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Fully Nonlinear Elliptic Equations on General Domains

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Abstract. By means of the Pucci operator, we construct a function u_0, which plays an essential role in our considerations, and give the existence and regularity theorems for the bounded viscosity solutions of the generalized Dirichlet problems of second order fully nonlinear elliptic equations on the general bounded domains, which may be irregular. The approximation method, the accretive operator technique and the Caffarelli's perturbation theory are used.

1 Introduction

Let Ω be a general bounded domain in $\mathbf{R}^n$, which may be with irregular boundary points. A boundary point will be called *regular* (with respect to the Pucci operator, see Section 2 below) if there exists a barrier function at that point. In this paper, we consider the problem of existence and regularity for the viscosity solutions of the fully nonlinear second order elliptic equations of the form

$$F(D^2u, x) = f(x), \quad \text{in } \Omega \tag{1}$$

with zero Dirichlet boundary condition in some weak sense, where D^2u is the Hessian matrix of the function u, $F(M, x)$ and $f(x)$ are the continuous functions in $\mathbf{R}^{n\times n} \times \Omega$ and Ω respectively. In our results no assumption is made about the domain Ω.

The main impetus for studying the fully nonlinear second order uniformly elliptic equations arose through the stochastic control problem and the stochastic differential game theory. The important examples are the Hamilton-Jacobi-Bellman equations

$$\inf_{\alpha\in\mathcal{A}} \{L_\alpha u - f_\alpha(x)\} = 0,$$

and the Isaacs equations

$$\sup_{\beta\in\mathcal{B}} \inf_{\alpha\in\mathcal{A}} \{L_{\alpha\beta} u - f_{\alpha\beta}(x)\} = 0,$$

where $\mathcal{A}$, $\mathcal{B}$ are two index sets. For each $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$, L_α, $L_{\alpha\beta}$ are linear uniformly elliptic operators with bounded measurable coefficients, and f_α, $f_{\alpha\beta}$ are real functions. See [CC, Section 2.3] or [CIL, Section 1].

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The existence of the viscosity solutions for the Dirichlet problems of the equation (1) with the usual Dirichlet boundary condition

$$u = 0, \quad \text{on } \partial\Omega \tag{2}$$

in regular domains has been treated by several authors, notably Ishii [I] using the Perron's method and the uniqueness result for the $C^0(\bar{\Omega})$ solutions, Evans [E2] and Bian [Bi] by the accretive operator methods for the $W^{2,p}(\Omega)$ and $W^{1,\infty}(\Omega)$ solutions respectively, and Trudinger [T] by the continuity methods for the $C^{2,\alpha}(\bar{\Omega})$ solutions. However, there seem to be few results in general bounded domains. For an irregular bounded domain Ω, the boundary condition (2) is too strong a hypothesis. Generally, one cannot prescribe boundary values of the solutions at every point of $\partial\Omega$.

Berestycki, Nirenberg and Varadhan [BNV] worked with a refined version for the linear elliptic equations

$$\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u + c(x) u = f(x) \tag{3}$$

in general bounded domains. They introduced a function $u_0 \in W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ for all $p > 1$, satisfying

$$\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u_0 + \sum_{i=1}^{n} b^i(x) D_i u_0 = -1, \quad u_0 > 0, \text{ in } \Omega,$$

and used the notion

$$u \stackrel{u_0}{=} 0, \quad \text{on } \partial\Omega \tag{4}$$

to describe zero Dirichlet boundary condition in the following weak sense, which went back to [SV].

Definition 1 **(Weak Boundary Condition)** For a sequence $x^l \to \partial\Omega$, we say $x^l \stackrel{u_0}{\to} \partial\Omega$ if $u_0(x^l) \to 0$. Given $u \in C(\Omega)$, the notation $u \stackrel{u_0}{=} 0$ on $\partial\Omega$ means: along any sequence $x^l \stackrel{u_0}{\to} \partial\Omega$, we have $u(x^l) \to 0$.

In [BNV], the existence of the generalized Dirichlet problems (3), (4) was obtained, by formulating the refined maximum principle. Padilla [P] presented an extension of their results to the Riemann manifolds. Recently, we give in [B2] the necessary and sufficient conditions for the solvability with positive solutions in $W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ of the semilinear elliptic equations

$$\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^i(x) D_i u = f(x, u)$$

with the boundary conditions (4) on general bounded domains. Our methods are the refined a priori estimates and the degree theory. See [CJ] for the case of the linear parabolic equations.

Since the existence of the viscosity solutions is solved successfully, their regularity has attracted wide interest. In 1989, Caffarelli [C] developed a general technique using polynomial approximation for obtaining regularity of the viscosity solutions. In particular, he proved that if the homogeneous constant coefficient equation

$$F(D^2u, 0) = 0 \tag{5}$$

has $C^{2,\bar{\alpha}}$ estimates, and the $L^n(B_r)$ averages of f and the oscillation of F in x with respective to 0 are small compared to r^α for $\alpha \in (0, \bar{\alpha})$, then any viscosity solution of (1) in B_1 is actually $C^{2,\alpha}$ at $x = 0$.

In the linear theory, when one considers the C^2 solutions of the Poisson equations

$$\Delta u = f(x)$$

with $f(x)$ a Dini continuous function, the modulus of continuity of D^2u is well known, see [Bu] and [K2]. But if $f(x)$ is only continuous, then a counterexample [GT, Problem 4.9(a)] shows that it is possible for the Poisson equations to have no C^2 solution.

In the fully nonlinear setting, Kovats [K1] considered the C^2 solutions of the constant coefficient equation

$$F(D^2u, 0) = f(x)$$

with the L^∞-norm of $f(x)$ being Dini continuous, and proved that D^2u has modulus of continuity. He used the methods of polynomial approximation and maximum principle. In [K2] Kovats applied a result in Dini-Campanto space to discuss the regularity of the classical solutions for the equations (1) when the L^n averages of f satisfy the Dini conditions. But he depended on the global $C^{1,1}$ estimates of the corresponding homogeneous constant coefficient equations (5), which is a key mistake. In fact, (5) has only the $C^{2,\bar{\alpha}}$ interior estimates. Recently, Chen and Zou got Kovats's results in the case of parabolic equations, see [CZ].

The main purposes of the present paper are

(a) to extend the study of the existence in [BNV] for the $W^{2,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ strong solutions of the linear problems (3), (4) to the $C^\alpha(\Omega) \cap L^\infty(\Omega)$ viscosity solutions of the fully nonlinear elliptic boundary value problems:

$$F(D^2u, x) = f(x), \quad \text{in } \Omega, \tag{6}$$

$$u \overset{u_0}{=} 0, \quad \text{on } \partial\Omega, \tag{7}$$

and formulate the related existence theorem in the fully nonlinear case, where we use the Pucci operator to introduce the function u_0, which is totally different from the linear case;

(b) to generalize the results in [C] for the $C^{2,\alpha}(\Omega)$ regularity of the viscosity solutions of the fully nonlinear elliptic equations (6) to the $C^{2,\psi}(\Omega)$ regularity, where ψ is a modulus of continuity. Not as in [K1], our condition is only that $f(x)$ is Dini continuous in the weaker L^n sense. Moreover, our conditions are weaker than [K2].

