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Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two

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A B S T R A C T

We establish upper bounds on the blow-up rate of the gradients of solutions of the Lamé system with partially infinite coefficients in dimensions greater than two as the distance between the surfaces of discontinuity of the coefficients of the system tends to zero.

1. Introduction and main results

In this paper, we establish upper bounds on the blow-up rate of the gradients of solutions of the Lamé system with partially infinite coefficients in dimensions greater

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than two as the distance between the surfaces of discontinuity of the coefficients of the system tends to zero. This work is stimulated by the study of Babuška, Andersson, Smith and Levin in [10] concerning initiation and growth of damage in composite materials. The Lamé system is assumed and they computationally analyzed the damage and fracture in composite materials. They observed numerically that the size of the strain tensor remains bounded when the distance ϵ, between two inclusions, tends to zero. This was proved by Li and Nirenberg in [31]. Indeed such ϵ-independent gradient estimates were established there for solutions of divergence form second order elliptic systems, including linear systems of elasticity, with piecewise Hölder continuous coefficients in all dimensions. See Bonnetier and Vogelius [16] and Li and Vogelius [32] for corresponding results on divergence form elliptic equations.

The estimates in [31] and [32] depend on the ellipticity of the coefficients. If ellipticity constants are allowed to deteriorate, the situation is very different. Consider the scalar equation

$$\begin{cases} \nabla \cdot \Big(a_k(x) \nabla u_k \Big) = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^d$, $d \geq 2$, containing two ϵ-apart convex inclusions D_1 and D_2, $\varphi \in C^2(\partial\Omega)$ is given, and

$$a_k(x) = \begin{cases} k \in (0, \infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \Omega \setminus \overline{D_1 \cup D_2}. \end{cases}$$

When $k = \infty$, the L^∞-norm of $|\nabla u_\infty|$ for the solutions u_∞ of (1.1) generally becomes unbounded as ϵ tends to 0. The blow up rate of $|\nabla u_\infty|$ is respectively $\epsilon^{-1/2}$ in dimension $d = 2$, $(\epsilon |\ln \epsilon|)^{-1}$ in dimension $d = 3$, and ϵ^{-1} in dimension $d \geq 4$. See Bao, Li and Yin [11], as well as Budiansky and Carrier [18], Markenscoff [36], Ammari, Kang and Lim [2], Ammari, Kang, Lee, Lee and Lim [4] and Yun [41,42]. Further, more detailed, characterizations of the singular behavior of ∇u_∞ have been obtained by Ammari, Ciraolo, Kang, Lee and Yun [7], Ammari, Kang, Lee, Lim and Zribi [6], Bonnetier and Triki [14,15], Gorb and Novikov [24] and Kang, Lim and Yun [26,27]. For related works, see [3,5,8,9,12,15,17,19–22,25,28–30,33–35,37,39,40] and the references therein.

In this paper, we mainly investigate the gradient estimates for the Lamé system with partially infinite coefficients in dimension $d = 3$, a physically relevant dimension. This paper is a continuation of [13], where the estimate for dimension $d = 2$, another physically relevant dimension, is established. We prove that $(\epsilon |\ln \epsilon|)^{-1}$ is an upper bound of the blow up rate of the strain tensor in dimension three, the same as the scalar equation case mentioned above. New difficulties need to be overcome, and a number of refined estimates, via appropriate iterations, are used in our proof. We also prove that ϵ^{-1} is an upper bound of the blow up rate of the strain tensor in dimension $d \geq 4$, which is also the same as the scalar equation case. Note that it has been proved in [11] that these upper bounds in dimension $d \geq 3$ are optimal in the scalar equation case.

We consider the Lamé system in linear elasticity with piecewise constant coefficients, which is stimulated by the study of composite media with closely spaced interfacial boundaries. Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^2 boundary, and D_1 and D_2 are two disjoint convex open sets in Ω with $C^{2,\gamma}$ boundaries, $0 < \gamma < 1$, which are ϵ apart and far away from $\partial\Omega$, that is,

$$\begin{aligned} &\overline{D}_1, \overline{D}_2 \subset \Omega, \quad \text{the principle curvatures of } \partial D_1, \partial D_2 \geq \kappa_0 > 0, \\ &\epsilon := \text{dist}(D_1, D_2) > 0, \quad \text{dist}(D_1 \cup D_2, \partial\Omega) > \kappa_1 > 0, \end{aligned} \tag{1.2}$$

where κ_0, κ_1 are constants independent of ϵ. We also assume that the $C^{2,\gamma}$ norms of ∂D_i are bounded by some constant independent of ϵ. This implies that each D_i contains a ball of radius r_0^* for some constant $r_0^* > 0$ independent of ϵ. Denote

$$\widetilde{\Omega} := \Omega \setminus \overline{D_1 \cup D_2}.$$

Assume that $\widetilde{\Omega}$ and $D_1 \cup D_2$ are occupied, respectively, by two different isotropic and homogeneous materials with different Lamé constants (λ, μ) and (λ_1, μ_1). Then the elasticity tensors for the inclusions and the background can be written, respectively, as $\mathbb{C}^1$ and $\mathbb{C}^0$, with

$$C^1_{ij\,kl} = \lambda_1 \delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

and

$$C^0_{ij\,kl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.3}$$

where $i, j, k, l = 1, 2, 3$ and δ_{ij} is the Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$. Let $u = (u_1, u_2, u_3)^T : \Omega \to \mathbb{R}^3$ denote the displacement field. For a given vector valued function φ, we consider the following Dirichlet problem for the Lamé system

$$\begin{cases} \nabla \cdot \Big(\left(\chi_{\widetilde{\Omega}} \mathbb{C}^0 + \chi_{D_1 \cup D_2} \mathbb{C}^1\right) e(u) \Big) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where χ_D is the characteristic function of $D \subset \mathbb{R}^3$,

$$e(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

is the strain tensor.

Assume that the standard ellipticity condition holds for (1.4), that is,

$$\mu > 0, \quad 3\lambda + 2\mu > 0; \qquad \mu_1 > 0, \quad 3\lambda_1 + 2\mu_1 > 0. \tag{1.5}$$

For $\varphi \in H^1(\Omega;\mathbb{R}^3)$, it is well known that there exists a unique solution $u \in H^1(\Omega;\mathbb{R}^3)$ of the Dirichlet problem (1.4), which is also the minimizer of the energy functional

$$J_1[u] = \frac{1}{2}\int_\Omega \Big(\big(\chi_{\widetilde{\Omega}}\mathbb{C}^0 + \chi_{D_1\cup D_2}\mathbb{C}^1\big)\, e(u), e(u)\Big) dx$$

on

$$H^1_\varphi(\Omega;\mathbb{R}^3) := \left\{ \, u \in H^1(\Omega;\mathbb{R}^3) \;\middle|\; u-\varphi \in H^1_0(\Omega;\mathbb{R}^3) \, \right\}.$$

More details can be found in the Appendix in [13].

Introduce the linear space of rigid displacement in $\mathbb{R}^3$,

$$\Psi := \Big\{ \psi \in C^1(\mathbb{R}^3;\mathbb{R}^3) \;\big|\; \nabla\psi + (\nabla\psi)^T = 0 \,\Big\},$$

equivalently,

$$\Psi = \operatorname{span}\left\{ \psi^1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \psi^2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ \psi^3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \right.$$
$$\left. \psi^4 = \begin{pmatrix}x_2\\-x_1\\0\end{pmatrix},\ \psi^5 = \begin{pmatrix}x_3\\0\\-x_1\end{pmatrix},\ \psi^6 = \begin{pmatrix}0\\x_3\\-x_2\end{pmatrix} \right\}.$$

If $\xi \in H^1(D;\mathbb{R}^3)$, $e(\xi) = 0$ in D, and $D \subset \mathbb{R}^3$ is a connected open set, then ξ is a linear combination of $\{\psi^\alpha\}$ in D. If an element ξ in Ψ vanishes at three non-collinear points, then $\xi \equiv 0$, see Lemma 6.1.

For fixed λ and μ satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$, denoting u_{λ_1,μ_1} the solution of (1.4). Then, as proved in the Appendix in [13],

$$u_{\lambda_1,\mu_1} \to u \quad \text{in } H^1(\Omega;\mathbb{R}^3) \quad \text{as} \quad \min\{\mu_1, 3\lambda_1 + 2\mu_1\} \to \infty,$$

where u is a $H^1(\Omega;\mathbb{R}^3)$ solution of

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u := \nabla\cdot\big(\mathbb{C}^0 e(u)\big) = 0, & \text{in } \widetilde{\Omega}, \\ u\big|_+ = u\big|_-, & \text{on } \partial D_1 \cup \partial D_2, \\ e(u) = 0, & \text{in } D_1 \cup D_2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu_0}\big|_+ \cdot \psi^\alpha = 0, & i = 1,2, \quad \alpha = 1,2,\cdots,6, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where

$$\frac{\partial u}{\partial \nu_0}\bigg|_+ := \big(\mathbb{C}^0 e(u)\big)\,\vec{n} = \lambda\,(\nabla\cdot u)\,\vec{n} + \mu\left(\nabla u + (\nabla u)^T\right)\vec{n}$$

and $\vec{n}$ is the unit outer normal of D_i, $i = 1, 2$. Here and throughout this paper the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. In this paper we study solutions of (1.6), a Lamé system with infinite coefficients in $D_1 \cup D_2$.

The existence, uniqueness and regularity of weak solutions of (1.6), as well as a variational formulation, can be found in the Appendix in [13]. In particular, the H^1 weak solution is in $C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^3) \cap C^1(\overline{D_1 \cup D_2}; \mathbb{R}^3)$. The solution is also the unique function which has the least energy in appropriate functional spaces, characterized by

$$I_\infty[u] = \min_{v \in \mathcal{A}} I_\infty[v],$$

where

$$I_\infty[v] := \frac{1}{2} \int_{\widetilde{\Omega}} \left(\mathbb{C}^{(0)} e(v), e(v) \right) dx,$$

and

$$\mathcal{A} := \left\{ u \in H^1_\varphi(\Omega; \mathbb{R}^3) \;\middle|\; e(u) = 0 \quad \text{in } D_1 \cup D_2 \right\}.$$

It is well known, see [38], that for any open set O and $u, v \in C^2(O)$,

$$\int_O \left(\mathbb{C}^0 e(u), e(v) \right) dx = - \int_O \left(\mathcal{L}_{\lambda,\mu} u \right) \cdot v + \int_{\partial O} \frac{\partial u}{\partial \nu_0} \bigg|_+ \cdot v. \tag{1.7}$$

A calculation gives

$$\left(\mathcal{L}_{\lambda,\mu} u \right)_k = \mu \Delta u_k + (\lambda + \mu) \partial_{x_k} \left(\nabla \cdot u \right), \qquad k = 1, 2, 3. \tag{1.8}$$

We assume that for some $\delta_0 > 0$,

$$\delta_0 \le \mu, 3\lambda + 2\mu \le \frac{1}{\delta_0}. \tag{1.9}$$

Since D_1 and D_2 are two strictly convex subdomains of Ω, there exist two points $P_1 \in \partial D_1$ and $P_2 \in \partial D_2$ such that

$$\operatorname{dist}(P_1, P_2) = \operatorname{dist}(\partial D_1, \partial D_2) = \epsilon. \tag{1.10}$$

Use $\overline{P_1 P_2}$ to denote the line segment connecting P_1 and P_2. Throughout the paper, unless otherwise stated, C denotes a constant, whose values may vary from line to line, depending only on $d, \kappa_0, \kappa_1, \gamma, \delta_0$, and an upper bound of the C^2 norm of $\partial\Omega$ and the $C^{2,\gamma}$ norms of ∂D_1 and ∂D_2, but not on ϵ. Also, we call a constant having such dependence a *universal constant*. The main result of this paper is for dimension three.

Theorem 1.1. *Assume that Ω, D_1, D_2, ϵ are defined in (1.2), λ and μ satisfy (1.9) for some $\delta_0 > 0$, and $\varphi \in C^2(\partial\Omega; \mathbb{R}^3)$. Let $u \in H^1(\Omega; \mathbb{R}^3) \cap C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^3)$ be the solution of (1.6). Then for $0 < \epsilon < 1/2$, we have*

$$\|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^3)} \leq \frac{C}{\epsilon|\ln\epsilon|}\|\varphi\|_{C^2(\partial\Omega;\mathbb{R}^3)}, \tag{1.11}$$

where C is a universal constant.

Remark 1.1. The proof of Theorem 1.1 actually gives the following stronger estimates:

$$|\nabla u(x)| \leq \left(\frac{C}{|\ln\epsilon|\big(\epsilon + \mathrm{dist}^2(x, \overline{P_1P_2})\big)} + \frac{C\mathrm{dist}(x, \overline{P_1P_2})}{\epsilon + \mathrm{dist}^2(x, \overline{P_1P_2})}\right)\|\varphi\|_{C^2(\partial\Omega;\mathbb{R}^3)}, \quad x \in \widetilde{\Omega}, \tag{1.12}$$

and

$$|\nabla u(x)| \leq C\|\varphi\|_{C^2(\partial\Omega;\mathbb{R}^3)}, \qquad x \in D_1 \cup D_2. \tag{1.13}$$

Remark 1.2. The strict convexity assumption on ∂D_1 and ∂D_2 can be replaced by a weaker relative strict convexity assumption, see (3.5) in Section 3.

Remark 1.3. Here $\varphi \in C^2(\partial\Omega; \mathbb{R}^3)$ can be replaced by $\varphi \in H^{1/2}(\partial\Omega; \mathbb{R}^3)$. Indeed, the H^1 norm of the solution u in Ω is bounded by a universal constant. Then standard elliptic estimates give a universal bound of u in C^2 norm in $\left\{x \in \Omega \;\middle|\; \frac{\kappa_1}{4} < \mathrm{dist}(x, \partial\Omega) < \frac{\kappa_1}{2}\right\}$. We apply the theorem in $\Omega' := \left\{x \in \Omega \;\middle|\; \mathrm{dist}(x, \partial\Omega) > \frac{\kappa_1}{3}\right\}$ with $\varphi' := u\big|_{\partial\Omega'}$.

