have for any $\varepsilon > 0$

$$\begin{aligned}&\int_\Omega \left(\eta^2\, u^{q-\alpha}\left((u^{\alpha-1}|f|)^2 + u^{\alpha-1}|g|\right)\right)dx\\&\le \left(\int_\Omega \left((u^{\alpha-1}|f|)^2 + u^{\alpha-1}|g|\right)^{\frac{r}{2}} dx\right)^{\frac{2}{r}} \left(\int_\Omega (\eta^2 u^{q-\alpha})^{\frac{r}{r-2}} dx\right)^{\frac{r-2}{r}}\\&\le K^2\left(\varepsilon\left(\int_\Omega \eta^{\frac{2n}{n-2}} u^{\frac{n(q-\alpha)}{n-2}} dx\right)^{\frac{n-2}{n}} + \varepsilon^{-\frac{n}{r-n}} \int_\Omega \eta^2 u^{q-\alpha} dx\right),\end{aligned}$$

if $\mathrm{supp}\eta \subset B_{2R}$. Hence by the choice of

$$\varepsilon = \frac{1}{2CK^2},$$

and substitution into (13), we obtain

$$\int_\Omega \eta^{\frac{2n}{n-2}} u^{\frac{n(q-\alpha)}{n-2}} dx \le C\left(\int_\Omega u^{q+\beta}|D\eta|^2 dx + K^{\frac{2r}{r-n}} \int_\Omega \eta^2 u^{q-\alpha} dx\right)^{\frac{n}{n-2}}.$$

Let $q = p + \frac{n}{2}(\alpha+\beta) - \beta$, we have

$$\frac{n(q-\alpha)}{n-2} = \frac{n}{n-2}\left(p + \frac{n-2}{2}(\alpha+\beta)\right) = \frac{np}{n-2} + \frac{n}{2}(\alpha+\beta),$$

and may rewrite by (11) the above inequality

$$\int_\Omega \eta^{\frac{2n}{n-2}} u^{\frac{np}{n-2}+\frac{n}{2}(\alpha+\beta)} dx \le C\left(\int_\Omega u^{p+\frac{n}{2}(\alpha+\beta)}\left(|D\eta|^2 + \eta^2 K^{\frac{2r}{r-n}}\right)dx\right)^{\frac{n}{n-2}}, \tag{14}$$

where C is a positive constant, which depends only on $n, \lambda, \Lambda, \alpha, \beta$ and p, and is bounded if p is bounded away from

$$1 + \beta - \frac{n}{2}(\alpha+\beta).$$

This inequality (14) can now be iterated to yield the desired estimates. We specify the cutoff function η and the exponent p more precisely. Writing $\kappa = \frac{n}{n-2}$. For $R > 0$, $j = 0, 1, 2, \cdots$, and

$$p > \max\left\{0, 1 + \beta - \frac{n}{2}(\alpha+\beta)\right\},$$

set

$$R_j = (1 + 2^{-j})R, \quad p_j = p\kappa^j,$$
$$\eta_j \equiv 1 \text{ in } B_{R_j}, \quad \eta_j \equiv 0 \text{ in } \Omega \setminus B_{R_{j-1}},$$

with

$$0 \le \eta_j \le 1, \quad |D\eta_j| \le \frac{2}{R_j - R_{j-1}} = \frac{2^{j+1}}{R} \text{ in } \Omega.$$

By the inequality (14), we have

$$\int_{B_{R_j}} u^{p_j+\frac{n}{2}(\alpha+\beta)} dx \le C\left(\frac{2^{j+1}}{R} + K^{\frac{r}{r-n}}\right)^{2\kappa}\left(\int_{B_{R_{j-1}}} u^{p_{j-1}+\frac{n}{2}(\alpha+\beta)} dx\right)^{\kappa},$$

$$\left(\int_{B_{R_j}} u^{p_j+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p_j}} \le \left(2^{j-1}C(R,K)\right)^{\frac{2}{p_{j-1}}}\left(\int_{B_{R_{j-1}}} u^{p_{j-1}+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p_{j-1}}}, \tag{15}$$

