GLOBAL SOLUTIONS AND EXTERIOR DIRICHLET PROBLEM FOR MONGE-AMPÈRE EQUATION IN $\mathbb{R}^2$

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Abstract. Monge-Ampère equation $\det(D^2u) = f$ in two dimensional spaces is different in nature from their counterparts in higher dimensional spaces. In this article we employ new ideas to establish two main results for the Monge-Ampère equation defined either globally in $\mathbb{R}^2$ or outside a convex set. First we prove the existence of a global solution that satisfies a prescribed asymptotic behavior at infinity, if f is asymptotically close to a positive constant. Then we solve the exterior Dirichlet problem if data are given on the boundary of a convex set and at infinity.

1. Introduction

The aim of this article is to study convex, viscousity solutions of

$$\det(D^2u) = f \tag{1.1}$$

either globally defined in $\mathbb{R}^2$ or defined outside a convex set.

The research of global solutions dates back to 1950s. A classical result of Jörgens (for $n = 2$ [20]), Calabi ($n \leq 5$ [5]), and Pogorelov ($n \geq 2$, [24]) states that any classical convex solution of

$$\det(D^2u) = 1, \quad \text{in } \mathbb{R}^n$$

is a quadratic polynomial. Another proof in the line of affine geometry was given by Cheng-Yau [11]. Caffarelli [6] gave a proof for viscosity solutions.

If (1.1) is defined outside a strictly convex, bounded subset in $\mathbb{R}^n$ and $f \equiv 1$, Caffarelli-Li [8] proved that the solution u is asymptotically close to a quadratic polynomial at infinity for $n \geq 3$. Similarly for $n = 2$ and $f \equiv 1$, using complex analysis Ferrer-Martìnez-Milán [14, 15] and Delanoë [13] proved that u is asymptotically close to a quadratic polynomial plus a logarithmic term.

These asymptotics results were extended by the authors in [4] for f being a perturbation of 1 at infinity. Namely, for $n \geq 3$ and f being an optimal perturbation of 1, u is asymptotically close to a quadratic polynomial at infinity. For $n = 2$ and

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f being the optimal perturbation of 1, u is close to a quadratic polynomial plus a logarithmic term at infinity.

Two natural questions are related to the asymptotic behavior of u at infinity. First, given a prescribed asymptotic behavior at infinity, can one find a global solution u that satisfies the asymptotic behavior? The second question is: Let D be an open, bounded, strictly convex subset of $\mathbb{R}^n$ with smooth boundary. Given $\phi \in C^2(\partial D)$ and a prescribed asymptotic behavior of u at infinity, can one find u of (1.1) defined in $\mathbb{R}^n \setminus D$ that satisfies the boundary data at ∂D and infinity?

These questions for $n \geq 3$ are solved in [8] for $f \equiv 1$ and [4] for f being a perturbation of 1. However for $n = 2$, all the approaches used for higher dimensional cases failed. The purpose of this article is to employ a new method that solves the existence of global solution for (1.1) in $\mathbb{R}^2$ and a corresponding exterior Dirichlet problem.

First we consider convex viscosity solutions of

$$\det(D^2 u) = f, \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

where we assume f to satisfy

$$\begin{cases} \dfrac{1}{c_0} \leq f(x) \leq c_0, & \forall x \in \mathbb{R}^2, \\ \left|D^j(f(x)-1)\right| \leq \dfrac{c_0}{(1+|x|)^{\beta+j}}, & j = 0,1,..,k, \ \forall x \in \mathbb{R}^2, \end{cases} \tag{1.3}$$

for some $c_0 > 0, \beta > 2$ and $k \geq 3$.

Remark 1.1. *The assumption $\beta > 2$ in (1.3) is sharp, as the readers may see counter examples in the authors' previous work* [4].

Let $\mathbb{M}^{2\times 2}$ be the set of the real valued, 2×2 matrices and

$$\mathcal{A} := \left\{ A \in \mathbb{M}^{2\times 2} : \ A \text{ is symmetric, positive definite and } \det(A) = 1 \right\}.$$

Our first main theorem is on the existence of global solution with prescribed asymptotic behavior at infinity:

Theorem 1.1. *Suppose (1.3) holds for f. Given $A \in \mathcal{A}$, $b \in \mathbb{R}^2$ and $c \in \mathbb{R}$, there exists $\epsilon_0(A, c_0) > 0$ such that if*

$$\left| D^m \left(f\left(\sqrt{A}^{-1} y\right) - \fint_{\partial B(0,|y|)} f\left(\sqrt{A}^{-1} x\right) dS \right) \right| \leq \epsilon_0, \quad \forall y \in \mathbb{R}^2, \quad m = 0,1, \tag{1.4}$$

then there exists a unique solution u to (1.2) satisfying

$$\limsup_{|x|\to\infty} |x|^{j+\sigma} \left| D^j \left(u(x) - \left(\frac{1}{2} x' A x + b \cdot x + d \log \sqrt{x' A x} + c \right) \right) \right| < \infty \tag{1.5}$$

for $j = 0,1,..,k+1$, $\sigma \in (0, \min\{\beta - 2, 2\})$ and $d = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f-1)$.

Remark 1.2. *It is easy to observe that (1.4) follows from (1.3) if $|y|$ is large. On the other hand $f_1(x) := f\left(\sqrt{A}^{-1} x\right)$ could be very different from 1 when $|x|$ is not large, even though it is very close to a radial function.*

Throughout the article we shall use $B(x_0, r)$ to denote the disk centered at x_0 with radius r. If x_0 is the origin we may use B_r.

If the dimension is higher than 2, the analogue of Theorem 1.1 can be proved using a standard upper-lower solutions method: In order to find a global solution of $\det(D^2u) = f$ for f close to 1 at infinity, one can solve for $\det(D^2u_R) = \overline{f}$ and $\det(D^2U_R) = \underline{f}$ in B_R, where $\overline{f}$ and $\underline{f}$ are radial functions greater than f and smaller than f respectively. Both $\underline{f}$ and $\overline{f}$ are close to 1 at infinity and the difference between u_R and U_R is only $O(1)$ if they take the same value on ∂B_R. Thus it is easy to obtain a global solution of $\det(D^2u) = f$ in $\mathbb{R}^n$ by a sequence of local solutions. However for $n = 2$, such a process is completely destroyed by a logarithmic term. In order for a limiting process to work, it is crucial to obtain a point-wise, uniform estimate for the Hessian matrix of a sequence of approximating solutions. Because of the logarithmic term, the shapes of certain level sets cannot be determined and almost all estimates that work so well for higher dimensional equations fail.