The space $C^{2,\psi}(\Omega)$ is defined in the obvious generalization of the Hölder spaces, namely the space of all $u \in C^2(\Omega)$ with seminorm

$$\sup_{x,y\in\Omega} d_{x,y}^2 \psi(d_{x,y}) \frac{\|D^2u(x) - D^2u(y)\|}{\psi(|x-y|)} < +\infty,$$

where $d_{x,y} = \min(d_x, d_y)$ and $d_x = \operatorname{dist}(x, \partial\Omega)$.

We assume throughout this paper that F is a uniformly elliptic operator, *i.e.*, there are two positive constants θ and Θ such that for any $M \in \mathbf{R}^{n\times n}$ and $x \in \Omega$

$$\theta\|N\| \le F(M+N, x) - F(M, x) \le \Theta\|N\|, \quad \forall\, N \ge 0, \tag{8}$$

where $\|M\| = \sup_{|x|=1} |Mx|$. In following, without loss of generality, we also may assume

$$F(0, x) \equiv 0, \quad \text{in } \Omega. \tag{9}$$

For otherwise, we may replace $F(M, x)$ and $f(x)$ by $F(M, x) - F(0, x)$ and $f(x) - F(0, x)$ respectively.

To conclude this section, we give the definition of viscosity solutions, which take the place that the Dirichlet principle and the concept of variational solution enjoy in divergence form theory, of the equations (6).

Definition 2 **(Viscosity Solution)** A continuous function u in Ω is a viscosity subsolution (resp. viscosity supersolution) of (6) in Ω, when the following condition holds: if $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$ and $u - \varphi$ has a local maximum at x_0 then $F(D^2\varphi(x_0), x_0) \ge f(x_0)$ (resp. if $u - \varphi$ has a local minimum at x_0 then $F\big(D^2\varphi(x_0), x_0\big) \le f(x_0)$).

We say that u is a viscosity solution of (6) when it is subsolution and supersolution.

We say that $F(D^2u, x) \ge$ (resp. $\le, =$) $f(x)$ in the viscosity sense in Ω whenever u is a viscosity subsolution (resp. supersolution, solution) of (6) in Ω.

The C^2 classical solutions or $W^{2,q}$ $(q > n)$ strong solutions of (6) are viscosity solutions. Reversely, the C^2 or $W^{2,q}$ $(q > n)$ viscosity solutions of (6) are also classical or strong solutions.

The rest part of the paper is organized as follows: In the next section we introduce u_0 by means of the Pucci operator, and prove that this is reasonable. It is shown that $u \overset{u_0}{=} 0$ is the same as $u = 0$ to all boundary points admitting a barrier function. Section 3 is devoted to the existence theorem of the viscosity solutions to the usual Dirichlet problems (1), (2) by the m-accretive operator technique, and then to generalized Dirichlet problems (6), (7) by the approximation method. In the last section, the interior $C^{2,\psi}$ regularity of the viscosity solutions for the equations (6) is obtained, where the Caffarelli's perturbation method is used.

From now on, C will denote a positive constant, depending only on the space dimension n, the ellipticity constants θ, Θ and the diameter of the domain Ω; it may be different in each inequality or formula.

2 Construction of u_0

For the construction of u_0, we introduce the Pucci extremal operator

$$P^{+}(M) = \Theta \sum_{e_i>0} e_i + \frac{\theta}{n} \sum_{e_i<0} e_i,$$

where $e_1, e_2, \dots, e_n$ are the eigenvalues of $M \in \mathbf{R}^{n\times n}$. The usefulness of the Pucci extremal operator is in avoiding linearization of the equations (6).

It is clear that $P^{+}(M)$ is a uniformly elliptic (with ellipticity constants θ/n, $n\Theta$) and convex operator since its subadditivity

$$P^{+}(M_1 + M_2) \le P^{+}(M_1) + P^{+}(M_2), \quad M_1, M_2 \in \mathbf{R}^{n\times n},$$

see [CC, Lemma 2.10]. Moreover, the following Alexandroff-Bakelman-Pucci estimate from [CC, Theorem 3.6] holds for $P^{+}(M)$.

Theorem 3 (ABP **Estimate**) *Assume that $u \in C^0(\bar{\Omega})$ satisfies $P^{+}(D^2u) \ge f(x)$ in the viscosity sense in Ω, and $u \le 0$ on $\partial\Omega$. Then*

$$\sup_{\Omega} u \le C\|f^{-}\|_{L^n(\Omega)},$$

where $f^{-} = \max(-f, 0)$.

We also need the existence theorem of the classical solutions for the homogeneous constant coefficient equations on the smooth domains, which is easy to obtain from [CC, Proposition 9.8 and Remark 2].

Theorem 4 (**Existence**) *Let $\partial\Omega$ be smooth, $G(M)$ be convex or concave and $g \in C^0(\partial\Omega)$. If G satisfies (8), then there exists a unique solution $v \in C^{2,\bar{\alpha}}(\Omega) \cap C^0(\bar{\Omega})$ to the Dirichlet problem*

$$G(D^2v) = 0, \quad \textit{in}\ \Omega,\ v = g(x), \quad \textit{on}\ \partial\Omega,$$

and v satisfies

$$\|v\|^{*}_{C^{2,\bar{\alpha}}(\bar{B}_{r/2})} \le \bar{C}\left(\sup_{B_r}|v| + r^2|G(0)|\right),$$

for any ball $B_r \subset \Omega$, where $\bar{\alpha} \in (0,1)$ and $\bar{C} > 0$ are the constants depending only on n, θ and Θ. Here $\|v\|^{*}_{C^{2,\bar{\alpha}}(\overline{B_r})}$ denotes the adimensional $C^{2,\bar{\alpha}}(\overline{B_r})$ norm:

$$\|v\|^{*}_{C^{2,\bar{\alpha}}(\overline{B_r})} = \sup_{B_r}|v| + r\sup_{B_r}|Dv| + r^2\sup_{B_r}\|D^2v\| + r^{2+\bar{\alpha}}\sup_{x,y\in B_r}\frac{\|D^2v(x) - D^2v(y)\|}{|x-y|^{\bar{\alpha}}}.$$

Now, we begin to construct the function u_0 satisfying $P^{+}(D^2u_0) = -1$ in Ω and $u_0 = 0$, in some sense, on $\partial\Omega$.