Remark 1.4. Since the blow up rate of $|\nabla u_\infty|$ for solutions of the scalar equation (1.1) when $k = \infty$ is known to reach the magnitude $(\epsilon|\ln\epsilon|)^{-1}$ in dimension three, see [11], estimate (1.11) is expected to be optimal.

Following arguments in the proof of Theorem 1.1, we establish the corresponding estimates for higher dimensions $d \geq 4$. Let $\Omega \subset \mathbb{R}^d$, $d \geq 4$ be a bounded open set with C^2 boundary, and D_1 and D_2 are two disjoint convex open sets in Ω with $C^{2,\gamma}$ boundaries, satisfying (1.2). Let $\mathbb{C}^0$ be given by (1.3) with $i, j, k, l = 1, 2, \cdots, d$, where λ and μ satisfy

$$\mu > 0, \quad d\lambda + 2\mu > 0,$$

and

$$\Psi := \left\{\psi \in C^1(\mathbb{R}^d; \mathbb{R}^d) \;\middle|\; \nabla\psi + (\nabla\psi)^T = 0\right\} \tag{1.14}$$

be the linear space of rigid displacement in $\mathbb{R}^d$. With $e_1, \cdots, e_d$ denoting the standard basis of $\mathbb{R}^d$,

$$\left\{ \; e_i, \; x_j e_k - x_k e_j \;\middle|\; 1 \le i \le d, \; 1 \le j < k \le d \; \right\}$$

is a basis of Ψ. Denote the basis of Ψ as $\{\psi^\alpha\}$, $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$. Consider

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u := \nabla \cdot \big(\mathbb{C}^0 e(u)\big) = 0, & \text{in } \widetilde{\Omega}, \\ u\big|_+ = u\big|_-, & \text{on } \partial D_1 \cup \partial D_2, \\ e(u) = 0, & \text{in } D_1 \cup D_2, \\ \int_{\partial D_i} \frac{\partial u}{\partial \nu_0}\big|_+ \cdot \psi^\alpha = 0, & i = 1, 2, \quad \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Then we have

Theorem 1.2. *Assume as above, and $\varphi \in C^2(\partial\Omega; \mathbb{R}^d)$, $d \ge 4$. Let $u \in H^1(\Omega; \mathbb{R}^d) \cap C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^d)$ be the solution of (1.15). Then for $0 < \epsilon < 1/2$, we have*

$$\|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^d)} \le \frac{C}{\epsilon} \|\varphi\|_{C^2(\partial\Omega;\mathbb{R}^d)}, \tag{1.16}$$

where C is a universal constant.

Remark 1.5. The proof of Theorem 1.2 actually gives the following stronger estimate in dimension $d \ge 4$:

$$|\nabla u(x)| \le \begin{cases} \dfrac{C}{\epsilon + \mathrm{dist}^2(x, \overline{P_1 P_2})} \|\varphi\|_{C^2(\partial\Omega;\mathbb{R}^d)}, & x \in \widetilde{\Omega}, \\ C \|\varphi\|_{C^2(\partial\Omega;\mathbb{R}^d)}, & x \in D_1 \cup D_2. \end{cases}$$

We also have Remarks 1.2–1.4 accordingly.

The rest of this paper is organized as follows. In Section 2, we first introduce a setup for the proof of Theorem 1.1. Then we state a proposition, Proposition 2.1, containing key estimates, and deduce Theorem 1.1 from the proposition. In Sections 3 and 4, we prove Proposition 2.1. The proof of Theorem 1.2 is given in Section 5. A linear algebra lemma, Lemma 6.2, used in the proof of Theorem 1.1, is given in Section 6.

2. Outline of the Proof of Theorem 1.1

The proof of Theorem 1.1 makes use of the following decomposition. By the third line of (1.6), u is a linear combination of $\{\psi^\alpha\}$ in D_1 and D_2, respectively. Since it is clear that $\mathcal{L}_{\lambda,\mu}\xi = 0$ in $\widetilde{\Omega}$ and $\xi = 0$ on $\partial\widetilde{\Omega}$ imply that $\xi = 0$ in $\widetilde{\Omega}$, we decompose the solution of (1.6), as in [13], as follows:

$$u = \begin{cases} \sum\limits_{\alpha=1}^{6} C_1^\alpha \psi^\alpha, & \text{in } \overline{D}_1, \\ \sum\limits_{\alpha=1}^{6} C_2^\alpha \psi^\alpha, & \text{in } \overline{D}_2, \\ \sum\limits_{\alpha=1}^{6} C_1^\alpha v_1^\alpha + \sum\limits_{\alpha=1}^{6} C_2^\alpha v_2^\alpha + v_0, & \text{in } \widetilde{\Omega}, \end{cases} \tag{2.1}$$

where $v_i^\alpha \in C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^3)$, $i = 1, 2$, $\alpha = 1, 2, \cdots, 6$, and $v_0 \in C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^3)$ are respectively the solution of

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_i^\alpha = 0, & \text{in } \widetilde{\Omega}, \\ v_i^\alpha = \psi^\alpha, & \text{on } \partial D_i, \\ v_i^\alpha = 0, & \text{on } \partial D_j \cup \partial\Omega, \ j \neq i, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_0 = 0, & \text{in } \widetilde{\Omega}, \\ v_0 = 0, & \text{on } \partial D_1 \cup \partial D_2, \\ v_0 = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The constants $C_i^\alpha := C_i^\alpha(\epsilon)$, $i = 1, 2$, $\alpha = 1, 2, \cdots, 6$, are uniquely determined by the fourth line of (1.6), see (4.5) below.

By the decomposition (2.1), we write

$$\nabla u = \sum_{\alpha=1}^{3} (C_1^\alpha - C_2^\alpha) \nabla v_1^\alpha + \sum_{\alpha=1}^{3} C_2^\alpha \nabla(v_1^\alpha + v_2^\alpha) + \sum_{i=1}^{2} \sum_{\alpha=4}^{6} C_i^\alpha \nabla v_i^\alpha + \nabla v_0, \quad \text{in } \widetilde{\Omega}, \tag{2.4}$$

then

$$|\nabla u| \leq \sum_{\alpha=1}^{3} |C_1^\alpha - C_2^\alpha| |\nabla v_1^\alpha| + \sum_{\alpha=1}^{3} |C_2^\alpha| |\nabla(v_1^\alpha + v_2^\alpha)| + \sum_{i=1}^{2} \sum_{\alpha=4}^{6} |C_i^\alpha| |\nabla v_i^\alpha| + \big|\nabla v_0\big|, \quad \text{in } \widetilde{\Omega}. \tag{2.5}$$

The proof of Theorem 1.1 can be reduced to the following proposition. Without loss of generality, we only need to prove Theorem 1.1 for $\|\varphi\|_{C^2(\partial\Omega)} = 1$, and for the general case by considering $u/\|\varphi\|_{C^2(\partial\Omega)}$ if $\|\varphi\|_{C^2(\partial\Omega)} > 0$. If $\varphi\big|_{\partial\Omega} = 0$, then $u = 0$.

Proposition 2.1. *Under the hypotheses of Theorem 1.1, and the normalization $\|\varphi\|_{C^2(\partial\Omega)} = 1$, let v_i^α and v_0 be the solution to (2.2) and (2.3), respectively. Then for $0 < \epsilon < 1/2$, we have*

$$\left\|\nabla v_0\right\|_{L^\infty(\widetilde{\Omega})} \le C; \tag{2.6}$$

$$\left\|\nabla(v_1^\alpha + v_2^\alpha)\right\|_{L^\infty(\widetilde{\Omega})} \le C, \quad \alpha = 1,2,3; \tag{2.7}$$

$$\left|\nabla v_i^\alpha(x)\right| \le \frac{C}{\epsilon + \mathrm{dist}^2(x, \overline{P_1P_2})}, \quad i = 1,2, \quad \alpha = 1,2,3, \quad x \in \widetilde{\Omega}; \tag{2.8}$$

$$\left|\nabla v_i^\alpha(x)\right| \le \frac{C\mathrm{dist}(x, \overline{P_1P_2})}{\epsilon + \mathrm{dist}^2(x, \overline{P_1P_2})} + C, \quad i = 1,2, \quad \alpha = 4,5,6, \quad x \in \widetilde{\Omega}; \tag{2.9}$$

and

$$\left|C_i^\alpha\right| \le C, \quad i = 1,2, \ \alpha = 1,2,\cdots,6; \tag{2.10}$$

$$\left|C_1^\alpha - C_2^\alpha\right| \le \frac{C}{|\ln\epsilon|}, \quad \alpha = 1,2,3. \tag{2.11}$$

Proof of Theorem 1.1 by using Proposition 2.1. Clearly, we only need to prove the theorem under the normalization $\|\varphi\|_{C^2(\partial\Omega)} = 1$.

Since

$$\nabla u = \begin{pmatrix} 0 & C_i^4 & C_i^5 \\ -C_i^4 & 0 & C_i^6 \\ -C_i^5 & -C_i^6 & 0 \end{pmatrix}, \quad \text{in } D_i, \quad i = 1,2,$$

estimate (1.13) follows from (2.10).

By (2.5) and Proposition 2.1, we have

$$\begin{aligned} \left|\nabla u(x)\right| &\le \sum_{\alpha=1}^{3} |C_1^\alpha - C_2^\alpha|\,|\nabla v_1^\alpha(x)| + \sum_{i=1}^{2}\sum_{\alpha=4}^{6} |C_i^\alpha|\,|\nabla v_i^\alpha| + C \\ &\le \frac{C}{|\ln\epsilon|\left(\epsilon + \mathrm{dist}^2(x, \overline{P_1P_2})\right)} + \frac{C\mathrm{dist}(x, \overline{P_1P_2})}{\epsilon + \mathrm{dist}^2(x, \overline{P_1P_2})}. \end{aligned} \tag{2.12}$$

Theorem 1.1 follows immediately. □

To complete this section, we recall some properties of the tensor $\mathbb{C}$. For the isotropic elastic material, let

$$\mathbb{C} := (C_{ij\,kl}) = \left(\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right), \quad \mu > 0, \quad d\lambda + 2\mu > 0. \tag{2.13}$$

The components $C_{ij\,kl}$ satisfy the following symmetric condition:

$$C_{ij\,kl} = C_{kl\,ij} = C_{klj\,i}, \quad i,j,k,l = 1,2,\cdots,d. \tag{2.14}$$

We will use the following notations:

$$(\mathbb{C}A)_{ij} = \sum_{k,l=1}^{d} C_{ij\,kl} A_{kl}, \quad \text{and} \quad (A, B) := A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij},$$

for every pair of $d \times d$ matrices $A = (A_{ij})$, $B = (B_{ij})$. By the symmetric condition (2.14), we have

$$(\mathbb{C}A, B) = (A, \mathbb{C}B). \tag{2.15}$$

For an arbitrary $d \times d$ real symmetric matrix $\eta = (\eta_{ij})$, we have

$$C_{ij\,kl}\,\eta_{kl}\eta_{ij} = \lambda\,\eta_{ii}\eta_{kk} + 2\mu\,\eta_{kj}\eta_{kj}.$$

It follows from (2.13) that $\mathbb{C}$ satisfies the ellipticity condition

$$\min\Big\{2\mu, d\lambda + 2\mu\Big\}|\eta|^2 \leq C_{ij\,kl}\,\eta_{kl}\eta_{ij} \leq \max\Big\{2\mu, d\lambda + 2\mu\Big\}|\eta|^2, \tag{2.16}$$

where $|\eta|^2 = \sum\limits_{i,j=1}^{d} \eta_{ij}^2$. In particular,

$$\min\Big\{2\mu, d\lambda + 2\mu\Big\}\left|A + A^T\right|^2 \leq \left(\mathbb{C}\left(A + A^T\right), \left(A + A^T\right)\right). \tag{2.17}$$

3. Estimates of $|\nabla v_0|$, $|\nabla v_i^\alpha|$, and $|\nabla(v_1^\alpha + v_2^\alpha)|$

We first fix notations. Use (x_1, x_2, x_3) to denote a point in $\mathbb{R}^3$ and $x' = (x_1, x_2)$. By a translation and rotation if necessary, we may assume without loss of generality that the points P_1 and P_2 in (1.10) satisfy

$$P_1 = \Big(0', \frac{\epsilon}{2}\Big) \in \partial D_1, \quad \text{and} \quad P_2 = \Big(0', -\frac{\epsilon}{2}\Big) \in \partial D_2.$$

Fix a small universal constant R, such that the portion of ∂D_1 and ∂D_2 near P_1 and P_2, respectively, can be represented by

$$x_3 = \frac{\epsilon}{2} + h_1(x'), \quad \text{and} \quad x_3 = -\frac{\epsilon}{2} + h_2(x'), \quad \text{for} \quad |x'| < 2R. \tag{3.1}$$

Then by the smoothness assumptions on ∂D_1 and ∂D_2, the functions $h_1(x')$ and $h_2(x')$ are of class $C^{2,\gamma}(B_R(0'))$, satisfying

$$\frac{\epsilon}{2} + h_1(x') > -\frac{\epsilon}{2} + h_2(x'), \quad \text{for} \quad |x'| < 2R,$$

$$h_1(0') = h_2(0') = 0, \quad \nabla h_1(0') = \nabla h_2(0') = 0, \tag{3.2}$$

$$\nabla^2 h_1(0') \geq \kappa_0 I, \quad \nabla^2 h_2(0') \leq -\kappa_0 I, \tag{3.3}$$

and

$$\|h_1\|_{C^{2,\gamma}(B'_{2R})} + \|h_2\|_{C^{2,\gamma}(B'_{2R})} \leq C. \tag{3.4}$$

In particular, we only use a weaker relative strict convexity assumption of ∂D_1 and ∂D_2, that is

$$h_1(x') - h_2(x') \geq \kappa_0 |x'|^2, \quad \text{if} \ \ |x'| < 2R. \tag{3.5}$$

For $0 \leq r \leq 2R$, denote

$$\Omega_r := \left\{ \ (x', x_3) \in \mathbb{R}^3 \ \Big| \ -\frac{\epsilon}{2} + h_2(x') < x_3 < \frac{\epsilon}{2} + h_1(x'), \ |x'| < r \ \right\}.$$

For $0 \leq |z'| < R$, let

$$\widehat{\Omega}_s(z') := \left\{ \ (x', x_3) \in \mathbb{R}^3 \ \Big| \ -\frac{\epsilon}{2} + h_2(x') < x_3 < \frac{\epsilon}{2} + h_1(x'), \ |x' - z'| < s \ \right\}. \tag{3.6}$$

3.1. Estimates of $|\nabla v_0|$, $|\nabla v_i^\alpha|$ *for* $\alpha = 1, 2, 3$, *and* $|\nabla(v_1^\alpha + v_2^\alpha)|$

Lemma 3.1.

$$\|v_0\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_0\|_{L^\infty(\widetilde{\Omega})} \leq C. \tag{3.7}$$

$$\|v_1^\alpha + v_2^\alpha\|_{L^\infty(\widetilde{\Omega})} + \|\nabla(v_1^\alpha + v_2^\alpha)\|_{L^\infty(\widetilde{\Omega})} \leq C, \quad \alpha = 1, 2, \cdots, 6. \tag{3.8}$$

The proof of Lemma 3.1 is essentially the same as in [13] for dimension two. We omit it here. By Lemma 3.1, (2.6) and (2.7) is proved.