The proof of Theorem 1.1 is as follows. First we look for a radial solution of $\det(D^2u) = \tilde{f}_1(r)$, where $\tilde{f}_1(r) := \fint_{\partial B_r} f_1(x)dS$, and take this solution as the first term in our approximation. As we look for more terms down the road we treat the additional terms as solutions to the linearized equation of the Monge-Ampère equation expanded at the radial solution. In order to make all the additional terms proportionally smaller, we need to use the structure of Monge-Ampère equation and a sharp estimate of the Green's function corresponding to the linearized equation. Standard estimates for Green's functions are not enough for our purpose because the iteration process requires a very sharp form. What makes it worse is the ellipticity of the linearized equation could be very bad near the origin, since f_1 could be very different from 1 near the origin. The proof in Lemma 2.2, which relies heavily on results of Kenig-Ni and Cordes-Nirenberg for $n = 2$, overcomes this difficulty by estimating the Green's function over "good regions" first and then use the maximum principle to control the "bad region".

The second main theorem is on the exterior Dirichlet problem proposed in the previous work of the authors [4]. We look to solve the following exterior Dirichlet problem: Let D be a bounded, strictly convex set with smooth boundary in $\mathbb{R}^2$. Suppose $\varphi \in C^2(\partial D)$ and u is a solution of

$$\begin{cases} \det(D^2u) = f(x), \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \\ u \in C^0(\mathbb{R}^2 \setminus D) \text{ is a locally convex viscosity solution,} \\ \\ u = \varphi(x), \qquad\qquad \text{on } \partial D. \end{cases} \tag{1.6}$$

In [4] we conjectured that for any $\varphi \in C^2(\partial D)$, as long as

$$d > \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus D}(f-1) - \frac{1}{2\pi}\,\text{area}(D),$$

there is always a locally convex solution to

$$\begin{cases} \det(D^2u) = f(x), & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u = \varphi(x), & \text{on } \partial D, \\ \limsup_{|x|\to\infty} |x|^{j+\sigma} \left| D^j \left(u(x) - \left(\frac{1}{2} x' A x + b \cdot x + d \log \sqrt{x'Ax} + c_d \right) \right) \right| < \infty \end{cases}$$

for $j = 0, 1, ..., k$ ($k \geq 3$), $\sigma \in (0, \min\{\beta - 2, 2\})$, $c_d \in \mathbb{R}$ is uniquely determined, where φ is a given smooth function on ∂D, $A \in \mathcal{A}$, $b \in \mathbb{R}^2$.

Because of the additional assumption (1.4) we are not able to prove this conjecture for arbitrary convex domain D. However since we are using a new approach we can weaken the assumption of ϕ to being Hölder continuous:

Theorem 1.2. *Let $r_0 > 0$, $\phi \in C^\alpha(\partial B_{r_0})$ for some $\alpha \in (0, 1)$ and f satisfy (1.3). Then for any $d > \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus B_{r_0}} (f - 1) - \frac{1}{2} r_0^2$, there exists $\epsilon_0(r_0, d, \alpha) > 0$ such that if (1.4) holds for f and*

$$\sup_{x,y \in \partial B_{r_0}} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha} \leq \epsilon_0,$$

a unique u to (1.6) exists (for $D = B_{r_0}$) and satisfies

$$\limsup_{|x|\to\infty} |x|^{j+\sigma} \left| D^j \Big(u(x) - \big(\frac{1}{2} |x|^2 + d \log |x| + c_d \big) \Big) \right| < \infty \tag{1.7}$$

for $j = 0, .., k + 1$ and $\sigma \in (0, \min\{\beta - 2, 2\})$, $c_d \in \mathbb{R}$ is uniquely determined by ϕ, d, f and r_0.

The organization of this article is as follows. The proof of Theorem 1.1, which is by an iteration method, is arranged in section two. The proof of Theorem 1.2 in section three is based on a Perron's method. Theorem 1.1 plays an essential role in the proof of Theorem 1.2. Here we further remark that in order to use Theorem 1.1 in the proof of Theorem 1.2, it is crucial to assume that f_1 is very close to its spherical average rather than 1. Finally the proof of Theorem 1.2 also relies on a result (Lemma 3.1) of the authors' previous paper [4] to determine the unique constant in the expansion.

2. Proof of Theorem 1.1

Denote

$$f_1(y) := f(\sqrt{A}^{-1} y), \quad \text{and} \quad \tilde{f}_1(y) := \frac{1}{2\pi|y|} \int_{\partial B(0,|y|)} f_1(x) dS.$$

We only need to determine $v(y)$, which satisfies

$$\det(D^2 v(y)) = f_1(y), \quad y \in \mathbb{R}^2$$

and

$$\limsup_{|y|\to\infty} |y|^{j+\sigma} \left| D^j \Big(v(y) - \frac{1}{2} |y|^2 - d \log |y| - c \Big) \right| = 0$$

for $j = 0, ..., k+1$ and $\sigma \in (0, \min\{\beta - 2, 2\})$, where

$$d = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f_1 - 1)dx = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f-1)dx.$$

Once such v is found, we let

$$u(x) = v\left(\sqrt{A}x\right) + b \cdot x.$$

Then we see that (1.5) holds for u.

2.1. **Radial solutions and some elementary estimates.** Before we set out to find v, we first construct a radial solution of

$$\det(D^2 U) = \tilde{f}_1, \quad \text{in} \quad \mathbb{R}^2. \tag{2.1}$$

Let

$$U(r) = \int_0^r \left(\int_0^s 2t\tilde{f}_1(t)dt\right)^{\frac{1}{2}} ds, \quad r = |y|,$$

then one can verify easily that

$$U'(r) = \left(\int_0^r 2t\tilde{f}_1(t)dt\right)^{\frac{1}{2}}, \quad U''(r) = \frac{r\tilde{f}_1(r)}{\left(\int_0^r 2s\tilde{f}_1(s)ds\right)^{\frac{1}{2}}},$$

and consequently

$$\det(D^2U) = \partial_{11}U\partial_{22}U - \partial_{12}U^2 = U''(r)\frac{U'(r)}{r} = \tilde{f}_1(r), \quad r > 0.$$

