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Liouville theorem and isolated singularity of fractional Laplacian system with critical exponents

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ABSTRACT

This paper is devoted to the fractional Laplacian system with critical exponents. We use the method of moving spheres to derive a Liouville Theorem with at most three radial solutions, and then prove the solutions in $\mathbb{R}^n\backslash\{0\}$ are radially symmetric and monotonically decreasing. Together with blow up analysis, we get the upper bound of the local solutions in $B_1\backslash\{0\}$. Our results is an extension of the classical works by Caffarelli et al. (1989), Caffarelli et al. (2014), Chen and Lin (2015) and Guo and Liu (2008).

1. Introduction

The semilinear elliptic equation

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \quad \mathbb{R}^n, \\ u > 0 \quad \text{and} \quad u \in C^2(\mathbb{R}^n), \end{cases} \tag{1.1}$$

with critical exponent has been studied in many papers, where $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian and $n \geq 3$. The following celebrated Liouville Theorem was established by Caffarelli–Gidas–Spruck [4] using the method of moving planes. There exist a positive constant ε and $y \in \mathbb{R}^n$ such that any C^2 solution of (1.1) has to be the form

$$U_1(x) := (n(n-2))^{\frac{n-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{n-2}{2}}. \tag{1.2}$$

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With reasonable behavior at infinity, namely $u(x) = O(|x|^{2-n})$ for large $|x|$, the result was obtained earlier by Obata [41] and Gidas–Ni–Nirenberg [21]. The proof of Obata was more geometric, while the proof of Gidas–Ni–Nirenberg was by the method of moving planes. Li–Zhang [33] developed a rather systematic, and simpler approach to Liouville Theorem using the method of moving spheres. They can catch the form of solutions directly, instead of reducing it to the radially symmetry of u and concluding by using ordinary differential equation.

Such Liouville Theorem has played a fundamental role in the study of semilinear elliptic equations with critical exponent, which include the Yamabe problem and the Nirenberg problem. Eq. (1.1) is conformally invariant in the sense that if u is a solution, then after a conformal mapping $x \longmapsto y$ the function

$$v(y) := u(x)J^{\frac{2-n}{2n}}(x),$$

where J is the Jacobian, is also a solution. In view of conformal geometry, a solution u of (1.1) defines a conformal flat metric $g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}$ with constant scalar curvature. It is worth mentioning that the classification of solutions of higher order conformally invariant equations was studied by Wei–Xu [49], Lu–Wei–Xu [39] and Lu–Wang–Zhu [38].

The works of Schoen and Yau [42–44] on conformal flat manifolds and the Yamabe problem have highlighted the significance of studying solutions of (1.1) with a nonempty singular set. The issues related to

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \ \ \mathbb{R}^n\backslash\{0\}, \\ u > 0 \quad \text{and} \quad u \in C^2(\mathbb{R}^n\backslash\{0\}) \end{cases} \tag{1.3}$$

have received great interest and have been widely studied in [4,13–15,19,30,37,46] and references therein, since they are the simplest examples of singular equations. All C^2 positive radial solutions of (1.3) had been described by Fowler [19]. A classification of global solutions was given by Caffarelli–Gidas–Spruck in [4] that if 0 is a non-removable singular point of u, then u is radially symmetry about the origin.

Remark that the full understanding asymptotical behavior of

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in} \ \ B_1\backslash\{0\}, \\ u > 0 \quad \text{and} \quad u \in C^2(B_1\backslash\{0\}) \end{cases}$$

is crucial to construct singular solutions on Riemannian manifolds, where B_1 is the unit ball of $\mathbb{R}^n$. Caffarelli–Gidas–Spruck also proved in the same paper [4] that u is asymptotically symmetric with respect to the origin and furthermore,

$$u(x) = u_0(x)(1 + o(1)) \quad \text{near} \quad x = 0, \tag{1.4}$$

where $u_0(x)$ is a radially symmetric solution of (1.3). Thus, a corollary of (1.4) is that there exist two positive constants c_1 and c_2 such that

$$c_1|x|^{-\frac{n-2}{2}} \le u(x) \le c_2|x|^{-\frac{n-2}{2}} \quad \text{near} \quad x = 0.$$

Moreover, García–Huidobro–Manásevich–Mitidieri–Yarur [20] obtained the existence and nonexistence of positive singular solutions for a class of semilinear elliptic systems in $B_1 \setminus \{0\}$.

It is worthy of studying coupled exponent problem, including various scalar equations and coupled systems, because they are challenging in view of mathematics. Classifications of solutions to the system

$$\begin{cases} -\Delta u = \alpha_1 u^{\frac{n+2}{n-2}} + \beta u^{\frac{2}{n-2}} v^{\frac{n}{n-2}} \qquad \text{in} \quad \mathbb{R}^n, \\ -\ \Delta v = \alpha_2 v^{\frac{n+2}{n-2}} + \beta v^{\frac{2}{n-2}} u^{\frac{n}{n-2}} \qquad \text{in} \quad \mathbb{R}^n, \\ u, v > 0 \ \text{ and } \ u, v \in C^2(\mathbb{R}^n), \end{cases} \tag{1.5}$$

where α_1, α_2 and β are positive constants, were studied respectively Chen–Li [8] and Guo–Liu [25]. From [8,25], it follows that u and v are radially symmetric with respect to the same point. Moreover, $(u, v) = (kU_1, lU_1)$, where $k, l > 0$ satisfies

$$\alpha_1 k^{\frac{4}{n-2}} + \beta k^{\frac{4-n}{n-2}} l^{\frac{n}{n-2}} = 1, \quad \alpha_2 l^{\frac{4}{n-2}} + \beta l^{\frac{4-n}{n-2}} k^{\frac{n}{n-2}} = 1,$$

and U_1 is defined as (1.2). In fact, more general systems were considered in [8,25] and (1.5) is a special case of their problems. Moreover, Wei–Weth [48] also studied the other types of systems of two-coupled Schrödinger equations. Existence and nonexistence of positive radial solutions for a class of semilinear elliptic system in R^n was discussed by Mitidieri [40] and Chen–Lu [17].

Studying the properties of positive singular solutions to a two-coupled elliptic system with critical exponents is closely related to coupled nonlinear Schrödinger equations for nonlinear optics and Bose–Einstein condensates. Stimulated by this, Chen–Lin [16] proved that if (u, v) is a singular solution of

$$\begin{cases} -\Delta u = \alpha_1 u^{\frac{n+2}{n-2}} + \beta u^{\frac{2}{n-2}} v^{\frac{n}{n-2}} & \text{in} \quad \mathbb{R}^n\backslash\{0\}, \\ -\Delta v = \alpha_2 v^{\frac{n+2}{n-2}} + \beta v^{\frac{2}{n-2}} u^{\frac{n}{n-2}} & \text{in} \quad \mathbb{R}^n\backslash\{0\}, \\ u, v > 0 \ \text{ and } \ u, v \in C^2(\mathbb{R}^n\backslash\{0\}), \end{cases}$$

then both u and v are radially symmetric about the origin and are strictly decreasing with respect to $r = |x| > 0$. With some additional conditions, they also obtained that either 0 is a removable singular point, or there exist two positive constants c_1 and c_2 such that

$$c_1|x|^{-\frac{n-2}{2}} \le u(x), \quad v(x) \le c_2|x|^{-\frac{n-2}{2}} \quad \text{near} \quad x = 0.$$

In recent years, the fractional Laplacian has more and more applications in Physics, Chemistry, Biology, Probability, and Finance, and has drawn more and more attention from the mathematical community.

It can be understood as the infinitesimal generator of a stable Lévy process [2]. In particular, the fractional Laplacian with the critical exponent arises in contexts such as the Euler–Lagrangian equations of Sobolev inequalities [12,31,36], a fractional Yamabe problem [22,23,29], a fractional Nirenberg problem [26,27] and so on.

The fractional Laplacian takes the form

$$(-\Delta)^\sigma u(x) := C_{n,\sigma} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\backslash B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2\sigma}} dy, \tag{1.6}$$

where $\sigma \in (0, 1)$, $n \ge 2$, and

$$C_{n,\sigma} := \frac{2^{2\sigma}\sigma\Gamma(\frac{n}{2} + \sigma)}{\pi^{\frac{n}{2}}\Gamma(1-\sigma)}$$

with the gamma function Γ. The operator $(-\Delta)^\sigma$ is well defined in the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$.

One can also define the fractional Laplacian acting on spaces of functions with weaker regularity. Let the space

$$L_\sigma(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\sigma}} dx < \infty \right\},$$

endowed with the norm

$$\|u\|_{L_\sigma(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\sigma}} dx.$$

We can verify that if $u \in C^2(\mathbb{R}^n) \cap L_\sigma(\mathbb{R}^n)$, the integral on the right hand side of (1.6) is well defined in $\mathbb{R}^n$.

A feature of

$$\begin{cases} (-\Delta)^{\sigma} u = u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in} \;\; \mathbb{R}^n, \\ u > 0 \quad \text{and} \quad u \in C^2(\mathbb{R}^n) \cap L_{\sigma}(\mathbb{R}^n) \end{cases} \tag{1.7}$$

is conformal invariant, and one may refer to [7,24] for its connections to conformal geometry. Since the radially symmetry property is essential for the development of symmetrization techniques for fractional elliptic and parabolic partial differential equations, a lot of people are interested in the radially symmetry results. More recently, Jin–Li–Xiong showed in [26] that a solution of (1.7) has to be the form

$$U_{\sigma}(x) := \left(\frac{\Gamma(n+\frac{\sigma}{2})}{\Gamma(n-\frac{\sigma}{2})} \right)^{\frac{n-2\sigma}{4\sigma}} \left(\frac{2\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{n-2\sigma}{2}}, \tag{1.8}$$

where ε is a positive constant and $y \in \mathbb{R}^n$. By the fact that

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-\frac{1}{2})} = \frac{n(n-2)}{4},$$

we know that the result is consistent with (1.2). On the other hand, via the corresponding integral equations Chen–Li–Ou [11,12] classified the solution of the fractional and higher order semilinear partial differential equations. Lately, Chen–Li–Li [9] developed the approach to study a more general fractional operators on a bounded domain.

Caffarelli–Jin–Sire–Xiong studied the global behaviors of solutions of the fractional Yamabe equation

$$\begin{cases} (-\Delta)^{\sigma} u = u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in} \;\; \mathbb{R}^n \backslash \{0\}, \\ u > 0 \quad \text{and} \quad u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap L_{\sigma}(\mathbb{R}^n) \end{cases} \tag{1.9}$$

with an isolated singularity at the origin. They proved in [5] that if the origin is a non-removable isolated singularity, then the solution u of (1.9) is radially symmetric with respect to the origin and strictly decreasing with respect to $r = |x| > 0$. It is consistent with the result of Caffarelli–Gidas–Spruck [4] on Laplacian. Jin–de Queiroz–Sire–Xiong [28] obtained the similar result if Eq. (1.9) is defined in $\mathbb{R}^n \setminus \mathbb{R}^k$ $(1 \le k \le n - 2\sigma)$ and there exists $x_0 \in \mathbb{R}^k$ such that $\limsup_{x \to (x_0, 0, \ldots, 0)} u = +\infty$.