To estimate (2.8), we introduce a scalar function $\bar{u} \in C^2(\mathbb{R}^3)$, such that $\bar{u} = 1$ on ∂D_1, $\bar{u} = 0$ on $\partial D_2 \cup \partial\Omega$,

$$\bar{u}(x) = \frac{x_3 - h_2(x') + \frac{\epsilon}{2}}{\epsilon + h_1(x') - h_2(x')}, \quad \text{in} \ \ \Omega_{2R}, \tag{3.9}$$

and

$$\|\bar{u}\|_{C^2(\mathbb{R}^3 \setminus \Omega_R)} \leq C. \tag{3.10}$$

Define

$$\bar{u}_1^\alpha = \bar{u}\psi^\alpha, \quad \alpha = 1, 2, 3, \quad \text{in} \ \ \widetilde{\Omega}, \tag{3.11}$$

then $\bar{u}_1^\alpha = v_1^\alpha$ on $\partial\widetilde{\Omega}$.

Similarly, we define

$$\bar{u}_2^\alpha = \underline{u}\psi^\alpha, \quad \alpha = 1, 2, 3, \quad \text{in} \ \ \widetilde{\Omega}, \tag{3.12}$$

such that $\bar{u}_2^\alpha = v_2^\alpha$ on $\partial\widetilde{\Omega}$, where $\underline{u}$ is a scalar function in $C^2(\mathbb{R}^3)$ satisfying $\underline{u} = 1$ on ∂D_2, $\underline{u} = 0$ on $\partial D_1 \cup \partial\Omega$,

$$\underline{u}(x) = \frac{-x_3 + h_1(x') + \frac{\epsilon}{2}}{\epsilon + h_1(x') - h_2(x')}, \quad \text{in} \quad \Omega_{2R}, \tag{3.13}$$

and

$$\|\underline{u}\|_{C^2(\mathbb{R}^3 \setminus \Omega_R)} \leq C. \tag{3.14}$$

In order to prove (2.8), it suffices to prove the following proposition.

Proposition 3.2. *Assume the above, let $v_i^\alpha \in H^1(\widetilde{\Omega}; \mathbb{R}^3)$ be the weak solution of (2.2) with $\alpha = 1, 2, 3$. Then for $i = 1, 2,\ \alpha = 1, 2, 3$,*

$$\int\limits_{\widetilde{\Omega}} |\nabla(v_i^\alpha - \bar{u}_i^\alpha)|^2\, dx \leq C; \tag{3.15}$$

and

$$\|\nabla v_i^\alpha\|_{L^\infty(\widetilde{\Omega} \setminus \Omega_R)} \leq C, \tag{3.16}$$

$$|\nabla(v_i^\alpha - \bar{u}_i^\alpha)(x)| \leq \begin{cases} \dfrac{C}{\sqrt{\epsilon}}, & |x'| \leq \sqrt{\epsilon}, \\ \dfrac{C}{|x'|}, & \sqrt{\epsilon} < |x'| \leq R, \end{cases} \qquad \forall \ \ x \in \Omega_R. \tag{3.17}$$

Consequently,

$$|\nabla v_i^\alpha(x)| \leq \frac{C}{\epsilon + |x'|^2}, \qquad \forall \ \ x \in \Omega_R, \tag{3.18}$$

and

$$|\nabla_{x'} v_i^\alpha(x)| \leq \begin{cases} \dfrac{C}{\sqrt{\epsilon}}, & |x'| \leq \sqrt{\epsilon}, \\ \dfrac{C}{|x'|}, & \sqrt{\epsilon} < |x'| \leq R. \end{cases} \tag{3.19}$$

A direct calculation gives, in view of (3.2)–(3.5), that

$$|\partial_{x_k} \bar{u}(x)| \leq \frac{C|x_k|}{\epsilon + |x'|^2}, \quad k = 1, 2, \qquad |\partial_{x_3} \bar{u}(x)| \leq \frac{C}{\epsilon + |x'|^2}, \qquad x \in \Omega_R. \tag{3.20}$$

Thus

$$|\nabla \bar{u}_i^\alpha(x)| \leq \frac{C}{\epsilon + |x'|^2}, \quad i = 1, 2,\ \alpha = 1, 2, 3, \quad x \in \Omega_R. \tag{3.21}$$

For $k, l = 1, 2$,

$$|\partial_{x_k x_l} \bar{u}(x)| \leq \frac{C}{\epsilon + |x'|^2}, \quad |\partial_{x_k x_3} \bar{u}(x)| \leq \frac{C|x'|}{(\epsilon + |x'|^2)^2}, \quad \partial_{x_3 x_3} \bar{u}(x) = 0, \quad x \in \Omega_R. \tag{3.22}$$

For $\bar{u}_i^\alpha$, defined by (3.11) and (3.12), making use of (1.8) and (3.22), we have, for $i = 1, 2,\ \alpha = 1, 2, 3$,

$$|\mathcal{L}_{\lambda,\mu} \bar{u}_i^\alpha(x)| \leq C \sum_{k+l<6} |\partial_{x_k x_l} \bar{u}(x)| \leq \frac{C}{\epsilon + |x'|^2} + \frac{C|x'|}{(\epsilon + |x'|^2)^2}, \quad x \in \Omega_R. \tag{3.23}$$

For $|z'| \leq 2R$, we always use δ to denote

$$\delta := \delta(z') = \frac{\epsilon + h_1(z') - h_2(z')}{2}. \tag{3.24}$$

By (3.2)–(3.5),

$$\frac{1}{C} \left(\epsilon + |z'|^2\right) \leq \delta(z') \leq C \left(\epsilon + |z'|^2\right). \tag{3.25}$$

Proof of Proposition 3.2. Let

$$w_i^\alpha := v_i^\alpha - \bar{u}_i^\alpha, \qquad i = 1, 2,\ \alpha = 1, 2, 3. \tag{3.26}$$

For simplicity, denote

$$w := w_i^\alpha, \qquad \text{and} \quad \tilde{u} = \bar{u}_i^\alpha, \qquad i = 1, 2,\ \alpha = 1, 2, 3.$$

The proof is divided into four steps.

STEP 1. Proof of (3.15) and (3.16).

By (3.26) and (2.2),

$$\begin{cases} \mathcal{L}_{\lambda,\mu} w = -\mathcal{L}_{\lambda,\mu} \tilde{u}, & \text{in } \widetilde{\Omega}, \\ w = 0, & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{3.27}$$

Multiplying the equation in (3.27) by w and integrating by parts, we have

$$\int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e(w), e(w)\right) dx = \int_{\widetilde{\Omega}} w \left(\mathcal{L}_{\lambda,\mu} \tilde{u}\right) dx. \tag{3.28}$$

By the Poincaré inequality,

$$\|w\|_{L^2(\widetilde{\Omega}\setminus\Omega_R)} \leq C \|\nabla w\|_{L^2(\widetilde{\Omega}\setminus\Omega_R)}. \tag{3.29}$$

Note that the above constant C is independent of ϵ. By the Sobolev trace embedding theorem,

$$\int\limits_{\substack{|x'|=R,\\ -\epsilon/2+h_2(x')<x_3<\epsilon/2+h_1(x')}} |w| \leq C \left(\int\limits_{\widetilde{\Omega}\setminus\Omega_R} |\nabla w|^2 \, dx \right)^{1/2}. \tag{3.30}$$

It follows from the first Korn's inequality, (2.17), (3.28) and the definition of $\tilde{u}$ that

$$\begin{aligned}
\int\limits_{\widetilde{\Omega}} |\nabla w|^2 \, dx &\leq 2 \int\limits_{\widetilde{\Omega}} |e(w)|^2 dx \\
&\leq C \Big| \int\limits_{\Omega_R} w \left(\mathcal{L}_{\lambda,\mu}\tilde{u}\right) dx \Big| + C \Big| \int\limits_{\widetilde{\Omega}\setminus\Omega_R} w \left(\mathcal{L}_{\lambda,\mu}\tilde{u}\right) dx \Big| \\
&\leq C \Big| \int\limits_{\Omega_R} w \left(\mathcal{L}_{\lambda,\mu}\tilde{u}\right) dx \Big| + C \int\limits_{\widetilde{\Omega}\setminus\Omega_R} |w| dx \\
&\leq C \Big| \int\limits_{\Omega_R} w \left(\mathcal{L}_{\lambda,\mu}\tilde{u}\right) dx \Big| + C \left(\int\limits_{\widetilde{\Omega}\setminus\Omega_R} |\nabla w|^2 dx \right)^{1/2},
\end{aligned} \tag{3.31}$$

while, using (1.8) and (3.30),

$$\begin{aligned}
\Big| \int\limits_{\Omega_R} w \left(\mathcal{L}_{\lambda,\mu}\tilde{u}\right) dx \Big| &\leq C \sum_{k+l<6} \Big| \int\limits_{\Omega_R} w \partial_{x_k x_l} \tilde{u} \, dx \Big| \\
&\leq C \int\limits_{\Omega_R} |\nabla w| \left|\nabla_{x'}\tilde{u}\right| dx + \int\limits_{\substack{|x'|=R,\\ -\epsilon/2+h_2(x')<x_3<\epsilon/2+h_1(x')}} C\,|w| \\
&\leq C \left(\int\limits_{\Omega_R} |\nabla w|^2 dx \right)^{1/2} \left(\int\limits_{\Omega_R} \left|\nabla_{x'}\tilde{u}\right|^2 dx \right)^{1/2} \\
&\quad + C \left(\int\limits_{\widetilde{\Omega}\setminus\Omega_R} |\nabla w|^2 \, dx \right)^{1/2}.
\end{aligned} \tag{3.32}$$

Using

$$\int\limits_{\Omega_R} \left|\nabla_{x'}\tilde{u}\right|^2 dx \leq C,$$

we have, from the above,

$$\int_{\widetilde{\Omega}} |\nabla w|^2\,dx \le C\left(\int_{\widetilde{\Omega}} |\nabla w|^2 dx\right)^{1/2}.$$

This estimate yields (3.15).

A consequence of (3.15) and (3.10) is

$$\int_{\widetilde{\Omega}\setminus\Omega_{R/2}} |\nabla v_i^\alpha|^2 dx \le 2\int_{\widetilde{\Omega}\setminus\Omega_{R/2}} \left(|\nabla \bar{u}_i^\alpha|^2 + |\nabla(v_i^\alpha - \bar{u}_i^\alpha)|^2\right)dx \le C.$$

Applying classical elliptic estimates, we obtain (3.16).

STEP 2. Proof of

$$\int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2\,dx \le \begin{cases} C\epsilon^2, & 0\le |z'|\le \sqrt{\epsilon}, \\ C|z'|^4, & \sqrt{\epsilon}<|z'|\le R, \end{cases} \tag{3.33}$$

where $\delta=\delta(z')$ is defined by (3.24).

For $0<t<s<R$, let η be a smooth function satisfying $0\le\eta(x')\le 1$, $\eta(x')=1$ if $|x'-z'|<t$, $\eta(x')=0$ if $|x'-z'|>s$, and $|\nabla\eta(x')|\le\frac{2}{s-t}$. Multiplying the equation in (3.27) by $w\eta^2$ and integrating by parts leads to

$$\int_{\widehat{\Omega}_s(z')} \left(\mathbb{C}^0 e(w), e(w\eta^2)\right)dx = \int_{\widehat{\Omega}_s(z')} (w\eta^2)\mathcal{L}_{\lambda,\mu}\tilde{u}\,dx. \tag{3.34}$$

For the left hand side of (3.34), using the first Korn's inequality and some standard arguments, we have

$$\int_{\widehat{\Omega}_s(z')} \left(\mathbb{C}^0 e(w), e(w\eta^2)\right)dx \ge \frac{1}{C}\int_{\widehat{\Omega}_s(z')} |\nabla(w\eta)|^2 dx - C\int_{\widehat{\Omega}_s(z')} |w|^2|\nabla\eta|^2 dx,$$

and for the right hand side of (3.34),

$$\begin{aligned}\left|\int_{\widehat{\Omega}_s(z')} (w\eta^2)\mathcal{L}_{\lambda,\mu}\tilde{u}\,dx\right| &\le \left(\int_{\widehat{\Omega}_s(z')} |w|^2 dx\right)^{1/2}\left(\int_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2\,dx\right)^{1/2} \\ &\le \frac{1}{(s-t)^2}\int_{\widehat{\Omega}_s(z')} |w|^2 dx + (s-t)^2\int_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2\,dx.\end{aligned}$$

It follows that

$$\int_{\widehat{\Omega}_t(z')} |\nabla w|^2 dx \leq \frac{C}{(s-t)^2} \int_{\widehat{\Omega}_s(z')} |w|^2 dx + C(s-t)^2 \int_{\widehat{\Omega}_s(z')} \left|\mathcal{L}_{\lambda,\mu}\tilde{u}\right|^2 dx. \tag{3.35}$$

Case 1. Estimate (3.33) for $\sqrt{\epsilon} \leq |z'| \leq R$.