Let $\{\Omega_k\}$ be a sequence of smooth domains such that

$$\Omega_k \subset \overline{\Omega_k} \subset \Omega_{k+1}, \quad k = 1, 2, \ldots, \bigcup_{k=1}^{\infty} \Omega_k = \Omega. \tag{10}$$

By Theorem 3 and 4, there is a unique solution $u_k \in C^{2,\bar{\alpha}}(\Omega_k) \cap C^0(\overline{\Omega_k})$ of

$$P^+(D^2 u_k) = -1, \quad \text{in } \Omega_k,$$
$$u_k = 0, \quad \text{on } \partial\Omega_k,$$

and

$$0 < u_k(x) \le C, \quad x \in \Omega_k, \|u_k\|_{C^{2,\bar{\alpha}}(\overline{\Omega'})} \le C(\Omega'), \quad \Omega' \subset\subset \Omega,$$

where $\bar{\alpha} \in (0, 1)$, and the constant $C(\Omega') > 0$ depends only on n, θ, Θ, diam Ω and dist$(\Omega', \partial\Omega)$. Noting

$$P^+\big(D^2(u_k - u_{k+1})\big) \ge P^+(D^2 u_k) - P^+(D^2 u_{k+1}) = 0, \quad \text{in } \Omega_k,$$
$$u_k - u_{k+1} \le 0, \quad \text{on } \partial\Omega_k,$$

and using Theorem 3 again we find the sequence $\{u_k(x)\}$ is strictly monotone increasing in k. Consequently, $u_k \to u_0$ in $C^2(\Omega)$, and $u_0 \in C^{2,\bar{\alpha}}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$P^+(D^2 u_0) = -1, \quad 0 < u_0 \le C, \text{ in } \Omega. \tag{11}$$

Next, we shall explain that such definition of u_0 is reasonable and natural.

Remark 5 It is easily seen that the function u_0 defined above is independent of the choices of the subdomains Ω_k, which implies that u_0 depends only on n, θ, Θ and Ω.

In fact, if $\{\Omega'_l\}$ is another sequence of smooth domains such that

$$\Omega'_l \subset \overline{\Omega'_l} \subset \Omega'_{l+1}, \quad l = 1, 2, \ldots, \bigcup_{l=1}^{\infty} \Omega'_l = \Omega,$$

and $u'_l \in C^{2,\bar{\alpha}}(\Omega'_l) \cap C^0(\overline{\Omega'_l})$ is a unique positive solution of

$$P^+(D^2 u'_l) = -1, \quad \text{in } \Omega'_l,$$
$$u'_l = 0, \quad \text{on } \partial\Omega'_l,$$

and $u'_l \to u'_0$ in $C^2(\Omega)$, then

$$P^+\big(D^2(u_k - u'_l)\big) \ge P^+(D^2 u_k) - P^+(D^2 u'_l) = 0, \quad \text{in } \Omega_k,$$
$$u_k - u'_l \le 0, \quad \text{on } \partial\Omega_k,$$

when $\Omega_k \subset \Omega_l'$. Hence we get from Theorem 3 $u_k \le u_l'$ in Ω_k and $u_0 \le u_0'$ in Ω. Similarly, $u_0' \le u_0$ in Ω.

Remark 6 When $\partial\Omega$ satisfies the exterior cone condition, $u \overset{u_0}{=} 0$ on $\partial\Omega$ is the same as u continuous on $\partial\Omega$ and $u(x) = 0$ for each $x \in \partial\Omega$.

We recall that $\partial\Omega$ satisfies an exterior cone condition at $y \in \partial\Omega$ if there is a truncated spherical cone with vertex at y:

$$C_y = \{y\} \cup \{x \in \mathbf{R}^n \mid 0 < |x-y| \le r, \theta_0 \le \theta(x) \le \pi\}$$

lying outside $\bar{\Omega}$. Here $r > 0$, $\theta_0 \in (\pi/2, \pi)$, and

$$\theta(x) = \arccos \frac{(x-y)\cdot\eta}{|x-y|} \in [0,\pi], \quad y \ne x,$$

for some unit vector $\eta \in \mathbf{R}^n$. Michael [M] has shown that such point y admits a strong barrier function for the linear elliptic operators. In particular, he proved [M, Theorem 2.1]

Theorem 7 *There exists a positive function $p \in C^2[0,\pi]$ and constants $\lambda \in (0,1)$, $\nu > 0$, $K > 0$, such that*

$$0 < h(x) = 1 - e^{-K|x-y|^{\lambda} p(\theta(x))} \in C^2(\mathbf{R}^n \setminus C_y)$$

has the property that

$$\sum_{i,j=1}^{n} a^{ij}(x) D_{ij} h \le -\lambda |x-y|^{\lambda-2} e^{-K|x-y|^{\lambda} p(\theta(x))}$$

for all $x \in \mathbf{R}^n \setminus C_y$, and every real symmetric matrix $\left(a^{ij}(x)\right)$ for which

$$\frac{\theta}{n}|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le \Theta|\xi|^2, \quad \xi \in \mathbf{R}^n.$$

Therefore, it means that in $U = \Omega \cap B_r(y)$ the above positive function $h \in C^2(U)$ satisfies

$$P^{+}(D^2 h) = \sup\Big\{ \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} h \Big\} \le -\lambda|x-y|^{\lambda-2} \mathrm{e}^{-K|x-y|^{\lambda} p(\theta(x))} \le -1,$$

for r small enough, where the sup is taken over all symmetric matrices $\left(a^{ij}(x)\right)$ whose eigenvalues belong to $[\theta/n, \Theta]$, see [CC, (2.5)].

Set $U_k = \Omega_k \cap B_r(y)$, and

$$h_k(x) = u_k(x) - A\left(h(x) + \frac{|x-y|^2}{4n\Theta} \right).$$

We may fix $A > 0$ depending only on $n, \theta, \Theta, \Omega$ and r, so that

$$P^+(D^2 h_k) \geq P^+(D^2 u_k) - A\Big(P^+(D^2 h) + \frac{1}{2}\Big) = -1 + \frac{A}{2} \geq 0, \quad \text{in } U_k,$$
$$h_k(x) = -A\left(h(x) + \frac{|x-y|^2}{4n\Theta}\right) \leq 0, \quad \text{on } \partial\Omega_k \cap B_r(y),$$
$$h_k(x) = u_k(x) - A\Big(h(x) + \frac{r^2}{4n\Theta}\Big) \leq C - \frac{Ar^2}{4n\Theta} \leq 0, \quad \text{on } \partial B_r(y) \cap \Omega_k.$$

Theorem 3 implies that $h_k \leq 0$ in U_k, *i.e.*

$$0 < u_k(x) \leq A\left(h(x) + \frac{|x-y|^2}{4n\Theta}\right) \quad \text{in } U_k.$$

Keeping x fixed and letting $k \to +\infty$, we find

$$0 < u_0(x) \leq A\left(h(x) + \frac{|x-y|^2}{4n\Theta}\right) \quad \text{in } U,$$

and

$$u_0(x) \to 0, \quad \text{as } x \to y.$$

This remark follows from Definition 1.

