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LIOUVILLE PROPERTY AND REGULARITY OF A HESSIAN QUOTIENT EQUATION

By Jiguang Bao, Jingyi Chen, Bo Guan, and Min Ji

Abstract. We are concerned with properties of (convex) solutions to the Hessian quotient equation $S_{n,k}(D^2u) = \varphi$, $1 \leq k < n$. As our first main result we prove some regularity of strong solutions, while the second states that for $\varphi \equiv 1$ an entire convex solution with a quadratic growth bound must be a quadratic polynomial.

1. Introduction. In this paper we consider local and global properties of solutions to the Hessian quotient equation in $\mathbb{R}^n$

$$S_{n,k}(D^2u) = 1, \quad 1 \leq k < n. \tag{1.1}$$

Here the operator $S_{n,k}(D^2u)$ is defined as follows. Let $S_k(\lambda)$ be the kth elementary symmetric function of $\lambda \in \mathbb{R}^n$ and

$$S_{n,k}(\lambda) = \frac{S_n(\lambda)}{S_k(\lambda)}, \quad \lambda \in \mathbb{R}^n.$$

Then $S_{n,k}(D^2u) = S_{n,k}(\lambda[D^2u])$ where $\lambda[D^2u] = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of the Hessian, D^2u, of a function u defined in $\mathbb{R}^n$.

Equation (1.1) is elliptic at a solution u if $D^2u > 0$, i.e., u is (locally) strictly convex. It belongs to an important class of fully nonlinear elliptic equations which has been studied by many authors (cf. [5] and [23]). Such equations are closely related to problems in differential geometry. In [15], the Hessian quotient equation on $\mathbb{S}^n$ was studied in connection with a geometric problem of prescribing curvatures. When $n = 3$ and $k = 1$, equation (1.1) arises in special Lagrangian geometry: if u is a solution of (1.1), the graph of Du over $\mathbb{R}^3$ in $\mathbb{C}^3$ is a special Lagrangian submanifold in $\mathbb{C}^3$, i.e., its mean curvature vanishes everywhere and the complex structure on $\mathbb{C}^3$ sends the tangent space of the graph to the normal space at every point. This special case has received much attention.

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In this paper we will first prove a regularity result for strong solutions to (1.1). Following [4] and [11], we call a function $u \in W^{2,p}_{loc}(\Omega)$ for $p > \frac{n}{2}$ an *admissible strong solution* of (1.1) in a domain Ω in $\mathbb{R}^n$ if $D^2u > 0$ and (1.1) is satisfied almost everywhere in Ω. By a classical theorem of Calderón and Zygmund [7] functions in $W^{2,p}_{loc}(\Omega)$, $p > \frac{n}{2}$, are pointwise twice differentiable almost everywhere (cf. [4]).

Throughout this paper, we set $\gamma = (n-1)(n-k)$ if $1 \le k \le n-2$, and $\gamma = 2(n-1)$ if $k = n-1$. Note that $\gamma > \frac{n}{2}$ and $\gamma \ge 2k$ for all $n \ge 2$ and $k < n$.

THEOREM 1.1. *Let Ω be a domain in $\mathbb{R}^n$ and $p > \gamma$. Then every admissible strong solution of (1.1) in $W^{2,p}_{loc}(\Omega)$ is smooth.*

In Section 2 we will prove Theorem 1.1 for admissible strong solutions of the more general equation

$$S_{n,k}(D^2u) = \varphi \text{ in } \Omega, \quad 1 \le k < n \tag{1.2}$$

where $\varphi \in C^\infty(\Omega)$ and $\varphi > 0$.

It would be interesting to determine sharp lower bounds for p in Theorem 1.1. For the Monge-Ampère equation ($k = 0$), the optimal bound is $n(n-1)/2$; see [1], [2] and [24]. Related results for the Hessian equation

$$S_k(D^2u) = \varphi \tag{1.3}$$

may be found in [25] and [26]. We point out that Theorem 1.1 fails in general if $p < n$, as we will see at the end of Section 2 where we show that a radially symmetric admissible solution to (1.1) must be either a quadratic polynomial or a strong solution in $W^{2,q}_{loc}$ for all $q < n$ but not in $W^{2,n}_{loc}$.

The second part of this paper concerns global solutions of equation (1.1). A well known theorem due to Jörgens [17] ($n = 2$), Calabi [6] ($n = 3, 4, 5$) and Pogorelov [22] ($n \ge 2$) asserts that a smooth strictly convex solution to the Monge-Ampère equation $\det D^2u = 1$ over the entire space $\mathbb{R}^n$ must be a quadratic polynomial. (A more general result is due to Cheng-Yau [9].) This was extended to viscosity solutions by Caffarelli [3]. In this article we will prove the following Liouville property of global solutions to (1.1).

THEOREM 1.2. *Let $u \in C^\infty(\mathbb{R}^n)$ be a strictly convex solution of (1.1) satisfying*

$$u(x) \le A(1 + |x|^2) \quad \forall x \in \mathbb{R}^n \tag{1.4}$$

where A is a constant. Then u is a quadratic polynomial.