Furthermore, Caffarelli–Jin–Sire–Xiong [5] also derived the local behaviors of positive solutions of the fractional Yamabe equation

$$\begin{cases} (-\Delta)^{\sigma} u = u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in} \;\; B_1 \backslash \{0\}, \\ u > 0 \quad \text{and} \quad u \in C^2(B_1 \backslash \{0\}) \cap L_{\sigma}(\mathbb{R}^n) \end{cases} \tag{1.10}$$

with an isolated singularity at the origin: either 0 is a removable singular point of u, or there exist two positive constants c_1 and c_2 such that

$$c_1 |x|^{-\frac{n-2\sigma}{2}} \le u(x) \le c_2 |x|^{-\frac{n-2\sigma}{2}} \quad \text{near} \quad x = 0.$$

Inspired by the work on Laplacian, the Liouville Theorem for the system

$$\begin{cases} (-\Delta)^{\sigma} u = \alpha_1 u^{\frac{n+2\sigma}{n-2\sigma}} + \beta u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}} & \text{in} \quad \mathbb{R}^n, \\ (-\Delta)^{\sigma} v = \alpha_2 v^{\frac{n+2\sigma}{n-2\sigma}} + \beta v^{\frac{2\sigma}{n-2\sigma}} u^{\frac{n}{n-2\sigma}} & \text{in} \quad \mathbb{R}^n, \\ u, v > 0 \;\; \text{and} \;\; u, v \in C^2(\mathbb{R}^n) \cap L_{\sigma}(\mathbb{R}^n) \end{cases} \tag{1.11}$$

has been studied in [32,51], where α_1, α_2 and β are positive constants. By the method of moving planes in the integral form, Zhuo–Chen–Cui–Yuan obtained in [51] that the positive solution of (1.11) is radially symmetry. But they do not give the form of the solution of the system. Chen–Li [17], and Chen–Li–Ou [10] had obtained the classification of solutions for a system of integral equations which is closely related to the system (1.11). And recently Dai–Liu–Lu [18] studied the Liouville Theorem about the fractional Laplacian system in half space.

In the present paper, we study the fractional Laplacian systems with critical exponents, and obtain the Liouville Theorem, the radially symmetry and the upper bound estimates near the isolated singularity. We point out that the non-locality of the fractional Laplacian makes it difficult to investigate. To deal with it, we will make use of the extension method which was introduced by Caffarelli–Silvestre [6]. Furthermore, when the system is concerned, the main difficulty stems from the fact that it is not clear for us to know where to start moving spheres, since the nonlinearities are coupled each other. In order to start the method, we make a Kelvin transformation. Such conformal invariance allows us to use the moving spheres method introduced by Li–Zhu [34]. This observation has also been used in [5,26,28,34,50].

The first main result of our paper is that we shall first catch the exact form and the number of the solutions using the method of moving spheres, which is consistent with the work of Chen–Li [8] and Guo–Liu [25] on Laplacian.

Theorem 1.1. *Let (u, v) be a solution of (1.11), then $(u, v) = (kU_\sigma, lU_\sigma)$, where U_σ is given by (1.8), and at most three pairs $(k, l) \in (0, +\infty) \times (0, +\infty)$ satisfy*

$$\alpha_1 k^{\frac{n+2\sigma}{n-2\sigma}} + \beta k^{\frac{2\sigma}{n-2\sigma}} l^{\frac{n}{n-2\sigma}} = k, \quad \alpha_2 l^{\frac{n+2\sigma}{n-2\sigma}} + \beta l^{\frac{2\sigma}{n-2\sigma}} k^{\frac{n}{n-2\sigma}} = l. \tag{1.12}$$

Next, we shall investigate the radially symmetry of singular solutions of

$$\begin{cases} (-\Delta)^\sigma u = \alpha_1 u^{\frac{n+2\sigma}{n-2\sigma}} + \beta u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}} & \text{in} \quad \mathbb{R}^n \backslash \{0\}, \\ (-\Delta)^\sigma v = \alpha_2 v^{\frac{n+2\sigma}{n-2\sigma}} + \beta v^{\frac{2\sigma}{n-2\sigma}} u^{\frac{n}{n-2\sigma}} & \text{in} \quad \mathbb{R}^n \backslash \{0\}, \\ u, v > 0 \quad \text{and} \quad u, v \in C^2(\mathbb{R}^n \backslash \{0\}) \cap L_\sigma(\mathbb{R}^n), \end{cases} \tag{1.13}$$

by analogy with the approach taken in Theorem 1.1, which is an extension of Chen–Lin [16] works on Laplacian.

Theorem 1.2. *Let (u, v) be a solution of (1.13) and*

$$\limsup_{x \to 0} u + \limsup_{x \to 0} v = \infty,$$

then both u and v are radially symmetric and monotonically decreasing with respect to $|x|$.

Finally, we are also interested in analyzing the local behaviors of solutions of system in a punctured ball

$$\begin{cases} (-\Delta)^\sigma u = \alpha_1 u^{\frac{n+2\sigma}{n-2\sigma}} + \beta u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}} & \text{in} \quad B_1 \backslash \{0\}, \\ (-\Delta)^\sigma v = \alpha_2 v^{\frac{n+2\sigma}{n-2\sigma}} + \beta v^{\frac{2\sigma}{n-2\sigma}} u^{\frac{n}{n-2\sigma}} & \text{in} \quad B_1 \backslash \{0\}, \\ u, v > 0 \quad \text{and} \quad u, v \in C^2(B_1 \backslash \{0\}) \cap L_\sigma(\mathbb{R}^n). \end{cases} \tag{1.14}$$

Theorem 1.3. *Let (u, v) be a solution of (1.14), then there exists a positive constant $C = C(n, \sigma)$ such that*

$$u(x) \le C|x|^{-\frac{n-2\sigma}{2}}, \quad v(x) \le C|x|^{-\frac{n-2\sigma}{2}} \quad \textit{near} \quad x = 0.$$

So far we cannot get the lower bound of the local solutions in $B_1 \backslash \{0\}$. On the one hand, in this case $\sigma = 1$, Chen–Lin [16] obtained the lower bound if the system defined in $\mathbb{R}^n \backslash \{0\}$ with $n \ge 5$. By proving the radially symmetry of positive singular solution, they reduced the system to an ODE by the classical change of variables from Fowler [19]. As a result, they established the upper and lower bound using the standard ODE theory. In our case, we consider the system only in $B_1\backslash\{0\}$, and we have no idea to reduce this system to an ODE. On the other hand, compared with the single equation (1.10) our problem is a coupled system, which turns out to be much more difficult and complicated.

Our paper is organized as follows. Section 2 includes some definitions of basic space and elementary propositions on nonlinear boundary system which will be used in our following proof. Section 3 is devoted to obtain the Liouville Theorem with at most three radial solutions. Theorem 1.2 on symmetry of global solutions of (1.13) is proved in Section 4. Finally, we obtain the upper bound and prove Theorem 1.3 in Section 5.

2. Preliminaries

2.1. The extension method

The traditional methods on local differential operators, such as on Laplacian, may not work on the nonlocal operator $(-\Delta)^\sigma$. To circumvent this difficulty, Caffarelli and Silvestre [6] introduced the extension method that reduced this nonlocal equation into a local one in higher dimensions with the conormal derivative boundary condition.

More precisely, for $u\in C^2(\mathbb{R}^n)\cap L_\sigma(\mathbb{R}^n)$, define

$$U(x,t):=\int_{\mathbb{R}^n}\mathcal{P}_\sigma(x-\xi,t)u(\xi)d\xi, \tag{2.1}$$

where

$$\mathcal{P}_\sigma(x,t):=\frac{\beta(n,\sigma)t^{2\sigma}}{(|x|^2+t^2)^{(n+2\sigma)/2}}$$

with a constant $\beta(n,\sigma)$ such that $\int_{\mathbb{R}^n}\mathcal{P}_\sigma(x,1)dx=1$. It follows that

$$U\in C^2(\mathbb{R}^{n+1}_+)\cap C(\overline{\mathbb{R}^{n+1}_+}),\quad t^{1-2\sigma}\partial_t U(x,t)\in C(\overline{\mathbb{R}^{n+1}_+}),$$

and U satisfies

$$\operatorname{div}(t^{1-2\sigma}\nabla U)=0\quad\text{in}\quad \mathbb{R}^{n+1}_+, \tag{2.2}$$

$$U(\cdot,0)=u\qquad\text{on}\quad \partial\mathbb{R}^{n+1}_+.$$

In addition, by works of Caffarelli and Silvestre [6], it is known that up to a constant,

$$\frac{\partial U}{\partial\nu^\sigma}=(-\Delta)^\sigma u\quad\text{on}\quad\partial\mathbb{R}^{n+1}_+,$$

where

$$\frac{\partial}{\partial\nu^\sigma}U(x,0):=-\lim_{t\to0^+}t^{1-2\sigma}\partial_t U(x,t).$$

From this and (u,v) is a solution of (1.11), we have

$$\frac{\partial U}{\partial\nu^\sigma}=\alpha_1u^{\frac{n+2\sigma}{n-2\sigma}}+\beta u^{\frac{2\sigma}{n-2\sigma}}v^{\frac{n}{n-2\sigma}}\qquad\text{on}\quad\partial\mathbb{R}^{n+1}_+. \tag{2.3}$$

In order to study the behavior of the solutions of (1.11), we just need to study the behaviors of U defined by (2.1).

2.2. A weight Sobolev space

In the proof of main theorems, we shall need some propositions, where weak solutions suffice. Therefore, we introduce the definition of a Weight Sobolev Space.

Let D be a domain in $\mathbb{R}^{n+1}_+$. Denote by $L^2(t^{1-2\sigma}, D)$ the Banach space of all measurable functions U defined on D, for which

$$\|U\|_{L^2(t^{1-2\sigma},D)} := \left(\int_D t^{1-2\sigma}|U|^2 dX\right)^{\frac{1}{2}} < \infty,$$

and $X := (x,t) \in \mathbb{R}^{n+1}_+$. We say that $U \in W^{1,2}(t^{1-2\sigma}, D)$ if $U \in L^2(t^{1-2\sigma}, D)$, and its weak derivatives ∇U exist and belong to $L^2(t^{1-2\sigma}, D)$. The norm of U in $W^{1,2}(t^{1-2\sigma}, D)$ is given by

$$\|U\|_{W^{1,2}(t^{1-2\sigma},D)} := \left(\int_D t^{1-2\sigma}|U|^2 dX + \int_D t^{1-2\sigma}|\nabla U|^2 dX\right)^{\frac{1}{2}}.$$

We denote $\mathcal{B}_R(X)$ as the ball in $\mathbb{R}^{n+1}$ with radius R and center X, $\mathcal{B}^+_R(X)$ as $\mathcal{B}_R(X) \cap \mathbb{R}^{n+1}_+$, and $B_R(x)$ as the ball in $\mathbb{R}^n$ with radius R and center x. We also write $\mathcal{B}_R(0)$, $\mathcal{B}^+_R(0)$, $B_R(0)$ as $\mathcal{B}_R$, $\mathcal{B}^+_R$, B_R for short respectively. For a domain $D \subset \mathbb{R}^{n+1}_+$ with boundary ∂D, we denote $\partial' D := \partial D \cap \partial\mathbb{R}^{n+1}_+$ and $\partial'' D := \partial D \cap \mathbb{R}^{n+1}_+$. It is easy to see that $\partial'\mathcal{B}^+_R(X) = \{X = (x,0) : |x| < R\}$ and $\partial''\mathcal{B}^+_R(X) = \{X = (x,t) : |X| = R, t > 0\}$.