where

$$C(R,K) = C\left(\frac{1}{R} + K^{\frac{r}{r-n}}\right)$$

and C is independent of j. By iterating (15) we obtain

$$\begin{aligned}
&\left(\int_{B_{R_j}} u^{p_j+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p_j}}\\
&\le \left(2^{j-1}C(R,K)\right)^{\frac{2}{p_{j-1}}}\left(2^{j-2}C(R,K)\right)^{\frac{2}{p_{j-2}}}\left(\int_{B_{R_{j-2}}} u^{p_{j-2}+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p_{j-2}}}\\
&\le \cdots\\
&\le \left(\prod_{i=1}^{j}\left(2^{i-1}C(R,K)\right)^{\frac{2}{p_{i-1}}}\right)\left(\int_{B_{R_0}} u^{p_0+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p_0}}\\
&= 2^{\frac{2}{p}\sum_{i=1}^{j}\frac{i-1}{\kappa^{i-1}}}C(R,K)^{\frac{2}{p}\sum_{i=1}^{j}\frac{1}{\kappa^{i-1}}}\left(\int_{B_{2R}} u^{p+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p}}\\
&\le 2^{\frac{2}{p(\kappa-1)^2}}C(R,K)^{\frac{2\kappa}{p(\kappa-1)}}\left(\int_{B_{2R}} u^{p+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p}}\\
&= 2^{\frac{(n-2)^2}{2p}}\left(C(R,K)^n\int_{B_{2R}} u^{p+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p}}.
\end{aligned}$$

By means of (5) and the definition of $C(R,K)$, we have

$$\begin{aligned}
\left(\int_{B_R} u^{p_j}dx\right)^{\frac{1}{p_j}} &\le C\left(\int_{B_{R_j}} u^{p_j+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p_j}}\\
&\le C\left(\left(R^{-n}+K^{\frac{rn}{r-n}}\right)\int_{B_{2R}} u^{p+\frac{n}{2}(\alpha+\beta)}dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Consequently, letting j tend to infinity, we finally establish the estimate (11) since p_j tends to infinity, and

$$\sup_{B_R} u = \lim_{p\to\infty}\left(\int_{B_R} u^p dx\right)^{\frac{1}{p}}.$$

3 Some Estimates to Hessian Equations

In this section, we apply the local maximum principle Theorem 2 to the Hessian equations (1), and prove Theorem 1, a second derivative bound. Rewrite the equations (1) as

$$F(D^2u) := \left(S_k(D^2u)\right)^{\frac{1}{k}} = f(x) \quad \text{in } \Omega.$$

In its proof, we introduce the following notations:

$$F^{ij} = \frac{\partial F}{\partial u_{ij}}(D^2u), \qquad S_k^{ij} = \frac{\partial S_k}{\partial u_{ij}}(D^2u).$$

First we collect here some results concerning $S_k(D^2u)$.

Proposition 4 Suppose u is a C^2 k-convex function, then

(a) $\sum_{i=1}^{n} S_k^{ii}(D^2u) = (n-k+1)S_{k-1}(D^2u)$;

(b) The Newton-Maclaurin inequality

$$\left(\frac{S_{k-l+1}(D^2u)}{C_n^{k-l+1}}\right)^{\frac{1}{k-l+1}} \le \left(\frac{S_{k-l}(D^2u)}{C_n^{k-l}}\right)^{\frac{1}{k-l}} \quad \text{for } l=1,2,\cdots,k-1;$$

(c) $\dfrac{\left(S_{k-l+1}^{ij}(D^2u)\right)}{S_{k-l+1}(D^2u)} \ge \dfrac{\left(S_{k-l}^{ij}(D^2u)\right)}{S_{k-l}(D^2u)}$, in the sense of matrices, for $l=1,2,\cdots,k-1$;

(d) The Reilly formula

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(S_k^{ij}(D^2u)) = 0,\ i=1,2,\cdots,n,\ \text{if } u\in C^3.$$

Here

$$C_n^l = \frac{n!l!}{(n-l)!}, \qquad l=1,2,\cdots,n.$$

From Proposition 4 we obtain immediately the following result.

