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# Fractional Hardy–Hénon equations on exterior domains ☆

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**Abstract**

In this paper, we consider the fractional Hardy–Hénon equations with an isolated singularity. If the isolated singularity is located at the origin, we give a classification of solutions to this equation. If the isolated singularity is located at infinity, in the case of exterior domains, we provide decay estimates of solutions and their gradients at infinity. Our results are an extension of the classical work by Caffarelli, Gidas et al.


*MSC:* 35R11; 35B06

*Keywords:* Fractional Laplacian; Isolated singularity; Radially symmetric; Rate estimate

## 1. Introduction

The global behaviors of

$$-\Delta u = |x|^{\tau} u^{p} \qquad \text{in} \quad \mathbb{R}^{n} \setminus \{0\}, \tag{1}$$

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with one singularity at the origin have aroused great attention, where $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian, $\tau > -2$, $p > 1$ and $n \geq 3$. Equation (1) is traditionally called the Hénon (resp., Hardy, or Lane–Emden) equation for $\tau > 0$ (resp., $\tau < 0$, $\tau = 0$).

In the special case of $\tau = 0$ and $\frac{n}{n-2} \leq p \leq \frac{n+2}{n-2}$, Caffarelli–Gidas–Spruck [7] proved that if the origin is a non-removable singularity, then a nonnegative $C^2$ solution of (1) is radially symmetric about the origin. Notice that we say the origin 0 is a non-removable singularity of solution $u$ if $\limsup_{x\to 0} u(x) = +\infty$. See Harrell and Simon [14] for $-2 < \tau < 2$ and $1 < p < \frac{n+\tau}{n-2}$, Aviles [1,2] for $-2 < \tau < 2$ and $p = \frac{n+\tau}{n-2}$, Gidas–Spruck [16] for $-2 < \tau < 2$ and $\frac{n+\tau}{n-2} < p < \frac{n+2}{n-2}$.

In the classical paper [16], Gidas–Spruck also studied the isolated singularity located at infinity. They obtained that if $u$ is a $C^2$ positive solution of

$$-\Delta u = |x|^{\tau} u^p \qquad \text{in} \quad \mathbb{R}^n \backslash \overline{B_1}, \tag{2}$$

with $\tau > -2$, $n \geq 3$, $1 < p < \frac{n+2}{n-2}$, and $\mathbb{R}^n \backslash \overline{B_1} := \{x \in \mathbb{R}^n; |x| > 1\}$, then

$$u(x) \leq C|x|^{-\frac{2+\tau}{p-1}}, \quad |\nabla u(x)| \leq C|x|^{-\frac{1+\tau+p}{p-1}} \quad \text{as} \ x \to +\infty. \tag{3}$$

Recently, for the estimate (3) of the problem (2), Phan–Souplet [28] present a simpler proof. Indeed, Gidas–Spruck [16] studied a more general case and (2) is a special case of it. In particular, for the case $\tau = 0$, Serrin–Zou [30] extended the first estimate of (3) to degenerate elliptic equations

$$-\triangle_m u = u^p \qquad \text{in} \quad \mathbb{R}^n \setminus \overline{B_1}. \tag{4}$$

Here $1 < m < n$, $m - 1 < p < \frac{n(m-1)+m}{n-m}$, and

$$\triangle_m u := \operatorname{div}(|\nabla u|^{m-2} \nabla u)$$

is the well-known $m$-Laplace operator. Serrin–Zou [30] obtained that

$$u(x) \leq C|x|^{-\frac{m}{p+1-m}}, \quad \text{as} \ x \to +\infty.$$

After that, Polácik–Quittner–Souplet [29] proved that

$$u(x) \leq C|x|^{-\frac{m}{p+1-m}}, \quad |\nabla u(x)| \leq C|x|^{-\frac{p+1}{p+1-m}} \quad \text{as} \ x \to +\infty.$$

Notice that for $m = 2$, $\triangle_m u$ stands for the classical Laplace operator. It is consistent with the estimate of (3) for (4). We may also see [21,20,26,27] for more details about the isolated singularity.

The above results stimulate us to study the behaviors of solutions for the fractional Hardy–Hénon equations with an isolated singularity. This paper is aiming at studying the global behaviors of positive solutions of

$$(-\Delta)^{\sigma} u = |x|^{\tau} u^p \qquad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \tag{5}$$

where $0 < \sigma < 1$, $\tau \in \mathbb{R}$, $p > 1$, $n \geq 2$, and $(-\Delta)^\sigma$ is the fractional Laplacian taking the form

$$\begin{aligned}(-\Delta)^\sigma u(x) :&= C_{n,\sigma} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy \\ &= C_{n,\sigma} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy,\end{aligned} \tag{6}$$

here P.V. stands for the Cauchy principal value and

$$C_{n,\sigma} := \frac{2^{2\sigma} \sigma \Gamma(\frac{n}{2} + \sigma)}{\pi^{\frac{n}{2}} \Gamma(1 - \sigma)}$$

with the gamma function $\Gamma$. In recent years, there has been a great deal of interest in using the fractional Laplacian $(-\Delta)^\sigma$ to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars (see [4,10,13,34] and the references therein).