Definition 2.1. We say U, $V \in W^{1,2}(t^{1-2\sigma}, D)$ is a weak solution (resp. supersolution, subsolution) of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in} \quad D, \\ \operatorname{div}(t^{1-2\sigma}\nabla V) = 0 & \text{in} \quad D, \\ \dfrac{\partial U}{\partial \nu^\sigma} = \alpha_1 U^{\frac{n+2\sigma}{n-2\sigma}} + \beta U^{\frac{2\sigma}{n-2\sigma}} V^{\frac{n}{n-2\sigma}} & \text{on} \quad \partial' D, \\ \dfrac{\partial V}{\partial \nu^\sigma} = \alpha_2 V^{\frac{n+2\sigma}{n-2\sigma}} + \beta V^{\frac{2\sigma}{n-2\sigma}} U^{\frac{n}{n-2\sigma}} & \text{on} \quad \partial' D, \end{cases} \tag{2.4}$$

if for every (resp. nonnegative) $\Phi \in C^\infty_c(D \cup \partial' D)$,

$$\int_D t^{1-2\sigma}\nabla U \nabla \Phi dX = (\text{resp. } \geq, \leq) \int_{\partial' D} (\alpha_1 U^{\frac{n+2\sigma}{n-2\sigma}} + \beta U^{\frac{2\sigma}{n-2\sigma}} V^{\frac{n}{n-2\sigma}})\Phi dx,$$

and

$$\int_D t^{1-2\sigma}\nabla V \nabla \Phi dX = (\text{resp. } \geq, \leq) \int_{\partial' D} (\alpha_2 V^{\frac{n+2\sigma}{n-2\sigma}} + \beta V^{\frac{2\sigma}{n-2\sigma}} U^{\frac{n}{n-2\sigma}})\Phi dx.$$

2.3. Preliminary results for system (2.4) with nonlinear boundary conditions

The preliminary results for single equations with linear boundary conditions have been established in [3,26,45,47]. The following propositions are a version of these results for system (2.4) with nonlinear boundary conditions.

Proposition 2.2 ($W^{1,2}$ *Estimates near Boundary*)**.** *If* U, $V \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}^+_R)$ *is a positive weak solution of (2.4) in* $\mathcal{B}^+_R$, *there exist a positive constant* C *depending only on* n, σ, α_1, α_2, β, $\|U(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$, *and* $\|V(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$ *such that*

$$\|\nabla(U+V)\|_{L^2(t^{1-2\sigma},\mathcal{B}^+_{R/2})} \leq CR^{-1}\|U+V\|_{L^2(t^{1-2\sigma},\mathcal{B}^+_R)}. \tag{2.5}$$

Proof. By scaling $U_R(X) := R^{\frac{n-2\sigma}{2}}U(RX)$, $V_R(X) := R^{\frac{n-2\sigma}{2}}V(RX)$, one can assume $R = 1$. From the condition, it follows that $U + V$ is the solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U+V)) = 0 & \text{in } \mathcal{B}_1^+, \\ \dfrac{\partial(U+V)}{\partial\nu^\sigma} = \alpha_1 U^{\frac{n+2\sigma}{n-2\sigma}} + \alpha_2 V^{\frac{n+2\sigma}{n-2\sigma}} + \beta V^{\frac{2\sigma}{n-2\sigma}}U^{\frac{2\sigma}{n-2\sigma}}(U+V) & \text{on } \partial'\mathcal{B}_1^+. \end{cases}$$

Let $\eta \in C_c^\infty(\mathcal{B}_1^+ \cup \partial'\mathcal{B}_1^+)$ be a cut-off function which is equal to 1 in $\mathcal{B}_{1/2}^+$ and is supported in $\mathcal{B}_{3/4}^+$. By a density argument, we can choose $\eta^2(U+V)$ as a test function, then

$$\begin{aligned} &\int_{\mathcal{B}_1^+} t^{1-2\sigma}\nabla(U+V)\nabla(\eta^2(U+V)) \\ &= \int_{\partial'\mathcal{B}_1^+} (\alpha_1 U^{\frac{n+2\sigma}{n-2\sigma}} + \alpha_2 V^{\frac{n+2\sigma}{n-2\sigma}} + \beta V^{\frac{2\sigma}{n-2\sigma}}U^{\frac{2\sigma}{n-2\sigma}}(U+V))\eta^2(U+V)dx \\ &\le \int_{\partial'\mathcal{B}_1^+} (\alpha_1 U^{\frac{4\sigma}{n-2\sigma}} + \alpha_2 V^{\frac{4\sigma}{n-2\sigma}} + \beta U^{\frac{2\sigma}{n-2\sigma}}V^{\frac{2\sigma}{n-2\sigma}})\eta^2(U+V)^2 \\ &=: \int_{\partial'\mathcal{B}_1^+} a\eta^2(U+V)^2. \end{aligned}$$

Using Cauchy-inequality, we obtain that

$$\int_{\mathcal{B}_1^+} t^{1-2\sigma}\eta^2|\nabla(U+V)|^2 dX \le 4\int_{\mathcal{B}_1^+} t^{1-2\sigma}(U+V)^2|\nabla\eta|^2 + 2\int_{\partial'\mathcal{B}_1^+} a\eta^2(U+V)^2.$$

From the Trace inequality [26, Proposition 2.1], we have $U(\cdot,0), V(\cdot,0) \in L^{\frac{2n}{n-2\sigma}}(B_1)$. As a result, $a \in L^{\frac{n}{2\sigma}}(B_1)$. The Hölder inequality gives that

$$\begin{aligned} \int_{\partial'\mathcal{B}_1^+} a\eta^2(U+V)^2 &\le \|a\|_{L^{\frac{n}{2\sigma}}(B_1)}\|\eta(U+V)\|^2_{L^{\frac{2n}{n-2\sigma}}(B_1)} \\ &\le C\|a\|_{L^{\frac{n}{2\sigma}}(B_1)}\|\nabla(\eta(U+V))\|^2_{L^2(t^{1-2\sigma},B_1)}. \end{aligned}$$

If $\|a\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta$ for some sufficiently small δ, then the conclusion follows immediately. If not, for any $x_0 \in B_{1/2}$, $X_0 = (x_0, 0)$, we can choose r small such that $\|a\|_{L^{\frac{n}{2\sigma}}(B_r(x_0))} < \delta$. Applying the above result in $\mathcal{B}_r^+(x_0)$, we have

$$\|U+V\|_{W^{1,2}(t^{1-2\sigma},\mathcal{B}_{r/2}^+(X_0))} \le C\|U+V\|_{L^2(t^{1-2\sigma},\mathcal{B}_1^+)},$$

which implies that

$$\|\nabla(U+V)\|_{L^2(t^{1-2\sigma},\mathcal{B}_{r/2}^+(X_0))} \le C\|U+V\|_{L^2(t^{1-2\sigma},\mathcal{B}_1^+)}.$$

From the Universal Coverage Theorem (2.5) follows for $R = 1$. Applying the result to $U_R(X)$ and $V_R(X)$, we finish the proof. □

Proposition 2.3 (*Local Maximum Principle*)**.** *If U, $V \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+)$ is a positive weak subsolution of (2.4), then for all $p > 0$, there exists a positive constant C depending only on n, σ, p, α_1, α_2, β, $\|U(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$, and $\|V(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$ such that*

$$\sup_{\mathcal{B}_{R/2}^+}(U+V) \le CR^{-(n+2-2\sigma)/p}\|U+V\|_{L^p(t^{1-2\sigma},\mathcal{B}_{3R/4}^+)}. \tag{2.6}$$

Proof. We also assume $R = 1$. As the above, a straightforward calculation shows that $U+V$ is a weak subsolution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U+V)) = 0 & \text{in } \mathcal{B}_1^+, \\ \dfrac{\partial(U+V)}{\partial\nu^\sigma} = a(U+V) & \text{on } \partial'\mathcal{B}_1^+, \end{cases}$$

where

$$a := \alpha_1 U^{\frac{4\sigma}{n-2\sigma}} + \beta U^{\frac{2\sigma}{n-2\sigma}} V^{\frac{2\sigma}{n-2\sigma}} + \alpha_2 V^{\frac{4\sigma}{n-2\sigma}} \in L^{\frac{n}{2\sigma}}(B_1).$$

By [26, Lemma 2.8], there exists a sufficiently small $\delta > 0$ which depends only on n and σ such that if $\|a\|_{L^{\frac{n}{2\sigma}}(B_1)} < \delta$, then

$$\|(U+V)(\cdot,0)\|_{L^{p_0}(B_{3/4})} \le C\|U+V\|_{W(t^{1-2\sigma},\mathcal{B}_1^+)} \tag{2.7}$$

for some $p_0 > \frac{2n}{n-2\sigma}$, where C is a positive constant and depends only on n, σ, $\|U(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_1)}$, and $\|V(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_1)}$. If not, like the proof of Proposition 2.2, which implies that (2.7) follows. As an easy consequence of $U > 0$, $V > 0$, we have $U(\cdot,0)$, $V(\cdot,0) \in L^{p_0}(B_{3R/4})$. It follows that $a \in L^{q_0}(B_{3R/4})$ for some $q_0 > \frac{n}{2\sigma}$.

