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Hessian equations on exterior domain ☆

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ABSTRACT

In the paper, we consider the Hessian equation $\sigma_k(\lambda(D^2u)) = f(x)$ where f is a positive function outside a bounded domain of $\mathbb{R}^n$, $n \geq 3$ and $f(x) = 1 + O(|x|^{-\beta})$ for some $\beta > 2$ at infinity. Using the Perron's method we prove the existence and uniqueness for viscosity solutions of exterior Dirichlet problem with prescribed asymptotic behavior at infinity. There are examples to show that the result is optimal. This is an extension of the theorems given by Bao–Li–Li in [2] for $f \equiv 1$ and Bao–Li–Zhang in [3] for $k = n$.

1. Introduction

In this paper, we consider the exterior Dirichlet problem for Hessian equations

$$\sigma_k(\lambda(D^2u)) = f(x), \quad \text{in} \quad \mathbb{R}^n \setminus \overline{D}, \tag{1.1}$$

$$u = \varphi(x), \quad \text{on} \quad \partial D, \tag{1.2}$$

where D is a bounded open set in $\mathbb{R}^n$, $n \geq 3$, $\varphi \in C^2(\partial D)$, $\lambda(D^2u)$ denotes the eigenvalues $\lambda_1, \cdots, \lambda_n$ of the Hessian matrix of u,

$$\sigma_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k-th elementary symmetric function, $k = 1, \cdots, n$, $f \in C^0(\mathbb{R}^n \setminus D)$ is a perturbation of 1 near infinity, which satisfies:

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$$f \geq 0, \quad \text{on} \quad \mathbb{R}^n \setminus D, \tag{1.3}$$

$$\text{there exists a constant } \beta > 2 \text{ such that } \limsup_{|x|\to\infty} |x|^\beta |f(x) - 1| < \infty. \tag{1.4}$$

For $k = 1$, (1.1) corresponds to Poisson's equation, which is linear. The interior Dirichlet problem of Poisson's equation has been widely discussed. For exterior Dirichlet problem, there also is extensive literature. For example, Meyers and Serrin [16] gave a sufficient condition for existence and uniqueness of a classical solution.

For $k = n$, (1.1) corresponds to the Monge–Ampère equation, which is fully nonlinear. The interior Dirichlet problem of the Monge–Ampère equations is adequately discussed through the works of Aleksandrov [1], Nirenberg [17], Calabi [9], Cheng and Yau [10], Caffarelli, Nirenberg and Spruck [7], Trudinger and Wang [21], Urbas [22] and many other works and the references therein. In contrast to the traditional Dirichlet problems there is an exterior Dirichlet problem of the Monge–Ampère equation. For example, Caffarelli and Li [6] proved an existence result for the exterior Dirichlet problem with the prescribed asymptotic behavior at infinity

$$\limsup_{|x|\to\infty} \left(|x|^{n-2} \left| u(x) - (\frac{1}{2}x^T Ax + b \cdot x + c) \right| \right) < \infty \tag{1.5}$$

for (1.1), (1.2) when $f \equiv 1$, where $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ and A is an $n \times n$ real symmetric positive definite matrix with $\det A = 1$. They also gave out the asymptotic behavior of solutions for (1.1) at infinity. Later Bao, Li and Zhang [3] extended Caffarelli and Li's results to the Dirichlet problem $\det(D^2 u) = f$ where f is a perturbation of 1 at infinity. Note that the above asymptotic results are extension for a classical theorem of Jörgens [14], Calabi [9], and Pogorelov [18], which states that any classical convex solution of $\det(D^2 u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. More extensive proofs and outstanding results of this classical problem are given by Cheng and Yau [11], Caffarelli [4], Jost and Xin [15] and many other followers.

For $2 \leq k \leq n$, the Hessian equation (1.1) is an important class of second-order fully nonlinear elliptic equations. There have been many well-known results on Hessian equations in the bounded domain. For instance, Caffarelli, Nirenberg and Spruck demonstrated the classical solvability of the interior Dirichlet problem in [8]. Urbas [23] established the existence of viscosity solutions. In particular, Wang reviewed the existence and regularity theory for solutions of the associated Dirichlet problem in [24]. Corresponding to the interior Dirichlet problem mentioned above, the exterior Dirichlet problem also captured the attention of many researchers. For the case $A = c^* I$, where $c^* = (C_n^k)^{-\frac{1}{k}}$ and I is the $n \times n$ identity matrix, the exterior Dirichlet problem (1.1), (1.2), (1.5) of Hessian equation has been investigated in [12]. Bao, Li and Li [2] extend this to a more general A.

In this paper we intend to deal with the existence and uniqueness of viscosity solutions for problem (1.1), (1.2), (1.5) in exterior domain for further study, where f is a perturbation of 1 at infinity, and A is a positive definite matrix.

To work in the realm of elliptic equations, we have to restrict ourselves to a suitable class of functions, that is, the admissible (or k-convex) functions. We say that a function $u \in C^2(\mathbb{R}^n \setminus \overline{D})$ is admissible (or k-convex) if $\lambda(D^2 u) \in \overline{\Gamma}_k$ in $\mathbb{R}^n \setminus \overline{D}$, where Γ_k is the connected component of $\{\lambda \in \mathbb{R}^n |\ \sigma_k(\lambda) > 0\}$ containing

$$\Gamma^+ = \{\lambda \in \mathbb{R}^n |\ \lambda_i > 0,\ i = 1, \cdots, n\}.$$

Moreover,

$$\Gamma_k = \{\lambda \in \mathbb{R}^n |\ \sigma_j > 0,\ 1 \leq j \leq k\}.$$

See [8,20].

For the reader's convenience, we recall the definition of viscosity solutions to Hessian equations (see [5,23] and the references therein). We also use the following definitions, which can be found in [19].

For $\Omega \subset \mathbb{R}^n$, we use $\mathrm{USC}(\Omega)$ and $\mathrm{LSC}(\Omega)$ to denote respectively the set of upper and lower semicontinuous real valued functions on Ω.

Definition 1.1. A function $u \in \mathrm{USC}(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity subsolution of equation (1.1) in $\mathbb{R}^n \setminus \overline{D}$ (or say that u satisfies $\sigma_k(\lambda(D^2u)) \geq f(x)$ in $\mathbb{R}^n \setminus \overline{D}$ in the viscosity sense) if for any function $\psi \in C^2(\mathbb{R}^n \setminus \overline{D})$ and point $\bar{x} \in \mathbb{R}^n \setminus \overline{D}$ satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \geq u \quad \text{on} \quad \mathbb{R}^n \setminus \overline{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\bar{x}))) \geq f(\bar{x}).$$

A function $u \in \mathrm{LSC}(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity supersolution of equation (1.1) in $\mathbb{R}^n \setminus \overline{D}$ (or say that u satisfies $\sigma_k(\lambda(D^2u)) \leq f(x)$ in $\mathbb{R}^n \setminus \overline{D}$ in the viscosity sense) if for any k-convex function $\psi \in C^2(\mathbb{R}^n \setminus \overline{D})$ and point $\bar{x} \in \mathbb{R}^n \setminus \overline{D}$ satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \leq u \quad \text{on} \quad \mathbb{R}^n \setminus \overline{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\bar{x}))) \leq f(\bar{x}).$$

A function $u \in C^0(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity solution of (1.1) if it is both a viscosity subsolution and supersolution of (1.1).

It is well known that a function $u \in C^2(\mathbb{R}^n \setminus \overline{D})$ is a k-convex classical solution (respectively, subsolution, supersolution) of (1.1) if and only if it is a C^2 viscosity solution (respectively, subsolution, supersolution).

We also give out the definition of viscosity solutions to the exterior Dirichlet problem for Hessian equations.

Definition 1.2. Let $\varphi \in C^0(\partial D)$. A function $u \in \mathrm{USC}(\mathbb{R}^n \setminus D)$ ($u \in \mathrm{LSC}(\mathbb{R}^n \setminus D)$) is said to be a viscosity subsolution (supersolution) of the Dirichlet problem (1.1), (1.2), if u is a viscosity subsolution (supersolution) of (1.1) in $\mathbb{R}^n \setminus \overline{D}$ and $u \leq (\geq)\ \varphi$ on ∂D. A function $u \in C^0(\mathbb{R}^n \setminus D)$ is said to be a viscosity solution of (1.1), (1.2) if it is both a viscosity subsolution and supersolution.