The operator $(-\Delta)^\sigma$ is well defined in the Schwartz space of rapidly decaying $C^\infty$ functions in $\mathbb{R}^n$. One can also define the fractional Laplacian acting on spaces of functions with weaker regularity. $L_\sigma(\mathbb{R}^n)$ is the space defined as

$$L_\sigma(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\sigma}} dx < \infty \right\},$$

with the norm

$$\|u\|_{L_\sigma(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\sigma}} dx.$$

We can verify that if $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap L_\sigma(\mathbb{R}^n)$, the integral on the right hand side of (6) is well defined in $\mathbb{R}^n \setminus \{0\}$. Moreover, from [31, Proposition 2.4], it follows that

$$\begin{aligned}&(-\Delta)^\sigma u \in C^{1,1-2\sigma}(\mathbb{R}^n \setminus \{0\}), \qquad \text{if} \quad 0 < \sigma < \tfrac{1}{2}, \\ &(-\Delta)^\sigma u \in C^{0,2-2\sigma}(\mathbb{R}^n \setminus \{0\}), \qquad \text{if} \quad \tfrac{1}{2} \leq \sigma < 1.\end{aligned}$$

Problems concerning the fractional Laplacian $(-\Delta)^\sigma$ with an isolated singularity at the origin have attracted a lot of attention. In particular, Caffarelli–Jin–Sire–Xiong [8] studied the global behaviors of positive solutions of the fractional Yamabe equations

$$(-\Delta)^\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}} \qquad \text{in } \mathbb{R}^n \setminus \{0\} \tag{7}$$

with an isolated singularity at the origin. By the method of moving spheres, they obtained that if the origin is a non-removable isolated singularity, then the solution $u$ of (7) is radially symmetric with respect to the origin and monotonically decreasing radially. It is consistent with the result of

Caffarelli–Gidas–Spruck [7] work on Laplacian. Jin–de Queiroz–Sire–Xiong [18] further studied the equation (7) in $\mathbb{R}^n \setminus \mathbb{R}^k$ with condition that there exists $x_0 \in \mathbb{R}^k$ such that $\limsup_{x\to(x_0,0)} u = +\infty$. And they obtained the solution $u$ of (7) is radially symmetric with respect to the origin. Sun–Xiong [32] recently studied for higher order fractional case.

The first of our main results concerns the global behaviors of (5) with a singularity located at the origin.

**Theorem 1.1.** *Let $-2\sigma < \tau \le 0$, $1 < p \le \frac{n+2\sigma+2\tau}{n-2\sigma}$. If $u \in C^2(\mathbb{R}^n \setminus \{0\}) \cap L_\sigma(\mathbb{R}^n)$ is a positive solution of* (5)*, and suppose that the origin* 0 *is a non-removable singularity, then $u(x)$ is radially symmetric and monotonically decreasing radially.*

Theorem 1.1 extends results from [7] and [8]. It is known that the fractional Laplacian operator is nonlocal. To overcome the problem, we will make use of the extension method which is introduced by Caffarelli–Silvestre [9]. After that, by the method of moving spheres, which has been widely used and has become a powerful and user-friendly tool in the study of nonlinear partial differential equations (see [11,22–24,35]), and Kelvin transformation, we can finish the proof of Theorem 1.1.

The second main result of the paper is to treat the isolated singularity located at infinity. We provide decay estimates and the gradient estimates at infinity.