Remark 1.3. When $k = n-1$, Theorem 1.2 holds without assumption (1.4). For $1 \le k \le n-2$, however, it is not clear to the authors whether it is enough to merely assume that u is strictly convex.

Theorem 1.2 will be proved in Section 3. From the proof we will obtain a similar characterization of global solutions to the Hessian equation (1.3) when the right-hand side function is constant; see Theorem 3.2. We also mention some related work on Bernstein type results for global special Lagrangian graphs. (We refer the reader to Osserman's article [21] for a survey on classical Bernstein theorems for minimal surfaces in $\mathbb{R}^n$.) Recently, Yuan [27] proved that any convex entire solution to the special Lagrangian equation

$$F(D^2u) := \tan^{-1}\lambda_1 + \cdots + \tan^{-1}\lambda_n = c \quad \text{in } \mathbb{R}^n,$$

where $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the Hessian D^2u, is a quadratic polynomial for all $n \geq 2$; the case $n = 2$ was considered earlier by Fu [13]. In [18], Jost-Xin treated the problem using a different method under the assumption that $|D^2u|$ is bounded. Note that the Hessian quotient equation (1.1) is different from the special Lagrangian equation except when $n = 3, k = 1$.

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2. Local bounds for second derivatives of strong solutions. In this section we will prove a slightly more general result that includes Theorem 1.1. The main step in our argument is to derive a local bound for the second derivatives of $W^{2,p}$ admissible strong solutions when $p > \gamma$. We will need the notion of upper contact set and the Alexandrov maximum principle. Let Ω be a bounded domain in $\mathbb{R}^n$ and $v \in C^0(\Omega)$. We recall (see [14]) that the *upper contact set* of v, denoted $\Gamma_v^+(\Omega)$, is defined to be the subset of Ω where the graph of v lies below a support hyperplane in $\mathbb{R}^{n+1}$, that is,

$$\Gamma_v^+(\Omega) = \{x \in \Omega : v(z) \leq v(x) + \nu \cdot (z - x), \text{ for all } z \in \Omega, \text{ for some } \nu \in \mathbb{R}^n\}. \tag{2.1}$$

We have the following form of the Alexandrov maximum principle.

LEMMA 2.1. *Let (a^{ij}) be an $n \times n$ matrix which is positive definite a.e. in a bounded domain $\Omega \subset \mathbb{R}^n$, and $v \in W^{2,q}_{loc}(\Omega) \cap C^0(\bar{\Omega})$ with $v = 0$ on $\partial\Omega$, where $q > n$. Then*

$$\sup_{\Omega} v \leq Cd \left(\int_{\Gamma_v^+(\Omega)} \frac{(-\, a^{ij} D_{ij} v)^n}{\det(a^{ij})}\, dx \right)^{\frac{1}{n}}, \tag{2.2}$$

provided that $(a^{ij}D_{ij}v)/\det(a^{ij})^{\frac{1}{n}} \in L^n(\Omega)$, *where* $d = diam\,\Omega$ *and* C *is a constant depending only on* n.

This is a slight extension of Lemma 9.3 in [14], which assumes $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. A detailed proof can be found in [8]; so we omit it here. Note that by the Sobolev embedding theorem $W^{2,q}_{loc}(\Omega) \subset C^{1,\alpha}(\Omega)$ for $\alpha = 1 - \frac{n}{q}$ when $q > n$.

We now state the main estimate of this section.

THEOREM 2.2. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and* $u \in W^{2,p}_{loc}(\Omega)$ *an admissible strong solution of (1.2) where* $p > \gamma$, $\varphi \in C^{1,1}(\Omega)$, *and* $\varphi > 0$ *in* Ω. *Then* $u \in C^{1,1}(\Omega)$ *and, for any* $y \in \Omega$ *and* $0 < R < 1$ *with* $\overline{B_{3R}(y)} \subset \Omega$,

$$\sup_{B_R(y)} |D^2u| \leq C + C(R^{-n}\|\Delta u\|^p_{L^p(B_{3R}(y))})^{\frac{1}{q}} \tag{2.3}$$

where $q = \min\{n, p-\gamma\}$ *and* $C > 0$ *is a constant depending on* n, k, p, $\|\varphi\|_{C^{1,1}(\overline{B_{3R}(y)})}$ *and* $\inf_{B_{3R}(y)} \varphi$.