Let

$$\mathcal{A}_k = \{A|\ A \text{ is a real } n \times n \text{ symmetric positive definite matrix, with } \sigma_k(\lambda(A)) = 1\}.$$

Our main result is

Theorem 1.1. *Let D be a smooth bounded strictly convex open subset in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi \in C^2(\partial D)$. Suppose that f satisfies (1.3) and (1.4). Then for any given $b \in \mathbb{R}^n$ and $A \in \mathcal{A}_k$ with $2 \leq k \leq n$, there exists some constant c_*, depending only on n, b, A, D, f, and $||\varphi||_{C^2(\partial D)}$, such that for every $c > c_*$ there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of (1.1), (1.2) and*

$$\limsup_{|x|\to\infty} \left(|x|^{min\{\frac{k}{H_k},\ \beta\}-2} \cdot \left| u(x) - (\frac{1}{2}x^T Ax + b \cdot x + c) \right| \right) < \infty, \quad \text{if} \quad \beta \neq \frac{k}{H_k}, \tag{1.6}$$

or

$$\limsup_{|x|\to\infty}\left(|x|^{\frac{k}{H_k}-2}(\ln|x|)^{-1}\cdot\left|u(x)-(\frac{1}{2}x^TAx+b\cdot x+c)\right|\right)<\infty,\quad \textit{if}\ \beta=\frac{k}{H_k}, \tag{1.7}$$

where

$$H_k=\max\left\{\lambda_i(A)\frac{\partial}{\partial\lambda_i}\sigma_k(\lambda(A))\mid i=1,\cdots,n\right\}.$$

Remark 1.1. Theorem 1.1 is a general extension of the case $f\equiv 1$ in [2]. For the case $k=n$, the Monge–Ampère equations, Theorem 1.1 has been proved by Bao, Li and Zhang [3], but they missed the case (1.7). Thus we only need to prove for $2\le k\le n-1$.

Remark 1.2. The following example shows that the assumption $\beta>2$ in (1.4) is optimal and the constant c_* cannot be removed. Let $A=c^*I$, where $c^*=(C_n^k)^{-\frac{1}{k}}$ and I is the $n\times n$ identity matrix. It is obvious that $\lambda(A)=(c^*,\cdots,c^*)$. By computation,

$$\begin{aligned}\lambda_i\frac{\partial}{\partial\lambda_i}\sigma_k(\lambda(A))&=\lambda_i\sum_{1\le j_1<\cdots<j_{k-1}\le n,j_l\ne i}\lambda_{j_1}\cdots\lambda_{j_{k-1}}\\&=\lambda_i\sum_{1\le j_1<\cdots<j_{k-1}\le n,j_l\ne i}(c^*)^{k-1}\\&=C_{n-1}^{k-1}(c^*)^k\\&=\frac{k}{n}\end{aligned}$$

is independent of i, thus $H_k=\frac{k}{n}$. Set $D=B_1$, and $f=1+r^{-\beta}$, $r=|x|\ge 1$. Suppose that u is a radial solution of $\sigma_k(D^2u)=f$, that is

$$C_{n-1}^{k-1}u''(r)(\frac{u'(r)}{r})^{k-1}+C_{n-1}^k(\frac{u'(r)}{r})^k=f(r).$$

We have

$$\begin{aligned}C_n^kr^{n-k}(u'(r))^k&=C_n^k(u'(1))^k+\int_1^r\left(nt^{n-1}+nt^{n-\beta-1}\right)dt\\&=\begin{cases}r^n+\frac{n}{n-\beta}r^{n-\beta}+d_1, & \beta\ne\frac{k}{H_k}=n,\\ r^n+n\ln r+d_2, & \beta=\frac{k}{H_k}=n,\end{cases}\end{aligned}$$

where

$$d_1=C_n^k(u'(1))^k-1-\frac{n}{n-\beta},\quad d_2=C_n^k(u'(1))^k-1$$

are constants.

For $\beta\ne\frac{k}{H_k}=n$,

$$u(r)=u(1)+\int_1^r c^*t(1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}dt.$$

Since

$$\lim_{r\to\infty}\frac{u(r)}{\frac{c^*}{2}r^2}=\lim_{r\to\infty}\frac{c^*r(1+\frac{n}{n-\beta}r^{-\beta}+d_1r^{-n})^{\frac{1}{k}}}{c^*r}=1,$$

we rewrite

$$u(r)=\frac{c^*}{2}r^2+u(1)-\frac{c^*}{2}+\int_1^r c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt. \tag{1.8}$$

For $\beta>2$,

$$u(r)=\frac{c^*}{2}r^2+C_1-\int_r^\infty c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt,$$

where

$$C_1=u(1)-\frac{c^*}{2}+\int_1^\infty c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt<\infty.$$

By computation, we have

$$\lim_{r\to+\infty}\frac{-\int_r^\infty c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt}{r^{2-\beta}}=\frac{nc^*}{k(2-\beta)(n-\beta)},\quad \text{for } \beta<n,$$

and

$$\lim_{r\to+\infty}\frac{-\int_r^\infty c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt}{r^{2-n}}=\frac{c^*d_1}{k(2-n)},\quad \text{for } \beta>n.$$

Then for $d_1\neq 0$, there exists a constant $C\neq 0$ such that

$$-\int_r^\infty c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt=Cr^{2-\min\{n,\beta\}}+o(r^{2-\min\{n,\beta\}}),\quad r\to+\infty.$$

Thus we have

$$u(r)=\frac{c^*}{2}r^2+C_1+Cr^{2-\min\{n,\beta\}}+o(r^{2-\min\{n,\beta\}}),\quad r\to+\infty.$$

In the case of $d_1=0$, it follows from (1.8) that

$$u(r)=\frac{c^*}{2}r^2+C_1+\frac{nc^*}{k(2-\beta)(n-\beta)}r^{2-\beta}+o(r^{2-\beta}),\quad r\to+\infty.$$

For $\beta=2$,

$$\begin{aligned}
&\lim_{r\to\infty}\frac{\int_1^r c^*t\Big((1+\frac{n}{n-2}t^{-2}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt}{\ln r}\\
&=\lim_{r\to\infty}\frac{c^*r\Big((1+\frac{n}{n-2}r^{-2}+d_1r^{-n})^{\frac{1}{k}}-1\Big)}{r^{-1}}\\
&=\lim_{r\to\infty}\frac{c^*\Big((1+\frac{n}{n-2}r^{-2}+d_1r^{-n})^{\frac{1}{k}}-1\Big)}{r^{-2}}\\
&=\lim_{r\to\infty}\frac{c^*\frac{1}{k}(1+\frac{n}{n-2}r^{-2}+d_1r^{-n})^{\frac{1}{k}-1}(-\frac{2n}{n-2}r^{-3}-nd_1r^{-n-1})}{-2r^{-3}}\\
&=\lim_{r\to\infty}\frac{c^*(\frac{2n}{n-2}+nd_1r^{2-n})}{2k}\\
&=\frac{c^*n}{k(n-2)}.
\end{aligned}$$

Therefore

$$u(r)=\frac{c^*}{2}r^2+\frac{c^*n}{k(n-2)}\ln r+C_2+O(r^{2-\min\{4,n\}})\quad \text{as}\quad r\to\infty,$$

where

$$C_2=u(1)-\frac{c^*}{2}+\int_1^\infty\left(c^*t\Big((1+\frac{n}{n-2}t^{-2}+d_1t^{-n})^{\frac{1}{k}}-1\Big)-\frac{c^*n}{k(n-2)}t^{-1}\right)dt<\infty.$$

For $0<\beta<2$,

$$\begin{aligned}
&\lim_{r\to\infty}\frac{\int_1^r c^*t\Big((1+\frac{n}{n-\beta}t^{-\beta}+d_1t^{-n})^{\frac{1}{k}}-1\Big)dt}{r^{2-\beta}}\\
&=\lim_{r\to\infty}\frac{c^*r\Big((1+\frac{n}{n-\beta}r^{-\beta}+d_1r^{-n})^{\frac{1}{k}}-1\Big)}{(2-\beta)r^{1-\beta}}\\
&=\lim_{r\to\infty}\frac{c^*\Big((1+\frac{n}{n-\beta}r^{-\beta}+d_1r^{-n})^{\frac{1}{k}}-1\Big)}{(2-\beta)r^{-\beta}}\\
&=\lim_{r\to\infty}\frac{c^*\frac{1}{k}(1+\frac{n}{n-\beta}r^{-\beta}+d_1r^{-n})^{\frac{1}{k}-1}(-\frac{\beta n}{n-\beta}r^{-\beta-1}-nd_1r^{-n-1})}{(2-\beta)(-\beta)r^{-\beta-1}}\\
&=\lim_{r\to\infty}\frac{c^*(\frac{n\beta}{n-\beta}+nd_1r^{\beta-n})}{k\beta(2-\beta)}\\
&=\frac{c^*n}{k(n-\beta)(2-\beta)}.
\end{aligned}$$