Moreover

$$U(r) = \frac{1}{2}r^2 + d\log r + c_d + U(0) + O(r^{-\delta}), \quad \text{as } r \to \infty,$$

where $\delta = \min\{\beta - 2, 2\}$, using (1.3) and the definitions of $\tilde{f}_1$ and f_1,

$$d = \lim_{r\to+\infty}\frac{U(r) - \frac{r^2}{2}}{\log r} = \int_0^\infty r\left(\tilde{f}_1(r) - 1\right)dr = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f_1 - 1)dx,$$

and

$$\begin{aligned} c_d &= \lim_{r\to+\infty}\left(U(r) - \frac{r^2}{2} - d\log\left(r + \sqrt{r^2+d}\right) + d\log\frac{r + \sqrt{r^2+d}}{r}\right) \\ &= \int_0^\infty \left(\left(\int_0^s 2tf_1(t)dt\right)^{\frac{1}{2}} - s - \frac{d}{\sqrt{s^2+d}}\right)ds + d\log 2. \end{aligned}$$

Note that f_1 may not be close to 1 for $|y|$ not large, but it is close to $\tilde{f}_1$ when ϵ_0 in (1.4) is small.

Next, we will give some estimates for f_1 and $\tilde{f}_1$. We observe that in addition to (1.4), f_1 also satisfies

$$\begin{cases} \frac{1}{c_0} \le f_1(y) \le c_0, & \forall y \in \mathbb{R}^2, \\ \left|D^j(f_1(y) - 1)\right| \le \frac{C_0(c_0, A)}{(1+|y|)^{\beta+j}}, & j = 0, 1..., k. \end{cases} \tag{2.2}$$

It is easy to check that in polar coordinates

$$|\partial_r f_1| + \frac{1}{r}|\partial_\theta f_1| \le \frac{C(c_0, A)}{r^{\beta+1}}, \quad r \ge 1, \tag{2.3}$$

and

$$|\partial_{rr} f_1| + \frac{1}{r}|\partial_{r\theta} f_1| + \frac{1}{r^2}|\partial_{\theta\theta} f_1| \le \frac{C(c_0, A)}{r^{\beta+2}}, \quad r \ge 1. \tag{2.4}$$

Now we claim that

$$|D^j(f_1 - \tilde{f}_1)(y)| \le \frac{C(c_0, A)}{(1+|y|)^{\beta+j}}, \quad y \in \mathbb{R}^2, \quad j = 0, 1, 2. \tag{2.5}$$

Obviously, we just need to verify (2.5) for $r = |y| \ge 1$. Indeed, writing $f_1 - \tilde{f}_1$ as

$$\begin{aligned} f_1(y) - \tilde{f}_1(r) &= f_1(re^{i\psi}) - \frac{1}{2\pi}\int_0^{2\pi} f_1(re^{i\theta})d\theta \qquad (y = re^{i\psi}) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big(f_1(re^{i\psi}) - f_1(re^{i\theta})\Big)d\theta. \end{aligned} \tag{2.6}$$

We first use the estimate on $\partial_\theta f_1$ in (2.3) to obtain

$$|f_1(y) - \tilde{f}_1(r)| \le \frac{C(c_0, A)}{(1+r)^\beta}.$$

Then, for $j = 1$, we have

$$\Big|D(f_1 - \tilde{f}_1)(y)\Big| \le C\left(|\partial_r f_1| + \frac{1}{r}|\partial_\theta f_1|\right) \le \frac{C(c_0, A)}{(1+r)^{\beta+1}}.$$

Finally, for $j = 2$, it is easy to see from (2.6) that

$$\Big|\partial_{rr}(f_1 - \tilde{f}_1)\Big| \le \frac{C(c_0, A)}{(1+r)^{\beta+2}}.$$

Since $\tilde{f}_1$ is radial,

$$\partial_{r\theta}(f_1 - \tilde{f}_1) = \partial_{r\theta} f_1, \quad \partial_{\theta\theta}(f_1 - \tilde{f}_1) = \partial_{\theta\theta} f_1.$$

Therefore, by (2.4),

$$\Big|D^2(f_1 - \tilde{f}_1)(x)\Big| \le C\left(\Big|\partial_{rr}(f_1 - \tilde{f}_1)\Big| + \frac{|\partial_{r\theta} f_1|}{r} + \frac{|\partial_{\theta\theta} f_1|}{r^2}\right) \le \frac{C}{(1+r)^{\beta+2}}.$$

Thus, (2.5) is established. Combining (1.4) and (2.5), we obtain

$$\Big|D^m(f_1 - \tilde{f}_1)(y)\Big| \le \frac{\epsilon_1(\epsilon_0, A, c_0, \beta)}{(1+|y|)^{\beta_1}}, \quad y \in \mathbb{R}^2, \ m = 0, 1, \tag{2.7}$$

where $\beta_1 = \frac{\beta}{2} + 1 \in (2, \beta)$ and $\epsilon_1 \to 0$ as $\epsilon_0 \to 0$.

We further obtain, by simple computations, that

$$\partial_{11} U = F_1 + F_2\cos(2\theta), \quad \partial_{22} U = F_1 - F_2\cos(2\theta), \quad \partial_{12} U = F_2\sin(2\theta) \tag{2.8}$$

where

$$F_1 := \frac{1}{2}(U''(r) + U'(r)/r), \quad F_2 := \frac{1}{2}(U''(r) - U'(r)/r).$$

It follows from (2.2), (1.4) and (2.5) that there exists $c_1(c_0, A) > 0$ such that

$$\begin{cases} \left|D^j\,(\partial_{22}U-1)\,(y)\right| \le \dfrac{c_1}{(1+|y|)^{2+j}}, \quad y\in\mathbb{R}^2, \\ \\ \left|D^j(\partial_{11}U-1)(y)\right| \le \dfrac{c_1}{(1+|y|)^{2+j}}, \quad y\in\mathbb{R}^2, \\ \\ \left|D^j(\partial_{12}U)(y)\right| \le \dfrac{c_1}{(1+|y|)^{2+j}}, \quad y\in\mathbb{R}^2, \end{cases} \tag{2.9}$$

for $j=0,1,2$. It is easy to verify (2.9) for y large since $\tilde{f}_1$ is close to f_1 and f_1 is close to 1 when $|y|$ is large. For $|y|$ not large (2.9) certainly holds.