Before presenting the proof of Theorem 2.2 we will derive two technical lemmas. Following the notations in [20] we set

$$S_{k;i}(\lambda) = S_k(\lambda)|_{\lambda_i=0} \quad \text{for fixed } i \in \{1, \ldots, n\}.$$

For any i we have

$$\frac{\partial S_k}{\partial \lambda_i}(\lambda) = S_{k-1;i}(\lambda). \tag{2.4}$$

It follows that

$$\frac{\partial S_{n,k}}{\partial \lambda_i}(\lambda) = \frac{S_{n-1;i}(\lambda)S_k(\lambda) - S_n(\lambda)S_{k-1;i}(\lambda)}{(S_k(\lambda))^2}. \tag{2.5}$$

LEMMA 2.3. *For all* $1 \leq i \leq n$, $1 \leq k < n$, *we have*

$$S_{n-1;i}(\lambda)S_k(\lambda) - S_n(\lambda)S_{k-1;i}(\lambda) = S_{n-1;i}(\lambda)S_{k;i}(\lambda) \quad \text{on } \mathbb{R}^n.$$

Proof. We have

$$\begin{aligned} S_{n-1;i}(\lambda)S_k(\lambda) &= S_{n-1;i}(\lambda)(\lambda_i S_{k-1;i}(\lambda) + S_{k;i}(\lambda)) \\ &= S_n(\lambda)S_{k-1;i}(\lambda) + S_{n-1;i}(\lambda)S_{k;i}(\lambda) \end{aligned}$$

from which the desired identity follows. □

By (2.5) and Lemma 2.3 we see that if $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n > 0$ then

$$\frac{\partial S_{n,k}}{\partial \lambda_1}(\lambda) \leq \cdots \leq \frac{\partial S_{n,k}}{\partial \lambda_n}(\lambda). \tag{2.6}$$

LEMMA 2.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ satisfy $S_{n,k}(\lambda) \geq 1$ and $\lambda_1 \geq \cdots \geq \lambda_n > 0$. Then*

$$\left(\frac{\partial S_{n,k}}{\partial \lambda_n}(\lambda)\right)^n \leq (S_1(\lambda))^\gamma \prod_{i=1}^n \frac{\partial S_{n,k}}{\partial \lambda_i}(\lambda), \tag{2.7}$$

where $\gamma = (n-1)(n-k)$ if $1 \leq k \leq n-2$, and $\gamma = 2(n-1)$ if $k = n-1$.

Proof. Note that for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $\lambda_1 \geq \cdots \geq \lambda_n > 0$,

$$S_{k;i}(\lambda) \geq \begin{cases} \lambda_1 \cdots \lambda_{i-1}\lambda_{i+1} \cdots \lambda_{k+1}, & \text{if } i \leq k, \\ \lambda_1 \cdots \lambda_k, & \text{if } i > k. \end{cases}$$

Thus,

$$\begin{aligned} \prod_{i=1}^{n-1} S_{k;i}(\lambda) &= \prod_{i=1}^{k} S_{k;i}(\lambda) \cdot \prod_{i=k+1}^{n-1} S_{k;i}(\lambda) \\ &\geq (\lambda_1 \cdots \lambda_k)^{k-1}(\lambda_{k+1})^k \cdot (\lambda_1 \cdots \lambda_k)^{n-k-1} \\ &= (\lambda_1 \cdots \lambda_k)^{n-2}(\lambda_{k+1})^k, \end{aligned} \tag{2.8}$$

and by $S_{n,k}(\lambda) \geq 1$,

$$\prod_{i=1}^{n} S_{n-1;i}(\lambda) = (S_n(\lambda))^{n-1} \geq (S_k(\lambda))^{n-1} \geq (S_{k;n}(\lambda))^{n-1}. \tag{2.9}$$

When $k < n-1$, we have by (2.8) and (2.9) that

$$\begin{aligned} \frac{(S_{n-1;n}(\lambda)S_{k;n}(\lambda))^n}{\prod_{i=1}^n (S_{n-1;i}(\lambda)S_{k;i}(\lambda))} &\leq \frac{(\lambda_1 \cdots \lambda_{n-1})^n S_{k;n}(\lambda)}{\prod_{i=1}^n S_{k;i}(\lambda)} \\ &= \frac{(\lambda_1 \cdots \lambda_{n-1})^n}{\prod_{i=1}^{n-1} S_{k;i}(\lambda)} \\ &\leq \frac{(\lambda_1 \cdots \lambda_{n-1})^n}{(\lambda_1 \cdots \lambda_k)^{n-2}(\lambda_{k+1})^k} \\ &= (\lambda_1 \cdots \lambda_k)^2(\lambda_{k+1})^{n-k}(\lambda_{k+2} \cdots \lambda_{n-1})^n \\ &\leq (S_1(\lambda))^{(n-1)(n-k)}, \end{aligned}$$

while for $k = n-1$,

$$\begin{aligned}\frac{(S_{n-1;n}(\lambda)S_{k;n}(\lambda))^n}{\prod_{i=1}^n (S_{n-1;i}(\lambda)S_{k;i}(\lambda))} &= \left(\frac{(S_{n-1;n}(\lambda))^n}{\prod_{i=1}^n S_{n-1;i}(\lambda)}\right)^2 \\ &\leq (S_{n-1;n}(\lambda))^2 \\ &\leq (S_1(\lambda))^{2(n-1)}\end{aligned}$$

by (2.9). Now (2.7) follows from (2.5) and Lemma 2.3. □

Proof of Theorem 2.2. For convenience we write equation (1.2) in the form

$$F(D^2u) \equiv (S_{n,k}(\lambda[D^2u]))^{\frac{1}{n-k}} = \varphi^{\frac{1}{n-k}}. \tag{2.10}$$

It is known that F is a concave function of D^2u when u is convex (cf. [23]).