By the Local Maximum Principle for single equations with linear boundary conditions [26, Proposition 2.6], we conclude that for any $p > 0$, (2.6) holds for $R = 1$. □

Proposition 2.4 (*Weak Harnack Inequality*). *If U, $V \in W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+)$ is a positive weak supersolution of (2.4), then for all $0 < \widetilde{p} \le (n+1)/n$, there exists a positive constant C depending only on n, σ, $\widetilde{p}$, α_1, α_2, β, $\|U(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$, and $\|V(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$ such that*

$$\inf_{\mathcal{B}_{R/2}^+}(U+V) \ge CR^{-(n+2-2\sigma)/\widetilde{p}}\|U+V\|_{L^{\widetilde{p}}(t^{1-2\sigma},\mathcal{B}_{3R/4}^+)}. \tag{2.8}$$

Proof. Assume $R = 1$. If (U,V) is a weak supersolution of (2.4), then $U+V$ is a weak supersolution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U+V)) = 0 & \text{in } \mathcal{B}_1^+, \\ \dfrac{\partial(U+V)}{\partial\nu^\sigma} = \beta U^{\frac{2\sigma}{n-2\sigma}} V^{\frac{2\sigma}{n-2\sigma}}(U+V) & \text{on } \partial'\mathcal{B}_1^+, \end{cases}$$

applying [26, Proposition 2.6] as the above, we obtain (2.8). □

Proposition 2.5 (*Harnack Inequality*). *If U, $V \in W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+)$ is a positive weak solution of (2.4), then there exists a positive constant C depending only on n, σ, α_1, α_2, β, $\|U(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$, and $\|V(\cdot,0)\|_{L^{\frac{2n}{n-2\sigma}}(B_R)}$, such that*

$$\sup_{\mathcal{B}_{R/2}^+}(U+V) \le C \inf_{\mathcal{B}_{R/2}^+}(U+V). \tag{2.9}$$

Proof. The Harnack Inequality follows from Propositions 2.3 and 2.4. □

Proposition 2.6 (C^α *Estimates near Boundary*). *If U, $V \in W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+)$ is a positive weak solution of (2.4), there exist $\alpha \in (0,1)$ and a positive constant C, both depending only on n, σ, α_1, α_2, β, R, $\|U(\cdot,0)\|_{L^\infty(B_{3R/4})}$, and $\|V(\cdot,0)\|_{L^\infty(B_{3R/4})}$ such that*

$$\|U\|_{C^\alpha(\overline{\mathcal{B}_{R/2}^+})} + \|V\|_{C^\alpha(\overline{\mathcal{B}_{R/2}^+})} \le C. \tag{2.10}$$

Proof. Assume $R=1$. Consider

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U)=0 & \text{in } \mathcal{B}_1^+, \\ \dfrac{\partial U}{\partial \nu^\sigma}=\alpha_1 U^{\frac{n+2\sigma}{n-2\sigma}}+\beta U^{\frac{2\sigma}{n-2\sigma}}V^{\frac{n}{n-2\sigma}} & \text{on } \partial'\mathcal{B}_1^+, \end{cases}$$

and

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla V)=0 & \text{in } \mathcal{B}_1^+, \\ \dfrac{\partial V}{\partial \nu^\sigma}=\alpha_2 V^{\frac{n+2\sigma}{n-2\sigma}}+\beta V^{\frac{2\sigma}{n-2\sigma}}U^{\frac{n}{n-2\sigma}} & \text{on } \partial'\mathcal{B}_1^+, \end{cases}$$

respectively. From the Harnack inequality, we conclude that $U,V\in L^\infty_{\mathrm{loc}}(\mathcal{B}_1^+\cup\partial'\mathcal{B}_1^+)$. Then (2.10) obtained with the help of [26, Proposition 2.6]. □

3. Liouville theorem

In Section 3, we prove Theorem 1.1. To obtain the symmetry and the exact form of solution of (1.11), we only need to prove by [33, Lemma 11.1] that for any $x\in\mathbb{R}^n$, there is a positive constant $\lambda:=\lambda(x)$ so that

$$u_{x,\lambda}=u,\quad v_{x,\lambda}=v\quad \text{in } \mathbb{R}^n\backslash\{x\}. \tag{3.1}$$

In addition, (3.1) can be reduced to

$$U_{X,\lambda}=U,\quad V_{X,\lambda}=V\quad \text{in } \mathbb{R}^{n+1}_+, \tag{3.2}$$

where $X:=(x,0)$, U, V are defined as (2.1), and

$$U_{X,\lambda}(\xi):=\left(\frac{\lambda}{|\xi-X|}\right)^{n-2\sigma}U\left(x+\frac{\lambda^2(\xi-X)}{|\xi-X|^2}\right)$$

is the Kelvin transformation of U with respect to the ball $\mathcal{B}_\lambda^+(X)$.

In order to prove (3.2), we introduce

$$\bar\lambda(x):=\sup\big\{\mu>0\;\big|\;U_{X,\lambda}\le U, V_{X,\lambda}\le V\quad \text{in } \mathbb{R}^{n+1}_+\backslash\mathcal{B}_\lambda^+(X),\ \forall\,\lambda\in(0,\mu)\big\}.$$

First, the following lemma is necessary to guarantee that the set over which we are taking the supremum is non-empty so that $\bar\lambda(x)$ is well defined.

Lemma 3.1. *For all $x\in\mathbb{R}^n$ there exists $\lambda_0:=\lambda_0(x)>0$ such that for all $\lambda\in(0,\lambda_0)$,*

$$U_{X,\lambda}\le U,\quad V_{X,\lambda}\le V\quad \text{in } \mathbb{R}^{n+1}_+\backslash\mathcal{B}_\lambda^+(X). \tag{3.3}$$

Obviously, for a fixed x either $\bar\lambda(x)<\infty$ or $\bar\lambda(x)=\infty$.

If $\bar\lambda(x)=+\infty$ for some x, we then obtain that for all $\lambda>0$, $U_{X,\lambda}\le U$ in $\mathbb{R}^{n+1}_+\backslash\mathcal{B}_\lambda^+(X)$. It is not hard to see

$$\lim_{\xi\to+\infty}|\xi|^{n-2\sigma}U_{X,\lambda}(\xi)=\lim_{\xi\to+\infty}\frac{|\xi|^{n-2\sigma}\lambda^{n-2\sigma}}{|\xi-X|^{n-2\sigma}}U\left(X+\frac{\lambda^2(\xi-X)}{|\xi-X|^2}\right)=\lambda^{n-2\sigma}U(X). \tag{3.4}$$

We thus conclude that

$$\lambda^{n-2\sigma}U(X)\le\limsup_{\xi\to+\infty}|\xi|^{n-2\sigma}U(\xi).$$

Consequently, the arbitrariness of $\lambda>0$ implies

$$\limsup_{\xi\to+\infty}|\xi|^{n-2\sigma}U(\xi)=+\infty.$$

If $\bar\lambda(\widehat{x})<+\infty$ for some $\widehat{x}$, it follows from

Lemma 3.2. *If $\bar{\lambda}(x) < +\infty$ for some $x \in \mathbb{R}^n$, then $U_{X,\bar{\lambda}(x)} = U$, $V_{X,\bar{\lambda}(x)} = V$ in $\mathbb{R}^{n+1}_+$.*

and (3.4) that

$$\limsup_{\xi\to+\infty} |\xi|^{n-2\sigma} U(\xi) = \limsup_{\xi\to+\infty} |\xi|^{n-2\sigma} U_{\widehat{X},\bar{\lambda}(\widehat{x})}(\xi) = \bar{\lambda}^{n-2\sigma}(\widehat{x}) U(\widehat{X}) < +\infty,$$

which is a contradiction. Hence, either $\bar{\lambda}(x) < +\infty$ for all $x \in \mathbb{R}^n$, or $\bar{\lambda}(x) = +\infty$ for all $x \in \mathbb{R}^n$.

In the case $\bar{\lambda}(x) = +\infty$ for all $x \in \mathbb{R}^n$, we have by [33, Lemma 11.3] that both U and V are positive constants. It is clear to see that this case never happens due to Eqs. (1.11). We now conclude that $\bar{\lambda}(x) < +\infty$ for all $x \in \mathbb{R}^n$. This case establishes (3.2) in view of Lemma 3.2 and completes the proof of the first part of Theorem 1.1.

We now intend to prove Lemmas 3.1 and 3.2. After that, we shall prove the second part of Theorem 1.1, that is, (1.11) admits at most three solutions.

Proof of Lemma 3.1. For simplicity, we prove (3.3) only for U, since the proof on V is similar. We shall prove that for fixed x there exist $\mu := \mu(x)$ and $\lambda_0 := \lambda_0(x)$ satisfying $0 < \lambda_0 < \mu$ such that for $\lambda \in (0, \lambda_0)$,

$$U_{X,\lambda} \le U \quad \text{in} \ \mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{B}^+_\mu(X)}, \tag{3.5}$$

and

$$U_{X,\lambda} \le U \quad \text{in} \ \overline{\mathcal{B}^+_\mu(X)} \backslash \mathcal{B}^+_\lambda(X). \tag{3.6}$$

As the first step, we prove (3.5). For every $0 < \mu < 1$, define

$$\phi(\xi) := \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U,$$

which satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\phi) = 0 & \text{in} \ \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X), \\ \dfrac{\partial\phi}{\partial\nu^\sigma} = 0 & \text{on} \ \partial'(\mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X)) \end{cases}$$

and

$$\phi = \inf_{\partial''\mathcal{B}^+_\mu(X)} U \le U \quad \text{on} \ \partial''\mathcal{B}^+_\mu(X).$$

It is easy to see that $\lim_{\xi\to+\infty} \phi(\xi) = 0 \le \lim_{\xi\to+\infty} U(\xi)$. By the standard Maximum principle, we conclude that

$$\left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U \le U(\xi) \quad \text{in} \ \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X). \tag{3.7}$$

Let

$$\lambda_0 := \mu \min\left\{ \left(\inf_{\partial''\mathcal{B}^+_\mu(X)} U \Big/ \sup_{\mathcal{B}^+_\mu(X)} U \right)^{\frac{1}{n-2\sigma}}, \left(\inf_{\partial''\mathcal{B}^+_\mu(X)} V \Big/ \sup_{\mathcal{B}^+_\mu(X)} V \right)^{\frac{1}{n-2\sigma}} \right\}. \tag{3.8}$$

Then for all $\xi \in \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X)$ and $\lambda \in (0, \lambda_0)$, it follows from (3.7) and (3.8)

$$\begin{aligned} U_{X,\lambda}(\xi) &= \left(\frac{\lambda}{|\xi - X|}\right)^{n-2\sigma} U(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}) \\ &\le \left(\frac{\lambda_0}{|\xi - X|}\right)^{n-2\sigma} \sup_{\mathcal{B}^+_\mu(X)} U \\ &\le \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U \\ &\le U(\xi). \end{aligned}$$

As the second step, we shall prove (3.6). For all $\lambda \in (0, \lambda_0)$, the above inequality gives that $U_{X,\lambda} = U$ on $\partial''\mathcal{B}_\mu^+(X)$. Together with $U_{X,\lambda} = U$ on $\partial''\mathcal{B}_\lambda^+(X)$ implies that for all $\lambda \in (0, \lambda_0)$,

$$U_{X,\lambda} \le U \qquad \text{on} \qquad \partial''\mathcal{B}_\mu^+(X) \cup \partial''\mathcal{B}_\lambda^+(X). \tag{3.9}$$

We will make use of the narrow domain technique of Berestycki and Nirenberg from [1], and show that for $\lambda \in (0, \lambda_0)$, $U_{X,\lambda} \le U$ in $D := \mathcal{B}_\mu^+(X) \backslash \mathcal{B}_\lambda^+(X)$.