2.2. **The first step of iteration.** Suppose that the solution u of (1.2) is of the form

$$u = U + \phi.$$

Clearly ϕ satisfies

$$\partial_{11}\phi\partial_{22}U + \partial_{22}\phi\partial_{11}U - 2\partial_{12}\phi\partial_{12}U + \det(D^2\phi) = f_1 - \tilde{f}_1, \quad \text{in } \mathbb{R}^2. \tag{2.10}$$

Let

$$a^*_{11} := \partial_{22}U, \quad a^*_{22} := \partial_{11}U, \quad a^*_{12} := -\partial_{12}U,$$

then by (2.9),

$$c_1^{-1}I \le (a^*_{ij})_{2\times 2} \le c_1 I.$$

It is well known that the first part of (2.10) can be written as a divergence form.

$$L\phi := \partial_i(a^*_{ij}\partial_j\phi) = \partial_{22}U\partial_{11}\phi + \partial_{11}U\partial_{22}\phi - 2\partial_{12}U\partial_{12}\phi, \quad \forall\phi\in C^2(\mathbb{R}^2),$$

because $\partial_i a^*_{ij} = 0$ for $j = 1,2$. Then (2.10) can be written as

$$\partial_i(a^*_{ij}\partial_j\phi) + \det(D^2\phi) = f_1 - \tilde{f}_1, \quad \text{in } \mathbb{R}^2. \tag{2.11}$$

Let G be the fundamental solution of $-L$ on $\mathbb{R}^2$

$$-\partial_{y_i}(a^*_{ij}(y)\partial_{y_j}G(x,y)) = \delta_x, \quad \text{in } \mathbb{R}^2,$$

where δ_x is the Dirac mass at x. According to the theory of Kenig-Ni [21] there exists $c_2(c_0, A)$ such that

$$|G(x,y)| \le \begin{cases} c_2\left|\log|x-y|\right|, & y\in B(x,\frac{1}{2}), \\ \\ c_2\left(\left|\log|x-y|\right|+1\right) & y\in\mathbb{R}^2\setminus B(x,\frac{1}{2}). \end{cases} \tag{2.12}$$

In the following, we will start our iteration process. We first solve

$$L\phi^0 = f_1 - \tilde{f}_1, \quad \text{in } \mathbb{R}^2 \tag{2.13}$$

by letting

$$\phi^0(x) = \int_{\mathbb{R}^2} G(x,y)(\tilde{f}_1(y) - f_1(y))dy. \tag{2.14}$$

The estimates of ϕ^0 are stated in the following. The proof will be given in subsection 2.4.

Proposition 2.1. *There exists $c_3 > 0$ only depending on c_0, A and β such that ϕ^0 satisfies*

$$\begin{cases} \left|D^j\phi^0(x)\right| \le \dfrac{c_3\epsilon_1}{(1+|x|)^{j+\tau}}, \quad \forall x \in \mathbb{R}^2,\ j=0,1,2 \\ \left|D^2\phi^0(y) - D^2\phi^0(z)\right| \le c_3\epsilon_1|y-z|^\alpha, \quad \forall\, y,z \in B_1, \\ \left|D^2\phi^0(y) - D^2\phi^0(z)\right| \le \dfrac{c_3\epsilon_1}{|x|^{2+\tau+\alpha}}|y-z|^\alpha,\ \forall\, y,z \in B_{\frac{3|x|}{2}} \setminus B_{\frac{|x|}{2}},\ |x|>1, \end{cases} \tag{2.15}$$

where $\tau \in \left(0, \frac{\beta}{2}-1\right)$,$\alpha \in (0,1)$ depends on c_0, A, β.

Once we have the estimate for ϕ_0 from Proposition 2.1, we let

$$\psi^1(x) = \int_{\mathbb{R}^2} G(x,y)\det(D^2\phi^0(y))dy,$$

then ψ^1 solves

$$L\psi^1 = -\det(D^2\phi^0), \quad \text{in } \mathbb{R}^2. \tag{2.16}$$

Since

$$\det(D^2\phi^0) = \partial_1\left(\partial_1\phi^0\partial_{22}\phi^0\right) - \partial_2\left(\partial_{12}\phi^0\partial_1\phi^0\right),$$

we write ψ^1 as

$$\psi^1(x) = \int_{\mathbb{R}^2}\left(-\partial_{y_1}G(x,y)\partial_1\phi^0(y)\partial_{22}\phi^0(y) + \partial_{y_2}G(x,y)\partial_1\phi^0(y)\partial_{12}\phi^0(y)\right)dy.$$

It is easy to use the decay rate of $D^2\phi^0$ in (2.15) to obtain

$$\left|\psi^1(x)\right| \le C(c_0, A, \beta)(c_3\epsilon_1)^2, \quad x \in B_{2R_0}. \tag{2.17}$$

Then from (2.17) and elliptic estimate we have

$$\left\|\psi^1(x)\right\|_{C^{2,\alpha}(B_{R_0})} \le C(c_0, A, \beta)c_3^2\epsilon_1^2. \tag{2.18}$$

For $|x| > R_0$, we decompose $\mathbb{R}^2$ into $E_1 \cup E_2$. For the integral on $E_1 = B(0, \frac{|x|}{2})$, we use Proposition 2.1 to get

$$\begin{aligned} &\left|\int_{E_1}\left(\partial_{y_1}G(x,y)\partial_1\phi^0(y)\partial_{22}\phi^0(y) - \partial_{y_2}G(x,y)\partial_1\phi^0(y)\partial_{12}\phi^0(y)\right)dy\right| \\ \le& C(c_0,\beta,A)(c_3\epsilon_1)^2\frac{\log|x|}{|x|^{2+2\tau}} \le \frac{C(c_0,A,\beta)(c_3\epsilon_1)^2}{(1+|x|)^\tau}. \end{aligned}$$

Remark 2.1. *Writing* $\det(D^2\phi^0)$ *in the divergence form leads to differentiation on G and thus we avoid a logarithmic term from the integration over E_1. This is exactly like the corresponding estimate for ϕ^0. Here we further remark that the estimate for ψ^1 is exactly like that for ϕ^0, as the estimate of G is the same, the Hölder norm of the elliptic operator in the scaling part still has the same bound.*

Using the rough estimate of G, (2.12), and estimates of ϕ^0, we obtain easily

$$\begin{aligned}&\left|\int_{E_2}\left(\partial_{y_1}G(x,y)\partial_1\phi^0(y)\partial_{22}\phi^0(y)-\partial_{y_2}G(x,y)\partial_1\phi^0(y)\partial_{12}\phi^0(y)\right)dy\right|\\ \leq&\frac{C(c_0,A,\beta)(c_3\epsilon_1)^2}{|x|^{4+2\tau}}\leq\frac{C(c_0,\beta,A)(c_3\epsilon_1)^2}{(1+|x|)^\tau}.\end{aligned}$$