3 Existence of Viscosity Solution

The purpose of this section is to prove the existence theorem of the $C^\alpha(\Omega) \cap L^\infty(\Omega)$ viscosity solutions to the problems (6), (7) in the general domains. There are the Hölder estimate and the compactness result to be needed, see [CC, Section 4.3].

Theorem 8 **(Hölder Continuity)** *Let u be a viscosity solution of (1). Then for any $\Omega' \subset\subset \Omega$, we have $u \in C^\alpha(\overline{\Omega'})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega'})} \leq C(\Omega')(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^n(\Omega)}),$$

where $\alpha \in (0,1)$, and $C(\Omega') > 0$ is a constant depending only on n, θ, Θ and $dist(\Omega', \partial\Omega)$.

If Ω satisfies a uniform exterior sphere (of radius R) condition and u is a viscosity solution of (1), (2). Then $u \in C^\alpha(\bar{\Omega})$ and

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C,$$

for some constants $\alpha \in (0,1)$ and $C > 0$, which depend only on n, θ, Θ, R, Ω, $\|f\|_{L^n(\Omega)}$ and $\|u\|_{L^\infty(\Omega)}$.

Theorem 9 **(Compactness)** *Let $\{F_k\}_{k\geq 1}$ be a sequence of uniformly elliptic operators with ellipticity constants θ and Θ, and $\{u_k\}_{k\geq 1}$ be viscosity solutions in Ω of $F_k(D^2u_k, x) = f(x)$. Assume that $\{F_k\}_{k\geq 1}$ converges uniformly in compact sets of $\mathbf{R}^{n\times n} \times \Omega$ to F, and that $\{u_k\}_{k\geq 1}$ is uniformly bounded in compact sets of Ω. Then there exists $u \in C(\Omega)$ and a subsequence of $\{u_k\}_{k\geq 1}$ that converges uniformly to u in compact sets of Ω. Moreover, $F(D^2u, x) = f(x)$ in the viscosity sense in Ω.*

First we prove the existence theorem for the usual Dirichlet problems (1), (2) on the smooth domains by using the accretive operator methods as in [B1]. The basic facts about accretive operators may be found in [E1, Section 5] or [E2, Section 8]. The book of Barbu [Ba] contains their proofs and more explanation.

Theorem 10 **(Existence on Smooth Domain)** *Let $\partial\Omega \in C^3$, $F \in C^3(\mathbf{R}^{n\times n} \times \bar{\Omega})$, $f \in C^0(\bar{\Omega})$, and suppose that F satisfy (8) and*

$$|F_{Mx}(M,x)| + |F_{Mxx}(M,x)| \leq F_0, \quad (M,x) \in \mathbf{R}^{n\times n} \times \bar{\Omega},$$

for some positive constant F_0. Then the problems (1), (2) have a viscosity solution $u \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$.

Proof Define

$$\bar{F}(M,x) = -F(-M,x) - \frac{\theta}{2}\operatorname{trace}(M), \quad (M,x) \in \mathbf{R}^{n\times n} \times \Omega,$$

and define also for each $(y,z) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n}$

$$a^{ij}(x;y,z) = \int_0^1 \bar{F}_{M_{ij}}\big((1-t)y + tz, x\big)\, dt, \quad x \in \Omega,$$

$$f(x;y,z) = \bar{F}(y,x) - \sum_{i,j=1}^{n} a^{ij}(x;y,z)y_{ij}, \quad x \in \Omega,$$

and the linear uniformly elliptic operator

$$L^{y,z}u = -\sum_{i,j=1}^{n} a^{ij}(x;y,z)D_{ij}u$$

for

$$u \in D(L^{y,z}) = \{u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \mid L^{y,z}u \in C^0(\bar{\Omega})\}, \quad q > 2n.$$

Then the equation (1) can be rewritten into

$$-\frac{\theta}{2}\triangle u + \bar{F}(-D^2u, x) = -f(x), \quad x \in \Omega.$$

By the quasilinearization representation of $\bar{F}$ [E2, Lemma 2.2]

$$\bar{F}(-D^2u, x) = \max_{y\in\mathbf{R}^{n\times n}} \min_{z\in\mathbf{R}^{n\times n}} \{L^{y,z}u + f(x; y, z)\}.$$

According to the standard elliptic theory the operator

$$L^{y,z}\colon D(L^{y,z}) \subset C^0(\bar{\Omega}) \to C^0(\bar{\Omega})$$

is m-accretive in $C^0(\bar{\Omega})$. Fix $t > 0$ and let

$$J_t^{y,z} = (I + tL^{y,z})^{-1}$$

be the resolvents of $L^{y,z}$, and

$$L_t^{y,z} = \frac{I - J_t^{y,z}}{t}$$

its Yosida approximations. We know that each $L_t^{y,z}$ is an everywhere defined, Lipschitz, accretive operator on $C^0(\bar{\Omega})$.

Next choose $T > 1$ and select a C^1 function $\beta_T(s)$ such that

$$\beta_T(s) = \begin{cases} s, & \text{for } |s| \le T-1, \\ T, & \text{for } |s| \ge T, \end{cases} \tag{12}$$

$$0 \le \beta_T'(s) \le 1, \quad s \in \mathbf{R}^1,$$

and define

$$B_{t,T}(u) = \beta_T\Big(\max_{\|y\|\le 1/t} \min_{z\in\mathbf{R}^{n\times n}} \big(L_t^{y,z}u + f(x; y, z)\big)\Big).$$

Then $B_{t,T}$ is also an everywhere defined, Lipschitz, accretive operator on $C^0(\bar{\Omega})$.

Hence the Perturbation Lemma [E2, Lemma 8.1] implies the existence of a unique $u_{t,T} \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ solving

$$tu_{t,T} - \frac{\theta}{2}\triangle u_{t,T} + B_{t,T}(u_{t,T}) = -f(x). \tag{13}$$

By (12) and L^q theory, we have

$$\|u_{t,T}\|_{W^{2,q}(\Omega)} \le C(T), \quad \text{for each } q > 2n,$$

where $C(T) = C(n, \theta, q, \Omega, \sup_\Omega |f|, T)$. Owing to the above estimate there exists a sequence $t_k \to 0$ and a function $u_T \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ such that

$$\begin{aligned} &u_{t_k,T} \rightharpoonup u_T \text{ weakly in } W^{2,q}(\Omega), \\ &u_{t_k,T} \to u_T \text{ in } C^1(\bar{\Omega}). \end{aligned} \tag{14}$$