Let $h > 0$ be sufficiently small and $\xi \in \mathbb{R}^n$ a fixed unit vector. We introduce the second order difference quotient

$$\Delta_{\xi\xi}^h u(x) = \frac{u(x+h\xi) - 2u(x) + u(x-h\xi)}{h^2}, \quad a.e.\ x \in \Omega_h,$$

where $\Omega_h \equiv \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\}$. By the concavity of F we obtain from (2.10) that for a.e. $x \in \Omega_h$

$$\begin{aligned}\varphi^{\frac{1}{n-k}}(x \pm h\xi) - \varphi^{\frac{1}{n-k}}(x) &= F(D^2u(x \pm h\xi)) - F(D^2u(x)) \\ &\leq F^{ij}(D^2u(x))(D_{ij}u(x \pm h\xi) - D_{ij}u(x)).\end{aligned} \tag{2.11}$$

Here

$$F^{ij}(M) = \frac{\partial F}{\partial m_{ij}}(M)$$

for any $n \times n$ matrix $M = (m_{ij})$.

Let $y \in \Omega$ with $\overline{B_{3R}(y)} \subset \Omega$. Without loss of generality we may assume $\varphi \geq 1$ on $B_{3R}(y)$. For simplicity we will write $B_r = B_r(y)$ for $r > 0$ and $F^{ij} := F^{ij}(D^2u(x))$ in the rest of this proof. It follows from (2.11) that

$$F^{ij}D_{ij}(\Delta_{\xi\xi}^h u(x)) \geq -\|\varphi^{\frac{1}{n-k}}\|_{C^{1,1}(\overline{B_{3R}})}, \quad a.e.\ x \in B_{2R} \tag{2.12}$$

when $h \leq R$. Consider the function

$$v = \eta\Delta_{\xi\xi}^h u,$$

where

$$\eta(x) = \left(1 - \frac{|x-y|^2}{4R^2}\right)^{\beta},$$

and $\beta > 2$ is a constant to be determined later. Direct calculation leads to

$$|D\eta| \le \frac{\beta}{R}\eta^{1-\frac{1}{\beta}}, \quad |D^2\eta| \le \frac{C(n,\beta)}{R^2}\eta^{1-\frac{2}{\beta}} \tag{2.13}$$

and

$$\begin{aligned} F^{ij}D_{ij}v &= F^{ij}(\eta D_{ij}(\Delta^h_{\xi\xi}u) + 2D_i\eta D_j(\Delta^h_{\xi\xi}u) + (\Delta^h_{\xi\xi}u)D_{ij}\eta) \\ &\ge -\eta\|\varphi^{\frac{1}{n-k}}\|_{C^{1,1}(\overline{B_{3R}})} + 2F^{ij}D_i\eta D_j(\Delta^h_{\xi\xi}u) + (\Delta^h_{\xi\xi}u)F^{ij}D_{ij}\eta \\ &\ge -\eta\|\varphi^{\frac{1}{n-k}}\|_{C^{1,1}(\overline{B_{3R}})} - \Lambda_F(2|D\eta||D(\Delta^h_{\xi\xi}u)| + |\Delta^h_{\xi\xi}u||D^2\eta|) \\ &\ge -C\eta - \frac{C(n,\beta)\Lambda_F}{R^2\eta^{\frac{2}{\beta}}}(v + R\eta^{1+\frac{1}{\beta}}|D(\Delta^h_{\xi\xi}u)|), \quad a.e.\ x \in B_{2R} \end{aligned} \tag{2.14}$$

by (2.12) and (2.13), where Λ_F is the maximum eigenvalue of the matrix (F^{ij}).

Next, for $x \in \Gamma^+_v(B_{2R})$ we take $z \in \partial B_{2R}$ with

$$\frac{z-x}{|z-x|} = -\frac{Dv(x)}{|Dv(x)|}.$$

Since $v = 0$ on ∂B_{2R}, it follows from (2.1) that

$$v(x) \ge v(z) - Dv(x)\cdot(z-x) = |z-x||Dv(x)| \ge R\eta^{\frac{1}{\beta}}|Dv(x)|$$

as

$$|x-z| \ge 2R - |x-y| \ge R\eta^{\frac{1}{\beta}}.$$

Consequently on $\Gamma^+_v(B_{2R})$

$$\eta|D(\Delta^h_{\xi\xi}u)| = |Dv - (\Delta^h_{\xi\xi}u)D\eta| \le |Dv| + (\Delta^h_{\xi\xi}u)|D\eta| \le \frac{(1+\beta)v}{R\eta^{\frac{1}{\beta}}} \tag{2.15}$$