A direct calculation gives that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U_{X,\lambda} - U)) = 0 & \text{in } D, \\ \dfrac{\partial(U_{X,\lambda} - U)}{\partial\nu^\sigma} = \alpha_1(u_{x,\lambda}^{\frac{n+2\sigma}{n-2\sigma}} - u^{\frac{n+2\sigma}{n-2\sigma}}) + \beta(u_{x,\lambda}^{\frac{2\sigma}{n-2\sigma}} v_{x,\lambda}^{\frac{n}{n-2\sigma}} - u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}}) & \text{on } \partial' D. \end{cases} \tag{3.10}$$

Let $(U_{X,\lambda} - U)^+ := \max(0, U_{X,\lambda} - U)$ which equals to 0 on $\partial'' D$ by (3.9). Multiplying the first equation in (3.10) by $(U_{X,\lambda} - U)^+$ and integrating by parts in D. With the help of the Mean Value Theorem, we have

$$\begin{aligned} &\int_D t^{1-2\sigma}|\nabla(U_{X,\lambda} - U)^+|^2 \\ &= \int_{\partial' D} \beta(u_{x,\lambda}^{\frac{2\sigma}{n-2\sigma}} v_{x,\lambda}^{\frac{n}{n-2\sigma}} - u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}})(u_{x,\lambda} - u)^+ + \int_{\partial' D} \alpha_1(u_{x,\lambda}^{\frac{n+2\sigma}{n-2\sigma}} - u^{\frac{n+2\sigma}{n-2\sigma}})(u_{x,\lambda} - u)^+ \\ &=: I_1 + I_2. \end{aligned} \tag{3.11}$$

By the Mean Value Theorem, there exists $\theta \in (0, 1)$ such that

$$\begin{aligned} I_1 \le & \frac{2\sigma\beta}{n-2\sigma} \int_{\partial' D} (\theta u_{x,\lambda} + (1-\theta)u)^{\frac{4\sigma-n}{n-2\sigma}} (\theta v_{x,\lambda} + (1-\theta)v)^{\frac{n}{n-2\sigma}} ((u_{x,\lambda} - u)^+)^2 \\ & + \frac{n\beta}{n-2\sigma} \int_{\partial' D} (\theta u_{x,\lambda} + (1-\theta)u)^{\frac{2\sigma}{n-2\sigma}} (\theta v_{x,\lambda} + (1-\theta)v)^{\frac{2\sigma}{n-2\sigma}} (v_{x,\lambda} - u)^+ (v_{x,\lambda} - u)^+. \end{aligned}$$

Since

$$x + \frac{\lambda^2(y-x)}{|y-x|^2} \in B_\lambda(x) \subset B_\mu(x) \subset \overline{B_1}(x) \quad \text{on } \partial' D,$$

it follows that

$$I_1 \le C \int_{\partial' D} ((u_{x,\lambda} - u)^+)^2 + (v_{x,\lambda} - v)^+ (u_{x,\lambda} - u)^+,$$

where C is a positive constant depending on n, σ, β and the upper and lower bounds for u, v on $\overline{B_1}(x)$. Applying the Hölder inequality, we discover that

$$\begin{aligned} I_1 \le & C|B_\mu(x)|^{\frac{2\sigma}{n}} \left(\int_{\partial' D} \left((u_{x,\lambda} - u)^+\right)^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}} \\ & + C|B_\mu(x)|^{\frac{2\sigma}{n}} \left(\int_{\partial' D} \left((u_{x,\lambda} - u)^+\right)^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{2n}} \left(\int_{\partial' D} \left((v_{x,\lambda} - v)^+\right)^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{2n}}. \end{aligned}$$

Therefore, the Trace inequality [26, Proposition 2.1] gives that

$$\begin{aligned} I_1 \le & C\mu^{2\sigma} \int_D t^{1-2\sigma}|\nabla(U_{X,\lambda} - U)^+|^2 \\ & + C\mu^{2\sigma} \left(\int_D t^{1-2\sigma}|\nabla(U_{X,\lambda} - U)^+|^2\right)^{\frac{1}{2}} \left(\int_D t^{1-2\sigma}|\nabla(V_{X,\lambda} - V)^+|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.12}$$

Here C is a positive constant independent on μ.

By the similar argument, it follows that

$$I_2 \le C\mu^{2\sigma} \int_D t^{1-2\sigma}|\nabla(U_{X,\lambda} - U)^+|^2, \tag{3.13}$$

where C is a positive constant independent on μ. In view of (3.11)–(3.13), we have

$$(1 - C\mu^{2\sigma})\|\nabla(U_{X,\lambda} - U)^+\|_{L^2(t^{1-2\sigma},D)} \le C\mu^{2\sigma}\|\nabla(V_{X,\lambda} - V)^+\|_{L^2(t^{1-2\sigma},D)}.$$

It follows that from μ sufficiently small,

$$\|\nabla(U_{X,\lambda} - U)^+\|_{L^2(t^{1-2\sigma},D)} \le \frac{1}{2}\|\nabla(V_{X,\lambda} - V)^+\|_{L^2(t^{1-2\sigma},D)}. \tag{3.14}$$

By the same argument, we can also obtain that

$$\|\nabla(V_{X,\lambda} - V)^+\|_{L^2(t^{1-2\sigma},D)} \le \frac{1}{2}\|\nabla(U_{X,\lambda} - U)^+\|_{L^2(t^{1-2\sigma},D)}. \tag{3.15}$$

Combining (3.14) with (3.15), we have

$$\nabla(U_{X,\lambda} - U)^+ = 0 \quad \text{in } D.$$

Then returning to (3.9) we conclude that $(U_{X,\lambda} - U)^+ = 0$ on D. This implies (3.6) and Lemma 3.1 is proved. □

Proof of Lemma 3.2. From the definition of $\bar\lambda(x)$, it is obviously that for all $\lambda \in (0, \bar\lambda(x))$,

$$U_{X,\lambda} \le U \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X). \tag{3.16}$$

For $\xi \in \overline{\mathcal{B}^+_\lambda(X)}\backslash\{X\}$, we derive $X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \in \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X)$. Applying (3.16), we get

$$\left(\frac{|\xi - X|}{\lambda}\right)^{n-2\sigma} U(\xi) = U_{X,\lambda}\left(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}\right) \le U\left(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}\right),$$

that is,

$$U \le U_{X,\lambda} \quad \text{in } \overline{\mathcal{B}^+_\lambda(X)}\backslash\{X\}. \tag{3.17}$$

We prove Lemma 3.2 by contradiction. Without loss of generality, we suppose $U_{X,\bar\lambda(x)} \not\equiv U$. By the above argument, We shall prove that there exists a positive constant ε such that (3.17) is established for all $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon)$, which contradicts with the definition of $\bar\lambda(x)$. As a result, Lemma 3.2 follows.

We next claim that $U_{X,\bar\lambda(x)} \not\equiv U$ implies $V_{X,\bar\lambda(x)} \not\equiv V$. In fact, if $V_{X,\bar\lambda(x)} \equiv V$, a direct calculation gives that

$$0 = \frac{\partial(V_{X,\bar\lambda(x)} - V)}{\partial\nu^\sigma}(x, 0) = \beta\left(u_{x,\bar\lambda(x)}^{\frac{2\sigma}{n-2\sigma}} - u^{\frac{2\sigma}{n-2\sigma}}\right) v^{\frac{n}{n-2\sigma}} \ne 0,$$

which is a contradiction. Now, let us divide the region $\overline{\mathcal{B}^+_\lambda(X)}\backslash\{X\}$ into three parts,

$$K_1 := \left\{\xi \in \overline{\mathcal{B}^+_\lambda(X)} \mid 0 < |\xi - X| < \delta_1\right\},$$
$$K_2 := \left\{\xi \in \overline{\mathcal{B}^+_\lambda(X)} \mid \delta_1 \le |\xi - X| \le \bar\lambda - \delta_2\right\},$$
$$K_3 := \left\{\xi \in \overline{\mathcal{B}^+_\lambda(X)} \mid \bar\lambda - \delta_2 \le |\xi - X| \le \lambda\right\},$$

where δ_1, δ_2 will be fixed later. It suffices to prove that (3.17) holds respectively on K_1, K_2, K_3.

Combining

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U_{X,\bar\lambda(x)} - U)) = 0 & \text{in } \mathcal{B}^+_{\bar\lambda(x)}(X), \\ \dfrac{\partial(U_{X,\bar\lambda(x)} - U)}{\partial\nu^\sigma} \ge 0 & \text{on } \partial'\mathcal{B}^+_{\bar\lambda(x)}(X) \setminus \{X\}, \end{cases}$$

with the fact that $U_{X,\bar\lambda(x)} \not\equiv U$, we conclude in view of the strong Maximum principle that

$$U_{X,\bar\lambda(x)} - U > 0 \quad \text{in} \ \ \mathcal{B}^{+}_{\bar\lambda(x)}(X) \cup \partial' \mathcal{B}^{+}_{\bar\lambda(x)}(X) \backslash \{X\}.$$

By [26, Proposition 3.1], we have $\liminf_{\xi \to X}(U_{X,\bar\lambda}(\xi) - U(\xi)) > 0$. As a result, there exist two positive constants δ_1 and C_1 such that

$$U_{X,\bar\lambda(x)} - U > C_1 \quad \text{in} \ \ K_1. \tag{3.18}$$

Choose $\varepsilon_1 < \delta_1$ small such that for all $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon_1)$,

$$U_{X,\bar\lambda(x)}\left(X + \frac{\bar\lambda^2(x)}{\lambda^2}(\xi - X)\right) - U_{X,\bar\lambda(x)}(\xi) > -\frac{C_1}{2} \quad \text{in} \ \ K_1, \tag{3.19}$$

and

$$\left(\frac{\bar\lambda(x)}{\bar\lambda(x)+\varepsilon_1}\right)^{n-2\sigma}\left(U(\xi) + \frac{C_1}{2}\right) \geq U(\xi) + \frac{C_1}{4}. \tag{3.20}$$

A direct calculation gives

$$U_{X,\lambda}(\xi) = \left(\frac{\bar\lambda(x)}{\lambda}\right)^{n-2\sigma} U_{X,\bar\lambda(x)}\left(X + \frac{\bar\lambda^2(x)}{\lambda^2}(\xi - X)\right)$$

It follows that from (3.18)–(3.20),

$$\begin{aligned} U_{X,\lambda}(\xi) &\geq \left(\frac{\bar\lambda(x)}{\bar\lambda(x)+\varepsilon_1}\right)^{n-2\sigma} U_{X,\bar\lambda(x)}\left(X + \frac{\bar\lambda^2(x)}{\lambda^2}(\xi - X)\right) \\ &\geq \left(\frac{\bar\lambda(x)}{\bar\lambda(x)+\varepsilon_1}\right)^{n-2\sigma}\left(U(\xi) + \frac{C_1}{2}\right) \\ &\geq U(\xi) + \frac{C_1}{4}. \end{aligned}$$

Consequently, for any $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon_1)$,

$$U_{X,\lambda} \geq U \qquad \text{in} \ \ K_1. \tag{3.21}$$

Since K_2 is compact, there exists a positive constant C_2 such that $U_{X,\bar\lambda(x)} - U > C_2$ in K_2. By the uniform continuity of U on compact sets, there exists a positive constant ε_2 small such that for all $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon_2)$, $U_{X,\lambda} - U_{X,\bar\lambda(x)} > -C_2/2$ in K_2. Hence, for all $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon_2)$,

$$U_{X,\lambda} - U > C_2/2 \qquad \text{in} \ \ K_2. \tag{3.22}$$

Now let us focus on the region K_3. Using the narrow domain technique as that in Lemma 3.1, we can choose δ_2 small (notice that we can choose ε as small as we want less then ε_1 and ε_2 such that for $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon)$,

$$U \leq U_{X,\lambda}, \quad V \leq V_{X,\lambda} \qquad \text{in} \ \ K_3. \tag{3.23}$$

Together with (3.21)–(3.23), we can see that the moving spheres procedure may continue beyond $\bar\lambda(x)$ where we reach a contradiction. And we complete the proof of Lemma 3.2. □

Finally, we consider the solvability of the nonlinear algebraic system (1.12).