Corollary 5 For any $\Omega_0 \subset\subset \Omega$, we have

$$\frac{I}{C\Delta u} \le (S_k^{ij}(D^2u)) \le C(\Delta u)^{k-1}I \ \text{ in } \Omega_0,$$

in the sense of matrices, where I is the identify matrix, and C is a positive constant depending only on $n, k, \inf_{\Omega_0} f$.

Proof Using Proposition 4 (a) and (b), we have in the sense of matrices

$$\begin{aligned}(S_k^{ij}(D^2u)) &\le \left(\sum_{i=1}^{n} S_k^{ii}(D^2u)\right) I \\ &= (n-k+1)S_{k-1}(D^2u)I \\ &\le (n-k+1)C_n^{k-1}\left(\frac{S_1(D^2u)}{C_n^1}\right)^{k-1} I \\ &\le C(\Delta u)^{k-1}I.\end{aligned}$$

On the other hand, it follows from Proposition 4 (c) and the equation (1)

$$(S_k^{ij}(D^2u)) \ge \frac{S_k(D^2u)}{S_1(D^2u)}(S_1^{ij}(D^2u)) = \frac{f^k}{\Delta u}I \ge \frac{I}{C\Delta u}.$$

Denote the first order difference quotients

$$\Delta_l^h u(x) = \frac{u(x+he_l)-u(x)}{h},$$

and the second order difference quotients

$$\Delta_{ll}^h u(x) = \frac{u(x+he_l)-2u(x)+u(x-he_l)}{h^2},$$

where $h > 0$, e_l is the l-th coordinate direction, and $l = 1, 2, \cdots, n$.

Now we begin to prove Theorem 1.

Proof of Theorem 1 Let $x, x \pm he_l \in \Omega$ for sufficiently small $h > 0$, we use the concavity of F to conclude that

$$F(D^2u(x \pm he_l)) \le F(D^2u(x)) + F^{ij}(D^2u(x))(D_{ij}u(x \pm he_l) - D_{ij}u(x)). \tag{16}$$

We have by (1) and (16) that

$$F^{ij}D_{ij}(\Delta_{ll}^h u(x)) \ge \Delta_{ll}^h f(x) \tag{17}$$

for $x \in \Omega$ and $l = 1, 2, \cdots, n$. It follows from

$$F^{ij} = \frac{1}{k}S_k^{\frac{1}{k}-1}S_k^{ij} = \frac{1}{k}f^{1-k}S_k^{ij}$$

that

$$S_k^{ij}D_{ij}(\Delta_{ll}^h u(x)) \ge kf^{k-1}\Delta_{ll}^h f(x). \tag{18}$$

Thus for all nonnegative functions $v \in W_0^{1,s}(\Omega)$ with

$$s = \frac{n(k+1)}{n+k-1},$$

$$\int_\Omega v\left(S_k^{ij}D_{ij}\left(\sum_{l=1}^n \Delta_{ll}^h u\right) - kf^{k-1}\sum_{l=1}^n \Delta_{ll}^h f\right)dx \ge 0.$$

From Proposition 4(d), we see by integration and difference by part

$$\int_\Omega \left(D_j v S_k^{ij} D_i\left(\sum_{l=1}^n \Delta_{ll}^h u\right) - k\sum_{l=1}^n \Delta_l^h(vf^{k-1})\Delta_l^h f\right)dx \le 0.$$

Note

$$D_j v \in L^s(\Omega), \quad S_k^{ij}(D^2u), \quad f \in L_{\mathrm{loc}}^\infty(\Omega),$$
$$\Delta_{ll}^h u \in W_{\mathrm{loc}}^{1,\frac{s}{s-1}}(\Omega), \quad \Delta_l^h v \in L^s(\Omega), \quad \Delta_l^h f, \Delta_l^h f^{k-1} \in L_{\mathrm{loc}}^r(\Omega),$$

and

$$\frac{1}{s} + \frac{1}{r} \le \frac{n+k-1}{n(k+1)} + \frac{1}{n} \le 1.$$

Letting $h \to 0$, we have

$$\int_\Omega \left(D_j v\left(S_k^{ij}D_i(\Delta u) - kf^{k-1}D_j f\right) - k(k-1)vf^{k-2}|Df|^2\right)dx \le 0.$$