and, by (2.14) and the concavity of v on $\Gamma^+_v(B_{2R})$,

$$0 \le -F^{ij}D_{ij}v \le C\eta + \frac{C\Lambda_F v}{R^2\eta^{\frac{2}{\beta}}}, \quad a.e. \text{ in } \Gamma^+_v(B_{2R}), \tag{2.16}$$

where $C > 0$ is a constant depending only on n, β and $\|\varphi^{\frac{1}{n-k}}\|_{C^{1,1}(\overline{B_{3R}})}$. Note that $\{F^{ij}\}$ is positive definite with eigenvalues

$$\frac{\partial}{\partial \lambda_i}(S_{n,k}(\lambda[D^2u]))^{\frac{1}{n-k}}, \quad i = 1, \dots, n.$$

We have

$$\det(F^{ij}) = \prod_{i=1}^{n} \frac{\partial}{\partial \lambda_i}(S_{n,k}(\lambda[D^2u]))^{\frac{1}{n-k}} = \frac{\varphi^{\frac{n}{n-k}-n}}{(n-k)^n} \prod_{i=1}^{n} \frac{\partial}{\partial \lambda_i} S_{n,k}(\lambda[D^2u]) \tag{2.17}$$

by (2.10). Since $S_{n,k}(\lambda) \geq 1$,

$$\lambda_{k+1} \geq 1 \quad \text{and} \quad S_k(\lambda[D^2u]) \leq C\lambda_1 \cdots \lambda_k \leq CS_{k;n}(\lambda[D^2u]) \tag{2.18}$$

where $\lambda[D^2u] = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n > 0$. From (2.17), (2.5) and Lemma 2.3 we have

$$\begin{aligned} \det(F^{ij}) &= \frac{\varphi^{\frac{n}{n-k}-n}}{(n-k)^n S_k(\lambda[D^2u])^{2n}} \prod_{i=1}^{n} S_{n-1;i}(\lambda[D^2u]) S_{k;i}(\lambda[D^2u]) \\ &\geq \frac{S_{k;n}(\lambda[D^2u])}{CS_k(\lambda[D^2u])^{n+1}} \cdot (\lambda_1 \cdots \lambda_k)^{n-2} (\lambda_{k+1})^k \\ &\geq \frac{1}{CS_k(\lambda[D^2u])^2} \\ &\geq \frac{1}{C(\Delta u)^{2k}} \end{aligned} \tag{2.19}$$

by (2.8), (2.9), (2.18) and the Newton-Maclaurin inequality

$$\frac{1}{n} S_1(\lambda) \geq \cdots \geq \left(\frac{k!(n-k)!}{n!} S_k(\lambda) \right)^{\frac{1}{k}} \geq \cdots \geq (S_n(\lambda))^{\frac{1}{n}}.$$

Next, by Lemma 2.4, (2.6) and (2.17)

$$\frac{\Lambda_F^n}{\det(F^{ij})} \leq S_1(\lambda[D^2u])^{\gamma} = (\Delta u)^{\gamma}. \tag{2.20}$$

Inserting (2.19) and (2.20) into (2.16) we obtain

$$0 \leq \frac{-F^{ij} D_{ij} v}{\det(F^{ij})^{\frac{1}{n}}} \leq C\eta(\Delta u)^{\frac{2k}{n}} + \frac{Cv(\Delta u)^{\frac{\gamma}{n}}}{R^2 \eta^{\frac{2}{\beta}}} \quad \textit{a.e.} \text{ in } \Gamma_v^+(B_{2R}), \tag{2.21}$$

where C depends on n, β and $\|\varphi^{\frac{1}{n-k}}\|_{C^{1,1}(\overline{B_{3R}})}$.

Choosing $\beta = \frac{2n}{p-\gamma}$, we see from Lemma 2.1 and (2.21) that

$$\begin{aligned}\sup_{B_{2R}} v &\le CR\left(\int_{\Gamma_v^+(B_{2R})} \frac{(-F^{ij}D_{ij}v)^n}{\det(F^{ij})}\,dx\right)^{\frac{1}{n}}\\ &\le CR\left(\int_{\Gamma_v^+(B_{2R})} (\Delta u)^{2k}\,dx\right)^{\frac{1}{n}} + CR^{-1}\left(\int_{\Gamma_v^+(B_{2R})} (\Delta u)^{\gamma}(\eta^{-2/\beta}v)^n\,dx\right)^{\frac{1}{n}}\\ &\le CR\left(\int_{B_{2R}} (\Delta u)^{2k}\,dx\right)^{\frac{1}{n}} + CR^{-1}\left(\int_{B_{2R}} (\Delta u)^{\gamma}(\Delta_{\xi\xi}^h u)^{p-\gamma}\,dx\right)^{\frac{1}{n}}.\end{aligned}$$