**Theorem 1.2.** *Let $1 < p < \frac{n+2\sigma}{n-2\sigma}$. Suppose that $u \in C^2(\mathbb{R}^n \setminus \overline{B_1}) \cap L_\sigma(\mathbb{R}^n)$ is a positive solution of*

$$(-\Delta)^\sigma u = |x|^\tau u^p \qquad \text{in } \mathbb{R}^n \setminus \overline{B_1}, \tag{8}$$

*then there exists a positive constant $C = C(n, \sigma, \tau, p)$ such that*

$$u(x) \le C|x|^{-\frac{2\sigma+\tau}{p-1}}, \quad |\nabla u(x)| \le C|x|^{-\frac{2\sigma+\tau+p-1}{p-1}} \qquad \text{near } x \to +\infty. \tag{9}$$

Theorem 1.2 extends results from [16] and [28]. It is also an extension of the result on the Laplacian to the fractional Laplacian. Our proof of Theorem 1.2 is based on the observation that estimates (9) for given $p$, $\tau$ can be reduced to the Liouville property for the same $p$ but with $\tau$ replaced by 0. It is based on the important fact ([19, Remark 1.9], [12, Theorem 4]) that the only nonnegative solution of

$$(-\Delta)^\sigma u = u^p \qquad \text{in } \mathbb{R}^n,$$

with $1 < p < \frac{n+2\sigma}{n-2\sigma}$, is $u = 0$. With the help of the doubling property (see Section 3 below), we first obtain the uniform estimate with respect with the (Hölder bounded) coefficient $c(x)$. Then by a change of variable, we can replace the coefficient $|x|^\tau$ with a smooth function which is bounded and bounded away from 0 in a suitable domain. It follows that we finish the proof of this theorem.

The organization of this paper is as follows: In section 2, we first explain that in order to prove Theorem 1.1, it suffices to prove (10). For this purpose, we just need to obtain (17). After that, we will give a proof for Theorem 1.2 in Section 3. Besides, for readers' convenience, we also prove and collect some basic propositions which will be used in our proof in Appendix.

## 2. Global solutions with an isolated singularity

### 2.1. Proof of Theorem 1.1

**Proof.** In this section, we denote $B_R(x)$ as the ball in $\mathbb{R}^n$ with radius $R$ and center $x$. To prove that $u$ is radially and monotonically decreasing radially it suffices to show that $u$ is symmetrical about any hyperplane passing through the origin and it is monotone decreasing along the normal direction. Without loss of generality, let us prove that $u$ is symmetric about the hyperplane $\{y_1 = 0\}$ and it is monotone decreasing along the $y_1$ axis.

For all $x \in \mathbb{R}^n \setminus \{0\}$, assume that for any $\lambda \in (0, |x|)$,

$$u_{x,\lambda}(y) \leq u(y) \qquad \text{in} \quad \mathbb{R}^n \backslash (B_\lambda(x) \cup \{0\}), \tag{10}$$

where

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2\sigma} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

Let $t, s \in \mathbb{R}$ satisfy $t \leq s$, $t + s > 0$ and $m > \max\{s, \frac{st}{s+t}\}$, then $0 < (m-s)(m-t) < m^2$. With the help of (10), choosing $y = te_1$, $x = me_1$ and $\lambda^2 = (m-s)(m-t)$, we have

$$\left(\frac{\sqrt{(m-s)(m-t)}}{m-t}\right)^{n-2\sigma} u\left[\left(m + \frac{(m-s)(m-t)}{t-m}\right)e_1\right] \leq u(te_1),$$

where the unite vector $e_1 := (1, 0, \cdots, 0) \in \mathbb{R}^n$. That is,

$$\left(\frac{m-s}{m-t}\right)^{\frac{n-2\sigma}{2}} u\,(se_1) \leq u(te_1).$$

After sending $m \to \infty$, it follows that

$$u(se_1) \leq u(te_1). \tag{11}$$

For $s > 0$, let $t \to -s$, we obtain that

$$u(se_1) \leq u(-se_1). \tag{12}$$

By the same argument, choosing $y = -te_1$, $x = -me_1$ and $\lambda^2 = (m-s)(m-t)$, it follows that

$$u(-se_1) \leq u(-te_1).$$

For $s > 0$, let $t \to -s$, we have

$$u(-se_1) \leq u(se_1). \tag{13}$$

Combining (12) with (13), we deduce that $u$ is symmetric about the hyperplane $\{y_1 = 0\}$.

On the other hand, for $0 < t < s$, a consequence of (11) is that $u$ is monotone decreasing along the axis of $y_1$.

Therefore, to finish the proof of Theorem 1.1, we only need to get (10). □

### 2.2. *Proof of the inequality* (10)

It is known that the operator $(-\Delta)^\sigma$ is nonlocal, the traditional methods on local differential operators, such as on Laplacian, may not work on this nonlocal operator. To circumvent this difficulty, Caffarelli and Silvestre [9] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions with the conormal derivative boundary condition. The method has been used in many literatures, see [19,33].