Case 1: For $n = 4\sigma$, (1.12) can be written as

$$\alpha_1 k^2 + \beta l^2 = 1, \quad \alpha_2 l^2 + \beta k^2 = 1.$$

A direct calculation shows that

$$(\beta^2 - \alpha_1\alpha_2)l^2 = \beta - \alpha_1, \quad (\beta^2 - \alpha_1\alpha_2)k^2 = \beta - \alpha_2.$$

Obviously, the nonlinear algebraic system (1.12) has at most one positive solution.

Case 2: For $n > 4\sigma$, we first write (1.12) as

$$\alpha_1 + \beta k^{\frac{-n}{n-2\sigma}} l^{\frac{n}{n-2\sigma}} = k^{\frac{-4\sigma}{n-2\sigma}}, \quad \alpha_2 l^{\frac{n}{n-2\sigma}} k^{\frac{-n}{n-2\sigma}} + \beta = l^{\frac{n-4\sigma}{n-2\sigma}} k^{\frac{-n}{n-2\sigma}},$$

and set $x := (l/k)^{\frac{1}{n-2\sigma}}$. It follows that $(\alpha_1 + \beta x^n)/(\alpha_2 x^n + \beta) = x^{-n+4\sigma}$. Consider

$$f(x) := \alpha_1 + \beta x^n - \alpha_2 x^{4\sigma} - \beta x^{-n+4\sigma} \quad \text{in } (0, +\infty).$$

It suffices to prove that $f(x) = 0$ at most has three roots.

A direct calculation shows that

$$\begin{aligned} f'(x) &= n\beta x^{n-1} - 4\alpha_2\sigma x^{4\sigma-1} + (n - 4\sigma)\beta x^{-n+4\sigma-1} \\ &= x^{4\sigma-1}(n\beta x^{n-4\sigma} - 4\alpha_2\sigma + (n - 4\sigma)\beta x^{-n}) \\ &=: x^{4\sigma-1} g(x). \end{aligned}$$

Then we can get the monotonicity of $f(x)$ by analyzing the function $g(x)$. Since

$$g'(x) = n(n - 4\sigma)\beta x^{-1}(x^{n-4\sigma} - x^{-n}), \quad g'(1) = 0,$$

we have

$$g' < 0 \quad \text{in } (0, 1); \quad g' > 0 \quad \text{in } (1, +\infty).$$

In view of

$$g(0 + 0) = +\infty, \quad g(+\infty) = +\infty, \tag{3.24}$$

we see that if $g(1) \geq 0$, then $g \geq 0$, $f' \geq 0$ in $(0, +\infty)$. Together with

$$f(0 + 0) = -\infty, \quad f(+\infty) = +\infty, \tag{3.25}$$

we conclude that $f = 0$ has only one solution in $(0, +\infty)$. If $g(1) < 0$, we deduce by (3.24) that there exist $0 < x_1 < 1 < x_2$ such that $g(x_1) = g(x_2) = 0$. Thus

$$g > 0 \quad \text{in } (0, x_1); \quad g < 0 \quad \text{in } (x_1, x_2); \quad g(x) > 0 \quad \text{in } (x_2, +\infty).$$

That is

$$f' > 0 \quad \text{in } (0, x_1); \quad f' < 0 \quad \text{in } (x_1, x_2); \quad f' > 0 \quad \text{in } (x_2, +\infty).$$

We deduce using (3.25) that $f = 0$ has at most three solutions in $(0, +\infty)$.

Case 3: For $2\sigma < n < 4\sigma$, as in the previous proof, we also know that $f = 0$ has at most three solutions in $(0, +\infty)$.

4. Radially symmetry of singular global solutions

In the fourth section, we prove Theorem 1.2. Analogous to Section 3, we just give a proof for u. To prove that u is radially symmetric and monotonically decreasing it suffices to show that u is symmetric about any hyperplane which through the origin and it is monotone decreasing along the normal direction. Without loss of generality, we shall verify that u is symmetric about the hyperplane $\{y_1 = 0\}$ and it is monotone decreasing along the y_1 axis.

Assume that for all $x \in \mathbb{R}^n \setminus \{0\}$, $\lambda \in (0, |x|)$,

$$u_{x,\lambda} \leq u \qquad \text{in} \quad \mathbb{R}^n \backslash (B_\lambda(x) \cup \{0\}). \tag{4.1}$$

Let t, $s \in \mathbb{R}$ satisfy $t \leq s$, $t + s > 0$ and $m > \max\{s, \frac{st}{s+t}\}$, then $0 < (m-s)(m-t) < m^2$. With the help of (4.1), choosing $y = te_1$, $x = me_1$ and $\lambda^2 = (m-s)(m-t)$, we have

$$\left(\frac{\sqrt{(m-s)(m-t)}}{m-t} \right)^{n-2\sigma} u \left[\left(m + \frac{(m-s)(m-t)}{t-m} \right) e_1 \right] \leq u(te_1),$$

where the unite vector $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. That is,

$$\left(\frac{m-s}{m-t} \right)^{\frac{n-2\sigma}{2}} u\left(se_1\right) \leq u(te_1).$$

After sending $m \to \infty$, it follows that

$$u(se_1) \leq u(te_1),$$

which implies that u is monotone decreasing along the axis of y_1 due to $0 < t < s$. For $s > 0$, let $t \to -s$, we obtain that

$$u(se_1) \leq u(-se_1).$$

Instead of e_1 with $-e_1$, we have

$$u(-se_1) \leq u(se_1).$$

We deduce from the above inequality that u is symmetric about the hyperplane $\{y_1 = 0\}$.

Therefore, to finish the proof of Theorem 1.2, we only need to obtain (4.1).

Let U, V be defined as (2.1), and from the previous argument we first notice that U, $V \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+} \setminus \{0\})$ and

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \operatorname{div}(t^{1-2\sigma} \nabla V) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \dfrac{\partial U}{\partial \nu^\sigma} = \alpha_1 u^{\frac{n+2\sigma}{n-2\sigma}} + \beta u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}} & \text{on} \quad \partial\mathbb{R}^{n+1}_+ \backslash \{0\}, \\ \dfrac{\partial V}{\partial \nu^\sigma} = \alpha_2 v^{\frac{n+2\sigma}{n-2\sigma}} + \beta v^{\frac{2\sigma}{n-2\sigma}} u^{\frac{n}{n-2\sigma}} & \text{on} \quad \partial\mathbb{R}^{n+1}_+ \backslash \{0\}. \end{cases} \tag{4.2}$$

To obtain (4.1), it suffices to prove that for any $x \in \mathbb{R}^n \setminus \{0\}$, and any $\lambda \in (0, |x|)$

$$U_{X,\lambda} \leq U, \quad V_{X,\lambda} \leq V \qquad \text{in} \quad \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X), \tag{4.3}$$

where $X := (x, 0)$, and $U_{X,\lambda}$, $V_{X,\lambda}$ denote the Kelvin transformation of U, V.

For the sake of (4.3), define

$$\bar{\lambda}(x) := \sup\left\{ \mu \in (0, |x|) \;\middle|\; U_{X,\lambda} \leq U,\ V_{X,\lambda} \leq V \text{ in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X),\ \forall\, \lambda \in (0, \mu) \right\}.$$

The following lemma is necessary to guarantee that the set over which we are taking the supremum is non-empty and then $\bar{\lambda}(x)$ is well defined.

Lemma 4.1. *For all $x \in \mathbb{R}^n \backslash \{0\}$, there exists $\lambda_0 := \lambda_0(x) \in (0, |x|)$ such that for all $\lambda \in (0, \lambda_0)$,*

$$U_{X,\lambda} \leq U, \quad V_{X,\lambda} \leq V \qquad \textit{in} \quad \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X). \tag{4.4}$$

To get (4.3), it is just remaining to prove

Lemma 4.2. *For all $x \in \mathbb{R}^n \backslash \{0\}$,*

$$\bar{\lambda}(x) = |x|. \tag{4.5}$$

Now we begin to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. We are going to show that there exist $\mu := \mu(x)$ and $\lambda_0 := \lambda_0(x)$ satisfying $0 < \lambda_0 < \mu$ such that for all $\lambda \in (0, \lambda_0)$,

$$U_{X,\lambda} \le U \qquad \text{in } \mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{B}^+_\mu(X)},$$

and

$$U_{X,\lambda} \le U \qquad \text{in } \overline{\mathcal{B}^+_\mu(X)} \backslash \mathcal{B}^+_\lambda(X),$$

respectively. The proof of Lemma 4.1 follows almost exactly the proof of Lemma 3.1. Since there is an isolated singularity at the origin, we just need to notice that we conclude (3.7) using [35, Proposition 4.1]. □

Proof of Lemma 4.2. By Lemma 4.1, $\bar{\lambda}(x)$ is well defined, and we also know that for $x \neq 0$, $\bar{\lambda}(x) \le |x|$. We prove Lemma 4.2 by contradiction. Suppose $\bar{\lambda}(x) < |x|$ for some $x \neq 0$, we want to prove that there exists $\varepsilon \in (0, \frac{|x| - \bar{\lambda}(x)}{2})$ such that for any $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon)$,

$$U_{X,\lambda} \le U, \quad V_{X,\lambda} \le V \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X), \tag{4.6}$$

which contradicts with the definition of $\bar{\lambda}(x)$. Hence, Lemma 4.2 follows.

Without loss of generality, we assume that $\limsup_{x \to 0} u = \infty$. We now divide the region $\mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X)$ into two parts,

$$K_1 := \left\{ \xi \in \mathbb{R}^{n+1}_+ \;\middle|\; |\xi - X| \ge \bar{\lambda}(x) + \delta_2 \right\},$$
$$K_2 := \left\{ \xi \in \mathbb{R}^{n+1}_+ \;\middle|\; \lambda \le |\xi - X| \le \bar{\lambda}(x) + \delta_2 \right\},$$

where δ_2 will be fixed later. To obtain (4.6) it suffices to prove it holds respectively on K_1, K_2.