Consider now some given $\varphi \in C_0^4(\Omega)$, we claim

$$B_{t,T}(\varphi) \to \beta_T\big(\bar{F}(-D^2\varphi, x)\big) \tag{15}$$

uniformly on $\bar{\Omega}$ as $t \to 0$. Indeed, for any $y, z \in \mathbf{R}^{n\times n}$,

$$\begin{aligned}\|L_t^{y,z}\varphi - L^{y,z}\varphi\|_{C^0(\bar{\Omega})} &= \|J_t^{y,z}L^{y,z}\varphi - L^{y,z}\varphi\|_{C^0(\bar{\Omega})}\\ &\le t\|(L^{y,z})^2\varphi\|_{C^0(\bar{\Omega})} \le tC\|\varphi\|_{C^4(\bar{\Omega})}\end{aligned}$$

for $C = C(n, \Theta, F_0)$ independent of y, z. In the definition of $f(x; y, z)$, we observe by substituting $y = -D^2\varphi$,

$$\bar{F}(-D^2\varphi, x) = L^{-D^2\varphi, z}\varphi + f(x; -D^2\varphi, z), \quad z \in \mathbf{R}^{n\times n}.$$

Furthermore, we have

$$\begin{aligned}\bar{F}(-D^2\varphi, x) &= \min_{z\in\mathbf{R}^{n\times n}}\{L^{-D^2\varphi, z}\varphi + f(x; -D^2\varphi, z)\}\\ &\le \max_{\|y\|\le 1/t}\min_{z\in\mathbf{R}^{n\times n}}\{L^{y,z}\varphi + f(x; y, z)\} \le \bar{F}(-D^2\varphi, x)\end{aligned}$$

for $1/t \ge \sup_\Omega |D^2\varphi|$. Hence

$$\big|B_{t,T}(\varphi) - \beta_T\big(\bar{F}(-D^2\varphi, x)\big)\big| \le \max_{\|y\|\le 1/t}\min_{z\in\mathbf{R}^{n\times n}}|L_t^{y,z}\varphi - L^{y,z}\varphi| \le tC\|\varphi\|_{C^4(\bar{\Omega})},$$

and we obtain (15).

We define a kind of "partial inner product" for $f, g \in C^0(\bar{\Omega})$

$$[f, g]_+ = \inf_{t>0}\frac{\|f + tg\|_{C^0(\bar{\Omega})} - \|f\|_{C^0(\bar{\Omega})}}{t}.$$

The bracket $[\ ,\ ]_+$ has the characterization (see [E2, (8.14)])

$$[f, g]_+ = \max\{g(x_0)\cdot \operatorname{sgn} f(x_0) \mid x_0 \in \bar{\Omega}, |f(x_0)| = \|f\|_{C^0(\bar{\Omega})}\}, \quad f \not\equiv 0. \tag{16}$$

Now it follows from the accretiveness of $-\frac{\theta}{2}\triangle + B_{t,T}$, see [E2, (8.12)]

$$\Big[u_{t,T} - \varphi, \Big(-\frac{\theta}{2}\triangle u_{t,T} + B_{t,T}(u_{t,T})\Big) - \Big(-\frac{\theta}{2}\triangle\varphi + B_{t,T}(\varphi)\Big)\Big]_+ \ge 0$$

for any $\varphi \in C_0^2(\Omega)$. Then (13) implies

$$\Big[u_{t,T} - \varphi, -tu_{t,T} - f + \frac{\theta}{2}\triangle\varphi - B_{t,T}(\varphi)\Big]_+ \ge 0.$$

Let $t = t_k \to 0$, by (14), (15) and the upper-semicontinuity [E2, (8.11)] of $[\ ,\ ]_+$ with respect to uniform convergence,

$$\Big[u_T - \varphi, -f + \frac{\theta}{2}\triangle\varphi - \beta_T\big(\bar{F}(-D^2\varphi, x)\big)\Big]_+ \ge 0.$$

By (16) and the following technical lemma [E1, Lemma 2.2] or [E2, Lemma 9.1]

Lemma 11 *For a.e. $x_0 \in \Omega$ there exists a sequence $\{\varphi^k\} \subset C_0^4(\Omega)$, such that*

$$\varphi^k(x_0) \to u_T(x_0), \quad D\varphi^k(x_0) \to Du_T(x_0), \quad D^2\varphi^k(x_0) \to D^2u_T(x_0),$$

as $k \to \infty$, and

$$\varphi^k(x_0) - u_T(x_0) = \|\varphi^k - u_T\|_{C^0(\bar{\Omega})} > \varphi^k(x) - u_T(x),$$

for $x \in \Omega$, $x \neq x_0$, $k = 1, 2, \ldots$.

We have

$$-f(x_0) + \frac{\theta}{2}\triangle\varphi^k(x_0) - \beta_T\left(\bar{F}(-D^2\varphi^k(x_0), x_0)\right) \leq 0.$$

Let $k \to 0$ to find

$$-f(x_0) + \frac{\theta}{2}\triangle u_T(x_0) - \beta_T\left(\bar{F}(-D^2u_T(x_0), x_0)\right) \leq 0.$$

In the same way we can prove the reverse direction inequality. Therefore

$$\frac{\theta}{2}\triangle u_T - \beta_T\Big(-F(D^2u_T, x) + \frac{\theta}{2}\triangle u_T\Big) = f(x), \quad \text{a.e. } x \in \Omega.$$

Denote

$$F_T(M, x) = \frac{\theta}{2}\,\mathrm{trace}(M) - \beta_T\Big(-F(M, x) + \frac{\theta}{2}\,\mathrm{trace}(M)\Big).$$

It is clear that the ellipticity constants of F_T are $\theta/2, \Theta$, and $u_T \in W^{2,q}(\Omega)$ is a solution of

$$F_T(D^2u_T, x) = f(x), \quad \text{in } \Omega,\ u_T = 0, \quad \text{on } \partial\Omega.$$

Applying Theorem 3 and the following simple fact [CC, Proposition 2.13] to F_T

Lemma 12 *Let u satisfy $F(D^2u, x) \geq f(x)$ in the viscosity sense in Ω. Then $P^+(D^2u) \geq f(x)$ in Ω.*

We get

$$\sup_\Omega |u_T| \leq C.$$

By Theorem 8 we have

$$\|u_T\|_{C^\alpha(\bar{\Omega})} \leq C,$$

where C is independent of T. This estimate implies the existence of a subsequence (also denoted by u_T) and a function $u \in C^\alpha(\bar{\Omega})$ such that

$$u_T \to u \quad \text{in } C^0(\bar{\Omega}),$$

as $T \to \infty$. So according to Theorem 9 we know that that u is the viscosity solution of (1), (2).