In order to describe the method in a more precise way, let us give some notations. We use capital letters, such as $X = (x, t)$ to denote points in $\mathbb{R}^{n+1}_+$. We denote $\mathcal{B}_R(X)$ as the ball in $\mathbb{R}^{n+1}$ with radius $R$ and center $X$, $\mathcal{B}^+_R(X)$ as $\mathcal{B}_R(X) \cap \mathbb{R}^{n+1}_+$ and $B_R(x)$ as the ball in $\mathbb{R}^n$ with radius $R$ and center $x$. We also write $\mathcal{B}_R(0)$, $\mathcal{B}^+_R(0)$, $B_R(0)$ as $\mathcal{B}_R$, $\mathcal{B}^+_R$, $B_R$ for short respectively. For a domain $D \subset \mathbb{R}^{n+1}_+$ with boundary $\partial D$, we denote $\partial' D := \partial D \cap \partial\mathbb{R}^{n+1}_+$ and $\partial'' D := \partial D \cap \mathbb{R}^{n+1}_+$. In particular, $\partial'\mathcal{B}^+_R(X) := \partial\mathcal{B}^+_R(X) \cap \partial\mathbb{R}^{n+1}_+$ and $\partial''\mathcal{B}^+_R(X) := \partial\mathcal{B}^+_R(X) \cap \mathbb{R}^{n+1}_+$.

More precisely, for $u \in C^2(\mathbb{R}^n \backslash \{0\}) \cap L_\sigma(\mathbb{R}^n)$, define

$$U(x, t) := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - y, t) u(y) dy, \tag{14}$$

where

$$\mathcal{P}_\sigma(x, t) := \frac{\beta(n, \sigma) t^{2\sigma}}{(|x|^2 + t^2)^{(n+2\sigma)/2}}$$

with a constant $\beta(n, \sigma)$ such that $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x, 1) dx = 1$. Then

$$U \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+} \backslash \{0\}), \quad t^{1-2\sigma} \partial_t U(x, t) \in C(\mathbb{R}^n \backslash \{0\}),$$

and

$$\begin{aligned} \operatorname{div}(t^{1-2\sigma} \nabla U) &= 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\ U &= u \quad \text{on} \quad \mathbb{R}^n \backslash \{0\}. \end{aligned} \tag{15}$$

In order to study the behaviors of the solution $u$ of (5), we just need to study the behaviors of $U$ defined by (14). In addition, by works of Caffarelli and Silvestre [9], it is known that up to a constant,

$$\frac{\partial U}{\partial \nu^\sigma} = (-\Delta)^\sigma u \quad \text{on} \ \ \mathbb{R}^n \backslash \{0\},$$

where the conormal derivative

$$\frac{\partial U}{\partial \nu^\sigma} := -\lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x, t).$$

From this and (5), we have

$$\frac{\partial U}{\partial \nu^\sigma} = |x|^\tau u^p \quad \text{on } \mathbb{R}^n \backslash \{0\}. \tag{16}$$

For all $x \in \mathbb{R}^n \backslash \{0\}$, $X = (x, 0)$ and $\lambda > 0$, define the Kelvin transformation of $U$ as

$$U_{X,\lambda}(\xi) := \left(\frac{\lambda}{|\xi - X|}\right)^{n-2\sigma} U\left(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}\right) \quad \text{in } \mathbb{R}^{n+1}_+.$$

The aim is to show that for any $\lambda \in (0, |x|)$,

$$U_{X,\lambda}(\xi) \le U(\xi) \qquad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X). \tag{17}$$

In particular, choose $\xi = (y, 0)$, $y \in \mathbb{R}^n \setminus \{0\}$, then for any $\lambda \in (0, |x|)$,

$$u_{x,\lambda}(y) \le u(y) \qquad \text{in } \mathbb{R}^n \backslash (B_\lambda(x) \cup \{0\}),$$

that is (10).

## 2.3. *Proof of the inequality* (17)

For all $x \in \mathbb{R}^n \setminus \{0\}$, define

$$\bar{\lambda}(x) := \sup \left\{\lambda(x) \in (0, |x|) \;\middle|\; U_{X,\lambda}(\xi) \le U(\xi) \text{ in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X),\ \forall\, 0 < \lambda < \lambda(x)\right\}.$$

As the first step, we are going to make sure $\bar{\lambda}(x)$ is well defined, by proving

$$\left\{\lambda(x) \in (0, |x|) \;\middle|\; U_{X,\lambda}(\xi) \le U(\xi) \text{ in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X),\ \forall\, 0 < \lambda < \lambda(x)\right\} \ne \emptyset,$$

that is,

*Claim 1*: There exists $\lambda_0(x) \in (0, |x|)$ such that for any $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X). \tag{18}$$

In the second step, we give that

*Claim 2*:

$$\bar{\lambda}(x) = |x|. \tag{19}$$

Once (19) was obtained, Theorem 1.1 follows.