Therefore

$$u(r)=\frac{c^*}{2}r^2+\frac{c^*n}{k(n-\beta)(2-\beta)}r^{2-\beta}+u(1)-\frac{c^*}{2}+o(r^{2-\beta})\quad \text{as}\quad r\to\infty.$$

Furthermore, as $r \to \infty$, we have

$$u(r) = \begin{cases} \frac{c^*}{2}r^2 + \frac{c^* n}{k(n-\beta)(2-\beta)}r^{2-\beta} + C_3 + O(r^{2-\min\{n,2\beta\}}), & 1 < \beta < 2, \\ \frac{c^*}{2}r^2 + \frac{c^* n}{k(n-1)}r + \frac{c^*}{k}(\frac{1}{k}-1)(\frac{n}{n-1})^2 \ln r + C_4 + O(r^{2-n}), & \beta = 1, \\ \frac{c^*}{2}r^2 + \frac{c^* n}{k(n-\beta)(2-\beta)}r^{2-\beta} + \frac{c^*(k-1)}{2k^2(1-\beta)}(\frac{n}{n-\beta})^2 r^{2-2\beta} + C_5 + O(r^{2-n}), & 0 < \beta < 1, \end{cases}$$

where C_3, C_4, C_5 are some constants.

For $\beta = \frac{k}{H_k} = n$,

$$\begin{aligned} u(r) &= u(1) + \int_1^r c^* t(1 + nt^{-n}\ln t + d_2 t^{-n})^{\frac{1}{k}} dt \\ &= \frac{c^*}{2}r^2 + C_6 - \int_r^\infty c^* t\Big((1 + nt^{-n}\ln t + d_2 t^{-n})^{\frac{1}{k}} - 1\Big) dt, \end{aligned}$$

where

$$C_6 = u(1) - \frac{c^*}{2} + \int_1^\infty c^* t\Big((1 + nt^{-n}\ln t + d_2 t^{-n})^{\frac{1}{k}} - 1\Big) dt < \infty.$$

By computation, we have

$$\lim_{r\to+\infty} \frac{-\int_r^\infty c^* t\Big((1 + nt^{-n}\ln t + d_2 t^{-n})^{\frac{1}{k}} - 1\Big) dt}{r^{2-n}\ln r} = \frac{nc^*}{k(2-n)},$$

that is,

$$u(r) = \frac{c^*}{2}r^2 + C_3 + \frac{nc^*}{k(2-n)}r^{2-n}\ln r + o(r^{2-n}\ln r) \quad \text{as} \quad r \to \infty.$$

From the above example, we can see that $\beta > 2$ is optimal. A natural question rises.

Question 1. Does the problem (1.1), (1.2) with

$$u(r) = \frac{c^*}{2}r^2 + \frac{c^* n}{k(n-2)}\ln r + C_2 + O(r^{2-\min\{4,n\}}) \quad \text{as} \quad r \to \infty,$$

for $\beta = 2$, and

$$u(r) = \frac{c^*}{2}r^2 + \frac{c^* n}{k(n-\beta)(2-\beta)}r^{2-\beta} + u(1) - \frac{c^*}{2} + o(r^{2-\beta}) \quad \text{as} \quad r \to \infty,$$

for $0 < \beta < 2$ has a viscosity solution?

In spite of a coordinate translation, any real symmetric matrix A can be regarded as a real diagonal matrix, which has the same eigenvalues of A. Thus in this paper, without loss of generality, we always assume that A is diagonal. For $k = n$, we have $A \in \mathcal{A}_n$ and $\sigma_n(\lambda(A)) = 1$, and can find a diagonal matrix O such that $O^T AO = I \in \mathcal{A}_n$. Though $\lambda(A)$ may not be the same as $\lambda(I)$, we still have

$$\det(D_x^2 u) = \det(O^T D_y^2 u O) = \det(D_y^2 u)$$

under the transformation $y = Ox$. This is not true for $2 \le k \le n-1$ except the case $A = c_* I$. Thus we could only assume that A is diagonal but not of the form $A = c_* I$.

The organization of this paper is as follows. In Section 2 we construct a family of generalized symmetric smooth k-convex subsolutions of (1.1) in $\mathbb{R}^n \setminus \{0\}$ and in Section 3, we prove Theorem 1.1 using Perron's method.

2. Generalized symmetric functions and subsolutions

In this section, we construct a family of generalized symmetric smooth subsolutions of (1.1). First we have to give out the definition of generalized symmetric functions and generalized symmetric solutions.

Definition 2.1. For a diagonal matrix $A = \mathrm{diag}(a_1, \cdots, a_n)$, we call u a generalized symmetric function with respect to A if u is a function of

$$s = \frac{1}{2} x^T A x = \frac{1}{2} \sum_{i=1}^{n} a_i x_i^2.$$

If u is a generalized symmetric function with respect to A and u is a solution (subsolution, supersolution) of the Hessian equation (1.1), then we call u a generalized symmetric solution (subsolution, supersolution) of (1.1).

We abuse the notation slightly by writing $u(x) = u(\frac{1}{2} x^T A x)$ for a generalized symmetric function with respect to A when it is unambiguous.

Denote $a = (a_1, \cdots, a_n)$. If $A \in \mathcal{A}_k$, then we have $a_i > 0$ $(i = 1, \cdots, n)$ and $\sigma_k(a) = 1$. Here we first quote several notations for convenience. For any fixed t-tuple $\{i_1, \cdots, i_t\} \subset \{1, \cdots, n\}$, $1 \le t \le n-k$, we define

$$\sigma_{k;i_1 \cdots i_t}(a) = \sigma_k(a) \mid_{a_{i_1} = \cdots = a_{i_t} = 0};$$

that is, $\sigma_{k;i_1 \cdots i_t}(a)$ is the k-th order elementary symmetric function of the $n-t$ variables $\{ a_i \mid i \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_t\}\}$. The following properties of the functions σ_k will be used in this paper:

$$\sigma_k(a) = \sigma_{k;i}(a) + a_i \sigma_{k-1;i}(a), \quad i = 1, \cdots, n, \tag{2.1}$$

and

$$\sum_{i=1}^{n} a_i \sigma_{k-1;i}(a) = k \sigma_k(a). \tag{2.2}$$

Next we give out a useful proposition proved by Bao, Li and Li [2].

Proposition 2.1. *For any $A = \mathrm{diag}(a_1, \cdots, a_n)$, if $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A, then, with $a = (a_1, \cdots, a_n)$,*

$$\sigma_k(\lambda(D^2\omega)) = \sigma_k(a)(\omega')^k + \omega''(\omega')^{k-1} \sum_{i=1}^{n} \sigma_{k-1;i}(a)(a_i x_i)^2. \tag{2.3}$$

To prove Theorem 1.1 for $2 \le k \le n-1$, it suffices to obtain enough subsolutions with appropriate properties. We construct such subsolutions, which are generalized symmetric functions with respect to A.

Recall that $f \in C^0(\mathbb{R}^n \setminus D)$ can be continuously extended to the entire space, we assume that $f \geq 0$ is defined in $\mathbb{R}^n$. Let $\overline{f}$ be a smooth, positive function of $s = \frac{1}{2}x^T Ax$ such that $\overline{f} \geq f$ in $\mathbb{R}^n$ and satisfying (1.3) and (1.4). Without loss of generality, we can assume that $\overline{f} = 1 + C_0 s^{-\frac{\beta}{2}}$ for $s > s_0$, where C_0, s_0 are positive constants and s_0 is large enough. Then we only need to find enough solutions or subsolutions with appropriate properties of

$$\sigma_k(\lambda(D^2\omega)) = \overline{f}, \tag{2.4}$$

which are obviously subsolutions of (1.1).