By the inequality

$$\|\Delta_{\xi\xi}^h u\|_{L^p(B_{2R})} \le \|\Delta u\|_{L^p(B_{3R})}, \quad \text{if } h < R,$$

we have

$$\int_{B_{2R}} (\Delta u)^{\gamma}(\Delta_{\xi\xi}^h u)^{p-\gamma}\,dx \le \|\Delta u\|^{\gamma}_{L^p(B_{2R})}\|\Delta_{\xi\xi}^h u\|^{p-\gamma}_{L^p(B_{2R})} \le \|\Delta u\|^p_{L^p(B_{3R})}$$

and therefore,

$$\begin{aligned}\sup_{B_{2R}} v &\le CR\|\Delta u\|^{\frac{2k}{n}}_{L^{2k}(B_{2R})} + CR^{-1}(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\|\Delta u\|^{\frac{p}{n}}_{L^p(B_{3R})}\\ &\le C(R + R^{-1}(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}})\|\Delta u\|^{\frac{p}{n}}_{L^p(B_{3R})}\end{aligned} \tag{2.22}$$

where the constant C depends on n, k, p and $\|\varphi^{\frac{1}{n-k}}\|_{C^{1,1}(\overline{B_{3R}})}$. If $\sup_{B_{2R}} v < 1$, Theorem 2.2 holds trivially. So one may assume $\sup_{B_{2R}} v \ge 1$. If $p \le n+\gamma$ then $1 - \frac{2}{\beta} \ge 0$ and

$$\sup_{B_{2R}} v \le C(R + R^{-1})(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\|\Delta u\|^{\frac{p}{n}}_{L^p(B_{3R})}.$$

It follows that

$$\sup_{B_{2R}} v \le C(R + R^{-1})^{\frac{n}{q}}\|\Delta u\|^{\frac{p}{q}}_{L^p(B_{3R})}.$$

From (2.22) we see that this still holds when $p \ge n+\gamma$ as $1 - \frac{2}{\beta} \le 0$. We finally conclude that for $h < R$

$$\sup_{B_R} \Delta_{\xi\xi}^h u \le \left(\frac{4}{3}\right)^{\frac{2n}{p-\gamma}} \sup_{B_{2R}} v \le C + C(R^{-n}\|\Delta u\|^p_{L^p(B_{3R})})^{\frac{1}{q}}.$$

As ξ is an arbitrary unit vector in $\mathbb{R}^n$, this completes the proof of Theorem 2.2. □

Theorem 1.1 is a special case of the following.

THEOREM 2.5. *Let Ω be a domain in $\mathbb{R}^n$ and $u \in W^{2,p}_{loc}(\Omega)$ an admissible strong solution of (1.2) where $p > \gamma$, $\varphi \in C^\infty(\Omega)$, and $\varphi > 0$. Then $u \in C^\infty(\Omega)$.*

Proof. Let Ω' be a bounded subdomain of Ω, $\overline{\Omega'} \subset \Omega$. By Theorem 2.2 we have $u \in C^{1,1}(\overline{\Omega'})$. As (2.10) is concave and uniformly elliptic in a strictly convex solution with bounded second derivatives, the Evans-Krylov regularity theorem ([12], [19]) then implies that $u \in C^{2,\alpha}(\overline{\Omega'})$ for some $0 < \alpha < 1$. The smoothness of u now follows from the standard elliptic regularity theory. □

We conclude this section with a brief examination of radially symmetric solutions to (1.1). Let $u = u(r)$ be such a solution of (1.1). Then u satisfies

$$u'' \left(\frac{u'}{r}\right)^{n-k} = \frac{(n-1)!}{(k-1)!(n-k)!} u'' + \frac{(n-1)!}{k!(n-k-1)!} \frac{u'}{r}, \tag{2.23}$$

which we integrate to obtain

$$(u')^n - \frac{n!}{k!(n-k)!} r^{n-k} (u')^k = a, \tag{2.24}$$

where a is an arbitrary constant. If $a = 0$ then

$$u(r) = c \text{ or } u(r) = \frac{1}{2} \left(\frac{n!}{k!(n-k)!}\right)^{\frac{1}{n-k}} r^2 + c,$$

where c is a constant. Therefore u is a constant or quadratic polynomial in this case. Now assume $a > 0$. For r sufficiently small, take a positive solution u' for the polynomial equation (2.24). Letting $r \to 0$ in (2.24) and (2.23) we see that

$$\lim_{r\to 0} u'(r) = a^{\frac{1}{n}}, \lim_{r\to 0} u''(r) = \delta_{k(n-1)},$$

where $\delta_{k(n-1)}$ is the Kronecker delta for $k, n-1$. We can verify that $u'' > 0$ for small r, which implies that u is convex near the origin. It follows that

$$\lim_{r\to 0} r\Delta u = \lim_{r\to 0} (ru''(r) + (n-1)u'(r)) = (n-1)a^{\frac{1}{n}} \neq 0$$

and therefore, $\Delta u \in L^q_{loc}$ for all $q < n$ but $\Delta u \notin L^n_{loc}$. This also shows that Theorem 1.1 fails for $p < n$.