Note that for $\sqrt{\epsilon} \leq |z'| \leq R$, $0 < t < s < \frac{2|z'|}{3}$, we have

$$\begin{aligned}
\int_{\widehat{\Omega}_s(z')} |w|^2 dx &= \int_{|x'-z'|\leq s} \int_{-\frac{\epsilon}{2}+h_2(x')}^{\frac{\epsilon}{2}+h_1(x')} |w(x',x_3)|^2 dx_3 dx' \\
&\leq \int_{|x'-z'|\leq s} (\epsilon + h_1(x') - h_2(x'))^2 \int_{-\frac{\epsilon}{2}+h_2(x')}^{\frac{\epsilon}{2}+h_1(x')} |\partial_{x_3} w(x',x_3)|^2 dx_3 dx' \\
&\leq C|z'|^4 \int_{\widehat{\Omega}_s(z')} |\nabla w|^2 dx.
\end{aligned} \tag{3.36}$$

By (3.23), we have

$$\begin{aligned}
\int_{\widehat{\Omega}_s(z')} \left|\mathcal{L}_{\lambda,\mu}\tilde{u}\right|^2 dx &\leq \int_{\widehat{\Omega}_s(z')} \left(\frac{C}{\epsilon+|x'|^2} + \frac{C|x'|}{(\epsilon+|x'|^2)^2}\right)^2 dx \\
&\leq C \int_{|x'-z'|<s} \left(\frac{1}{\epsilon+|x'|^2} + \frac{|x'|^2}{(\epsilon+|x'|^2)^3}\right) dx' \\
&\leq \frac{Cs^2}{|z'|^4}, \qquad 0 < s < \frac{2|z'|}{3}.
\end{aligned} \tag{3.37}$$

Denote

$$\widehat{F}(t) := \int_{\widehat{\Omega}_t(z')} |\nabla w|^2 dx.$$

It follows from (3.35), (3.36) and (3.37) that

$$\widehat{F}(t) \leq \left(\frac{C_0|z'|^2}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2 \frac{s^2}{|z'|^4}, \qquad \forall\, 0 < t < s < \frac{2|z'|}{3}. \tag{3.38}$$

Set $t_i = \delta + 2C_0 i\,|z'|^2$, $i = 0, 1, 2, \cdots$. Then

$$\frac{C_0|z'|^2}{t_{i+1}-t_i}=\frac{1}{2}.$$

Let $k=\left[\frac{1}{4C_0|z'|}\right]$. Using (3.38) with $s=t_{i+1}$ and $t=t_i$, we have

$$\widehat{F}(t_i)\leq\frac{1}{4}\widehat{F}(t_{i+1})+\frac{C(t_{i+1}-t_i)^2t_{i+1}^2}{|z'|^4}\leq\frac{1}{4}\widehat{F}(t_{i+1})+C(i+1)^2|z'|^4,\qquad i=0,1,2,\cdots,k.$$

After k iterations, using (3.15), we obtain

$$\begin{aligned}\widehat{F}(t_0)&\leq\left(\frac{1}{4}\right)^k\widehat{F}(t_k)+C\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1}l^2|z'|^4\\&\leq\left(\frac{1}{4}\right)^k\widehat{F}(\frac{2|z'|}{3})+C|z'|^4\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1}l^2\leq C|z'|^4.\end{aligned}$$

This implies that

$$\int\limits_{\widehat{\Omega}_\delta(z')}|\nabla w|^2dx\leq C|z'|^4.$$

Case 2. Estimate (3.33) for $0\leq|z'|\leq\sqrt{\epsilon}$.

For $0\leq|z'|\leq\sqrt{\epsilon}$, $0<t<s<\sqrt{\epsilon}$, estimate (3.36) becomes

$$\int\limits_{\widehat{\Omega}_s(z')}|w|^2dx\leq C\epsilon^2\int\limits_{\widehat{\Omega}_s(z')}|\nabla w|^2dx,\qquad 0<s<\sqrt{\epsilon},\tag{3.39}$$

while estimate (3.37) becomes

$$\int\limits_{\widehat{\Omega}_s(z')}|\mathcal{L}_{\lambda,\mu}\tilde{u}|^2\leq\int\limits_{|x'-z'|<s}\left(\frac{C}{\epsilon+|x'|^2}+\frac{C|x'|^2}{(\epsilon+|x'|^2)^3}\right)dx'\leq\frac{Cs^2}{\epsilon^2}.\tag{3.40}$$

Estimate (3.38) becomes, in view of (3.35), (3.39) and (3.40),

$$\widehat{F}(t)\leq\left(\frac{C_0\epsilon}{s-t}\right)^2\widehat{F}(s)+C(s-t)^2\frac{s^2}{\epsilon^2},\quad\forall\,0<t<s<\sqrt{\epsilon}.\tag{3.41}$$

For $0\leq|z'|\leq\sqrt{\epsilon}$, let $t_i=\delta+2C_0i\epsilon$, $i=0,1,2,\cdots$. Thus

$$\frac{C_0\epsilon}{t_{i+1}-t_i}=\frac{1}{2}.$$

Let $k=\left[\frac{1}{4C_0\sqrt{\epsilon}}\right]$. By (3.41) with $s=t_{i+1}$ and $t=t_i$, we have

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C\epsilon^2 t_{i+1}^2}{\epsilon^2} \leq \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)^2\epsilon^2, \quad i = 0, 1, 2, \cdots, k.$$

After k iterations, we obtain

$$\begin{aligned}\widehat{F}(t_0) &\leq \left(\frac{1}{4}\right)^k \widehat{F}(t_k) + C\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1} l^2\epsilon^2 \\ &\leq \left(\frac{1}{4}\right)^k \widehat{F}(\sqrt{\epsilon}) + C\epsilon^2 \leq C\epsilon^2.\end{aligned}$$

This implies

$$\int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2 dx \leq C\epsilon^2.$$

STEP 3. Proof of (3.17).

Making a change of variables, for $0 \leq |z'| \leq R$,

$$\begin{cases} x' - z' = \delta y', \\ x_3 = \delta y_3, \end{cases} \tag{3.42}$$

the region $\widehat{\Omega}_\delta(z')$, becomes Q_1, where

$$Q_r = \left\{ y \in \mathbb{R}^3 \;\Big|\; -\frac{\epsilon}{2\delta} + \frac{1}{\delta}h_2(\delta y' + z') < y_3 < \frac{\epsilon}{2\delta} + \frac{1}{\delta}h_1(\delta y' + z'),\ |y'| < r \right\},$$
$$\text{for} \quad r \leq 1,$$

and the top and bottom boundaries of Q_r become

$$\begin{aligned} y_3 &= \hat{h}_1(y') =: \frac{1}{\delta}\left(\frac{\epsilon}{2} + h_1(\delta\, y' + z')\right), \quad \text{and} \\ y_3 &= \hat{h}_2(y') := \frac{1}{\delta}\left(-\frac{\epsilon}{2} + h_2(\delta\, y' + z')\right), \quad |y'| < 1, \end{aligned}$$

respectively. Thus

$$\hat{h}_1(0') - \hat{h}_2(0') = \frac{1}{\delta}\left(\epsilon + h_1(z') - h_2(z')\right) = 2,$$

and, by (3.2) and (3.3), for $|y'| \leq 1$,

$$\left|\nabla\hat{h}_1(y')\right| + \left|\nabla\hat{h}_2(y')\right| \leq C(\delta + |z'|), \qquad \left|\nabla^2\hat{h}_1(y')\right| + \left|\nabla^2\hat{h}_2(y')\right| \leq C\delta.$$

Since R is small, Q_1 is essentially $B_1(0') \times (-1, 1)$ as far as applications of the Sobolev embedding theory and classical L^p estimates for elliptic systems are concerned. Let

$$U(y',y_3) := \tilde{u}(x',x_3), \quad W(y',y_3) := w(x',x_3), \qquad y \in Q_1. \tag{3.43}$$

By (3.27),

$$\mathcal{L}_{\lambda,\mu} W = \mathcal{L}_{\lambda,\mu} U, \qquad y \in Q_1, \tag{3.44}$$

where

$$\mathcal{L}_{\lambda,\mu} U = \delta^2 \mathcal{L}_{\lambda,\mu} \tilde{u}.$$

Since $W = 0$ on the top and bottom boundaries of Q_1, we have, by the Poincaré inequality,

$$\|W\|_{H^1(Q_1)} \leq C \|\nabla W\|_{L^2(Q_1)}.$$

Using the interior and boundary $W^{2,p}$ estimates (see [1], and Theorem 2.5 in [23]) and the Sobolev embedding theorem, we have, for some $p > 3$,

$$\|\nabla W\|_{L^\infty(Q_{1/2})} \leq C \|W\|_{W^{2,p}(Q_{1/2})} \leq C \left(\|\nabla W\|_{L^2(Q_1)} + \|\mathcal{L}_{\lambda,\mu} U\|_{L^\infty(Q_1)} \right),$$

where C depends only on p and Q_1, but not on ϵ. Thus

$$\|\nabla w\|_{L^\infty(\widehat{\Omega}_{\delta/2}(z'))} \leq \frac{C}{\delta} \left(\delta^{-\frac{1}{2}} \|\nabla w\|_{L^2(\widehat{\Omega}_\delta(z'))} + \delta^2 \|\mathcal{L}_{\lambda,\mu} \tilde{u}\|_{L^\infty(\widehat{\Omega}_\delta(z'))} \right). \tag{3.45}$$

Case 1. (3.17) for $\sqrt{\epsilon} \leq |z'| \leq R$.

By (3.33),

$$\|\nabla w\|^2_{L^2(\widehat{\Omega}_\delta(z'))} = \int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2 \, dx \leq C|z'|^4.$$

By (3.23),

$$\delta \|\mathcal{L}_{\lambda,\mu} \tilde{u}\|_{L^\infty(\widehat{\Omega}_\delta(z'))} \leq \delta \left(\frac{C}{\epsilon + |z'|^2} + \frac{C|z'|}{(\epsilon + |z'|^2)^2} \right) \leq \frac{C}{|z'|}.$$

It follows from (3.45) that

$$|\nabla w(z',z_3)| \leq \frac{C|z'|^2}{\delta^{3/2}} + \frac{C}{|z'|} \leq \frac{C}{|z'|}, \qquad \forall \sqrt{\epsilon} \leq |z'| \leq R.$$

Case 2. (3.17) for $0 \leq |z'| \leq \sqrt{\epsilon}$.

Using (3.33), and (3.23), we have

$$\|\nabla w\|_{L^2(\widehat{\Omega}_\delta(z'))} \leq C\epsilon, \qquad \delta\,\|\mathcal{L}_{\lambda,\mu}\tilde{u}\|_{L^\infty(\widehat{\Omega}_\delta(z'))} \leq \frac{C|z'|}{\epsilon} + C,$$

and, using (3.45),

$$|\nabla w(z', z_3)| \leq \frac{C\epsilon}{\delta^{3/2}} + \frac{C|z'|}{\epsilon} + C \leq \frac{C}{\sqrt{\epsilon}}, \qquad \forall\;\; 0 \leq |z'| \leq \sqrt{\epsilon}.$$

STEP 4. Proof of (3.18) and (3.19).

Estimate (3.18) and (3.19) in Ω_R follows from (3.17) and (3.20).

Proposition 3.2 is established. □

3.2. Estimates of $|\nabla v_i^\alpha|$, $\alpha = 4, 5, 6$

Define

$$\bar{u}_1^\alpha = \bar{u}\psi^\alpha, \quad \text{and} \quad \bar{u}_2^\alpha = \underline{u}\psi^\alpha, \qquad \alpha = 4, 5, 6, \quad \text{in } \widetilde{\Omega}. \tag{3.46}$$

Clearly, $v_i^\alpha = \bar{u}_i^\alpha$ on $\partial\widetilde{\Omega}$, $i = 1, 2$, $\alpha = 4, 5, 6$.