Correspondingly elliptic estimates lead to estimates on higher derivatives. Therefore the following estimates have been obtained for ψ^1: for $x\in\mathbb{R}^2$, there exists $c_4(c_0,\beta,A)>0$ such that

$$\begin{cases}\left|D^j\psi^1(x)\right|\leq\dfrac{c_4c_3^2\epsilon_1^2}{(1+|x|)^{j+\tau}}, & \forall x\in\mathbb{R}^2,\ j=0,1,2\\ \left|D^2\psi^1(y)-D^2\psi^1(z)\right|\leq c_4c_3^2\epsilon_1^2|y-z|^\alpha, & \forall\, y,z\in B_1,\\ \left|D^2\psi^1(y)-D^2\psi^1(z)\right|\leq\dfrac{c_4c_3^2\epsilon_1^2}{|x|^{2+\tau+\alpha}}|y-z|^\alpha, & \forall\, y,z\in B_{\frac{3|x|}{2}}\setminus B_{\frac{|x|}{2}},\ |x|>1,\end{cases}\tag{2.19}$$

where $\alpha\in(0,1)$ is defined as in (2.15).

Remark 2.2. *The constant c_4 in (2.19) only depends on c_0,β,A and is obtained from evaluating the Green's representation formula and standard elliptic estimates. If the* $\det(D^2\phi^0)$ *is replaced by another function with fast decay at infinity, the constant c_4 does not change.*

2.3. **Completion of the proof of Theorem 1.1 by iteration.**

Proof of Theorem 1.1. We will prove it by iteration. Let

$$\phi^1:=\phi^0+\psi^1,$$

then, it is clear from (2.13) and (2.16) that

$$L\phi^1=L\phi^0+L\psi^1=f_1-\tilde{f}_1-\det(D^2\phi^0).\tag{2.20}$$

Rewrite it as

$$L\phi^1+\det(D^2\phi^1)=f_1-\tilde{f}_1+\det(D^2\phi^1)-\det(D^2\phi^0).$$

Let ψ^2 solve

$$L\psi^2:=\det(D^2\phi^0)-\det(D^2\phi^1).$$

In general, for $l\geq 2$, we define

$$\phi^l:=\phi^{l-1}+\psi^l,$$

and

$$L\psi^l:=\det(D^2\phi^{l-2})-\det(D^2\phi^{l-1}).$$

We will prove the following estimates for ϕ^l, $l \geq 0$:

$$
\begin{cases}
\left|D^j\phi^l(y)\right| \leq \dfrac{2c_3\epsilon_1}{(1+|y|)^{\tau+j}}, \quad y \in \mathbb{R}^2,\ j=0,1,2\\[2ex]
\left\|\phi^l\right\|_{C^{2,\alpha}(B_1)} \leq 2c_3\epsilon_1,\\[2ex]
|D^2\phi^l(y) - D^2\phi^l(z)| \leq \frac{2c_3\epsilon_1}{|x|^{\tau+2+\alpha}}|y-z|^\alpha,\ y,z \in B(x,\frac{|x|}{2}), |x|>1.
\end{cases} \tag{2.21}
$$

by using the following estimates for ψ^l, $l \geq 0$,

$$
\begin{cases}
\left|D^j\psi^{l+1}(x)\right| \leq \dfrac{2c_4(c_3\epsilon_1)^{l+2}}{(1+|x|)^{j+\tau}}, \quad \forall x \in \mathbb{R}^2,\ j=0,1,2\\[2ex]
\left|D^2\psi^{l+1}(y) - D^2\psi^l(z)\right| \leq 2c_4(c_3\epsilon_1)^{l+2}|y-z|^\alpha, \quad \forall\, y,z \in B_1,\\[2ex]
\left|D^2\psi^{l+1}(y) - D^2\psi^l(z)\right| \leq \dfrac{2c_4(c_3\epsilon_1)^{l+2}}{|x|^{2+\tau+\alpha}}|y-z|^\alpha,\ \forall\, y,z \in B_{\frac{3|x|}{2}} \setminus B_{\frac{|x|}{2}},\ |x|>1,
\end{cases} \tag{2.22}
$$

which can be proved by induction.

First, for $l = 0$, we have from (2.15) and (2.19) that (2.21) and (2.22) holds, respectively. Then, by the definition of ϕ^1, $\phi^1 = \phi^0 + \psi^1$, using the estimate of ϕ^0 and ψ^1, we immediately have

$$|D^j\phi^1(y)| \leq |D^j\phi^0(y)| + |D^j\psi^1(y)| \leq \frac{(c_3\epsilon_1 + c_4c_3^2\epsilon_1^2)}{(1+|y|)^{\tau+j}},$$

for $y \in \mathbb{R}^2$ and $j = 0,1,2$. The C^α estimate for the second derivatives are similar. If we choose ϵ_1 to satisfy $c_4c_3\epsilon_1 < \frac{1}{2}$ and $c_3\epsilon_1 < \frac{1}{2}$, then we obtain the estimate (2.21) holds for ϕ^1.

Since ψ^2 solve the linear equation, it has the expression

$$
\begin{aligned}
\psi^2(y) :&= \int_{\mathbb{R}^2} G(y,\eta)(\det(D^2\phi^1) - \det(D^2\phi^0))d\eta\\
&= \int_{\mathbb{R}^2} \partial_{\eta_1}G(y,\eta)\left(-\partial_1\phi^1\partial_{22}\phi^1 + \partial_1\phi^0\partial_{22}\phi^0\right)\\
&\qquad\qquad + \partial_{\eta_2}G(y,\eta)\left(-\partial_1\phi^0\partial_{12}\phi^0 + \partial_1\phi^1\partial_{12}\phi^1\right)d\eta.
\end{aligned}
$$

It is easy to see

$$
\begin{aligned}
\partial_1\phi^1\partial_{22}\phi^1 - \partial_1\phi^0\partial_{22}\phi^0 &= \partial_1\phi^0\partial_{22}\psi^1 + \partial_1\psi^1\partial_{22}\phi^0 + \partial_1\psi^1\partial_{22}\psi^1,\\
\partial_1\phi^1\partial_{12}\phi^1 - \partial_1\phi^0\partial_{12}\phi^0 &= \partial_1\phi^0\partial_{12}\psi^1 + \partial_1\psi^1\partial_{12}\phi^0 + \partial_1\psi^1\partial_{12}\psi^1.
\end{aligned}
$$

Thus ψ^2 can be evaluated as

$$
\begin{aligned}
\psi^2(y) = \int_{\mathbb{R}^2} \Big(& -\partial_{\eta_1}G(y,\eta)\left(\partial_1\phi^0\partial_{22}\psi^1 + \partial_1\psi^1\partial_{22}\phi^0 + \partial_1\psi^1\partial_{22}\psi^1\right)\\
&+ \partial_{\eta_2}G(y,\eta)\left(\partial_1\phi^0\partial_{12}\psi^1 + \partial_1\psi^1\partial_{12}\phi^0 + \partial_1\psi^1\partial_{12}\psi^1\right)\Big)d\eta.
\end{aligned}
$$

Using (2.15) and (2.19) we obtain (2.22) holds for ψ^2. That is, (2.22) holds for $l = 1$.