If $A = c^* I$, where $c^* = (C_n^k)^{-\frac{1}{k}}$, $2 \leq k \leq n$, then A satisfies $\sigma_k(\lambda(A)) = 1$ and (2.3) becomes

$$\sigma_k(\lambda(D^2\omega)) = \sigma_k(a)(\omega')^k + 2s\frac{k}{n}\omega''(\omega')^{k-1}.$$

By simple computation, the ordinary differential equation

$$\sigma_k(a)(\omega')^k + 2s\frac{k}{n}\omega''(\omega')^{k-1} = \overline{f}(s) \tag{2.5}$$

has a family of solutions

$$\omega_k(s) = \int_1^s \left(\eta^{-\frac{n}{2}} \int_0^\eta \frac{n}{2} t^{\frac{n}{2}-1}\overline{f}(t)dt + \alpha\eta^{-\frac{n}{2}}\right)^{\frac{1}{k}} d\eta,$$

where $\alpha > 0$, $s > 0$. That is, $\omega_k(\frac{c^*}{2}|x|^2)$ is a solution of (2.5) in $\mathbb{R}^n \setminus \{0\}$.

For $k = n$, $\omega_n(\frac{c^*}{2}|x|^2)$ is a solution of (2.4) in $\mathbb{R}^n \setminus \{0\}$ under affine transformations. Thus the Monge–Ampère equation has generalized symmetric solutions with respect to A, which are also radially symmetric for every $A \in \mathcal{A}_n$. However, for any given $A \in \mathcal{A}_k$ with $2 \leq k \leq n-1$, it is not enough to attain generalized symmetric solutions of (2.4) by only using these radially symmetric functions. A natural question is that whether (2.4) with $2 \leq k \leq n-1$ has generalized symmetric solutions with respect to A for every $A \in \mathcal{A}_k$?

To answer this question, we have

Proposition 2.2. *For any $A = diag(a_1, \cdots, a_n) \in \mathcal{A}_k$, $1 \leq k \leq n$. If there exists an $\omega \in C^2(\alpha, \beta)$ with $0 < \alpha < \beta < \infty$ such that $\omega(\frac{1}{2}x^T Ax)$ is a generalized symmetric solution of (2.4) in $\{x \in \mathbb{R}^n | \alpha < \frac{1}{2}x^T Ax < \beta\}$, then*

$$k = n \quad or \quad a_1 = \cdots = a_n = c^*$$

where $c^* = (C_n^k)^{-\frac{1}{k}}$.

Proof. We first deal with the case $\omega'' \equiv 0$. If $\omega'' \equiv 0$ holds, we have $\omega = c_1 s + c_2$ with constants $c_1, c_2 \in \mathbb{R}$. Combined with (2.3) we get $\overline{f} = (\omega')^k = (c_1)^k$. It is a contradiction. Thus in the following proof, we assume that $\omega'' \not\equiv 0 \in (\alpha, \beta)$, that is, there exists $\bar{s} \in (\alpha, \beta)$ such that $\omega''(\bar{s}) \neq 0$.

For $k = 1$ and $1 \leq i \leq n$, set $x = (0, \cdots, 0, \sqrt{\frac{2\bar{s}}{a_i}}, 0, \cdots, 0)$. Since $A \in \mathcal{A}_1$, we have

$$\begin{aligned}\overline{f}(\bar{s}) &= \Delta\omega(x) \\ &= \omega'(\bar{s})\Sigma_{j=1}^n a_j + \omega''(\bar{s})\sum_{j=1}^n (a_j x_j)^2 \\ &= \omega'(\bar{s}) + 2a_i\bar{s}\omega''(\bar{s}).\end{aligned}$$

It follows that

$$a_i = \frac{\overline{f}(\bar{s}) - \omega'(\bar{s})}{2\bar{s}\omega''(\bar{s})}$$

is independent of i. By $A \in \mathcal{A}_1$, we have $a_1 = \cdots = a_n = \frac{1}{n}$.

For $2 \le k \le n-1$ and $1 \le i \le n$, set $x = (0, \cdots, 0, \sqrt{\frac{2\bar{s}}{a_i}}, 0, \cdots, 0)$. We get

$$\begin{aligned}\overline{f}(\bar{s}) = \sigma_k(\lambda(D^2\omega)) &= \sigma_k(a)(\omega'(\bar{s}))^k + \omega''(\bar{s})(\omega'(\bar{s}))^{k-1}\sum_{j=1}^{n}\sigma_{k-1;j}(a)(a_j x_j)^2 \\ &= (\omega'(\bar{s}))^k + 2\bar{s}\omega''(\bar{s})(\omega'(\bar{s}))^{k-1}\sigma_{k-1;i}(a)a_i.\end{aligned}$$

It is obvious that $\omega' \neq 0$ in (α, β). It follows that

$$\sigma_{k-1;i}(a)a_i = \frac{\overline{f}(\bar{s}) - \big(\omega'(\bar{s})\big)^k}{2\bar{s}\omega''(\bar{s})\big(\omega'(\bar{s})\big)^{k-1}}.$$

We denote $A_k^i(a) = \sigma_{k-1;i}(a)a_i$, then $A_k^i(a)$ is independent of i. By $A \in \mathcal{A}_k$, for $i_1 \neq i_2$, we have

$$\begin{aligned}0 &= A_k^{i_1}(a) - A_k^{i_2}(a) \\ &= \sigma_{k-1;i_1}(a)a_{i_1} - \sigma_{k-1;i_2}(a)a_{i_2} \\ &= a_{i_1}(a_{i_2}\sigma_{k-2;i_1 i_2}(a) + \sigma_{k-1;i_1 i_2}(a)) - a_{i_2}(a_{i_1}\sigma_{k-2;i_1 i_2}(a) + \sigma_{k-1;i_1 i_2}(a)) \\ &= (a_{i_1} - a_{i_2})\sigma_{k-1;i_1 i_2}(a).\end{aligned}$$

Recall that $a_i > 0$, $i = 1, \cdots, n$, $\sigma_{k-1;i_1 i_2}(a) > 0$, which means $a_{i_1} = a_{i_2}$. By the arbitrariness of i_1 and i_2, we have $a_1 = \cdots = a_n = c^*$. $\square$

From Proposition 2.2 we find out that there is no generalized symmetric solution of (2.4) for $2 \le k \le n-1$ when $A \neq c^* I$. We will construct a family of generalized symmetric smooth functions satisfying

$$\omega'(s) > 0, \quad \omega''(s) < 0,$$

and

$$\sigma_k(\lambda(D^2\omega)) \ge \overline{f} \quad \text{and} \quad \sigma_m(\lambda(D^2\omega)) \ge 0,\ 1 \le m \le k-1.$$

For $A = \mathrm{diag}(a_1, \cdots, a_n) \in \mathcal{A}_k$, denote $a = (a_1, \cdots, a_n)$ and define

$$H_k(a) := \max_{1 \le i \le n} A_k^i(a). \tag{2.6}$$

Since $A_n^i(a) = a_i\sigma_{n-1;i}(a) = \sigma_n(a) = 1$ for every i, then $H_n(a) = 1$ holds. By (2.1) and (2.2) we have

$$A_k^i(a) = a_i\sigma_{k-1;i}(a) < \sigma_k(a) = 1, \quad 1 \le i \le n, \quad k \le n-1,$$

and

$$nH_k(a) \ge \sum_{i=1}^{n} A_k^i(a) = k\sigma_k(a) = k.$$

Thus we have

$$\frac{k}{n} \le H_k(a) < 1 \tag{2.7}$$

for $1 \le k \le n-1$, where "=" holds if and only if $a_1 = \cdots = a_n = c^*$. When $n \ge 3$ and $2 \le k \le n$, we know from above that

$$\frac{n}{2} \ge \frac{k}{2H_k(a)} > 1. \tag{2.8}$$

By a simple computation, the ordinary differential equation

$$(\omega')^k + 2sH_k(a)\omega''(\omega')^{k-1} = \overline{f}(s) \tag{2.9}$$

has a family of solutions

$$\omega_\alpha(s) = \beta_1 + \int_{s_0}^{s} \left(\eta^{-\frac{k}{2H_k(a)}} \int_2^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha\eta^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} d\eta,$$

where $\beta_1 \in \mathbb{R}$ and $s_0, \alpha, s > 0$. Then we have

$$\omega'_\alpha(s) = \left(s^{-\frac{k}{2H_k(a)}} \int_2^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha s^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} > 0,$$

$$\omega''_\alpha(s) = -\frac{1}{2H_k(a)} \left(s^{-\frac{k}{2H_k(a)}} \int_2^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha s^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}-1}$$
$$\cdot s^{-\frac{k}{2H_k(a)}-1} \left(\alpha + \int_2^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt - s^{\frac{k}{2H_k(a)}} \overline{f}(s) \right).$$

$\omega''_\alpha(s) \le 0$ holds if and only if

$$\alpha + \int_2^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt - s^{\frac{k}{2H_k(a)}} \overline{f}(s) \ge 0.$$

Clearly, in order to obtain $\omega''_\alpha(s) \le 0$ we only need to prove

$$\alpha + \int_{s_0}^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt - s^{\frac{k}{2H_k(a)}} \overline{f}(s) \ge 0$$

since $\overline{f} > 0$. Denote

$$F(s) = \alpha + \int_{s_0}^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt - s^{\frac{k}{2H_k(a)}} \overline{f}(s).$$

When $s < s_0$, if

$$\alpha \geq s_0^{\frac{k}{2H_k(a)}} \sup_{\mathbb{R}^n} \overline{f} + \int_0^{s_0} \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt,$$

then we have $F(s) \geq 0$.