Proposition 3.3. *Assume the above, let $v_i^\alpha \in H^1(\widetilde{\Omega}; \mathbb{R}^3)$ be the weak solution of (2.2) with $\alpha = 4, 5, 6$. Then for $i = 1, 2$, $\alpha = 4, 5, 6$,*

$$\int\limits_{\widetilde{\Omega}} |v_i^\alpha|^2\, dx + \int\limits_{\widetilde{\Omega}} |\nabla v_i^\alpha|^2\, dx \leq C, \tag{3.47}$$

and

$$\|\nabla v_i^\alpha\|_{L^\infty(\widetilde{\Omega}\setminus\Omega_R)} \leq C, \tag{3.48}$$

$$|\nabla(v_i^\alpha - \bar{u}_i^\alpha)(x', x_3)| \leq C, \qquad x \in \Omega_R. \tag{3.49}$$

Consequently,

$$|\nabla v_i^\alpha(x', x_3)| \leq \frac{C|x'|}{\epsilon + |x'|^2} + C, \qquad x \in \Omega_R. \tag{3.50}$$

Using (3.20) and (3.10), we have

$$|\nabla \bar{u}_i^\alpha(x)| \leq \frac{C|x'|}{\epsilon + |x'|^2} + C, \quad x \in \Omega_R, \tag{3.51}$$

and

$$|\nabla\bar{u}_i^\alpha(x)| \le C, \quad x \in \widetilde{\Omega} \setminus \Omega_R. \tag{3.52}$$

It follows from (3.46), (1.8), (3.20) and (3.22) that, for $i = 1, 2$, $\alpha = 4, 5, 6$,

$$|\mathcal{L}_{\lambda,\mu}\bar{u}_i^\alpha| \le C\left(|\nabla\bar{u}| + (\epsilon + |x'|)\sum_{k+l<6}|\partial_{x_k x_l}\bar{u}|\right) \le \frac{C}{\epsilon + |x'|^2}, \quad x \in \Omega_R. \tag{3.53}$$

Proof of Proposition 3.3. Denote

$$w_i^\alpha := v_i^\alpha - \bar{u}_i^\alpha, \qquad i = 1, 2, \ \alpha = 4, 5, 6. \tag{3.54}$$

For simplicity, we also use the notation

$$w := w_i^\alpha, \qquad \tilde{u} := \bar{u}_i^\alpha, \quad i = 1, 2, \quad \alpha = 4, 5, 6.$$

The proof is divided into three steps.

STEP 1. Proof of (3.47) and (3.48).

Similarly as Step 1 in the proof of Proposition 3.2, by (3.54) and (2.2) with $\alpha = 4, 5, 6$. Using (3.46), and (1.8), (3.30) again, (3.32) is replaced by

$$\begin{aligned}\int\limits_{\Omega_R} w\,(\mathcal{L}_{\lambda,\mu}\tilde{u})\,dx \le C\int\limits_{\Omega_R}|\nabla w||\nabla\tilde{u}|dx + \int\limits_{\substack{|x'|=R,\\ -\epsilon/2+h_2(x')<x_3<\epsilon/2+h_1(x')}} C\,|w| \\ \le C\left(\int\limits_{\Omega_R}|\nabla w|^2dx\right)^{1/2}\left(\int\limits_{\Omega_R}|\nabla\tilde{u}|^2dx\right)^{1/2} + C\left(\int\limits_{\widetilde{\Omega}\setminus\Omega_R}|\nabla w|^2\,dx\right)^{1/2}.\end{aligned} \tag{3.55}$$

Using (3.51), we have

$$\int\limits_{\Omega_R}|\nabla\tilde{u}|^2dx \le C. \tag{3.56}$$

It follows from (3.31) for this situation that

$$\int\limits_{\widetilde{\Omega}}|\nabla w|^2\,dx \le C\left(\int\limits_{\widetilde{\Omega}}|\nabla w|^2dx\right)^{1/2}.$$

This implies

$$\int_{\widetilde{\Omega}} |\nabla w|^2 \, dx \leq C.$$

By the Poincaré inequality,

$$\int_{\widetilde{\Omega}} |w|^2 \, dx + \int_{\widetilde{\Omega}} |\nabla w|^2 \, dx \leq C.$$

Combining with (3.56), we obtain (3.47).

Using (3.47) and recalling the definition of $\tilde{u}$, we apply the standard elliptic estimates (see [1]) to obtain (3.48).

STEP 2. Proof of

$$\int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2 dx \leq \begin{cases} C|z'|^6, & \sqrt{\epsilon} \leq |z'| < R, \\ C\epsilon^3, & 0 \leq |z'| < \sqrt{\epsilon}, \end{cases} \tag{3.57}$$

with $\delta = \delta(z')$ defined in (3.24).

The proof is similar to that of (3.33). We still have (3.35).

Case 1. Estimate (3.57) for $\sqrt{\epsilon} \leq |z'| \leq R$.

For $0 < t < s < \frac{2|z'|}{3}$, using (3.53), we have, instead of (3.37),

$$\int_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2 \, dx \leq \int_{|x'-z'|<s} \frac{C}{\epsilon + |x'|^2} \, dx' \leq \frac{Cs^2}{|z'|^2}. \tag{3.58}$$

Using (3.36), instead of (3.38), we have

$$\widehat{F}(t) \leq \left(\frac{C_0 |z'|^2}{s-t} \right)^2 \widehat{F}(s) + C(s-t)^2 \frac{s^2}{|z'|^2}, \quad \forall\, 0 < t < s < \frac{2|z'|}{3}. \tag{3.59}$$

We define $\{t_i\}$, k and iterate as in the proof of (3.33), right below formula (3.38), to obtain

$$\widehat{F}(t_0) \leq \left(\frac{1}{4}\right)^k \widehat{F}(\frac{3|z'|}{2}) + C|z'|^6 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} l^2 \leq C|z'|^6.$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2 dx \leq C|z'|^6.$$

Case 2. Estimate (3.57) for $0 \le |z'| \le \sqrt{\epsilon}$.

For $0 < t < s < \sqrt{\epsilon}$, estimate (3.39) remains the same. Estimate (3.40) becomes

$$\int_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2 \, dx \le \int_{|x'-z'|<s} \frac{C}{\epsilon + |x'|^2} \, dx' \le \frac{Cs^2}{\epsilon}, \qquad 0 < s < \sqrt{\epsilon}. \tag{3.60}$$

Estimate (3.41) becomes

$$F(t) \le \left(\frac{C_0\epsilon}{s-t}\right)^2 F(s) + \frac{C(s-t)^2 s^2}{\epsilon}, \quad \forall\ 0 < t < s < \sqrt{\epsilon}. \tag{3.61}$$

Define $\{t_i\}$, k and iterate as in the proof of (3.33), right below formula (3.41), to obtain

$$F(t_0) \le \left(\frac{1}{4}\right)^k \widehat{F}(\sqrt{\epsilon}) + C\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1} l^2\epsilon^3 \le C\epsilon^3.$$

This implies as before that

$$\int_{\widehat{\Omega}_s(z')} |\nabla w|^2 dx \le C\epsilon^3.$$

(3.57) is proved.

STEP 3. Proof of (3.49) and (3.50).

The proof is similar to that of (3.17). In Case 1, for $\sqrt{\epsilon} \le |z'| \le R$, using estimates (3.57) and (3.53),

$$\int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2 \, dx \le C|z'|^6, \quad \text{and} \quad \delta \, \|\mathcal{L}_{\lambda,\mu}\tilde{u}\|_{L^\infty(\widehat{\Omega}_\delta(z'))} \le C,$$

we obtain, using (3.45),

$$|\nabla w(z', z_3)| \le \frac{C|z'|^3}{\delta^{3/2}} + C \le C, \quad \text{for} \ \ \sqrt{\epsilon} \le |z'| \le R.$$

In Case 2, for $0 \le |z'| \le \sqrt{\epsilon}$, using estimates (3.57) and (3.53),

$$\int_{\widehat{\Omega}_\delta(z')} |\nabla w|^2 \, dx \le C\epsilon^3, \quad \text{and} \quad \delta \, \|\mathcal{L}_{\lambda,\mu}\tilde{u}\|_{L^\infty(\widehat{\Omega}_\delta(z'))} \le C,$$

we have, using again (3.45),

$$|\nabla w(z', z_3)| \leq \frac{C\epsilon^{3/2}}{\delta^{3/2}} + C \leq C, \quad \text{for} \ \ 0 \leq |z'| \leq \sqrt{\epsilon}.$$

Estimate (3.49) is established.

Estimate (3.50) follows from (3.49) and (3.51). □

4. Estimates of $|C_i|$ and $|C_1^\alpha - C_2^\alpha|$, $\alpha = 1, 2, 3$

In this section, we first prove that C_1^α and C_2^α are uniformly bounded with respect to ϵ, and then estimate the difference $C_1^\alpha - C_2^\alpha$.

Proposition 4.1. *Let C_i^α be defined in (2.1). Then*

$$|C_i^\alpha| \leq C, \qquad \forall\ i = 1, 2,\ \ \alpha = 1, 2, \cdots, 6; \tag{4.1}$$

and

$$|C_1^\alpha - C_2^\alpha| \leq \frac{C}{|\ln \epsilon|}, \quad \alpha = 1, 2, 3. \tag{4.2}$$

4.1. Boundedness of $|C_i|$

Proof of (4.1). Let u_ϵ be the solution of (1.6). By theorem 4.6 in the appendix in [13], u_ϵ is the minimizer of

$$I_\infty[u] := \frac{1}{2} \int\limits_{\widetilde{\Omega}} \Big(\mathbb{C}e(u), e(u) \Big)$$

on $\mathcal{A}$. It follows that

$$\|u_\epsilon\|^2_{H^1(\widetilde{\Omega})} \leq C \|e(u_\epsilon)\|^2_{L^2(\widetilde{\Omega})} \leq C I_\infty[u_\epsilon] \leq C.$$

By the Sobolev trace embedding theorem,

$$\|u_\epsilon\|_{L^2(\partial D_1 \setminus B_R)} \leq C.$$

Recalling that

$$u_\epsilon = \sum_{\alpha=1}^{6} C_1^\alpha \psi^\alpha, \qquad \text{on} \ \ \partial D_1.$$

If $C_1 := (C_1^1, C_1^2, \cdots, C_1^6)^T = 0$, there is nothing to prove. Otherwise

$$C \geq |C_1| \left\| \sum_{\alpha=1}^{6} \widehat{C}_1^\alpha \psi^\alpha \right\|_{L^2(\partial D_1 \setminus B_R)}, \tag{4.3}$$

where $\widehat{C}_1^\alpha = \frac{C_1^\alpha}{|C_1|}$ and $|\widehat{C}_1| = 1$. It is easy to see that there exists a universal constant $C > 0$ such that

$$\left\| \sum_{\alpha=1}^{6} \widehat{C}_1^\alpha \psi^\alpha \right\|_{L^2(\partial D_1 \setminus B_R)} \geq \frac{1}{C}. \tag{4.4}$$

Indeed, if not, along a subsequence $\epsilon \to 0$, $\widehat{C}_1^\alpha \to \overline{C}_1^\alpha$, and

$$\left\| \sum_{\alpha=1}^{6} \overline{C}_1^\alpha \psi^\alpha \right\|_{L^2(\partial D_1^* \setminus B_R)} = 0,$$

where ∂D_1^* is the limit of ∂D_1 as $\epsilon \to 0$ and $|\overline{C}_1| = 1$. This implies $\sum_{\alpha=1}^{6} \overline{C}_1^\alpha \psi^\alpha = 0$ on $\partial D_1^* \setminus B_R$. But $\left\{ \psi^\alpha |_{\partial D_1^* \setminus B_R} \right\}$ is easily seen to be linear independent, using Lemma 6.1, we must have $\overline{C}_1 = 0$. This is a contradiction. (4.1) follows from (4.3) and (4.4). □

4.2. Estimates of $|C_1^\alpha - C_2^\alpha|$, $\alpha = 1, 2, 3$

In the rest of this section, we prove (4.2). By the linearity of $e(u)$,

$$e(u) = \sum_{\alpha=1}^{6} C_1^\alpha e\left(v_1^\alpha\right) + \sum_{\alpha=1}^{6} C_2^\alpha e(v_2^\alpha) + e(v_0), \quad \text{in } \widetilde{\Omega}.$$

It follows from the fourth line of (1.6) that

$$\sum_{\alpha=1}^{6} C_1^\alpha \int_{\partial D_j} \frac{\partial v_1^\alpha}{\partial \nu_0}\bigg|_+ \cdot \psi^\beta + \sum_{\alpha=1}^{6} C_2^\alpha \int_{\partial D_j} \frac{\partial v_2^\alpha}{\partial \nu_0}\bigg|_+ \cdot \psi^\beta + \int_{\partial D_j} \frac{\partial v_0}{\partial \nu_0}\bigg|_+ \cdot \psi^\beta = 0,$$
$$j = 1, 2, \quad \beta = 1, 2, \cdots, 6. \tag{4.5}$$

Denote

$$a_{ij}^{\alpha\beta} = -\int_{\partial D_j} \frac{\partial v_i^\alpha}{\partial \nu_0}\bigg|_+ \cdot \psi^\beta, \qquad b_j^\beta = \int_{\partial D_j} \frac{\partial v_0}{\partial \nu_0}\bigg|_+ \cdot \psi^\beta, \quad i, j = 1, 2, \quad \alpha, \beta = 1, 2, \cdots, 6.$$

Multiplying the first line of (2.2) and (2.3), respectively, by v_j^β, and integrating by parts over $\widetilde{\Omega}$ leads to, in view of (1.7), that

$$a_{ij}^{\alpha\beta} = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e(v_i^\alpha), e(v_j^\beta) \right) dx, \quad b_j^\beta = -\int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e(v_0), e(v_j^\beta) \right) dx.$$

Then (4.5) can be written as

$$\begin{cases} \sum_{\alpha=1}^{6} C_1^{\alpha} a_{11}^{\alpha\beta} + \sum_{\alpha=1}^{6} C_2^{\alpha} a_{21}^{\alpha\beta} - b_1^{\beta} = 0, \\ \sum_{\alpha=1}^{6} C_1^{\alpha} a_{12}^{\alpha\beta} + \sum_{\alpha=1}^{6} C_2^{\alpha} a_{22}^{\alpha\beta} - b_2^{\beta} = 0, \end{cases} \qquad \beta = 1, 2, \cdots, 6. \tag{4.6}$$

For simplicity, we use a_{ij} to denote the 6×6 matrix $(a_{ij}^{\alpha\beta})$. To estimate $|C_1^{\alpha} - C_2^{\alpha}|$, $\alpha = 1, 2, 3$, we only use the first six equations in (4.6):

$$a_{11} C_1 + a_{21} C_2 = b_1, \tag{4.7}$$

where

$$C_1 = \Big(C_1^1, C_1^2, \cdots, C_1^6\Big)^T, \quad C_2 = \Big(C_2^1, C_2^2, \cdots, C_2^6\Big)^T, \quad b_1 = \Big(b_1^1, b_1^2, \cdots, b_1^6\Big)^T.$$

Set

$$p := b_1 - \Big(a_{11} + a_{21}\Big) C_2,$$

(4.7) can be rewritten as

$$a_{11}\Big(C_1 - C_2\Big) = p. \tag{4.8}$$

In order to prove (4.2), we first estimate the right hand side of (4.8).