Suppose that (2.21) and (2.22) holds for $l = k$, then by

$$\phi^{k+1} := \phi^k + \psi^{k+1} = \phi^0 + \sum_{l=1}^{k} \psi^l,$$

we have

$$\begin{aligned} \left|D^j \phi^{k+1}(y)\right| &\le \left|D^j \phi^0(y)\right| + \sum_{l=1}^{m} \left|D^j \psi^l\right| \\ &\le \frac{c_3\epsilon_1 + c_4(c_3\epsilon_1)^2 + 2c_4(c_3\epsilon_1)^3 + \cdots + 2c_4(c_3\epsilon_1)^{l+1}}{(1+|y|)^{j+\tau}} \\ &\le \frac{c_3\epsilon_1\left(1 + c_4(c_3\epsilon_1) + 2c_4(c_3\epsilon_1)^2 + \cdots + 2c_4(c_3\epsilon_1)^l\right)}{(1+|y|)^{j+\tau}} \\ &\le 2c_3\epsilon_1(1+|y|)^{-j-\tau}, \quad j = 0,1,2, \quad \text{in } \mathbb{R}^2. \end{aligned}$$

Similarly, we have (2.21) holds for ϕ^{k+1}. Continue this process, we can obtain (2.21) and (2.22) holds for any $l \ge 0$.

Notice that for all l, the estimates of ϕ^l satisfy the same bound as in (2.21), because the estimates for ψ^l use the same estimate for G and DG. The only difference is the right hand side: $\det(D^2\phi^l) - \det(D^2\phi^{l+1})$. Thus, for ϵ_1 small the process converges and ϕ^l converges to a solution of

$$\det(D^2 v) = f.$$

The estimates on the asymptotic behavior of u at infinity as well as their derivatives can be determined by the main theorem in [4]. Theorem 1.1 is established. □

2.4. **Proof of Proposition 2.1.** From (2.9) we see that

$$|D^j(a^*_{ij} - \delta_{ij})(y)| \le c_2(1+|y|)^{-2-j}, \quad j = 0,1,2, \quad \forall y \in \mathbb{R}^2. \tag{2.23}$$

So a^*_{ij} is very close to δ_{ij} when $|y|$ is large.

Before we present the proof of Proposition 2.1 we list two tools needed for this proof: Cordes-Nirenberg estimate and an estimate of the Green's function of L. The Cordes-Nirenberg estimate is stated in the following lemma (see e.g. [7]):

Lemma 2.1. *(Cordes-Nirenberg) For any h satisfying*

$$a_{ij}\partial_{ij} h = 0, \quad \text{in } B_1 \subset \mathbb{R}^n, \quad n \ge 2,$$

there exists an $\delta_0 > 0$ depending only on n such that if $|a_{ij} - \delta_{ij}| \le \delta_0$ for all $i, j = 1, ..., n$ the following estimate holds:

$$\|Dh\|_{C^{1/2}(B_{1/2})} \le C(n)\|h\|_{L^\infty(B_1)}.$$

The second tool is a gradient estimate of $G(x,y)$ for $|x| > 2R_0$ and $|y| \le |x|/2$. Here $R_0(c_0,\beta)$ is a large number that satisfies the following requirement: For any $R > R_0$, let

$$a^R_{ij}(y) := a^*_{ij}(Ry), \quad \frac{1}{2} < |y| < 2, \quad i, j = 1, 2$$

there holds

$$|a^R_{ij}(y) - \delta_{ij}| \le \delta_0 \quad \text{and} \quad \|a^R_{ij}(\cdot)\|_{C^\alpha(B_2\setminus B_{1/2})} \le 4. \tag{2.24}$$

where δ_0 is the absolute constant required in the Cordes-Nirenberg estimate. It is easy to see that (2.24) holds from (2.23) for R_0 large that only depends on c_0, β and A.

Lemma 2.2. *For $|x| > 2R_0$, there exists $C(\beta, c_0, A) > 0$ such that*

$$|D_y G(x,y)| \le C(\beta, c_0, A)\frac{\log|x|}{|x|}, \quad \forall y \in B(0, \frac{|x|}{2}).$$

Here D_y means the differentiation with respect to the component y.

Proof. Let $g(y) := G(x,y)$ for $|y| < \frac{9}{10}|x|$ and we write the equation for g in $B(0, \frac{9}{10}|x|)$ as

$$a^*_{ij}\partial_{ij} g = 0, \quad \text{in } B(0, \frac{9}{10}|x|). \tag{2.25}$$

we first estimate $|Dg|$ over $B(0, \frac{3}{4}|x|) \setminus B(0, \frac{1}{2}|x|)$. For any fixed y in this region, let $R = \frac{1}{10}|x|$ and

$$\overline{a}^R_{ij}(z) := a^*_{ij}(y + Rz), \quad g_R(z) := g(y + Rz), \quad |z| \le 1.$$

Clearly $|g_R(z)| \le C\log|x|$ by the estimate of Kenig-Ni and

$$\overline{a}^R_{ij}(z)\partial_{z_i z_j} g_R(z) = 0, \quad \text{in } B_1.$$

By the definition of R_0, we have $|\overline{a}^R_{ij} - \delta_{ij}| \le \delta_0$ where δ_0 is small enough for Lemma 2.1 to be applied. Using $|g_R(z)| \le C\log|x|$ and Lemma 2.1 we have

$$|Dg_R(z)| \le C\log|x|, \quad z \in B_{1/2},$$

which gives

$$|Dg| \le \frac{C\log|x|}{|x|}, \qquad \frac{9}{20}|x| \le |y| \le \frac{4}{5}|x|. \tag{2.26}$$