When $s > s_0$, for $\frac{k}{H_k(a)} \neq \beta$,

$$\begin{aligned} F(s) &= \alpha + \int_{s_0}^{s} \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1}(1 + C_0 t^{-\frac{\beta}{2}})dt - s^{\frac{k}{2H_k(a)}}(1 + C_0 s^{-\frac{\beta}{2}}) \\ &= \alpha - s_0^{\frac{k}{2H_k(a)}} + C_0 \frac{\frac{\beta}{2}}{\frac{k}{2H_k(a)} - \frac{\beta}{2}} s^{\frac{k}{2H_k(a)}-\frac{\beta}{2}} - C_0 \frac{\frac{k}{2H_k(a)}}{\frac{k}{2H_k(a)} - \frac{\beta}{2}} s_0^{\frac{k}{2H_k(a)}-\frac{\beta}{2}}. \end{aligned}$$

Set

$$\alpha > s_0^{\frac{k}{2H_k(a)}} - C_0 \frac{\frac{\beta}{2}}{\frac{k}{2H_k(a)} - \frac{\beta}{2}} s_0^{\frac{k}{2H_k(a)}-\frac{\beta}{2}} + C_0 \frac{\frac{k}{2H_k(a)}}{\frac{k}{2H_k(a)} - \frac{\beta}{2}} s_0^{\frac{k}{2H_k(a)}-\frac{\beta}{2}},$$

then we have $F(s) \geq 0$.

For $\frac{k}{H_k(a)} = \beta$,

$$\begin{aligned} F(s) &= \alpha + \int_{s_0}^{s} \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1}(1 + C_0 t^{-\frac{\beta}{2}})dt - s^{\frac{k}{2H_k(a)}}(1 + C_0 s^{-\frac{\beta}{2}}) \\ &= \alpha - s_0^{\frac{k}{2H_k(a)}} + \frac{k}{2H_k(a)} C_0(\ln s - \ln s_0) - C_0. \end{aligned}$$

Set $\alpha > s_0^{\frac{k}{2H_k(a)}} + C_0$, then we have $F(s) \geq 0$.

In conclusion, we can find a positive constant C_1 such that when $\alpha > C_1$, we have $F(s) \geq 0$, i.e., $\omega_\alpha''(s) \leq 0$ holds for $s > 0$, $\beta_1 \in \mathbb{R}$.

Then the ordinary differential equation

$$\begin{cases} (\omega'(s))^k + 2sH_k(a)\omega''(\omega')^{k-1} = \overline{f}(s), s > 0, \\ \omega'(s) > 0, s > 0, \\ \omega''(s) \leq 0, s > 0, \end{cases}$$

has a family of solutions

$$\omega_\alpha(s) = \beta_1 + \int_{s_0}^{s} \left(\eta^{-\frac{k}{2H_k(a)}} \int_2^{\eta} \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha \eta^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} d\eta, \tag{2.10}$$

where $\beta_1 \in \mathbb{R}$ and $s_0, s > 0$, $\alpha > C_1$.

It follows that

$$\begin{aligned} \omega_\alpha(s) &= \beta_1 + s - s_0 + \int_{s_0}^{s} \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_2^{\eta} \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha \eta^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} - 1 \right) d\eta \\ &= s + \mu(\alpha) - \int_{s}^{\infty} \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_2^{\eta} \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha \eta^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} - 1 \right) d\eta, \end{aligned}$$

where

$$\mu(\alpha)=\beta_1-s_0+\int_{s_0}^{\infty}\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_2^{\eta}\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\overline{f}(t)dt+\alpha\eta^{-\frac{k}{2H_k(a)}}\right)^{\frac{1}{k}}-1\right)d\eta.$$

For $\frac{k}{H_k(a)}\neq\beta$, we have

$$\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_2^{\eta}\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\overline{f}(t)dt+\alpha\eta^{-\frac{k}{2H_k(a)}}\right)^{\frac{1}{k}}-1\right)\sim\eta^{-\min\{\frac{k}{2H_k(a)},\frac{\beta}{2}\}}\quad\text{as}\quad\eta\to\infty,$$

while for $\frac{k}{H_k(a)}=\beta$,

$$\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_2^{\eta}\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\overline{f}(t)dt+\alpha\eta^{-\frac{k}{2H_k(a)}}\right)^{\frac{1}{k}}-1\right)\sim\eta^{-\frac{k}{2H_k(a)}}\ln\eta\quad\text{as}\quad\eta\to\infty.$$

Thus from (2.8) we have $\mu(\alpha)<\infty$ in both cases and

$$\omega_\alpha(s)=s+\mu(\alpha)+O(s^{1-\min\{\frac{k}{2H_k(a)},\frac{\beta}{2}\}})\quad\text{as}\quad s\to\infty \tag{2.11}$$

holds for $\beta\neq\frac{k}{H_k(a)}$,

$$\omega_\alpha(s)=s+\mu(\alpha)+O(s^{1-\frac{k}{2H_k(a)}}\ln s)\quad\text{as}\quad s\to\infty \tag{2.12}$$

holds for $\beta=\frac{k}{H_k(a)}$.

At the end of this section, we show that $\omega_\alpha(s)$ as above is a subsolution of (2.4) with properties we needed.

Proposition 2.3. *For $n\geq 3$, $2\leq k\leq n$, and $A\in\mathcal{A}_k$, let $\omega_\alpha(s)=\omega_\alpha(\frac{1}{2}x^TAx)$ be given in (2.10). Then ω_α is a smooth k-convex subsolution of (2.4) in $\mathbb{R}^n\setminus\{0\}$ satisfying*

$$\omega_\alpha(s)=s+\mu(\alpha)+O(s^{1-\min\{\frac{k}{2H_k(a)},\frac{\beta}{2}\}})\quad as\quad s\to\infty,\quad for\quad\beta\neq\frac{k}{H_k(a)}, \tag{2.13}$$

$$\omega_\alpha(s)=s+\mu(\alpha)+O(s^{1-\frac{k}{2H_k(a)}}\ln s)\quad as\quad s\to\infty,\quad for\quad\beta=\frac{k}{H_k(a)}. \tag{2.14}$$

Proof. Obviously (2.13) and (2.14) follow from (2.11) and (2.12). By discussion above, we have

$$\omega_\alpha'(s)=\left(s^{-\frac{k}{2H_k(a)}}\int_2^{s}\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\overline{f}(t)dt+\alpha s^{-\frac{k}{2H_k(a)}}\right)^{\frac{1}{k}}>0,$$

$$\omega_\alpha''(s)=\frac{(\omega_\alpha'(s))^{1-k}}{2H_k(a)s}\left(\overline{f}(s)-(\omega_\alpha'(s))^k\right)\leq 0.$$