Lemma 4.2.

$$\Big|a_{11}^{\alpha\beta} + a_{21}^{\alpha\beta}\Big| \leq C, \quad \alpha, \beta = 1, 2, \cdots, 6;$$
$$\Big|b_1^{\beta}\Big| \leq C, \quad \beta = 1, 2, \cdots, 6.$$

Consequently,

$$|p| \leq C. \tag{4.9}$$

Proof. For $\beta = 1, 2, 3$, using (3.16) and (3.18), we have

$$\int_{\widetilde{\Omega}} \Big|\nabla v_1^{\beta}\Big| dx \leq \int_{\Omega_R} \Big|\nabla v_1^{\beta}\Big| dx + \int_{\widetilde{\Omega} \setminus \Omega_R} \Big|\nabla v_1^{\beta}\Big| dx \leq \int_{\Omega_R} \frac{C dx}{\epsilon + |x'|^2} + C \leq C. \tag{4.10}$$

For $\beta = 4, 5, 6$, using (3.48) and (3.50), we have

$$\int_{\widetilde{\Omega}} \Big|\nabla v_1^{\beta}\Big| dx \leq \int_{\Omega_R} \frac{C(\epsilon + |x'|) dx}{\epsilon + |x'|^2} + C \leq C. \tag{4.11}$$

For $\alpha, \beta = 1, 2, \cdots, 6$, using (2.7), (4.10) and (4.11), we have

$$\left|a_{11}^{\alpha\beta} + a_{21}^{\alpha\beta}\right| = \left|\int\limits_{\tilde{\Omega}} \left(\mathbb{C}^0 e(v_1^\alpha + v_2^\alpha), e(v_1^\beta)\right) dx\right| \leq C \left\|\nabla(v_1^\alpha + v_2^\alpha)\right\|_{L^\infty(\tilde{\Omega})} \int\limits_{\tilde{\Omega}} \left|\nabla v_1^\beta\right| dx \leq C.$$

Similarly, it follows from (2.6), (4.10) and (4.11) that

$$\left|b_1^\beta\right| = \left|\int\limits_{\tilde{\Omega}} \left(\mathbb{C}^0 e(v_1^\beta), e(v_0)\right) dx\right| \leq C\|\nabla v_0\|_{L^\infty(\tilde{\Omega})} \int\limits_{\tilde{\Omega}} \left|\nabla v_1^\beta\right| dx \leq C, \quad \beta = 1, 2, \cdots, 6.$$

These estimates above, combining with (4.1), yield (4.9). $\square$

It can be proved that a_{11} is positive definite and therefore, recalling (4.8),

$$C_1 - C_2 = (a_{11})^{-1} p.$$

Given (4.9), estimate (4.2) would follow from the above if $\|(a_{11})^{-1}\| \leq \frac{C}{|\ln \epsilon|}$. However $\|(a_{11})^{-1}\| \geq \frac{1}{C} > 0$. We need to make more delicate estimate as below.

In view of the symmetry of $a_{11}^{\alpha\beta}$, we write it as a block matrix

$$a_{11} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11}^{14} & a_{11}^{15} & a_{11}^{16} \\ a_{11}^{24} & a_{11}^{25} & a_{11}^{26} \\ a_{11}^{34} & a_{11}^{35} & a_{11}^{36} \end{pmatrix}, \quad \text{and } D = \begin{pmatrix} a_{11}^{44} & a_{11}^{45} & a_{11}^{46} \\ a_{11}^{54} & a_{11}^{55} & a_{11}^{56} \\ a_{11}^{64} & a_{11}^{65} & a_{11}^{66} \end{pmatrix}.$$

Lemma 4.3. a_{11} *is positive definite, and*

$$\frac{|\ln \epsilon|}{C} \leq a_{11}^{\alpha\alpha} \leq C|\ln \epsilon|, \qquad \alpha = 1, 2, 3; \tag{4.12}$$

$$\frac{1}{C} \leq a_{11}^{\alpha\alpha} \leq C, \quad \alpha = 4, 5, 6; \tag{4.13}$$

and

$$\left|a_{11}^{\alpha\beta}\right| = \left|a_{11}^{\beta\alpha}\right| \leq C, \qquad \alpha, \beta = 1, 2, \cdots, 6, \ \alpha \neq \beta. \tag{4.14}$$

Moreover,

$$\frac{1}{C}I \leq D \leq CI, \tag{4.15}$$

where I is the 3×3 identity matrix, and

$$\frac{|\ln \epsilon|^3}{C} \leq \det a_{11} \leq C|\ln \epsilon|^3. \tag{4.16}$$

Remark 4.1. Roughly speaking, the estimates of A and B in Lemma 4.3 is that, for some positive constants c_1, c_2, c_3, independent of ϵ,

$$A \sim \begin{pmatrix} c_1|\ln \epsilon| & O(1) & O(1) \\ O(1) & c_2|\ln \epsilon| & O(1) \\ O(1) & O(1) & c_3|\ln \epsilon| \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} O(1) & O(1) & O(1) \\ O(1) & O(1) & O(1) \\ O(1) & O(1) & O(1) \end{pmatrix}.$$

We postpone the proof of Lemma 4.3 and first make use of it to prove (4.2).

Proof of (4.2). For convenience, we introduce notations

$$X_1 := \left(C_1^1 - C_2^1, C_1^2 - C_2^2, C_1^3 - C_2^3\right)^T, \quad X_2 := \left(C_1^4 - C_2^4, C_1^5 - C_2^5, C_1^6 - C_2^6\right)^T,$$

and

$$P_1 := (p_1, p_2, p_3)^T, \quad P_2 := (p_4, p_5, p_6)^T.$$

Now (4.8) can be rewritten as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}. \tag{4.17}$$

By Lemma 4.2 and Lemma 4.3, matrices A, B, D satisfy the assumptions of Lemma 6.2 in Appendix with $m = 3$, $\gamma = |\ln \epsilon|$ and $\theta = \frac{1}{C}$. Applying Lemma 6.2, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + O(\frac{1}{|\ln \epsilon|^2}) & O(\frac{1}{|\ln \epsilon|}) \\ * & * \end{pmatrix}.$$

It follows from (4.17) that

$$|X_1| = \sqrt{\sum_{\alpha=1}^{3} |C_1^\alpha - C_2^\alpha|} \leq \frac{C}{|\ln \epsilon|}.$$

Thus, the proof of Proposition 4.1 is completed. □

We are now in position to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. Estimates (2.6)–(2.7) have been proved in Lemma 3.1. Under assumption (1.2),

$$\frac{1}{C}(\epsilon+|x'|^2)\leq \epsilon+\mathrm{dist}^2(x,\overline{P_1P_2})\leq C(\epsilon+|x'|^2),\qquad x\in\Omega_R.$$

Estimate (2.8) in Ω_R follows from (3.17) and (3.21). Thus, using (3.16), (2.8) is proved. Combining (3.50) and (3.48) yields estimate (2.9). Estimate (2.10) and estimate (2.11) has been proved in Proposition 4.1. The proof of Proposition 2.1 is finished. □

Proof of Lemma 4.3. STEP 1. Proof of (4.12) and (4.13).

For any $\xi=(\xi_1,\xi_2,\cdots,\xi_6)^T\neq 0$, by (2.17),

$$\xi^T a_{11}\xi=\int\limits_{\widetilde{\Omega}}\left(\mathbb{C}^0 e\left(\xi_\alpha v_1^\alpha\right),e\left(\xi_\beta v_1^\beta\right)\right)dx\geq\frac{1}{C}\int\limits_{\widetilde{\Omega}}\left|e\left(\xi_\alpha v_1^\alpha\right)\right|^2dx>0. \tag{4.18}$$

In the last inequality we have used the fact that $e\left(\xi_\alpha v_1^\alpha\right)$ is not identically zero in $\widetilde{\Omega}$. Indeed, if $e\left(\xi_\alpha v_1^\alpha\right)=0$ in $\widetilde{\Omega}$, then $\sum_{\alpha=1}^6\xi_\alpha v_1^\alpha=\sum_{i=1}^6 a_i\psi^i$ in $\widetilde{\Omega}$, for some constants a_i, $i=1,2,\cdots,6$. On the other hand, $\sum_{i=1}^6\xi_\alpha v_1^\alpha=0$ on ∂D_2, so by Lemma 6.1, $a_i=0$, $\forall\, i$. Thus on ∂D_1, $\sum_{i=1}^6\xi_\alpha v_1^\alpha=\sum_{i=1}^6\xi_\alpha\psi^\alpha=0$, which implies, again using Lemma 6.1, that $\xi=0$. A contradiction. (4.18) implies that a_{11} is positive definite.

By (2.16) and (2.8), we have

$$a_{11}^{\alpha\alpha}=\int\limits_{\widetilde{\Omega}}\left(\mathbb{C}^0 e\left(v_1^\alpha\right),e\left(v_1^\alpha\right)\right)dx\leq C\int\limits_{\widetilde{\Omega}}\left|\nabla v_1^\alpha\right|^2dx\leq C|\ln\epsilon|,\qquad \alpha=1,2,3.$$

By (2.16) again, we have

$$a_{11}^{\alpha\alpha}=\int\limits_{\widetilde{\Omega}}\left(\mathbb{C}^0 e\left(v_1^\alpha\right),e\left(v_1^\alpha\right)\right)dx\geq\frac{1}{C}\int\limits_{\widetilde{\Omega}}\left|e\left(v_1^\alpha\right)\right|^2dx\geq\frac{1}{C}\int\limits_{\Omega_R}\left|\partial_{x_3}(v_1^\alpha)_\alpha\right|^2dx,\quad \alpha=1,2,3.$$

Notice that $(v_1^\alpha)_\alpha|_{\partial D_1}=\bar u|_{\partial D_1}=1$, $(v_1^\alpha)_\alpha|_{\partial D_2}=\bar u|_{\partial D_2}=0$, and recalling the definition of $\bar u$, (3.9), $\bar u(x',x_3)$ is linear in x_3 for fixed x', so $\bar u(x',\cdot)$ is harmonic, hence its energy is minimal, that is,

$$\int\limits_{h_2(x')-\frac{\epsilon}{2}}^{h_1(x')+\frac{\epsilon}{2}}\left|\partial_{x_3}(v_1^\alpha)_\alpha\right|^2dx_3\geq\int\limits_{h_2(x')-\frac{\epsilon}{2}}^{h_1(x')+\frac{\epsilon}{2}}|\partial_{x_3}\bar u|^2dx_3=\frac{1}{\epsilon+h_1(x')-h_2(x')}.$$

Integrating on $|x'|<R$, we obtain

$$\int\limits_{\Omega_R}\left|\partial_{x_3}(v_1^\alpha)_\alpha\right|^2dx=\int\limits_{|x'|<R}\int\limits_{h_2(x')-\frac{\epsilon}{2}}^{h_1(x')+\frac{\epsilon}{2}}\left|\partial_{x_3}(v_1^\alpha)_\alpha\right|^2dx_3dx_2dx_1$$

$$\geq \frac{1}{C} \int\limits_{|x'|<R} \frac{dx'}{\epsilon + |x'|^2} \geq \frac{|\ln \epsilon|}{C}.$$

Thus,

$$a_{11}^{\alpha\alpha} \geq \frac{|\ln \epsilon|}{C}, \quad \alpha = 1, 2, 3. \tag{4.19}$$

Estimate (4.12) is proved.

By (3.47), we have

$$a_{11}^{\alpha\alpha} = \int\limits_{\tilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^\alpha\right), e\left(v_1^\alpha\right)\right) dx \leq C, \quad \alpha = 4, 5, 6.$$

By the same argument, the claim (4.18) in [13] for higher dimensions still holds. Therefore

$$a_{11}^{\alpha\alpha} \geq \frac{1}{C} \int\limits_{\Omega_R \backslash \Omega_R} |e\left(v_1^\alpha\right)|^2 dx \geq \frac{1}{C} \int\limits_{\Omega_R \backslash \Omega_R} |\nabla v_1^\alpha|^2 dx \geq \frac{1}{C}, \quad \alpha = 4, 5, 6.$$

Estimate (4.13) is proved.

STEP 2. We deal with the cases $\alpha \neq \beta$. Proof of (4.14).