In view of the definition of $\bar{\lambda}(x)$, it is easy to see that

$$U_{X,\bar{\lambda}(x)} \le U, \quad V_{X,\bar{\lambda}(x)} \le V \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_{\bar{\lambda}(x)}(X). \tag{4.7}$$

Besides, by the fact

$$\begin{aligned} \lim_{\xi \to 0} U_{X,\bar{\lambda}(x)}(\xi) &= \lim_{\xi \to 0} \left(\frac{\bar{\lambda}(x)}{|\xi - X|} \right)^{n-2\sigma} U \left(X + \frac{\bar{\lambda}(x)^2 (\xi - X)}{|\xi - X|^2} \right) \\ &= \left(\frac{\bar{\lambda}(x)}{|X|} \right)^{n-2\sigma} U \left(X - \frac{\bar{\lambda}(x)^2 X}{|X|^2} \right) < \infty, \end{aligned}$$

we conclude from the strong Maximum principle that

$$U_{X,\bar{\lambda}(x)} < U, \quad V_{X,\bar{\lambda}(x)} < V \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \overline{\mathcal{B}^+_{\bar{\lambda}(x)}(X)}. \tag{4.8}$$

Via a calculation, we have

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla (U - U_{X,\bar{\lambda}(x)})) = 0 & \text{in } K_1, \\ \dfrac{\partial (U - U_{X,\bar{\lambda}(x)})}{\partial \nu^\sigma} = \alpha_1 (u^{\frac{n+2\sigma}{n-2\sigma}} - u_{x,\bar{\lambda}(x)}^{\frac{n+2\sigma}{n-2\sigma}}) + \beta (u^{\frac{2\sigma}{n-2\sigma}} v^{\frac{n}{n-2\sigma}} - u_{x,\bar{\lambda}(x)}^{\frac{2\sigma}{n-2\sigma}} v_{x,\bar{\lambda}(x)}^{\frac{n}{n-2\sigma}}) & \text{on } \partial' K_1. \end{cases}$$

Through a combination of [35, Proposition 4.1] and (4.8), we find that

$$(U-U_{X,\bar\lambda(x)})(\xi)\ge\left(\frac{\bar\lambda(x)+\delta_2}{|\xi-X|}\right)^{n-2\sigma}\inf_{\partial''\mathcal{B}^+_{\bar\lambda(x)+\delta_2}(X)}(U-U_{X,\bar\lambda(x)})>0\quad\text{ in }\ K_1. \tag{4.9}$$

We first claim that there exists $\varepsilon_1<\frac{|x|-\bar\lambda(x)}{2}$ small such that for any $\lambda\in(\bar\lambda(x),\bar\lambda(x)+\varepsilon_1)$,

$$|U_{X,\bar\lambda(x)}-U_{X,\lambda}|\le\frac{1}{2}\left(\frac{\bar\lambda(x)+\delta_2}{|\xi-X|}\right)^{n-2\sigma}\inf_{\partial''\mathcal{B}^+_{\bar\lambda(x)+\delta_2}(X)}(U-U_{X,\bar\lambda(x)})\quad\text{ in }\ K_1. \tag{4.10}$$

Indeed, notice that

$$\xi_{\bar\lambda(x)}:=X+\frac{\bar\lambda^2(x)(\xi-X)}{|\xi-X|^2},\quad \xi_\lambda:=X+\frac{\lambda^2(\xi-X)}{|\xi-X|^2}\in\overline{\mathcal{B}^+_{\frac{|x|+\bar\lambda(x)}{2}}(X)}$$

and

$$\begin{aligned}|U_{X,\bar\lambda(x)}(\xi)-U_{X,\lambda}(\xi)|&=\left|\left(\frac{\bar\lambda(x)}{|\xi-X|}\right)^{n-2\sigma}U(\xi_{\bar\lambda(x)})-\left(\frac{\lambda}{|\xi-X|}\right)^{n-2\sigma}U(\xi_\lambda)\right|\\&\le\left(\frac{\bar\lambda(x)}{|\xi-X|}\right)^{n-2\sigma}\left|U(\xi_{\bar\lambda(x)})-U(\xi_\lambda)\right|+\frac{\lambda^{n-2\sigma}-\bar\lambda^{n-2\sigma}(x)}{|\xi-X|^{n-2\sigma}}U(\xi_\lambda)\\&\le\left(\frac{\bar\lambda(x)+\delta_2}{|\xi-X|}\right)^{n-2\sigma}\left|U(\xi_{\bar\lambda(x)})-U(\xi_\lambda)\right|\\&\quad+\left(\frac{\bar\lambda(x)+\delta_2}{|\xi-X|}\right)^{n-2\sigma}\frac{\lambda^{n-2\sigma}-\bar\lambda^{n-2\sigma}(x)}{(\bar\lambda(x)+\delta_2)^{n-2\sigma}}U(\xi_\lambda).\end{aligned}$$

By the uniform continuity of U on compact sets, we can choose ε_1 sufficiently small, such that for all $\lambda\in(\bar\lambda(x),\bar\lambda(x)+\varepsilon_1)$,

$$\left|U(\xi_{\bar\lambda(x)})-U(\xi_\lambda)\right|\le\frac{1}{4}\inf_{\partial''\mathcal{B}^+_{\bar\lambda(x)+\delta_2}(X)}(U-U_{X,\bar\lambda(x)}),$$

and

$$\frac{\lambda^{n-2\sigma}-\bar\lambda^{n-2\sigma}(x)}{(\bar\lambda(x)+\delta_2)^{n-2\sigma}}U(\xi_\lambda)\le\frac{1}{4}\inf_{\partial''\mathcal{B}^+_{\bar\lambda(x)+\delta_2}(X)}(U-U_{X,\bar\lambda(x)}).$$

Then (4.10) follows.

We conclude from (4.9), (4.10) that for any $\lambda\in(\bar\lambda(x),\bar\lambda(x)+\varepsilon_1)$, $U_{X,\lambda}\le U$ in K_1. Hence, we have

$$U_{X,\lambda}\le U,\quad V_{X,\lambda}\le V\quad\text{ on }\ \partial' K_2.$$

Using the narrow domain technique as that the proof of (3.6), we can choose δ_2 small (notice that we can choose ε as small as we want less then ε_1 such that for $\lambda\in(\bar\lambda(x),\bar\lambda(x)+\varepsilon)$ such that

$$U_{X,\lambda}\le U,\quad V_{X,\lambda}\le V\quad\text{ in }\ K_2. \tag{4.11}$$

From the above argument, we can see that the moving spheres procedure may continue beyond $\bar\lambda(x)$ where we reach a contradiction. □

5. A upper bound near an isolated singularity

We shall prove Theorem 1.3 by the blow-up analysis and the method of moving spheres. Suppose the contrary that there exists a sequence $\{x_j\} \subset B_1$ such that $x_j \to 0$ as $j \to \infty$, and

$$|x_j|^{\frac{n-2\sigma}{2}}(u+v)(x_j) \to \infty \quad \text{as } j \to \infty. \tag{5.1}$$

Write $w := u + v$. Consider

$$h_j(x) := (|x_j|/2 - |x - x_j|)^{\frac{n-2\sigma}{2}} w(x) \quad \text{in } B_{|x_j|/2}(x_j).$$

Let $|\bar{x}_j - x_j| < |x_j|/2$ satisfy $h_j(\bar{x}_j) = \max_{|x-x_j|\le|x_j|/2} h_j(x)$, and define $2\mu_j := |x_j|/2 - |\bar{x}_j - x_j|$. Then

$$0 < 2\mu_j \le |x_j|/2 \quad \text{and} \quad |x_j|/2 - |x - x_j| \ge \mu_j \qquad \text{in } B_{\mu_j}(\bar{x}_j). \tag{5.2}$$

By the definition of h_j, we have

$$(2\mu_j)^{\frac{n-2\sigma}{2}} w(\bar{x}_j) = h_j(\bar{x}_j) \ge h_j \ge (\mu_j)^{\frac{n-2\sigma}{2}} w \qquad \text{in } B_{\mu_j}(\bar{x}_j). \tag{5.3}$$

Therefore,

$$w(\bar{x}_j)^{-1} w \le 2^{\frac{n-2\sigma}{2}} \qquad \text{in } B_{\mu_j}(\bar{x}_j). \tag{5.4}$$

On the other hand,

$$(2\mu_j)^{\frac{n-2\sigma}{2}} w(\bar{x}_j) = h_j(\bar{x}_j) \ge h_j(x_j) = (|x_j|/2)^{\frac{n-2\sigma}{2}} w(x_j) \to \infty \quad \text{as } j \to \infty. \tag{5.5}$$

Now, set

$$\begin{aligned} \widetilde{w}_j &:= w(\bar{x}_j)^{\frac{2}{n-2\sigma}}, \quad \bar{X}_j^w := (-\bar{x}_j w(\bar{x}_j)^{\frac{2}{n-2\sigma}}, 0), \\ U_j(y,t) &:= w^{-1}(\bar{x}_j) U\left(\bar{x}_j + y\widetilde{w}_j^{-1}, t\widetilde{w}_j^{-1}\right) \quad \text{in } \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}_j^w), \\ V_j(y,t) &:= w^{-1}(\bar{x}_j) V\left(\bar{x}_j + y\widetilde{w}_j^{-1}, t\widetilde{w}_j^{-1}\right) \quad \text{in } \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}_j^w), \end{aligned}$$

A calculation gives that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U_j) = 0 & \text{in } \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}_j^w), \\ \operatorname{div}(t^{1-2\sigma}\nabla V_j) = 0 & \text{in } \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}_j^w), \\ \dfrac{\partial U_j}{\partial \nu^\sigma} = \alpha_1 U_j^{\frac{n+2\sigma}{n-2\sigma}} + \beta U_j^{\frac{2\sigma}{n-2\sigma}} V_j^{\frac{n}{n-2\sigma}} & \text{on } \partial'\mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}_j^w)\setminus\{\bar{X}_j^w\}, \\ \dfrac{\partial V_j}{\partial \nu^\sigma} = \alpha_2 V_j^{\frac{n+2\sigma}{n-2\sigma}} + \beta V_j^{\frac{2\sigma}{n-2\sigma}} U_j^{\frac{n}{n-2\sigma}} & \text{on } \partial'\mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}_j^w)\setminus\{\bar{X}_j^w\}, \end{cases}$$

and $(U_j + V_j)(0) = 1$.

By Propositions 2.2 and 2.5, there exists some $\nu \in (0,1)$ such that for $R > 1$,

$$\|U_j\|_{W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+)} + \|V_j\|_{W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+)} + \|U_j\|_{C^\nu(\overline{\mathcal{B}_R^+})} + \|V_j\|_{C^\nu(\overline{\mathcal{B}_R^+})} \le C(R). \tag{5.6}$$

From (5.6), we obtain that after passing to a subsequence, there exist two nonnegative functions $\widetilde{U}, \widetilde{V} \in W^{1,2}_{\text{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+}) \cap C^\nu_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$ such that

$$\begin{cases} U_j \rightharpoonup \widetilde{U}, \quad V_j \rightharpoonup \widetilde{V} & \text{weakly in } W^{1,2}_{\text{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+}), \\ U_j \to \widetilde{U}, \quad V_j \to \widetilde{V} & \text{in } C^{\nu/2}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+}), \end{cases}$$

Moreover, $\widetilde{U}$ and $\widetilde{V}$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\widetilde{U})=0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \operatorname{div}(t^{1-2\sigma}\nabla\widetilde{V})=0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \dfrac{\partial\widetilde{U}}{\partial\nu^\sigma}=\alpha_1\widetilde{U}^{\frac{n+2\sigma}{n-2\sigma}}+\beta\widetilde{U}^{\frac{2\sigma}{n-2\sigma}}\widetilde{V}^{\frac{n}{n-2\sigma}} & \text{on} \quad \partial\mathbb{R}^{n+1}_+, \\ \dfrac{\partial\widetilde{V}}{\partial\nu^\sigma}=\alpha_2\widetilde{V}^{\frac{n+2\sigma}{n-2\sigma}}+\beta\widetilde{V}^{\frac{2\sigma}{n-2\sigma}}\widetilde{U}^{\frac{n}{n-2\sigma}} & \text{on} \quad \partial\mathbb{R}^{n+1}_+, \end{cases}$$

and $(\widetilde{U}+\widetilde{V})(0)=1$. By Theorem 1.1, we have

$$(\widetilde{u}+\widetilde{v})(x)=(\widetilde{U}+\widetilde{V})(x,0)=\left(\frac{1}{1+|x|^2}\right)^{\frac{n-2\sigma}{2}} \tag{5.7}$$

modulo some multiple, scaling and translation.