Then by Proposition 2.1 and above we have

$$\begin{aligned}\sigma_k(\lambda(D^2\omega_\alpha))&=\sigma_k(a)(\omega_\alpha')^k+\omega_\alpha''(\omega_\alpha')^{k-1}\sum_{i=1}^n\sigma_{k-1;i}(a)(a_ix_i)^2\\&\geq\sigma_k(a)(\omega_\alpha')^k+\omega_\alpha''(\omega_\alpha')^{k-1}\cdot 2H_k(a)s=\overline{f}(s)\geq f(x)\end{aligned}$$

and

$$\begin{aligned}\sigma_m(\lambda(D^2\omega_\alpha)) &= \sigma_m(a)(\omega'_\alpha)^m + \omega''_\alpha(\omega'_\alpha)^{m-1}\sum_{i=1}^n \sigma_{m-1;i}(a)(a_ix_i)^2 \\ &= \sigma_m(a)(\omega'_\alpha)^m + \frac{(\omega'_\alpha(s))^{m-k}}{2H_k(a)s}\left(\overline{f}(s) - (\omega'_\alpha(s))^k\right)\sum_{i=1}^n \sigma_{m-1;i}(a)(a_ix_i)^2 \\ &= (\omega'_\alpha)^m\left(\sigma_m(a) + \frac{(\omega'_\alpha(s))^{-k}}{2H_k(a)s}\left(\overline{f}(s) - (\omega'_\alpha(s))^k\right)\sum_{i=1}^n \sigma_{m-1;i}(a)(a_ix_i)^2\right) \\ &\geq (\omega'_\alpha)^m\left(\sigma_m(a) + \frac{(\omega'_\alpha(s))^{-k}}{2s}\left(\overline{f}(s) - (\omega'_\alpha(s))^k\right)\sum_{i=1}^n \frac{\sigma_{m-1;i}(a)(a_ix_i)^2}{\sigma_{k-1;i}(a)a_i}\right).\end{aligned}$$

It follows from (see [2])

$$\sigma_m(a)\sigma_{k-1;i}(a) \geq \sigma_{m-1;i}(a), \tag{2.15}$$

for each $1 \leq i \leq n$, that

$$\begin{aligned}&\sigma_m(a) + \frac{(\omega'_\alpha(s))^{-k}}{2s}\left(\overline{f}(s) - (\omega'_\alpha(s))^k\right)\sum_{i=1}^n \frac{\sigma_{m-1;i}(a)(a_ix_i)^2}{\sigma_{k-1;i}(a)a_i} \\ &\geq \sigma_m(a) + \frac{(\omega'_\alpha(s))^{-k}}{2s}\left(\overline{f}(s) - (\omega'_\alpha(s))^k\right)\sum_{i=1}^n \frac{\sigma_m(a)\sigma_{k-1;i}(a)(a_ix_i)^2}{\sigma_{k-1;i}(a)a_i} \\ &= \sigma_m(a) + \frac{(\omega'_\alpha(s))^{-k}}{2s}\left(\overline{f}(s) - (\omega'_\alpha(s))^k\right)\sigma_m(a)2s \\ &= \overline{f}(s)\sigma_m(a)(\omega'_\alpha(s))^{-k} > 0.\end{aligned}$$

We then immediately have $\sigma_m(\lambda(D^2\omega_\alpha)) > 0$ holds for $1 \leq m \leq k-1$. Thus ω_α is a k-convex subsolution of (2.4) in $\mathbb{R}^n \setminus \{0\}$. □

3. Proof of Theorem 1.1

Denote $E_s = \{x \in \mathbb{R}^n | \frac{1}{2}x^TAx < s\}$. Without loss of generality, we assume $E_2 \subset D \subset E_{s_0}$ for s_0 large enough. First we give out a lemma for $n \geq 3$, $2 \leq k \leq n$ and $A = \mathrm{diag}(a_1, \cdots, a_n) \in \mathcal{A}_k$.

Lemma 3.1. *Let $\varphi \in C^2(\partial D)$. Then there exists some constant C, depending only on f, n, $||\varphi||_{C^2(\partial D)}$, the upper bound of A, the diameter and the convexity of D, and the C^2 norm of ∂D, such that, for every $\xi \in \partial D$, there exists $\overline{x}(\xi) \in \mathbb{R}^n$ satisfying $|\overline{x}(\xi)| \leq C$, and*

$$\omega_\xi < \varphi \quad on \quad \partial D \setminus \{\xi\}, \quad \omega_\xi(\xi) = \varphi(\xi),$$

where

$$\omega_\xi(x) = \varphi(\xi) + \frac{F^{\frac{1}{k}}}{2}[(x - \overline{x}(\xi))^TA(x - \overline{x}(\xi)) - (\xi - \overline{x}(\xi))^TA(\xi - \overline{x}(\xi))], \ x \in \mathbb{R}^n, \tag{3.1}$$

and $F = \sup_{\mathbb{R}^n} \overline{f}$.

Proof. Let $\xi \in \partial D$. We may assume without loss of generality that $\xi = 0$ and ∂D can be locally represented by the graph of

$$x_n = \rho(x') = O(|x'|^2)$$

by a translation and a rotation, where $x' = (x_1, \cdots, x_{n-1})$. And φ has the local expansion

$$\begin{aligned}\varphi(x', \rho(x')) &= \varphi(0) + \sum_{i=1}^{n} \varphi_{x_i}(0)x_i + O(|x|^2)\\ &= \varphi(0) + \sum_{i=1}^{n-1} \varphi_{x_i}(0)x_i + O(|x'|^2),\end{aligned}$$

since D is bounded.

By (3.1), we have

$$\begin{aligned}\omega(x) &= \varphi(0) + \frac{F^{\frac{1}{k}}}{2}[(x-\overline{x})^T A(x-\overline{x}) - \overline{x}^T A\overline{x}]\\ &= \varphi(0) + \frac{F^{\frac{1}{k}}}{2}x^T Ax - F^{\frac{1}{k}}\overline{x}^T Ax.\end{aligned}$$

Recalling that $A \in \mathcal{A}_k$ we know that A is invertible. Thus we can find $\overline{x} = \overline{x}(t) \in \mathbb{R}^n$ such that

$$A\overline{x}(t) = F^{-\frac{1}{k}}(-\varphi_{x_1}(0), \cdots, -\varphi_{x_{n-1}}(0), t)^T,$$

where t will be chosen to fit our need later. Then

$$\omega(x) = \varphi(0) + \frac{F^{\frac{1}{k}}}{2}x^T Ax + \sum_{i=1}^{n-1} \varphi_{x_i}(0)x_i - tx_n.$$

It follows that

$$\begin{aligned}(\omega - \varphi)(x', \rho(x')) &= \frac{F^{\frac{1}{k}}}{2}x^T Ax - t\rho(x') + O(|x'|^2)\\ &\leq C(|x'|^2 + \rho(x')^2) - t\rho(x')\end{aligned}$$

where C depends only on the upper bound of A, $||\varphi||_{C^2(\partial D)}$ and the C^2 norm of ∂D. By the strict convexity of ∂D, there exists some constant $\epsilon > 0$ depending only on D such that

$$\rho(x') \geq \epsilon|x'|^2, \quad \text{for} \quad |x'| < \epsilon. \tag{3.2}$$

Clearly we can choose t large such that

$$(\omega - \varphi)(x', \rho(x')) < 0, \quad x \in \partial D \cap \{(x', \rho(x'))|\ 0 < |x'| < \epsilon\}.$$

On the other hand, by the strict convexity of ∂D and (3.2) we have

$$x_n \geq \epsilon^3, \quad \text{for} \quad x \in \partial D \setminus \{(x', \rho(x'))|\ |x'| < \epsilon\}.$$

By the boundedness of D and the discussion above, we can choose t large again such that

$$\omega(x) - \varphi(x) < 0, \quad x \in \partial D \setminus \{(x', \rho(x'))|\ |x'| < \epsilon\}.$$

Now we get a ω as required. The proof has been completed. □

Proof of Theorem 1.1. By an orthogonal transformation and by subtracting a linear function from u, we only need to prove Theorem 1.1 for the case that $2 \le k \le n-1$, $A = \mathrm{diag}(a_1, \cdots, a_n) \in \mathcal{A}_k$ and $b = 0$.