By definition,

$$a_{11}^{\alpha\beta} = a_{11}^{\beta\alpha} = \int\limits_{\tilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^\alpha\right), e(v_1^\beta)\right) dx = -\int\limits_{\partial D_1} \frac{\partial v_1^\alpha}{\partial \nu_0}\bigg|_+ \cdot \psi^\beta \, dS.$$

First,

$$\begin{aligned} a_{11}^{12} &= -\int\limits_{\partial D_1} \frac{\partial v_1^1}{\partial \nu_0}\bigg|_+ \cdot \psi^2 \, dS \\ &= -\int\limits_{\partial D_1} \left(\lambda\left(\nabla \cdot v_1^1\right) n_2 + \mu\left(\left(\nabla v_1^1 + (\nabla v_1^1)^T\right)\vec{n}\right)_2\right) dS \\ &= -\int\limits_{\partial D_1} \left(\lambda\Big(\sum_{k=1}^3 \partial_{x_k} (v_1^1)_k\Big) n_2 + \mu \sum_{l=1}^3 \left(\partial_{x_2} (v_1^1)_l + \partial_{x_l} (v_1^1)_2\right) n_l\right) dS. \end{aligned}$$

We only need to estimate the integral on the part $\partial D_1 \cap B_R$, because the rest is bounded. On boundary $\partial D_1 \cap B_R$, we have

$$\vec{n} = \frac{1}{\sqrt{1 + |\nabla_{x'} h_1|^2}} \Big(-\partial_{x_1} h_1, -\partial_{x_2} h_1, 1\Big).$$

Clearly, using (3.2)–(3.4),

$$|n_1| \leq C|x'|, \quad |n_2| \leq C|x'|, \quad n_3 = \frac{1}{\sqrt{1+|\nabla_{x'}h_1|^2}}.$$

Combining with the estimates (3.18), we have

$$\int\limits_{\partial D_1\cap B_R} \Big| \Big(\sum_{k=1}^{3} \partial_{x_k}(v_1^1)_k\Big) n_2 \Big| dS \leq \int\limits_{\partial D_1\cap B_R} \frac{C|x'|}{\epsilon+|x'|^2}\, dS \leq C.$$

Using the definition of $\bar{u}_1^1$, estimates (3.17), (3.18), (3.19), and

$$\Big|\int\limits_{\partial D_1\cap B_R} \partial_{x_3}(v_1^1)_2 n_3\, dS\Big| \leq \Big|\int\limits_{\partial D_1\cap B_R} \partial_{x_3}(\bar{u}_1^1)_2 n_3\, dS\Big| + \Big|\int\limits_{\partial D_1\cap B_R} \partial_{x_3}(w_1^1)_2 n_3\, dS\Big| \leq C,$$

we obtain

$$\int\limits_{\partial D_1\cap B_R} \Big| \sum_{l=1}^{3} \partial_{x_2}(v_1^1)_l n_l + \sum_{l=1}^{3} \partial_{x_l}(v_1^1)_2 n_l \Big| dS \leq C.$$

Therefore

$$|a_{11}^{12}| = |a_{11}^{21}| \leq C.$$

By the same way

$$\begin{aligned} a_{11}^{13} &= -\int\limits_{\partial D_1} \Big(\lambda\Big(\nabla\cdot v_1^1\Big)n_3 + \mu\Big(\Big(\nabla v_1^1 + (\nabla v_1^1)^T\Big)\vec{n}\Big)_3\Big)\, dS \\ &= -\int\limits_{\partial D_1} \Big(\lambda\Big(\sum_{i=1}^{3} \partial_{x_k}(v_1^1)_k\Big)n_3 + \mu\sum_{l=1}^{3}\Big(\partial_{x_3}(v_1^1)_l + \partial_{x_l}(v_1^1)_3\Big)n_l\Big)\, dS. \end{aligned}$$

For the terms $\partial_{x_k}(v_1^1)_l$, $k=1,2, l=1,2,3$, use the estimates (3.19), for $k=l=3$, use the definition $\bar{u}_1^1$ and the estimates (3.17) to obtain

$$\Big|\int\limits_{\partial D_1} \partial_{x_3}(v_1^1)_3 n_3\, dS\Big| = \Big|\int\limits_{\partial D_1} \partial_{x_3}(\bar{u}_1^1)_3 n_3\, dS\Big| + \Big|\int\limits_{\partial D_1} \partial_{x_3}(w_1^1)_3 n_3\, dS\Big| \leq C.$$

Therefore

$$|a_{11}^{13}| = |a_{11}^{31}| \leq C.$$

By the definition and the same reason,

$$a_{11}^{14} = -\int\limits_{\partial D_1} \left(\lambda\Big(\nabla\cdot v_1^1\Big)\vec{n} + \mu\Big(\nabla v_1^1 + (\nabla v_1^1)^T\Big)\vec{n}\right)\cdot\begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} dS$$

$$= -\int\limits_{\partial D_1} \left(\lambda\Big(\sum_{k=1}^{3}\partial_{x_k}(v_1^1)_k\Big)n_1 + \mu\sum_{l=1}^{3}\Big(\partial_{x_1}(v_1^1)_l + \partial_{x_l}(v_1^1)_1\Big)n_l\right) x_2\, dS$$

$$+ \int\limits_{\partial D_1} \left(\lambda\Big(\sum_{k=1}^{3}\partial_{x_k}(v_1^1)_k\Big)n_2 + \mu\sum_{l=1}^{3}\Big(\partial_{x_2}(v_1^1)_l + \partial_{x_l}(v_1^1)_2\Big)n_l\right) x_1\, dS$$

is bounded. $a_{11}^{15} = a_{11}^{51}$ and $a_{11}^{16} = a_{11}^{61}$ are also bounded, essentially the same as above.

$$a_{11}^{23} = -\int\limits_{\partial D_1} \left(\lambda\Big(\nabla\cdot v_1^2\Big)n_3 + \mu\Big(\Big(\nabla v_1^1 + (\nabla v_1^2)^T\Big)\vec{n}\Big)_3\right) dS$$

$$= -\int\limits_{\partial D_1} \left(\lambda\Big(\sum_{k=1}^{3}\partial_{x_k}(v_1^2)_k\Big)n_3 + \mu\sum_{l=1}^{3}\Big(\partial_{x_3}(v_1^2)_l + \partial_{x_l}(v_1^2)_3\Big)n_l\right) dS$$

is the same as a_{11}^{12}. While $a_{11}^{24} = a_{11}^{42}$ and $a_{11}^{34} = a_{11}^{43}$ are the same as a_{11}^{14}. $a_{11}^{25} = a_{11}^{52}$ and $a_{11}^{26} = a_{11}^{62}$, $a_{11}^{35} = a_{11}^{53}$ and $a_{11}^{36} = a_{11}^{63}$ are all the same.

$$a_{11}^{45} = -\int\limits_{\partial D_1} \left(\lambda\Big(\nabla\cdot v_1^4\Big)\vec{n} + \mu\Big(\nabla v_1^1 + (\nabla v_1^1)^T\Big)\vec{n}\right)\cdot\begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} dS$$

$$= -\int\limits_{\partial D_1} \left(\lambda\Big(\sum_{k=1}^{3}\partial_{x_k}(v_1^4)_k\Big)n_1 + \mu\sum_{l=1}^{3}\Big(\partial_{x_1}(v_1^4)_l + \partial_{x_l}(v_1^4)_1\Big)n_l\right) x_3\, dS$$

$$+ \int\limits_{\partial D_1} \left(\lambda\Big(\sum_{k=1}^{3}\partial_{x_k}(v_1^4)_k\Big)n_3 + \mu\sum_{l=1}^{3}\Big(\partial_{x_2}(v_1^4)_l + \partial_{x_l}(v_1^4)_3\Big)n_l\right) x_1\, dS$$

is much better. $a_{11}^{56} = a_{11}^{65}$ is the same as a_{11}^{45}. Estimate (4.14) is proved.

STEP 3. We will show

$$D \geq \frac{1}{C} I$$

for some constant C, independent of ϵ.

For $\xi \in \mathbb{R}^3$, $|\xi| = 1$, using (1.9), we have

$$\sum_{\alpha,\beta=4,5,6} a_{11}^{\alpha\beta}\xi_\alpha\xi_\beta = \int\limits_{\widetilde{\Omega}} \Big(\mathbb{C}^0 e\Big(\sum_{\alpha=4}^{6}\xi_\alpha v_1^\alpha\Big), e\Big(\sum_{\beta=4}^{6}\xi_\beta v_1^\beta\Big)\Big)dx \geq \frac{1}{C}\int\limits_{\widetilde{\Omega}} \Big|e\Big(\sum_{\alpha=4}^{6}\xi_\alpha v_1^\alpha\Big)\Big|^2 dx.$$

We claim that there exists a constant $C > 0$, independent of ϵ, such that

$$\int_{\widetilde{\Omega}} \Big| e\Big(\sum_{\alpha=4}^{6} \xi_\alpha v_1^\alpha\Big)\Big|^2 dx \geq \frac{1}{C}, \quad \forall\, \xi \in \mathbb{R}^3,\ |\xi| = 1. \tag{4.20}$$

Indeed, if not, then there exists $\epsilon_i \to 0^+$, $|\xi^i| = 1$, such that

$$\int_{\widetilde{\Omega}_{\epsilon_i}} \Big| e\Big(\sum_{\alpha=4}^{6} \xi_\alpha^i v_1^\alpha\Big)\Big|^2 dx \to 0. \tag{4.21}$$

Here and in the following proof we use the notations $D_1^* := (0, 0, -\frac{\epsilon}{2}) + D_1$, $D_2^* := (0, 0, \frac{\epsilon}{2}) + D_2$, $\widetilde{\Omega}^* := \Omega \setminus \overline{D_1^* \cup D_2^*}$, and $\widetilde{\Omega}_\epsilon = \Omega \setminus \overline{D_1 \cup D_2}$. The corresponding solution of (2.2) with $\alpha = 4, 5, 6$ is denoted as $v_1^\alpha(\epsilon)$. Since $v_1^\alpha(\epsilon_i) = 0$ on ∂D_2, it follows from the second Korn's inequality (see theorem 2.5 in [38]) that there exists a constant C, independent of ϵ, such that

$$\|v_1^\alpha(\epsilon_i)\|_{H^1(\widetilde{\Omega}_{\epsilon_i} \setminus B_{\bar{r}})} \leq C,$$

where $\bar{r} > 0$ is fixed. Then there exists a subsequence, we still denote $\{v_1^\alpha(\epsilon_i)\}$, such that

$$v_1^\alpha(\epsilon_i) \rightharpoonup \bar{v}_1^\alpha, \quad \text{in } H^1(\widetilde{\Omega}_{\epsilon_i} \setminus B_{\bar{r}}), \quad \text{as } \epsilon_i \to 0.$$

By the assumption (4.21), there exists $\bar{\xi}$ such that

$$\xi^i \to \bar{\xi}, \quad \text{as } \epsilon_i \to 0, \quad \text{with } |\bar{\xi}| = 1,$$

and

$$\int_{\widetilde{\Omega}^*} \Big| e\Big(\sum_{\alpha=4}^{6} \bar{\xi}_\alpha \bar{v}_1^\alpha\Big)\Big|^2 = 0.$$

This implies that

$$e\Big(\sum_{\alpha=4}^{6} \bar{\xi}_\alpha \bar{v}_1^\alpha\Big) = 0, \quad \text{in } \widetilde{\Omega}^*.$$

Hence, for some constants $\{b_\beta\}$, $\sum_{\alpha=4}^{6} \bar{\xi}_\alpha \bar{v}_1^\alpha = \sum_{\beta=1}^{6} b_\beta \psi^\beta$ in $\widetilde{\Omega}^*$. Since $\sum_{\beta=1}^{6} b_\beta \psi^\beta = 0$, on ∂D_2^*, it follows from Lemma 6.1 that $b_\beta = 0$, $\forall\ \beta$. Thus, $\sum_{\alpha=4}^{6} \bar{\xi}_\alpha \bar{v}_1^\alpha = 0$ in $\widetilde{\Omega}^*$. Restricted on ∂D_1^*, it says that $\sum_{\alpha=4}^{6} \bar{\xi}_\alpha \psi^\alpha = 0$ on ∂D_1^*. This yields, using again Lemma 6.1, $\bar{\xi}_\alpha = 0$, $\alpha = 4, 5, 6$, which contradicts with $|\bar{\xi}| = 1$.

(4.16) is immediately proved by using (4.12) and (4.14). The proof of Lemma 4.3 is finished. □

5. The proof of Theorem 1.2

Define v_i^α and v_0 by (2.2) and (2.3). By a decomposition similar to (2.1),

$$\nabla u = \sum_{i=1}^{2} \sum_{\alpha=1}^{\frac{d(d+1)}{2}} C_i^\alpha \nabla v_i^\alpha + \nabla v_0, \quad \text{in } \widetilde{\Omega}. \tag{5.1}$$

It follows that

$$|\nabla u| \leq \sum_{i=1}^{2} \sum_{\alpha=1}^{\frac{d(d+1)}{2}} |C_i^\alpha| \, |\nabla v_i^\alpha| + \big|\nabla v_0\big|, \quad \text{in } \widetilde{\Omega}. \tag{5.2}$$

As in Section 3, we write $x = (x', x_d)$, and let P_1, P_2, R be the same as in Section 3, and, instead of (3.1),

$$x_d = \frac{\epsilon}{2} + h_1(x'), \quad \text{and} \quad x_d = -\frac{\epsilon}{2} + h_2(x'), \quad \text{for } |x'| < 2R.$$

$\widehat{\Omega}_s(z')$ and $\Omega_s = \Omega_s(0')$ are defined accordingly. $\bar{u}$, $\underline{u}$ and $\bar{u}_i^\alpha$ are defined as in (3.9), (3.13), (3.11) and (3.12), with x_3 replaced by x_d, and $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$. We still have (3.10) and (3.14).

Proposition 5.1. *Assume the above, let $v_i^\alpha \in H^1(\widetilde{\Omega}; \mathbb{R}^d)$ be the weak solution of (2.2) with $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$. Then for $i = 1, 2,\ \alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$,*

$$\int_{\widetilde{\Omega}} |\nabla (v_i^\alpha - \bar{u}_i^\alpha)|^2 \, dx \leq C; \tag{5.3}$$

and

$$\|\nabla v_i^\alpha\|_{L^\infty(\widetilde{\Omega} \setminus \Omega_R)} \leq C, \tag{5.4}$$

$$|\nabla (v_i^\alpha - \bar{u}_i^\alpha)(x)| \leq \frac{C}{\epsilon + |x'|^2}, \qquad \forall \ x \in \Omega_R. \tag{5.5}$$

Consequently,

$$|\nabla v_i^\alpha(x)| \leq \frac{C}{\epsilon + |x'|^2}, \qquad \forall \ x \in \Omega_R. \tag{5.6}$$

Proof. The proof is similar to that of Proposition 3.2, and we only point out the main difference. The proof of (5.3) and (5.4) are the same as that of (3.15) and (3.16). We prove (5.5).