On the other hand, let us arbitrarily fix $x_0\in\mathbb{R}^n$, and $\lambda_0>0$. Then for all j large, we have $|x_0|<R_j/10, 0<\lambda_0<R_j/10$. For $X_0=(x_0,0)$, if we have proved that for any $\lambda\in(0,\lambda_0)$,

$$(U_j+V_j)_{X_0,\lambda}\le U_j+V_j \quad \text{in} \quad \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}^w_j)\backslash\mathcal{B}^+_\lambda(X_0). \tag{5.8}$$

Sending $j\to\infty$, by the arbitrariness of x_0, λ_0, we obtain that for any $x_0\in\mathbb{R}^n$ and any $\lambda>0$,

$$(\widetilde{u}+\widetilde{v})_{x_0,\lambda}\le\widetilde{u}+\widetilde{v} \quad \text{in} \quad \mathbb{R}^n\backslash B_\lambda(x_0).$$

An elementary calculus lemma [33, Lemma 11.3] gives that $\widetilde{u}+\widetilde{v}\equiv$ *constant*. This contradicts to (5.7). Therefore, we finish the proof of Theorem 1.3.

For simplicity, we denote $H_j:=U_j+V_j$, $H_{j,X_0,\lambda}:=(U_j+V_j)_{X_0,\lambda}$, and define

$$\bar{\lambda}(x):=\sup\left\{\mu\in(0,\lambda_0)\middle|H_{j,X_0,\lambda}\le H_j \ \text{ in } \ \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}^w_j)\backslash\mathcal{B}^+_\lambda(X_0),\ \forall\lambda\in(0,\mu)\right\}.$$

From what has been discussed in Sections 3 and 4 , it is clear to know that in order to obtain that (5.8), we just need to prove $\bar{\lambda}(x)=\lambda_0$. Thus, we first get

Lemma 5.1. *There exists $\lambda_1\in(0,\lambda_0)$ such that for any $\lambda\in(0,\lambda_1)$,*

$$H_{j,X_0,\lambda}\le H_j \quad \textit{in} \quad \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}^w_j)\backslash\mathcal{B}^+_\lambda(X_0).$$

Then we give that

Lemma 5.2.

$$\bar{\lambda}(x)=\lambda_0.$$

Proof of Lemma 5.1. The proof consists of two steps. In the first step, we prove $U_{j,X_0,\lambda}\le U_j$, $V_{j,X_0,\lambda}\le V_j$ in $\mathcal{B}^+_\mu(X_0)\backslash\mathcal{B}^+_\lambda(X_0)$. It follows that

$$H_{j,X_0,\lambda}\le H_j \quad \text{in} \quad \mathcal{B}^+_\mu(X_0)\backslash\mathcal{B}^+_\lambda(X_0).$$

We shall prove as the second step that

$$H_{j,X_0,\lambda}\le H_j \quad \text{in} \quad \mathcal{B}^+_{\widetilde{w}_j/2}(\bar{X}^w_j)\backslash\mathcal{B}^+_\mu(X_0).$$

The proof of the step 1 is almost precisely like that of (3.6) of Lemma 3.1. Notice that the difference is that we choose

$$\lambda_1 = \mu \min\left\{ \left(\frac{\inf_{\partial''\mathcal{B}^+_\mu(X)} U_j}{\sup_{\mathcal{B}^+_\mu(X)} U_j} \right)^{\frac{1}{n-2\sigma}}, \left(\frac{\inf_{\partial''\mathcal{B}^+_\mu(X)} V_j}{\sup_{\mathcal{B}^+_\mu(X)} V_j} \right)^{\frac{1}{n-2\sigma}}, \left(\frac{\inf_{\partial''\mathcal{B}^+_\mu(X)} H_j}{\sup_{\mathcal{B}^+_\mu(X)} H_j} \right)^{\frac{1}{n-2\sigma}} \right\},$$

instead of (3.8). We just prove the step 2.

Since

$$\inf_{\partial''\mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j)} H_j = \frac{1}{w(\bar{x}_j)} \inf_{\partial''\mathcal{B}^+_{1/2}} (U+V) \geq \frac{1}{w(\bar{x}_j)} \inf_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/4}} (U+V),$$

and from Proposition 2.5 that

$$\inf_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/4}} (U+V) \geq \frac{1}{C} \sup_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/4}} (U+V) \geq \frac{1}{C} \min_{\partial B_{1/2}} (u+v).$$

In addition, by the continuous of u and v, we have $u+v \geq C_0$ on $\partial B_{1/2}$ for some positive constant C_0. Hence, it follows that

$$H_j \geq \frac{C_0}{Cw(\bar{x}_j)} > \frac{C_0}{Cw^{3/2}(\bar{x}_j)} > 0 \quad \text{on } \partial''\mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j).$$

On the other hand, since $\frac{|x_j|}{2} \leq |\bar{x}_j| \leq \frac{3|x_j|}{2} \ll \frac{1}{2}$, for any $\xi \in \partial''\mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j)$, i.e., $\left|\bar{X}_j + \xi w(\bar{x}_j)^{\frac{-2}{n-2\sigma}}\right| = \frac{1}{2}$, we have

$$|\xi| \approx \frac{1}{2} w(\bar{x}_j)^{\frac{2}{n-2\sigma}} \to +\infty \quad \text{as} \quad j \to +\infty.$$

Thus,

$$\left(\frac{\mu}{|\xi - X_0|}\right)^{n-2\sigma} \approx \left(\frac{\mu}{2}\right)^{n-2\sigma} \frac{1}{w^2(\bar{x}_j)} \quad \text{on } \partial''\mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j).$$

A combination of the above argument, we get

$$H_j(\xi) > \left(\frac{\mu}{|\xi - X_0|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X_0)} H_j \quad \text{on } \partial''\mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j). \tag{5.9}$$

By the same argument as before, we obtain that

$$H_j(\xi) \geq \left(\frac{\mu}{|\xi - X_0|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X_0)} H_j \quad \text{in} \quad \mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j)\setminus\mathcal{B}^+_\mu(X_0). \tag{5.10}$$

Then for all $\xi \in \mathcal{B}^+_{\tilde{w}_j/2}(\bar{X}^w_j)\setminus\mathcal{B}^+_\mu(X_0)$, $\lambda \in (0, \lambda_1)$, we conclude using (5.10) that

$$\begin{aligned}
H_{j,X_0,\lambda}(\xi) &= \left(\frac{\lambda}{|\xi - X_0|}\right)^{n-2\sigma} H_j(X_0 + \frac{\lambda^2(\xi - X_0)}{|\xi - X_0|^2}) \\
&\leq \left(\frac{\lambda_1}{|\xi - X_0|}\right)^{n-2\sigma} \sup_{\mathcal{B}^+_\mu(X_0)} H_j \\
&\leq \left(\frac{\mu}{|\xi - X_0|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X_0)} H_j \leq H_j(\xi).
\end{aligned}$$

Lemma 5.1 is proved. □

Proof of Lemma 5.2. We argue by contradiction. Were that $\bar{\lambda}(x) < \lambda_0$, we shall show that there exists a positive constant ε such that for all $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon)$,

$$H_{j,X_0,\lambda} \leq H_j \quad \text{in} \quad \mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) \setminus \mathcal{B}^{+}_{\lambda}(X_0), \tag{5.11}$$

which contradicts with the definition of $\bar{\lambda}(x)$.

The region $\mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) \setminus \mathcal{B}^{+}_{\lambda}(X)$ is divided into three parts. For $\delta, \delta_1 > 0$ small, which will be fixed later, denote

$$K_1 := \{\xi \in \mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) | 0 < |\xi - \bar{X}^w_j| \leq \delta_1\},$$
$$K_2 := \{\xi \in \mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) | |\xi - \bar{X}^w_j| \geq \delta_1, |\xi - X_0| \geq \bar{\lambda}(x) + \delta\},$$
$$K_3 := \{\xi \in \mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) | \lambda \leq |\xi - X_0| \leq \bar{\lambda}(x) + \delta\}.$$

In order to obtain (5.11), we just need to prove that it holds respectively on K_1, K_2, K_3.

Similar to (5.9), we have

$$H_j \geq \frac{C_0}{C w(\bar{x}_j)} > \left(\frac{\lambda_0}{|\xi - X_0|}\right)^{n-2\sigma} \sup_{\mathcal{B}^{+}_{\lambda_0}(X_0)} H_j \geq H_{j,X_0,\bar{\lambda}(x)} \quad \text{on } \partial'' \mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j).$$

We say from the Strong Maximum Principle that

$$H_{j,X_0,\bar{\lambda}(x)} < H_j \quad \text{in} \quad \mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) \setminus \overline{\mathcal{B}^{+}_{\bar{\lambda}(x)}(X_0)}.$$

Using [35, Proposition 4.1], there exist two positive constants δ_1, C_1 such that $H_j - H_{j,X_0,\bar{\lambda}(x)} > C_1$ in K_1. Choose a positive constant ε_1 small such that for all $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_1)$, $H_{j,X_0,\bar{\lambda}(x)} - H_{j,X_0,\lambda} > -C_1/2$ in K_1. Hence,

$$H_j - H_{j,X_0,\lambda} > C_1/2 \quad \text{in} \quad K_1. \tag{5.12}$$

Together with $H_j > H_{j,X_0,\bar{\lambda}(x)}$ in K_2 and the compactness of K_2, there exists a positive constant C_2 such that $H_j - H_{j,X_0,\bar{\lambda}(x)} > C_2$ in K_2. By the uniform continuity of H_j on compact sets, there exists a positive constant ε_2 small such that for all $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_2)$, $H_{j,X_0,\bar{\lambda}(x)} - H_{j,X_0,\lambda} > -C_2/2$ in K_2. Therefore,

$$H_j - H_{j,X_0,\lambda} > C_2/2 \quad \text{in} \quad K_2. \tag{5.13}$$

Now let us focus on the region K_3. Using the narrow domain technique as that in Lemma 3.1, we can choose δ small (notice that we can choose ε as small as we want) such that

$$U_{j,X_0,\lambda} \leq U_j, \; V_{j,X_0,\lambda} \leq V_j \quad \text{in} \quad K_3.$$

Thus,

$$H_{j,X_0,\lambda} \leq H_j \quad \text{in} \quad K_3. \tag{5.14}$$

Combining (5.12)–(5.14), we obtain that there exists a positive constant ε such that for all $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon)$, $H_{j,X_0,\lambda} \leq H_j$ in $\mathcal{B}^{+}_{\tilde{w}_j/2}(\bar{X}^w_j) \setminus \mathcal{B}^{+}_{\lambda}(X_0)$, which contradicts with the definition of $\bar{\lambda}(x)$. Lemma 5.2 is proved. □

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