That is, Δu is a subsolution of the elliptic equation in the divergence form

$$D_j\left(S^{ij}D_i(\Delta u) - kf^{k-1}D_j f\right) + k(k-1)f^{k-2}|Df|^2 = 0,$$

and satisfies the following nonuniformly elliptic condition by Corollary 5

$$\frac{|\xi|^2}{C\Delta u} \le S_k^{ij}(D^2u)\xi_i\xi_j \le C(\Delta u)^{k-1}|\xi|^2$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

From Sobolev inequality

$$\|f^{k-1}Df\|_{L^r(B_{2R})} \le \left(\sup_{B_{2R}}|f|\right)^{k-1} \|Df\|_{L^r(B_{2R})} \le CR^{(k-1)(1-\frac{n}{r})}\|Df\|^k_{L^r(B_{2R})},$$

and

$$\begin{aligned}\|f^{k-2}|Df|^2\|_{L^{\frac{r}{2}}(B_{2R})} &\le \left(\sup_{B_{2R}}|f|\right)^{k-2} \|Df\|^2_{L^r(B_{2R})}\\ &\le CR^{(k-2)(1-\frac{n}{r})}\|Df\|^k_{L^r(B_{2R})},\end{aligned}$$

where C depends only on n, k, r. Now we can take in Theorem 2

$$\alpha = 1, \quad \beta = k-1, \quad f_j = kf^{k-1}D_jf, \quad g = k(k-1)f^{k-2}|Df|^2,$$

and so obtain the desired estimate (3).

Finally, we prove Theorem 3.

Proof of Theorem 3 Set

$$y = \frac{x}{t}, \qquad u_t(y) = \frac{u(ty)}{t^2}, \qquad t > 0.$$

We see that $D_x^2u = D_y^2u_t$ and u_t satisfies the same equation

$$S_k(D_y^2u_t) = c \ \text{ in } \mathbb{R}^n.$$

From Theorem 1, we have for any $R > 4$

$$\begin{aligned}\sup_{B_R(0)}|D_y^2u_t| &\le C\left(R^{-n}\int_{B_R(0)}|D_y^2u_t|^{\frac{nk}{2}+p}dy\right)^{\frac{1}{p}}\\ &\le C\left((tR)^{-n}\int_{B_{tR}(0)}|D_x^2u|^{\frac{nk}{2}+p}dx\right)^{\frac{1}{p}},\end{aligned}$$

where C is a positive constant depending only on n, k, p and c. By using (9), we get

$$\sup_{B_R(0)}|D_y^2u_t| \le C \tag{19}$$

for some positive constant C independent of t. The estimate (19) asserts that the equation (2) is uniformly elliptic. Recall that $S_k^{\frac{1}{k}}$ is concave. Now the interior $C^{2,\alpha}$ estimate

$$\|u_t\|_{C^{2,\alpha}(\overline{B_1(0)})} \le C \tag{20}$$

follows from Evans–Krylov Theorem[CC, Theorem 6.1], where $\alpha \in (0,1)$ and $C > 0$ are constants independent of t.

For $x \in \mathbb{R}^n$ and $t > |x|$, we obtain by (20)

$$|D_x^2u(x) - D_x^2u(0)| = |D_y^2u_t(y) - D_y^2u_t(0)| \le C|y|^\alpha = C\left|\frac{x}{t}\right|^\alpha.$$

Letting t go to ∞, we conclude $D_x^2u(x) = D_x^2u(0)$ for $x \in \mathbb{R}^n$, and hence u is a quadratic polynomial.

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半线性非一致椭圆方程的局部最大值原理及其在 Hessian 方程上的应用

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摘要: 本文获得了一个半线性散度型非一致椭圆方程的局部最大值原理,并由此导出了 Hessian 方程解的局部 $C^{1,1}$ 估计和一个 Bernstein 型结果.

关键词: 局部最大值原理; 非一致椭圆方程; 局部 $C^{1,1}$ 估计; Bernstein 型结果; Hessian 方程