For $\beta_1 \in \mathbb{R}$ and $\alpha > C_1$, set

$$\omega_\alpha(s) = \beta_1 + \int_{s_0}^{s} \left(\eta^{-\frac{k}{2H_k(a)}} \int_2^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha \eta^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} d\eta$$

as in (2.10). By Proposition 2.3 we have ω_α is a smooth k-convex subsolution of (2.4) in $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\omega_\alpha(s) = s + \mu(\alpha) + O(s^{1-\min\{\frac{k}{2H_k(a)}, \frac{\beta}{2}\}}) \quad \text{as} \quad s \to \infty, \quad \text{for} \quad \beta \neq \frac{k}{H_k(a)},$$
$$\omega_\alpha(s) = s + \mu(\alpha) + O(s^{1-\frac{k}{2H_k(a)}} \ln s) \quad \text{as} \quad s \to \infty, \quad \text{for} \quad \beta = \frac{k}{H_k(a)}.$$

Here

$$\mu(\alpha) = \beta_1 - s_0 + \int_{s_0}^{\infty} \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_2^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt + \alpha \eta^{-\frac{k}{2H_k(a)}} \right)^{\frac{1}{k}} - 1 \right) d\eta.$$

Obviously we see that $\mu(\alpha)$ is strictly increasing in α, and

$$\lim_{\alpha \to \infty} \mu(\alpha) = \infty. \tag{3.3}$$

We will fix the value of c_* in the proof. First we require that $c_* > \mu(C_1)$. Thus, by (3.3), for every $c > c_*$ there exists a unique $\alpha(c)$ such that

$$\mu(\alpha(c)) = c. \tag{3.4}$$

So $\omega_{\alpha(c)}$ satisfies

$$\omega_{\alpha(c)}(s) = s + c + O(s^{1-\min\{\frac{k}{2H_k(a)}, \frac{\beta}{2}\}}) \quad \text{as} \quad s \to \infty, \quad \text{for} \quad \beta \neq \frac{k}{H_k(a)},$$
$$\omega_{\alpha(c)}(s) = s + c + O(s^{1-\frac{k}{2H_k(a)}} \ln s) \quad \text{as} \quad s \to \infty, \quad \text{for} \quad \beta = \frac{k}{H_k(a)}.$$

Set

$$\underline{\omega}(x) = \max\{\omega_\xi(x) | \ \xi \in \partial D\},$$

where ω_ξ is given in Lemma 3.1. It is easy to see that $\underline{\omega}(x)$ is locally Lipschitz in $\mathbb{R}^n \setminus D$ and that $\underline{\omega} = \varphi$ on ∂D. By simple computation, we have

$$\sigma_k(\lambda(D^2 \omega_\xi)) = F\sigma_k(a) = F > \overline{f},$$

where $a = (a_1, \cdots, a_n)$. That is ω_ξ is a smooth k-convex subsolution of (2.4). Thus $\underline{\omega}$ is a viscosity subsolution of (2.4) in $\mathbb{R}^n \setminus D$.

Next we fix a number $\hat{\alpha} > 0$ such that

$$\min_{\partial E_{s_0+1}} \omega_{\hat{\alpha}} > \max_{\partial E_{s_0+1}} \underline{\omega}.$$

We require that c_* also satisfies $c_* > \mu(\hat{\alpha})$. For $c > c_*$, we have $\alpha(c) = \mu^{-1}(c) > \mu^{-1}(c_*) > \hat{\alpha}$ and

$$\omega_{\alpha(c)} > \omega_{\hat{\alpha}} > \underline{\omega} \quad \text{on} \quad \partial E_{s_0+1}. \tag{3.5}$$

Let

$$\beta_1 := \min\{\omega_\xi(x) |\ \xi \in \partial D,\ x \in \overline{E_{s_0}} \setminus D\}.$$

It follows that

$$\omega_\alpha \le \beta_1, \quad \text{in} \quad E_{s_0} \setminus \overline{D}, \tag{3.6}$$

for $\alpha > C_1$. Combined with the definition of $\underline{\omega}$, we have

$$\omega_{\alpha(c)} \le \beta_1 \le \underline{\omega} \quad \text{in} \quad E_{s_0} \setminus \overline{D}. \tag{3.7}$$

Now we define for $c > c_*$,

$$\underline{u}(x) = \begin{cases} \max\{\omega_{\alpha(c)}(x), \underline{\omega}(x)\}, & x \in E_{s_0+1} \setminus D, \\ \omega_{\alpha(c)}(x), & x \in \mathbb{R}^n \setminus E_{s_0+1}. \end{cases}$$

We know from (3.7) that $\underline{u} = \underline{\omega}$ in $E_{s_0} \setminus \overline{D}$ and in particular $\underline{u} = \underline{\omega} = \varphi$ on ∂D. We know from (3.5) that $\underline{u} = \omega_{\alpha(c)}$ in a neighborhood of ∂E_{s_0+1}. Therefore $\underline{u}$ is a locally Lipschitz function in $\mathbb{R}^n \setminus \overline{D}$. Since both $\omega_{\alpha(c)}$ and $\underline{\omega}$ are viscosity subsolutions of (2.4) in $\mathbb{R}^n \setminus \overline{D}$, so is $\underline{u}$.

For $c > c_*$, let $\underline{f}$ be a smooth function of $s = \frac{1}{2}x^T Ax$, where $A = \operatorname{diag}(a_1, \cdots, a_n) \in \mathcal{A}_k$, and denote that $a = (a_1, \cdots, a_n)$, such that $\underline{f} \le f$ in $\mathbb{R}^n$ and satisfies (1.3) and (1.4). Without loss of generality, we can assume that $\underline{f} < 1$ is a monotone increasing smooth function and $\underline{f} = 1 - C_0 s^{-\frac{\beta}{2}}$ for $s > s_0$, where C_0, s_0, β are the same as in the definition of $\overline{f}$.

Let

$$\overline{\omega}(s) = \beta_2 + \int_2^s \left(\eta^{-\frac{k}{2H_k(a)}} \int_0^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}} d\eta.$$

By simple computation we have, for $s > 0$,

$$\overline{\omega}\,'(s) = \left(s^{-\frac{k}{2H_k(a)}} \int_0^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}} > 0,$$

$$\begin{aligned} \overline{\omega}\,''(s) = {} & \frac{1}{2H_k(a)} \left(s^{-\frac{k}{2H_k(a)}} \int_0^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}-1} \\ & \cdot s^{-\frac{k}{2H_k(a)}-1} \left(s^{\frac{k}{2H_k(a)}} \underline{f}(s) - \int_0^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right) \\ = {} & \frac{1}{2H_k(a)} \left(s^{-\frac{k}{2H_k(a)}} \int_0^s \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \overline{f}(t)dt \right)^{\frac{1}{k}-1} \end{aligned}$$

$$\cdot s^{-\frac{k}{2H_k(a)}-1}\left(s^{\frac{k}{2H_k(a)}}\underline{f}(s)-\left(s^{\frac{k}{2H_k(a)}}\underline{f}(s)-\int_0^s t^{\frac{k}{2H_k(a)}}\underline{f}'(t)dt\right)\right)$$

$$=\frac{1}{2H_k(a)}\left(s^{-\frac{k}{2H_k(a)}}\int_0^s \frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\overline{f}(t)dt\right)^{\frac{1}{k}-1}s^{-\frac{k}{2H_k(a)}-1}\int_0^s t^{\frac{k}{2H_k(a)}}\underline{f}'(t)dt\geq 0,$$

and $\overline{\omega}$ satisfies the ordinary differential equation

$$(\omega'(s))^k+2sH_k(a)\omega''(\omega')^{k-1}=\underline{f}(s),\quad s>0.$$

Then we have

$$\begin{aligned}\sigma_k(\lambda(D^2\overline{\omega}))&=\sigma_k(a)(\overline{\omega}\,')^k+\overline{\omega}\,''(\overline{\omega}\,')^{k-1}\sum_{i=1}^n\sigma_{k-1;i}(a)(a_ix_i)^2\\&\leq\sigma_k(a)(\overline{\omega}\,')^k+\overline{\omega}\,''(\overline{\omega}\,')^{k-1}\cdot 2H_k(a)s=\underline{f}(s)\leq f(x).\end{aligned}$$

Since $a_i>0$, $\overline{\omega}\,'(s)>0$ and $\overline{\omega}\,''(s)\geq 0$, it is obvious that

$$\sigma_j(\lambda(D^2\overline{\omega}))=\sigma_j(a)(\overline{\omega}\,')^j+\overline{\omega}\,''(\overline{\omega}\,')^{j-1}\sum_{i=1}^n\sigma_{j-1;i}(a)(a_ix_i)^2\geq 0,\quad j=1,\cdots,k.$$

Thus $\overline{\omega}$ is a smooth k-convex supersolution of $\sigma_k(\lambda(D^2u))=f$ in $\mathbb{R}^n\setminus\{0\}$.