Now let

$$H(y) := \partial_1 g(y_1, y_2), \quad y = (y_1, y_2) \in B(0, \frac{|x|}{2}).$$

Differentiating (2.25) with respect to y_1:

$$a^*_{ij}\partial_{ij}H + \partial_1 a^*_{11}\partial_1 H + 2\partial_1 a^*_{12}\partial_2 H + \partial_1 a^*_{22}\partial_{22}F = 0, \quad \text{in } B(0, \frac{1}{2}|x|). \tag{2.27}$$

Using (2.25) again for the last term of (2.27), we have

$$\partial_{22} g = -\frac{a^*_{11}\partial_{11} g + 2a^*_{12}\partial_{12} g}{a^*_{22}}. \tag{2.28}$$

Combining (2.27) and (2.28) we have

$$a^*_{ij}\partial_{ij}H + \left(\partial_1 a^*_{11} - \frac{\partial_1 a^*_{22}}{a^*_{22}} a^*_{11}\right)\partial_1 H + \left(2\partial_1 a^*_{12} - \frac{2a^*_{12}}{a^*_{22}}\partial_1 a^*_{22}\right)\partial_2 H = 0$$

in $B(0, \frac{1}{2}|x|)$. Clearly maximum principle holds for H and it gives the desired bound for H. The estimate of $\partial_2 g(y)$ for $y \in B(0, |x|/2)$ is similar. Lemma 2.2 is established. □

Proof of Proposition 2.1. The estimate of ϕ^0 consists of two cases: $x \in B_{R_0}$ and $x \in \mathbb{R}^2 \setminus B_{R_0}$.

First for $x \in B_{R_0}$, it is easy to use (2.12) and (2.7) in (2.14) to obtain

$$|\phi^0(x)| \le \epsilon_1 C(c_0, \beta, A), \qquad \text{for } \ |x| < R_0.$$

The estimates for higher derivatives of ϕ^0 in B_{R_0} follow by standard elliptic estimate. Thus (2.15) is verified in B_{R_0}.

For the second case: $x \in \mathbb{R}^2 \setminus B_{R_0}$, we integrate over $E_1 = B(0, |x|/2)$ and $E_2 = \mathbb{R}^2 \setminus E_1$, respectively. The integration over E_1 can be written as

$$\left|\int_{E_1} (G(x,y) - G(x,0))(\tilde{f}_1 - f_1)dy\right| \le \int_{E_1} |D_2 G(x,\xi)| \cdot |y| \cdot \left| f_1(y) - \tilde{f}_1(|y|)\right| dy,$$

where ξ is on the segment oy, because the integration of $f_1 - \tilde{f}_1$ over E_1 is zero. By Lemma 2.2 the integration over E_1 is bounded by $C(c_0, \beta, A)\epsilon_1 |x|^{2-\beta_1}\log|x|$. The integration over E_2 can be estimated by the rough bound of $G(x, \eta)$ and $f_1 - \tilde{f}_1$. Then one sees easily that the bound for this part is $C(\beta, c_0, A)\epsilon_1 |x|^{2-\beta_1}\log|x|$. Consequently for all $x \in \mathbb{R}^2$, we have

$$|\phi^0(x)| \le C(c_0, A, \beta)\epsilon_1 |x|^{2-\beta_1}\log|x| \le \frac{C(c_0, \beta, A)\epsilon_1}{(1+|x|)^\tau}, \tag{2.29}$$

for $\tau \in (0, \frac{\beta}{2} - 1)$. (2.15) is established for $j = 0$.

To prove (2.15) for $j \ge 1$ and $|x| > R_0$, we apply the following re-scaling argument: consider

$$\phi^0_R(y) := \phi^0(Ry), \qquad \frac{1}{4} \le |y| \le 2, \quad R = |x| > R_0.$$

Then direct computation gives

$$\partial_i \left(a^*_{ij}(Ry)\partial_j \phi^0_R(y)\right) = R^2\left(f_1(Ry) - \tilde{f}_1(Ry)\right), \quad \text{in } B_2 \setminus B_{1/4}.$$

The C^1 norm of the right hand side is $O(R^{2-\beta})$ and the coefficients $a^*_{ij}(Ry)$ is only $O(R^{-2})$ different from δ_{ij} in C^1 norm as well. Moreover, by (2.29), $\left|\phi^0_R\right| \le C\epsilon_1 R^{-\tau}$ in $B_2 \setminus B_{1/4}$. Thus standard elliptic estimate gives

$$\begin{aligned}\left\|\phi^0_R\right\|_{C^{2,\alpha}(B_{3/2}\setminus B_{1/2})} \le & C(c_0, A, \beta)\left(\sup_{B_2\setminus B_{1/4}}\left|\phi^0_R\right| + \left\|R^2(f_1 - \tilde{f}_1)(R\cdot)\right\|_{C^\alpha(B_{3/2}\setminus B_{1/2})}\right)\\ \le & \frac{C(c_0, A, \beta)\epsilon_1}{R^\tau}.\end{aligned}$$

Proposition 2.1 follows from the estimate above. □

Remark 2.3. *The use of $\tilde{f}_1$ is quite essential in the estimate over E_1. Otherwise a logarithmic term will occur.*

3. Proof of Theorem 1.2

Recall that the assumption on d is

$$d > \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_{r_0}}(f-1) - \frac{1}{2}r_0^2.$$

By choosing ϵ_0 sufficiently small, depending on r_0 and d, we can extend f to the whole $\mathbb{R}^2$ such that f satisfies (1.3), (1.4) and

$$d = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f-1).$$

By Theorem 1.1 we can find U to satisfy

$$\begin{cases} \det(D^2U) = f, \quad \text{in} \quad \mathbb{R}^2, \\ U(x) = \frac{1}{2}|x|^2 + d\log|x| + C + O(|x|^{-\sigma}), \quad |x| > 1 \\ U \text{ is close to a radial function }. \end{cases}$$

By adding a constant to U if necessary we can make

$$\|\phi - U\|_{C^\alpha(\partial B_{r_0})} \le \epsilon_1(\epsilon_0). \tag{3.1}$$

where $\epsilon_1 > 0$ depends on ϵ_0 and tends to 0 as $\epsilon_0 \to 0$.