3. Liouville property of solutions. In this section we prove Theorem 1.2. Let $u \in C^\infty(\mathbb{R}^n)$ be a strictly convex solution of (1.1) which satisfies (1.4). We assume that $u \geq 0$, $u(0) = 0$ without loss of any generality, and furthermore

$$u(x) \leq A|x|^2, \quad \forall x \in \mathbb{R}^n \tag{3.1}$$

where A is a constant. We will make use of some known interior estimates for solutions to the Hessian equations. The Hessian quotient and Hessian equations are related by the Legendre transformation. In order to define the Legendre transform of u on the entire space, it is necessary to have a positive lower bound on u which grows faster than linear functions.

LEMMA 3.1. *Under assumption (3.1), there exists a constant $a > 0$ depending only on A and n such that*

$$u(x) \geq a|x|^2, \quad \forall x \in \mathbb{R}^n. \tag{3.2}$$

When $k = n - 1$, this holds for $a = n/2$ without assumption (3.1).

Proof. By the Newton-Maclaurin inequality

$$\frac{1}{n}S_1(\lambda) \geq \cdots \geq \left(\frac{k!(n-k)!}{n!}S_k(\lambda)\right)^{\frac{1}{k}} \geq \cdots \geq (S_n(\lambda))^{\frac{1}{n}}$$

we obtain from (1.1)

$$\det(D^2u(x)) \geq 1, \quad \forall x \in \mathbb{R}^n. \tag{3.3}$$

For $L > 0$ let $\Omega = \{x \in \mathbb{R}^n : u(x) < L\}$. Since u is strictly convex Ω is a nonempty convex open set. Let Γ be the ellipsoid of smallest volume containing Ω. By John's lemma (cf. [1])

$$\Gamma' \equiv \frac{1}{n}\Gamma \subset \Omega \subset \Gamma.$$

Therefore $u \leq L$ in $\overline{\Gamma'}$.

Suppose Γ' is defined by

$$\sum_{i=1}^{n}\frac{(x_i - x_i^0)^2}{a_i^2} \leq 1, \quad a_1 \geq \cdots \geq a_n > 0,$$

where $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$. Consider the function

$$v(x) = \frac{1}{2}(a_1 \cdots a_n)^{\frac{2}{n}}\left(\sum_{i=1}^{n}\frac{(x_i - x_i^0)^2}{a_i^2} - 1\right), \quad x \in \mathbb{R}^n.$$

We have $\det(D^2 v) = 1$ in Γ' and $v = 0$ on $\partial\Gamma'$. Consequently, $u - L \leq v$ in $\overline{\Gamma'}$ by the comparison principle for Monge-Ampère equations (cf. [14]). In particular,

$$L \geq L - u(x^0) \geq -v(x^0) = \frac{1}{2}(a_1 \cdots a_n)^{\frac{2}{n}}. \tag{3.4}$$

By assumption (3.1) we have $B_\rho(0) \subset \Omega$ where $\rho = \sqrt{L/A}$. Thus $\rho \leq n a_i$ for all i. It follows from (3.4) that

$$a_1 \leq \frac{(2L)^{\frac{n}{2}}}{a_2 \cdots a_n} \leq \frac{(2L)^{\frac{n}{2}}}{(\rho/n)^{n-1}} \leq C_0 \sqrt{L} \tag{3.5}$$

where C_0 depends only on A and n. Note that $0 \in \Omega$ hence $|x| \leq 2na_1$ for all $x \in \Omega$. We obtain from (3.5) that

$$u(x) = L \geq \left(\frac{a_1}{C_0}\right)^2 \geq a|x|^2, \quad \forall x \in \partial\Omega,$$

for some positive constant a depending on A, n but not on L. Notice that the level set Ω depends on the level L which is arbitrary. This proves (3.2).

Finally, when $k = n - 1$ we note that by (1.1) all eigenvalues of $D^2 u$ are greater than or equal to one everywhere and therefore $u(x) \geq \frac{1}{2}|x|^2$. □

Let w be the Legendre transform of u, which is given by

$$w(y) := \sup_{x \in \mathbb{R}^n} \{x \cdot y - u(x)\}, \quad y \in \mathbb{R}^n.$$

By Lemma 3.1,

$$\sup_{x \in \mathbb{R}^n} \{x \cdot y - u(x)\} \leq \sup_{x \in \mathbb{R}^n} \{x \cdot y - a|x|^2\} = \frac{1}{4a}|y|^2. \tag{3.6}$$

So $w(y)$ is defined for all $y \in \mathbb{R}^n$. By the strict convexity of u we see that w is smooth and strictly convex. Moreover, we have $w(0) = 0$, $Dw(0) = 0$ and

$$D_y^2 w(y) = (D_x^2 u(x))^{-1} \tag{3.7}$$

where $x = Dw(y)$. Consequently,

$$S_{n-k}(D^2 w) = 1 \quad \text{in } \mathbb{R}^n \tag{3.8}$$

and, in order to prove Theorem 1.2 it is enough to show that w is a quadratic polynomial.