Our existence result on the general domains is

Theorem 13 **(Existence on General Domain)** *Assume that $f \in C^0(\Omega) \cap L^\infty(\Omega)$ and $F \in C^1(\mathbf{R}^{n\times n} \times \Omega)$. If F satisfies (8), then the Dirichlet problem (6), (7) has a viscosity solution $u \in C^\alpha(\Omega) \cap L^\infty(\Omega)$ for some $\alpha \in (0, 1)$.*

Proof Let two C^3 sequences $\{F_k\}$ and $\{f_k\}$ converge uniformly in the compact sets of $\mathbf{R}^{n\times n}\times\Omega$ and Ω to F and f respectively. There is no loss of generality in assuming that $\{F_k\}$ is with ellipticity constants θ,Θ and $\{f_k\}$ is uniformly bounded in Ω. Moreover we can also suppose that

$$F_k(M,x)\equiv 0,\quad \|M\|+|x|>k.$$

It follows directly from Theorem 10 that there exists a function $u^k\in C^\alpha(\overline{\Omega_k})$, which solves in the viscosity sense

$$F_k(D^2u^k,x)=f_k(x),\quad \text{in } \Omega_k, \tag{17}$$

$$u^k=0,\quad \text{on } \partial\Omega_k, \tag{18}$$

where $\{\Omega_k\}$ is defined by (10). If we can prove that there is a positive constant C (independent of k) such that

$$|u^k(x)|\le Cu_0(x),\quad x\in\Omega_k, \tag{19}$$

then by Theorem 8 we have

$$\|u^k\|_{C^\alpha(\Omega)}\le C.$$

We find a subsequence of $\{u^k\}$ (still denoted by itself), such that $u^k\to u$ in $C(\Omega)$ and

$$|u(x)|\le Cu_0(x),\quad x\in\Omega.$$

Applying Theorem 9 and Definition 1, $u\in C^\alpha(\Omega)\cap L^\infty(\Omega)$ is a viscosity solution of (6), (7).

Now we establish the estimate (19). By Lemma 12, we have

$$P^+(D^2u^k)\ge f_k(x),\quad \text{in } \Omega_k.$$

We claim that in the viscosity sense

$$P^+\big(D^2(u_k-Cu_0)\big)\ge 0,\quad \text{in } \Omega_k, \tag{20}$$

for some constant (independent of k) $C>0$. It follows immediately from Theorem 3 that $u^k-Cu_0\le 0$ in Ω_k.

To show (20), we need to prove that if $x_0\in\Omega_k$, $\varphi\in C^2(\Omega_k)$, and $u^k-Cu_0-\varphi$ has a local maximum at x_0, then

$$P^+\big(D^2\varphi(x_0)\big)\ge 0.$$

In fact, by the definition that u^k is a viscosity solution of (17) and Lemma 12, we get

$$P^+\big(D^2(Cu_0+\varphi)(x_0)\big)\ge f_k(x_0),$$

$$P^+\big(D^2\varphi(x_0)\big)\ge P^+\big(D^2(Cu_0+\varphi)(x_0)\big)-P^+(CD^2u_0)\ge f_k(x_0)+C\ge 0.$$

Here we have used (11).

Applying above result to $-u^k$, which satisfies

$$-F_k\big(-D^2(-u^k),x\big)=-f_k(x),$$

we obtain $-u^k-Cu_0\le 0$ in Ω_k. This existence theorem on the general domains is proved.

4 $C^{2,\psi}$ Regularity

Introduce the function

$$\beta_y(x) = \sup_{M \in \mathbf{R}^{n\times n}} \frac{|F(M,x) - F(M,y)|}{\|M\| + 1}, \quad x, y \in \Omega,$$

which measures the oscillation of F at y respect to x. We shall prove that if the $L^n(B_r)$ averages of f and β_y have a modulus of continuity ω, that is,

$$\frac{1}{r}\Big(\int_{B_r(y)} \beta_y^n(x)\,dx\Big)^{\frac{1}{n}} + \frac{1}{r}\Big(\int_{B_r(y)} |f(x) - f(y)|^n\,dx\Big)^{\frac{1}{n}} \le \omega(r), \tag{21}$$

for any $y \in \Omega$, $r \in (0, 1/2)$, $B_r \subset \Omega$, then under certain restrictions on ω, the viscosity solutions $u \in C^{2,\psi}(\Omega)$ of the equations (6), where

$$\psi(r) = \omega(r) + \int_0^r \frac{\omega(s)}{s}\,ds.$$

We require that ω satisfies the Dini condition

$$\int_0^1 \frac{\omega(r)}{r}\,dr < +\infty, \tag{22}$$

and the additional assumptions

$$\liminf_{\mu\to 0+} \sup_{0<r\le 1/2} \frac{\omega(\mu r)}{\omega(r)} = 0, \tag{23}$$

$$\liminf_{\lambda\to 0+} \sup_{k=0,1,2,\dots} \frac{\lambda^{\bar\alpha}\omega(\lambda^k)}{\omega(\lambda^{k+1})} = 0, \tag{24}$$

where $\bar\alpha \in (0,1)$ is given by Theorem 4.

Remark 14 The conditions (22), (23) and (24) are satisfied by

$$\omega(r) = r^\alpha\Big(\ln\frac{1}{r}\Big)^\tau, \quad 0 < \alpha < \bar\alpha,\ \tau \in \mathbf{R}.$$

This enables us to generalize the results for Hölder continuous functions in $C^\alpha(\Omega)$, $\alpha \in (0, \bar\alpha)$.

Remark 15 The conditions (23) and (24) are technical restrictions, required only in our proof of Theorem 16 and Lemma 19 below respectively. (24) is weaker than [($\star$), K1] and [(14), K2]:

$$\lim_{\mu\to 0+} \sup_{0<t\le 1/2} \frac{\mu^{\bar\alpha}[t^{\bar\alpha} + \omega(t)]}{(\mu t)^{\bar\alpha} + \omega(\mu t)} = 0.$$

In fact, we fix $t = 1/2$ in Kovats's condition, and get

$$\lim_{\mu\to 0+} \frac{\mu^{\bar{\alpha}}}{\omega(\mu)} = 0.$$

So (24) can be derived from the following inequality

$$\frac{\lambda^{\bar{\alpha}}\omega(\lambda^k)}{\omega(\lambda^{k+1})} \leq \frac{2\lambda^{\bar{\alpha}}\omega(\lambda^k)}{\lambda^{(k+1)\bar{\alpha}} + \omega(\lambda^{k+1})} \leq \frac{2\lambda^{\bar{\alpha}}[(\lambda^k)^{\bar{\alpha}} + \omega(\lambda^k)]}{(\lambda\cdot\lambda^k)^{\bar{\alpha}} + \omega(\lambda\cdot\lambda^k)},$$

if λ is small enough.