Now we look for a function $u = U + h$ to satisfy

$$\begin{cases} \det(D^2u) = f, \quad \text{in} \quad \mathbb{R}^2 \setminus B_{r_0}, \\ u = \varphi, \quad \text{on} \quad \partial B_{r_0} \\ u = \frac{1}{2}|x|^2 + d\log|x| + O(1), \quad |x| > 1. \end{cases}$$

Using the information of U we need to find h to satisfy

$$\begin{cases} \partial_i(a_{ij}\partial_j h) + \det(D^2h) = 0, \quad \text{in} \quad \mathbb{R}^2 \setminus B_{r_0}, \\ h = \varphi - U, \quad \text{on} \quad \partial B_{r_0}, \\ h = O(1), \quad \text{in} \quad |x| > r_0. \end{cases} \tag{3.2}$$

where $a_{11} = U_{22}$, $a_{22} = U_{11}$, $a_{12} = -U_{12}$. Just like in the proof of Theorem 1.1 we have

$$|D^m(a_{ij}(x) - \delta_{ij})| \le C|x|^{-2-m}, \quad m = 0, 1, 2.$$

For the remaining part of the proof we shall use

$$L = \partial_i(a_{ij}\partial_j) = a_{ij}(x)\partial_{x_ix_j}.$$

We first look for ψ_0 that satisfies

$$\begin{cases} L\psi_0 = 0, \quad \text{in} \quad \mathbb{R}^2 \setminus B_{r_0}, \\ \psi_0 = \varphi - U, \quad \text{on} \quad \partial B_{r_0}, \\ |\psi_0| \le \epsilon_1, \quad |D^j\psi_0| \le C\epsilon_1|x|^{-2-j}, \quad j = 1, 2, 3. \end{cases}$$

The function ψ_0 can be determined as follows: Let $y = x/|x|^2$ for $|x| > r_0$ and $|y| < r_0$. Let $\tilde{\psi}_0(y) = \psi_0(y/|y|^2)$. Direct computation yields

$$b_{kl}(y)\partial_{y_k y_l}\tilde{\psi}_0 + b_k(y)\partial_{y_k}\tilde{\psi}_0 = 0, \quad \text{in} \quad B_{1/r_0}$$

where

$$b_{kl} = \frac{1}{|y|^4}\frac{\partial y_k}{\partial x_i}a_{ij}(\frac{y}{|y|^2})\frac{\partial y_l}{\partial x_j} = (\delta_{ki} - 2\frac{y_k y_i}{|y|^2})a_{ij}(\frac{y}{|y|^2})(\delta_{lj} - 2\frac{y_l y_j}{|y|^2}),$$

and

$$b_k(y) = a_{ij}(\frac{y}{|y|^2})\frac{2\delta_{ki}y_l - 2\delta_{kl}y_i - 2y_k\delta_{il}}{|y|^2}(\delta_{lj} - \frac{2y_l y_j}{|y|^2}).$$

Because of the closeness between a_{ij} and δ_{ij} one verifies easily that b_{kl} is uniformly elliptic in B_{1/r_0} and the C^α norm of both b_{kl} and b_k in B_{1/r_0} is finite.

By Schauder's estimate

$$||\tilde{\psi}_0||_{C^{2,\alpha}(B_{1/r_0})} \le c_1(c_0, d, r_0)\epsilon_1.$$

Thus by the definition of $\tilde{\psi}_0$ and standard elliptic estimate

$$|D^m\psi_0(x)| \le C\epsilon_1|x|^{-2-m} \quad m = 0, 1, 2, 3 \quad |x| > r_0.$$

Next we solve

$$\begin{cases} L\psi_1 = -\det(D^2\psi_0), & \text{in} \quad |x| > r_0 \\ \psi_1 = 0, \quad \text{on} \quad \partial B_{r_0}, & \psi_1 = O(1) \text{ at } \quad \infty. \end{cases}$$

by the reflection method. Using the smallness of ψ_0 we have

$$|D^m\psi_1(x)| \le c_1(c_1\epsilon_1)^2|x|^{-2-m} = c_1^3\epsilon_1^2|x|^{-2-m}, \quad m = 0, 1, 2, 3, \quad |x| > r_0.$$

Let $h_0 = \psi_0$ and $h_1 = \psi_1 + \psi_0$. Then it is easy to see that h_1 satisfies

$$Lh_1 + det(D^2h_0) = 0, \quad |x| > r_0.$$

Then we move on to define

$$\begin{cases} L\psi_2 = \det(D^2h_0) - \det(D^2h_1), & |x| > r_0, \\ \psi_2 = 0, \quad \text{on} \quad \partial B_{r_0}, & \psi_2 = O(1) \text{ at infinity}. \end{cases}$$

Based on the estimates on h_0 and h_1 we have

$$|D^m\psi_2(x)| \le c_1^5\epsilon_1^3|x|^{-2-m}, \quad m = 0, 1, 2, 3, \quad |x| > r_0.$$

Let $h_2 = h_1 + \psi_2$. Then it is easy to verify that

$$Lh_2 + \det(D^2h_1) = 0, \quad |x| > r_0.$$

In general we determine ψ_k to satisfy

$$\begin{cases} L\psi_k = \det(D^2h_{k-2}) - \det(D^2h_{k-1}), & |x| > r_0, \\ \psi_k = 0, \quad \text{on} \quad \partial B_{r_0}, & \psi_k = O(1) \text{ at } \infty. \end{cases}$$

For ψ_k we have

$$|D^m\psi_k(x)| \le c_1^{2k+1}\epsilon_1^k|x|^{-2-m}, \quad m = 0, 1, 2, 3, \quad |x| > r_0.$$

Eventually we let $h = \sum_{k=1}^\infty \psi_k$ and all the derivatives of h are small and decay at infinity, which means $u = U + h$ is convex.

The following lemma in [4] proves that c is uniquely determined by other parameters.

Lemma 3.1. *Let u_1, u_2 be two locally convex smooth functions on $\mathbb{R}^2 \setminus \bar{D}$ where D satisfies the same assumption as in Theorem 1.2. Suppose u_1 and u_2 both satisfy*

$$\begin{cases} \det(D^2u) = f \ \text{in} \ \mathbb{R}^2 \setminus \bar{D}, \\ u = \varphi, \quad \text{on} \ \partial D \end{cases}$$

with f satisfying (1.3) *and for the same constant d*

$$u_i(x) - \frac{1}{2}|x|^2 - d\log|x| = O(1), \quad x \in \mathbb{R}^2 \setminus \bar{D}, \quad i = 1, 2. \tag{3.3}$$

Then $u_1 \equiv u_2$.

Since Lemma 3.1 uniquely determines the constant in the expansion, Theorem 1.2 is established. □

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