We rewrite $\overline{\omega}$ as the form

$$\begin{aligned}\overline{\omega}(s)&=\beta_2+s-2+\int_2^s\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_0^\eta\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\underline{f}(t)dt\right)^{\frac{1}{k}}-1\right)d\eta\\&=s+\left(\beta_2-2+\int_2^\infty\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_0^\eta\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\underline{f}(t)dt\right)^{\frac{1}{k}}-1\right)d\eta\right)\\&\quad-\int_s^\infty\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_0^\eta\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\underline{f}(t)dt\right)^{\frac{1}{k}}-1\right)d\eta.\end{aligned}$$

For $\beta\neq\frac{k}{H_k(a)}$, we have

$$\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_0^\eta\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\underline{f}(t)dt\right)^{\frac{1}{k}}-1\right)\sim\eta^{-\frac{\beta}{2}}\quad\text{as}\quad\eta\to\infty,$$

while for $\beta=\frac{k}{H_k(a)}$,

$$\left(\left(\eta^{-\frac{k}{2H_k(a)}}\int_0^\eta\frac{k}{2H_k(a)}t^{\frac{k}{2H_k(a)}-1}\underline{f}(t)dt\right)^{\frac{1}{k}}-1\right)\sim\eta^{-\frac{k}{2H_k(a)}}\ln\eta\quad\text{as}\quad\eta\to\infty.$$

Therefore, we have for $\beta\neq\frac{k}{H_k(a)}$,

$$-\int_s^\infty \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_0^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}} - 1 \right) d\eta = O(s^{1-\frac{\beta}{2}}) \quad \text{as} \quad s \to \infty,$$

and for $\beta = \frac{k}{H_k(a)}$,

$$-\int_s^\infty \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_0^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}} - 1 \right) d\eta = O(s^{1-\frac{\beta}{2}} \ln s) \quad \text{as} \quad s \to \infty.$$

It follows from (2.8) and $\beta > 2$ that

$$\int_2^\infty \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_0^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}} - 1 \right) d\eta < \infty.$$

Set

$$\beta_2 = 2 - \int_2^\infty \left(\left(\eta^{-\frac{k}{2H_k(a)}} \int_0^\eta \frac{k}{2H_k(a)} t^{\frac{k}{2H_k(a)}-1} \underline{f}(t)dt \right)^{\frac{1}{k}} - 1 \right) d\eta + c,$$

then we have

$$\overline{\omega}(s) = s + c + O(s^{1-\frac{\beta}{2}}) \quad \text{as} \quad s \to \infty, \quad \text{for} \quad \beta \neq \frac{k}{H_k(a)},$$
$$\overline{\omega}(s) = s + c + O(s^{1-\frac{\beta}{2}} \ln s) \quad \text{as} \quad s \to \infty, \quad \text{for} \quad \beta = \frac{k}{H_k(a)}.$$

Set

$$\hat{b} := \max\{\omega_\xi(x) |\ \xi \in \partial D,\ x \in \overline{E_{s_0}} \setminus D\}.$$

We further require that $c_* > \hat{b}$, and then fix the value of c_*. We know from $\underline{f} < 1$ that $\beta_2 > c$ and

$$\overline{\omega} > \beta_2 > c_* > \hat{b} > \beta_1 > \omega_{\alpha(c)}, \quad \text{on} \ \ \partial D, \tag{3.8}$$

and

$$\lim_{|x|\to\infty} (\overline{\omega} - \omega_{\alpha(c)}) = 0.$$

Thus, in view of the comparison principle of (1.1), we have

$$\overline{\omega} \geq \omega_{\alpha(c)}, \quad \text{in} \ \ \mathbb{R}^n \setminus D. \tag{3.9}$$

By (3.8) and (3.5), we have, for $c > c_*$,

$$\omega_\xi \leq \overline{\omega}, \quad \text{on} \ \ \partial(E_{s_0+1} \setminus D),\ \xi \in \partial D. \tag{3.10}$$

Again by the comparison principle for smooth k-convex solutions of (1.1), we have

$$\omega_\xi \leq \overline{\omega}, \quad \text{in}\ E_{s_0+1} \setminus D, \ \ \xi \in \partial D. \tag{3.11}$$

Thus

$$\underline{\omega} \leq \overline{\omega}, \quad \text{in } E_{s_0+1} \setminus D, \tag{3.12}$$

combined with (3.9), implies that

$$\underline{u} \leq \overline{\omega}, \text{ in } \mathbb{R}^n \setminus D. \tag{3.13}$$

For any $c > c_*$, let $\mathcal{S}_c$ denote the set of $v \in \mathrm{USC}(\mathbb{R}^n \setminus D)$, which are viscosity subsolutions of (1.1) in $\mathbb{R}^n \setminus D$ satisfying $v = \varphi$, on ∂D, and $\underline{u} \leq v \leq \overline{\omega}$, in $\mathbb{R}^n \setminus D$. Obviously, $\mathcal{S}_c$ is not empty and $\underline{u} \in \mathcal{S}_c$. Let

$$u := \sup\{v(x) \mid v \in \mathcal{S}_c\}, \quad x \in \mathbb{R}^n \setminus D.$$

By the above discussion about $\overline{\omega}$ and the definition of $\underline{u}$, for $\beta \neq \frac{k}{H_k(a)}$

$$u(x) \leq \overline{\omega}(s) = s + c + O(s^{1-\frac{\beta}{2}}) \quad \text{as} \quad s \to \infty,$$

and

$$u(x) \geq \omega_{\alpha(c)}(s) = s + c + O(s^{1-\min\{\frac{k}{2H_k(a)}, \frac{\beta}{2}\}}) \quad \text{as} \quad s \to \infty.$$

Then

$$u(x) = s + c + O(s^{1-\min\{\frac{k}{2H_k(a)}, \frac{\beta}{2}\}}) \quad \text{as} \quad s \to \infty.$$

For $\beta = \frac{k}{H_k(a)}$,

$$u(x) \leq \overline{\omega}(s) = s + c + O(s^{1-\frac{\beta}{2}} \ln s) \quad \text{as} \quad s \to \infty$$

and

$$u(x) \geq \omega_{\alpha(c)}(s) = s + c + O(s^{1-\frac{k}{2H_k(a)}} \ln s) \quad \text{as} \quad s \to \infty.$$

Then

$$u(x) = s + c + O(s^{1-\frac{k}{2H_k(a)}} \ln s) \quad \text{as} \quad s \to \infty.$$

This gives out the estimate (1.6) and (1.7).

Next, we prove that u satisfies the boundary condition. Since $\underline{u} = \underline{\omega} = \varphi$ on ∂D we have

$$\liminf_{x \to \xi} u(x) \geq \lim_{x \to \xi} \underline{u} = \varphi(\xi), \quad \xi \in \partial D.$$

Therefore we only need to prove

$$\limsup_{x \to \xi} u(x) \leq \varphi(\xi), \quad \xi \in \partial D.$$

Let $\omega_c^+ \in C^2(\overline{E_{s_0} \setminus D})$ be defined by

$$\begin{cases} \Delta\omega_c^+ = 0, & \text{in } E_{s_0} \setminus \overline{D} \\ \omega_c^+ = \varphi, & \text{on } \partial D \\ \omega_c^+ = \max_{\partial E_{s_0}} \overline{\omega}, & \text{on } \partial E_{s_0}. \end{cases}$$

By Newtonian inequalities, it is easy to see that a viscosity subsolution v of (1.1) satisfies $\Delta v \geq 0$ in the viscosity sense. Then for all $v \in \mathcal{S}_c$, we have $v \leq \omega_c^+$ on $\partial(E_{s_0} \setminus D)$. By comparison principle we get

$$v \leq \omega_c^+, \quad \text{in } E_{s_0} \setminus \overline{D}.$$

By the arbitrariness of v we have

$$u \leq \omega_c^+, \quad \text{in } E_{s_0} \setminus \overline{D}$$

and then

$$\limsup_{x\to\xi} u(x) \leq \lim_{x\to\xi} \omega_c^+(x) = \varphi(\xi), \quad \xi \in \partial D.$$

Finally, we can prove by Perron's method that $u \in C^0(\mathbb{R}^n \setminus D)$ is a viscosity solution of (1.1), (1.2). Details can be found in [13] and the references therein. Theorem 1.1 is established. □

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