We first note that if $k = n - 1$ then $\Delta w = 1$ on $\mathbb{R}^n$ and w is convex. This implies that $D^2_{\xi\xi}w$ is a nonnegative harmonic function on $\mathbb{R}^n$ for any unit vector ξ, hence $D^2_{\xi\xi}w$ is a constant by Liouville Theorem for harmonic functions. In turn, we conclude that w is a quadratic polynomial. In what follows we therefore assume $n \geq 3$ and $k \leq n - 2$. Similarly to (3.6), by (3.1) we have

$$w(y) = \sup_{x\in\mathbb{R}^n}\{x \cdot y - u(x)\} \geq \sup_{x\in\mathbb{R}^n}\{x \cdot y - A|x|^2\} = \frac{1}{4A}|y|^2, \quad \forall y \in \mathbb{R}^n. \tag{3.9}$$

For a fixed constant $R > 0$ we define $v \in C^\infty(\mathbb{R}^n)$ by

$$v(x) := R^{-2}w(Rx), \quad x \in \mathbb{R}^n$$

and for $t > 0$ set

$$\Omega_t := \{x \in \mathbb{R}^n : 4Av(x) < t\}.$$

We see that $\Omega_t \subset B_1$ for all $0 < t \leq 1$ by (3.9) and

$$\det(v_{ij}(x)) = \det(w_{ij}(Rx)) \leq 1, \quad \forall x \in \mathbb{R}^n \tag{3.10}$$

by (3.8) and the Newton-Maclaurin inequality.

It follows from Lemma 1 of [1] (see also Theorem 1.4.2 in [16]) that

$$v(x) \geq \frac{t}{4A} - C_0\{\operatorname{dist}(x, \partial\Omega_t)\}^{\frac{2}{n}}, \quad \forall x \in \Omega_t, \tag{3.11}$$

where C_0 is independent of $t \in (0, 1]$. In particular,

$$\{\operatorname{dist}(\Omega_{\frac{t}{2}}, \partial\Omega_t)\}^{\frac{2}{n}} \geq \frac{t}{8C_0A}. \tag{3.12}$$

Using the convexity of v we obtain from (3.11)

$$|Dv(x)| \leq \frac{t - 4Av(x)}{4A\operatorname{dist}(x, \partial\Omega_t)} \leq C_0\{\operatorname{dist}(x, \partial\Omega_t)\}^{\frac{2-n}{n}}, \quad x \in \Omega_t. \tag{3.13}$$

From (3.12) and (3.13) we derive an *a priori* gradient bound

$$|Dv| \leq C_1 \quad \text{in } \overline{\Omega_{\frac{1}{2}}}. \tag{3.14}$$

We next apply the interior second derivative estimates due to Chou-Wang [10], combined with (3.12) for $t = \frac{1}{2}$, to obtain

$$|D^2v| \leq C_2 \quad \text{in } \overline{\Omega_{\frac{1}{4}}}. \tag{3.15}$$

It follows from the Evans-Krylov theorem (and, again, using (3.12) for $t = \frac{1}{4}$) that

$$|D^2 v|_{C^\alpha(\overline{\Omega_{\frac{1}{8}}})} \leq C_3, 0 < \alpha < 1. \tag{3.16}$$

Finally, for $L > 0$ we take $R = \sqrt{32AL}$ to obtain from (3.16)

$$|D^2 w|_{C^\alpha(\{w \leq L\})} \leq CL^{-\frac{\alpha}{2}}, \tag{3.17}$$

where C is a constant independent of L. Letting $L \to \infty$, we have $|D^2 w|_{C^\alpha(\mathbb{R}^n)} = 0$. Consequently, w is a quadratic polynomial. This completes the proof of Theorem 1.2.

Indeed, we have proved the following characterization of global solutions to the Hessian equation.

THEOREM 3.2. *Let $u \in C^\infty(\mathbb{R}^n)$ be a convex solution of the Hessian equation*

$$S_k(D^2 u) = 1, \quad 1 \leq k \leq n. \tag{3.18}$$

Suppose

$$u(x) \geq b|x|^2 - B \quad \forall x \in \mathbb{R}^n \tag{3.19}$$

where $b, B > 0$ are constant. Then u is a quadratic polynomial.

It would be interesting to see if Theorem 3.2 remains valid under weaker or without growth assumptions, or for k-convex solutions.

In light of Theorem 2.2 we have:

COROLLARY 3.3. *Suppose $u \in W^{2,p}_{loc}(\mathbb{R}^n)$ is an entire admissible strong solution of (1.1) where $p > \gamma$. If there exists a positive constant C such that for all large R*

$$\int_{B_R(0)} |D^2 u|^p \, dx \leq CR^n,$$

then u is a quadratic polynomial.

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