Now we state the interior regularity theorem as following:

Theorem 16 **($C^{2,\psi}$ Regularity)** *Let $u \in C(\Omega)\cap L^\infty(\Omega)$ be a viscosity solution of (6) in Ω, where F, f satisfy (8), (9), (21) and F is a concave function of M, while ω satisfies (22), (23) and (24). Then $u \in C^{2,\psi}(\Omega)$.*

For simplicity, in proving our results, without loss of generality we can assume $\Omega = B_1$, $y = 0$, $\beta = \beta_0$, $f(0) = 0$, $|u| \leq 1$ in B_1 and only prove that u is $C^{2,\psi}$ at 0.

Before starting our proof, we point out that $\omega(r)$ has following property.

Lemma 17 *For any $\lambda \in (0, 1/e]$, we have*

$$\sum_{i=k+1}^{\infty} \omega(\lambda^{i-1}) \leq \psi(\lambda^k),$$

for $k = 0, 1, 2, \ldots$.

Proof

$$\sum_{i=k+1}^{\infty}\omega(\lambda^{i-1}) = \omega(\lambda^k) + \sum_{i=k+2}^{\infty}\int_{i-k-1}^{i-k}\omega(\lambda^{i-1})\,dx \leq \omega(\lambda^k) + \int_1^\infty \omega(\lambda^{x+k-1})\,dx$$
$$= \omega(\lambda^k) + \frac{1}{\ln\lambda}\int_{\lambda^k}^{0}\frac{\omega(s)}{s}\,ds \leq \psi(\lambda^k).$$

The approximation lemma [C, Lemma 11] is a key ingredient in our proof.

Lemma 18 **(Approximation Lemma)** *Let u be a viscosity solution of the equation (6) in B_1. Then there exists a positive constant $\gamma = \gamma(n, \theta, \Theta)$, such that if $\|\beta\|_{L^n(B_1)} \leq \varepsilon$, then*

$$\sup_{B_{1/2}} |u - v| \leq C(\varepsilon^\gamma + \|f\|_{L^n(B_1)}),$$

where $v \in C^{2,\bar{\alpha}}(B_{3/4}) \cap C(\overline{B_{3/4}})$ is a solution of

$$F(D^2 v, 0) = 0, \quad \textit{in } B_{3/4},\ v = u, \quad \textit{on } \partial B_{3/4}.$$

Next, we start proving the following two lemmas that will be used to prove Theorem 16.

Lemma 19 *If F is a concave function of M, and*

$$\|\beta\|_{L^n(B_r)} \leq \delta r\omega(r),\ \|f\|_{L^n(B_r)} \leq \delta r\omega(r), \quad r \in (0, 1/2). \tag{25}$$

Then there exist λ, $\delta \in (0, 1/e)$, depending only on n, θ, Θ, $\psi(1)$ and the behaviour of ω in the lim inf *arising in (24), and a family of second order polynomials*

$$P^k(x) = A^k + B^k x + \frac{1}{2}x^t C^k x$$

with

$$F(C^k, 0) = 0, \tag{26}$$

so that

$$\sup_{B_{\lambda^k}} |u - P^k| \leq \lambda^{2k}\omega(\lambda^k), \tag{27}$$

$$|A^k - A^{k-1}|,\ \lambda^{k-1}|B^k - B^{k-1}|,\ \lambda^{2(k-1)}\|C^k - C^{k-1}\| \leq 8\bar{C}\lambda^{2(k-1)}\omega(\lambda^{k-1}), \tag{28}$$

for $k = 0, 1, 2, \ldots$, where $P^0 \equiv P^{-1} \equiv 0$, and $\bar{C}$ is given by Theorem 4.

Proof We prove the lemma by induction. When $k = 0$, (26), (27) and (28) are clearly. Suppose now that they are true for $0, 1, 2, \ldots, k$. We shall prove that (26), (27) and (28) hold for $k + 1$.

Let

$$w(x) = \frac{(u - P^k)(\lambda^k x)}{\lambda^{2k}\omega(\lambda^k)}, \quad \text{for } x \in B_1,$$

then $\sup_{B_1} |w| \leq 1$, and

$$F^k(D^2 w, x) = f^k(x),$$

where

$$F^k(M, x) = \frac{1}{\omega(\lambda^k)}\left[F\big(\omega(\lambda^k)M + C^k, \lambda^k x\big) - F(C^k, \lambda^k x)\right],$$

$$f^k(x) = \frac{1}{\omega(\lambda^k)}[f^k(\lambda^k x) - F(C^k, \lambda^k x)],$$

and

$$\begin{aligned}\beta^k(x) &= \sup_{M\in\mathbf{R}^{n\times n}} \frac{|F^k(M,x)-F^k(M,0)|}{\|M\|+1}\\ &\le \frac{1}{\omega(\lambda^k)}\left[\sup_{M\in\mathbf{R}^{n\times n}} \frac{\left|F\big(\omega(\lambda^k)M+C^k,\lambda^k x\big)-F\big(\omega(\lambda^k)M+C^k,0\big)\right|}{\|M\|+1}\right.\\ &\qquad \left.+\sup_{M\in\mathbf{R}^{n\times n}}\frac{|F(C^k,\lambda^k x)-F(C^k,0)|}{\|M\|+1}\right]\\ &\le \frac{\beta(\lambda^k x)}{\omega(\lambda^k)}\left[\sup_{M\in\mathbf{R}^{n\times n}}\frac{\|\omega(\lambda^k)M+C^k\|+1}{\|M\|+1}+\sup_{M\in\mathbf{R}^{n\times n}}\frac{\|C^k\|+1}{\|M\|+1}\right]\\ &\le C\frac{\beta(\lambda^k x)}{\omega(\lambda^k)}.\end{aligned}$$

Here we have used that

$$\|C^k\| \le \sum_{i=1}^{k}\|C^i-C^{i-1}\| \le 8\bar{C}\sum_{i=1}^{k}\omega(\lambda^{i-1}) \le 8\bar{C}\psi(1)$$

by the inductive assumption (28) and Lemma 17.

Now, a direct calculation, (25) and (26) yield

$$\|\beta^k\|_{L^n(B_1)} \le \frac{C}{\lambda^k\omega(\lambda^k)}\|\beta\|_{L^n(B_{\lambda^k})} \le \frac{C}{\lambda^k\omega(\lambda^k)}\cdot\delta\lambda^k\omega(\lambda^k) = C\delta,$$

$$\begin{aligned}\|f^k\|_{L^n(B_1)} &\le \frac{1}{\lambda^k\omega(\lambda^k)}[\|f\|_{L^n(B_{\lambda^k})}+(1+\|C^k\|)\|\beta\|_{L^n(B_{\lambda^k})}]\\ &\le (2+\|C^k\|)\delta \le C\delta.\end{aligned}$$

By Theorem 4 and the Approximation Lemma (Noting F^k satisfies uniformly (8) and (9)), there is a function $v\in C^{2,\bar{\alpha}}(B_{3/4})\cap C(\overline{B_{3/4}})$ such that