(i) For $\alpha = 1, 2, \cdots, d$, the same as (3.21),

$$|\nabla \bar{u}_i^\alpha(x)| \le \frac{C}{\epsilon + |x'|^2}, \qquad x \in \Omega_R, \tag{5.7}$$

and, instead of (3.23),

$$|\mathcal{L}_{\lambda,\mu}\bar{u}_i^\alpha(x)| \le C \sum_{k+l<2d} |\partial_{x_k x_l}\bar{u}(x)| \le \frac{C}{\epsilon + |x'|^2} + \frac{C|x'|}{(\epsilon + |x'|^2)^2}, \qquad x \in \Omega_R. \tag{5.8}$$

Using (5.8), we have, instead of (3.37) and (3.38), for $\sqrt{\epsilon} < |z'| < R$, $0 < s < \frac{2|z'|}{3}$,

$$\int\limits_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\bar{u}_i^\alpha|^2 \, dx \le C \int\limits_{|x'-z'|<s} \left(\frac{1}{\epsilon + |x'|^2} + \frac{|x'|^2}{(\epsilon + |x'|^2)^3} \right) dx' \le \frac{Cs^{d-1}}{|z'|^4}, \tag{5.9}$$

and denoting $\widehat{F}(t) := \int_{\widehat{\Omega}_t(z')} |\nabla(v_i^\alpha - \bar{u}_i^\alpha)|^2 dx$,

$$\widehat{F}(t) \le \left(\frac{C_0|z'|^2}{s-t} \right)^2 \widehat{F}(s) + C(s-t)^2 \frac{s^{d-1}}{|z'|^4}, \qquad \forall\, 0 < t < s < \frac{2|z'|}{3}. \tag{5.10}$$

Similar as Case 1 of Step 2 in the proof of Proposition 3.2, set $t_i = \delta + 2C_0 i|z'|^2$, $i = 0, 1, 2, \cdots$, and let $k = \left[\frac{1}{4C_0|z'|} \right]$. Using (5.10) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \le \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C(t_{i+1} - t_i)^2 t_{i+1}^{d-1}}{|z'|^4} \le \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)^2|z'|^{2(d-1)},$$
$$i = 1, 2, \cdots, k.$$

After k iterations, we obtain

$$\int\limits_{\widehat{\Omega}_\delta(z')} |\nabla(v_i^\alpha - \bar{u}_i^\alpha)|^2 dx = \widehat{F}(t_0) \le C|z'|^{2(d-1)}, \quad \forall\, \sqrt{\epsilon} < |z'| < R.$$

Instead of (3.40) and (3.41), using (5.8), for $0 \le |z'| < \sqrt{\epsilon}$, $0 < s < \sqrt{\epsilon}$,

$$\int\limits_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2 \le \int\limits_{|x'-z'|<s} \left(\frac{C}{\epsilon + |x'|^2} + \frac{C|x'|^2}{(\epsilon + |x'|^2)^3} \right) dx' \le \frac{Cs^{d-1}}{\epsilon^2}, \tag{5.11}$$

and

$$\widehat{F}(t) \leq \left(\frac{C_0\epsilon}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2\frac{s^{d-1}}{\epsilon^2}, \quad \forall\, 0 < t < s < \sqrt{\epsilon}. \tag{5.12}$$

Let $t_i = \delta + 2C_0 i\epsilon$, $i = 0, 1, 2, \cdots$ and $k = \left[\frac{1}{4C_0\sqrt{\epsilon}}\right]$. By (5.12) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C\epsilon^2 t_{i+1}^{d-1}}{\epsilon^2} \leq \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)^2\epsilon^{d-1}, \quad i = 1, 2, \cdots, k.$$

After k iterations, we have

$$\int\limits_{\widehat{\Omega}_\delta(z')} |\nabla(v_i^\alpha - \bar{u}_i^\alpha)|^2 dx = \widehat{F}(t_0) \leq C\epsilon^{d-1}, \quad \forall\, 0 \leq |z'| < \sqrt{\epsilon}.$$

Therefore, we have, instead of (3.33),

$$\int\limits_{\widehat{\Omega}_\delta(z')} |\nabla w_i^\alpha|^2\, dx \leq \begin{cases} C\epsilon^{d-1}, & 0 \leq |z'| \leq \sqrt{\epsilon}, \\ C|z'|^{2(d-1)}, & \sqrt{\epsilon} < |z'| \leq R. \end{cases} \tag{5.13}$$

As in Step 3 of the proof of Proposition 3.2, we have, instead of (3.45),

$$\|\nabla(v_i^\alpha - \bar{u}_i^\alpha)\|_{L^\infty(\widehat{\Omega}_{\delta/2}(z'))} \leq \frac{C}{\delta}\left(\delta^{1-\frac{d}{2}}\|\nabla(v_i^\alpha - \bar{u}_i^\alpha)\|_{L^2(\widehat{\Omega}_\delta(z'))} + \delta^2\|\mathcal{L}_{\lambda,\mu}\bar{u}_i^\alpha\|_{L^\infty(\widehat{\Omega}_\delta(z'))}\right). \tag{5.14}$$

Using (5.13) and (5.8), we obtain

$$|\nabla(v_i^\alpha - \bar{u}_i^\alpha)(x)| \leq \begin{cases} \frac{C}{\sqrt{\epsilon}}, & |x'| \leq \sqrt{\epsilon}, \\ \frac{C}{|x'|}, & \sqrt{\epsilon} < |x'| \leq R. \end{cases} \tag{5.15}$$

Consequently, (5.6) follows from (5.7) immediately.

(ii) For $d \leq \alpha \leq \frac{d(d+1)}{2}$, we have

$$|\nabla\bar{u}_i^\alpha(x)| \leq \frac{C|x'|}{\epsilon + |x'|^2} + C, \qquad x \in \Omega_R, \tag{5.16}$$

and, instead of (3.53),

$$|\mathcal{L}_{\lambda,\mu}\bar{u}_i^\alpha| \leq C\left(|\nabla\bar{u}| + (\epsilon + |x'|)\sum_{k+l<2d}|\partial_{x_k x_l}\bar{u}|\right) \leq \frac{C}{\epsilon + |x'|^2}, \quad x \in \Omega_R. \tag{5.17}$$

Using (5.17), we obtain, for $\sqrt{\epsilon} \leq |z'| \leq R$, $0 < t < s < \frac{2|z'|}{3}$, instead of (3.58),

$$\int\limits_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2\,dx \le \int\limits_{|x'-z'|<s} \frac{C}{\epsilon+|x'|^2}\,dx' \le \frac{Cs^{d-1}}{|z'|^2}. \tag{5.18}$$

Thus, we have

$$\widehat{F}(t) \le \left(\frac{C_0|z'|^2}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2\frac{s^{d-1}}{|z'|^2}, \qquad \forall\ 0<t<s<\frac{2|z'|}{3}. \tag{5.19}$$

Taking the same iteration procedure as Case 1 of Step 2 in the proof of Proposition 3.2, set $t_i = \delta + 2C_0 i\,|z'|^2$, $i = 0,1,2,\cdots$, and let $k = \left[\frac{1}{4C_0|z'|}\right]$. Using (5.19) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \le \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C(t_{i+1}-t_i)^2 t_{i+1}^{d-1}}{|z'|^2} \le \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)^2|z'|^{2d}, \quad i=1,2,\cdots,k.$$

After k iterations, we obtain

$$\int\limits_{\widehat{\Omega}_\delta(z')} |\nabla(v_i^\alpha - \bar{u}_i^\alpha)|^2 dx = \widehat{F}(t_0) \le C|z'|^{2d}, \quad \forall\ \sqrt{\epsilon} < |z'| < R.$$

For $0 \le |z'| \le \sqrt{\epsilon}$, $0<t<s<\sqrt{\epsilon}$, using (5.17), we have, instead of (3.60),

$$\int\limits_{\widehat{\Omega}_s(z')} |\mathcal{L}_{\lambda,\mu}\tilde{u}|^2\,dx \le \int\limits_{|x'-z'|<s} \frac{C}{\epsilon+|x'|^2}\,dx' \le \frac{Cs^{d-1}}{\epsilon}, \qquad 0<s<\sqrt{\epsilon}. \tag{5.20}$$

Then similarly as before, we have

$$\widehat{F}(t) \le \left(\frac{C_0|z'|^2}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2\frac{s^{d-1}}{\epsilon}, \qquad \forall\ 0<t<s<\frac{2|z'|}{3}$$

and iteration formula

$$\widehat{F}(t_i) \le \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C(t_{i+1}-t_i)^2 t_{i+1}^{d-1}}{\epsilon} \le \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)^2\epsilon^d, \quad i=1,2,\cdots,k.$$

Thus, we obtain

$$\int\limits_{\widehat{\Omega}_\delta(z')} |\nabla w_i^\alpha|^2 dx \le \begin{cases} C|z'|^{2d}, & \sqrt{\epsilon} \le |z'| < R, \\ C\epsilon^d, & 0 \le |z'| < \sqrt{\epsilon}. \end{cases} \tag{5.21}$$

Therefore, as in the proof of Proposition 3.3, using (5.14), (5.21) and (5.8), we have, for $i=1,2,\ d \le \alpha \le \frac{d(d+1)}{2}$,

$$|\nabla(v_i^\alpha - \bar{u}_i^\alpha)(x', x_d)| \le C, \qquad x \in \Omega_R. \tag{5.22}$$

Consequently, using (5.16),

$$|\nabla v_i^\alpha(x', x_d)| \le \frac{C|x'|}{\epsilon + |x'|^2} + C, \qquad x \in \Omega_R. \tag{5.23}$$

The proof of Proposition 5.1 is completed. □

Proof of Theorem 1.2. By the same argument, using Lemma 6.1 for $d \ge 4$, we still have (4.1) for dimensions $d \ge 4$. Using Proposition 5.1, Theorem 1.2 follows. □

6. Appendix: lemmas on Ψ and matrices

We first give a lemma on the linear space of rigid displacement Ψ.

Lemma 6.1. *Let ξ be an element of Ψ, defined by (1.14) with $d \ge 2$. If ξ vanishes at d distinct points $\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^d$, which do not lie on a $(d-2)$-dimensional plane, then $\xi \equiv 0$.*

Proof. Since $\xi \in \Psi$, it follows that

$$\xi(x) = Ax + b,$$

for some $b \in \mathbb{R}^d$ and some $d \times d$ skew symmetric matrix A. Let

$$\bar{y}^i = \bar{x}^i - \bar{x}^d, \quad 1 \le i \le d-1.$$

By the assumption, $\bar{y}^1, \cdots, \bar{y}^{d-1}$ is linearly independent. It follows from $\xi(\bar{x}^i) = 0$ that

$$A\bar{y}^i = \xi(\bar{x}^i) - \xi(\bar{x}^d) = 0, \quad 1 \le i \le d-1.$$

Therefore Rank $A \le 1$. This, together with $A^T + A = 0$, implies $A = 0$. Recalling that $\xi(\bar{x}^1) = 0$, we have $b = 0$. So $\xi \equiv 0$. □

Here we prove a linear algebraic lemma used in the proof of Proposition 4.1. We will use notation $\|B\| = \left(\sum_{i,j} |B_{ij}|^2\right)^{1/2}$ for a matrix B.

Lemma 6.2. *For $m \ge 1$, let A, D be $m \times m$ invertible matrices and B and C be $m \times m$ matrices satisfying, for some $0 < \theta < 1$ and $\gamma > 1$,*

$$\|A^{-1}\| \le \frac{1}{\theta\gamma}, \qquad \|B\| + \|C\| + \|D^{-1}\| \le \frac{1}{\theta}. \tag{6.1}$$

Then there exists $\bar{\gamma} = \bar{\gamma}(m) > 1$ *and* $C(m) > 1$*, such that if* $\gamma \geq \frac{\bar{\gamma}(m)}{\theta^4}$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible. Moreover,

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{pmatrix} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} - \begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

satisfies

$$\|E_{11}\| \leq \frac{C(m)}{\theta^5\gamma^2}, \qquad \|E_{12}\| \leq \frac{C(m)}{\theta^3\gamma}, \quad \text{and} \quad \|E_{22}\| \leq \frac{C(m)}{\theta^5\gamma}.$$

Proof. Clearly

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix},$$

where I is the $m \times m$ identity matrix. Since

$$\left\|CA^{-1}B\right\| \leq \frac{C_1(m)}{\theta^3\gamma},$$

for some constant $C_1(m)$ depending only on m, there exists some constant $\gamma_1(m)$, depending only on m, such that for $\gamma \geq \frac{\gamma_1(m)}{\theta^4}$, $D - CA^{-1}B$ is invertible and

$$\left\|\left(D - CA^{-1}B\right)^{-1}\right\| \leq \frac{2}{\theta}. \tag{6.2}$$

Then

$$\begin{aligned}
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} &= \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}^{-1}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} A^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ 0 & \left(D - CA^{-1}B\right)^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1}CA^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -\left(D - CA^{-1}B\right)^{-1}CA^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{pmatrix}.
\end{aligned}$$

The estimates for $|E_{11}|$ and E_{12} follow from (6.1) and (6.2). For $|E_{22}|$, we have

$$\|E_{22}\| = \left\| \left((I - D^{-1}CA^{-1}B)^{-1} - I \right) D^{-1} \right\| \leq C(m) \left\| D^{-1}CA^{-1}B \right\| \left\| D^{-1} \right\| \leq \frac{C(m)}{\theta^5 \gamma}.$$

The proof is finished. □

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