(29) $$F^k(D^2v,0)=0,\quad \text{in } B_{3/4},\ v=w,\quad \text{on } \partial B_{3/4},$$

and

$$\sup_{B_{1/2}}|w-v| \le C\big((C\delta)^\gamma+\|f^k\|_{L^n(B_1)}\big) \le C\delta^\gamma.$$

Let

$$P(x)=v(0)+Dv(0)x+\frac{1}{2}x^tD^2v(0)x,$$

then applying the interior estimates in Theorem 4 we have

$$\sup_{B_\lambda}|w-P| \le \sup_{B_\lambda}|w-v| + \sup_{B_\lambda}|v-P|$$
$$\le C\delta^\gamma + [v]_{2,\bar{\alpha},B_{3/8}} \cdot \lambda^{2+\bar{\alpha}} \le C(\delta^\gamma + \lambda^{2+\bar{\alpha}}).$$

We first choose $\lambda \in (0, 1/e)$ by (24) and then δ, so that

$$\sup_{B_\lambda}|w-P| \le \lambda^2 \frac{\omega(\lambda^{k+1})}{\omega(\lambda^k)}.$$

Rescaling back, we get

$$\sup_{B_{\lambda^{k+1}}}\left| u(x) - P^k(x) - \lambda^{2k}\omega(\lambda^k)P\Big(\frac{x}{\lambda^k}\Big)\right| \le \lambda^{2(k+1)}\omega(\lambda^{k+1}).$$

Denote

$$P^{k+1}(x) = P^k(x) + \lambda^{2k}\omega(\lambda^k)P\Big(\frac{x}{\lambda^k}\Big),$$

we have by (29), the definition of F^k and (26)

$$F(C^{k+1}, 0) = F\big(C^k + \omega(\lambda^k)D^2v(0), 0\big) = F(C^k, 0) = 0,$$

and

$$|A^{k+1} - A^k|, \lambda^k|B^{k+1} - B^k|, \lambda^{2k}\|C^{k+1} - C^k\|$$
$$\le \lambda^{2k}\omega(\lambda^k)\big(|v(0)| + |Dv(0)| + \|D^2v(0)\|\big)$$
$$\le \Big(\frac{8}{3}\Big)^2 \bar{C}\lambda^{2k}\omega(\lambda^k).$$

Hence, $P^{k+1}(x)$ satisfies the required conditions.

Lemma 20 *Under the same conditions as Lemma 19, there exists a second polynomial $P^\infty(x)$, such that*

$$\sup_{B_r}|u - P^\infty| \le Cr^2\psi(r),$$

for any $r \in (0, 1/2)$.

Proof From (28) and Lemma 17, three series

$$\sum_{k=0}^{\infty}|A^k - A^{k-1}|, \quad \sum_{k=0}^{\infty}|B^k - B^{k-1}|, \quad \sum_{k=0}^{\infty}\|C^k - C^{k-1}\|$$

are convergent. Hence, A^k, B^k and C^k converge. Let

$$A = \lim_{k\to\infty} A^k, \quad B = \lim_{k\to\infty} B^k, \quad C = \lim_{k\to\infty} C^k,$$

and

$$P^\infty(x) = A + Bx + \frac{1}{2}x^t Cx.$$

Clearly, $\{P^k\}$ converges uniformly in B_1 to P^∞.

For $k = 0, 1, 2, \ldots$, we have by (27), (28) and Lemma 17,

$$\begin{aligned}
\sup_{B_{\lambda^k}} |u - P^\infty| &\le \sup_{B_{\lambda^k}} |u - P^k| + \sum_{i=k+1}^{\infty} \sup_{B_{\lambda^k}} |P^i - P^{i-1}| \\
&\le \lambda^{2k}\omega(\lambda^k) + \sum_{i=k+1}^{\infty} (|A^i - A^{i-1}| + \lambda^k |B^i - B^{i-1}| + \lambda^{2k}\|C^i - C^{i-1}\|) \\
&\le \lambda^{2k}\omega(\lambda^k) + 8\bar{C} \sum_{i=k+1}^{\infty} \omega(\lambda^{i-1})(\lambda^{2(i-1)} + \lambda^k \lambda^{i-1} + \lambda^{2k}) \\
&\le (1 + 24\bar{C})\lambda^{2k}\psi(\lambda^k).
\end{aligned}$$

For any given $r \in (0, 1/2)$, there is some k, such that $\lambda^{k+1} < r \le \lambda^k$. Therefore

$$\sup_{B_r} |u - P^\infty| \le \sup_{B_{\lambda^k}} |u - P^\infty| \le C\lambda^{2k}\psi(\lambda^k) \le 2C\lambda^{2(k+1)-3}\psi(\lambda^{k+1}) \le \frac{2C}{\lambda^3} r^2 \psi(r).$$

This completes the proof for Lemma 20.

We can now give the

Proof of Theorem 16 Introduce the function

$$\bar{u}(y) = \frac{u(\mu y)}{K\mu^2},\ K = \mu^{-2} \sup_{B_1} |u| + \delta^{-1} + 1, \quad \mu \in (0, 1).$$

It follows from (6)

$$\bar{F}(D^2\bar{u}, y) = \bar{f}(y), \quad |\bar{u}| \le 1,\ y \in B_1,$$

where

$$\bar{F}(M, y) = \frac{1}{K} F(KM, \mu y),$$

$$\bar{f}(y) = \frac{1}{K} f(\mu y),$$

and

$$\bar{\beta}(y) = \sup_{M \in \mathbf{R}^{n\times n}} \frac{|F(KM, \mu y) - F(KM, 0)|}{K(\|M\| + 1)} \le \beta(\mu y),$$

Using the condition (23), we have

$$\sup_{0<r\le 1/2} \frac{\omega(\mu r)}{\omega(r)} < \delta$$

for δ needed in (25) and some small $\mu \in (0,1)$, and by (21)

$$\|\bar{\beta}\|_{L^n(B_r)} \leq \frac{1}{\mu}\|\beta\|_{L^n(B_{\mu r})} \leq \frac{1}{\mu} \cdot \mu r\omega(\mu r) < \delta r\omega(r),$$
$$\|\bar{f}\|_{L^n(B_r)} \leq \frac{1}{K\mu}\|f\|_{L^n(B_{\mu r})} \leq \frac{1}{K\mu} \cdot \mu r\omega(\mu r) < \delta r\omega(r),$$

for all $r \in (0,1/2)$. Therefore

$$\sup_{B_r} |u - P^\infty| \leq Cr^2\psi(r), \quad r \in (0,1/2), \tag{30}$$

where $P^\infty(x)$ comes from Lemma 20, and the constant C depends only on n, θ, Θ, diam Ω, $\psi(1)$ and the behaviour of ω in the lim inf arising in (23), (24). The desired estimate (30) follows immediately that u is $C^{2,\psi}$ at 0, see [CC, Section 8.1].

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