**Proof of Claim 1.** First of all, we are going to show that there exist $\mu$ and $\lambda_0(x)$ satisfying $0 < \lambda_0(x) < \mu < |x|$ such that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \quad \text{in } \overline{\mathcal{B}^+_\mu(X)} \backslash \mathcal{B}^+_\lambda(X). \tag{20}$$

Then we shall prove that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \quad \text{in } \mathcal{B}^+_{3/4} \backslash \overline{\mathcal{B}^+_\mu(X)}. \tag{21}$$

For every $0 < \lambda < \mu < \frac{1}{2}|x|$, $\xi \in \partial'' \mathcal{B}^+_\mu(X)$, we have $X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \in \mathcal{B}^+_\mu(X)$. Thus we can choose

$$\lambda_0(x) = \mu \left( \frac{\inf\limits_{\partial'' \mathcal{B}^+_\mu(X)} U}{\sup\limits_{\mathcal{B}^+_\mu(X)} U} \right)^{\frac{1}{n-2\sigma}},$$

such that for every $0 < \lambda < \lambda_0(x) < \mu$,

$$\begin{aligned} U_{X,\lambda}(\xi) &= \left( \frac{\lambda}{|\xi - X|} \right)^{n-2\sigma} U\left( X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \right) \\ &\le \left( \frac{\lambda_0}{\mu} \right)^{n-2\sigma} \sup_{\mathcal{B}^+_\mu(X)} U \\ &= \inf_{\partial'' \mathcal{B}^+_\mu(X)} U \le U(\xi). \end{aligned}$$

The above inequality, together with

$$U_{X,\lambda}(\xi) = U(\xi) \qquad \text{on } \partial'' \mathcal{B}^+_\lambda(X),$$

implies that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \qquad \text{on } \partial'' \mathcal{B}^+_\mu(X) \cup \partial'' \mathcal{B}^+_\lambda(X). \tag{22}$$

We will make use of the "narrow domain technique" of Berestycki and Nirenberg from [3], and show that there exist some $\mu$ small enough such that for all $\lambda \in (0, \lambda_0(x))$,

$$U_{X,\lambda}(\xi) \le U(\xi) \qquad \text{in } \mathcal{B}^+_\mu(X) \backslash \mathcal{B}^+_\lambda(X).$$

It is a straightforward computation to show that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U_{X,\lambda}) = 0 & \text{in } \mathcal{B}^+_\mu(X) \backslash \overline{\mathcal{B}^+_\lambda(X)}, \\ \frac{\partial}{\partial \nu^\sigma} U_{X,\lambda} = \left( \frac{\lambda}{|y-x|} \right)^{p^\sharp} |y_\lambda|^\tau u^p_{x,\lambda}(y) & \text{on } \partial'(\mathcal{B}^+_\mu(X) \backslash \overline{\mathcal{B}^+_\lambda(X)}), \end{cases}$$

which yield

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U_{X,\lambda}-U))=0 & \text{in } \mathcal{B}^+_\mu(X)\backslash\overline{\mathcal{B}^+_\lambda(X)}, \\ \frac{\partial}{\partial\nu^\sigma}(U_{X,\lambda}-U)=\left(\frac{\lambda}{|y-x|}\right)^{p^\sharp}|y_\lambda|^\tau u^p_{x,\lambda}(y)-|y|^\tau u^p(y) & \text{on } \partial'(\mathcal{B}^+_\mu(X)\backslash\overline{\mathcal{B}^+_\lambda(X)}), \end{cases} \tag{23}$$

where $y_\lambda := x + \frac{\lambda^2(y-x)}{|y-x|^2}$, $p^\sharp := n+2\sigma - p(n-2\sigma)$.

Let $(U_{X,\lambda}-U)^+ := \max(0, U_{X,\lambda}-U)$ which equals to 0 on $\partial''(\mathcal{B}^+_\mu(X)\backslash\mathcal{B}^+_\lambda(X))$. Multiplying the equation in (23) by $(U_{X,\lambda}-U)^+$ and integrating by parts in $\mathcal{B}^+_\mu(X)\backslash\overline{\mathcal{B}^+_\lambda(X)}$, we have

$$\int\limits_{\mathcal{B}^+_\mu(X)\backslash\mathcal{B}^+_\lambda(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2$$

$$= \int\limits_{B_\mu(x)\backslash B_\lambda(x)} \left[\left(\frac{\lambda}{|y-x|}\right)^{p^\sharp}|y_\lambda|^\tau u^p_{x,\lambda}(y)-|y|^\tau u^p(y)\right](u_{x,\lambda}-u)^+.$$

Combining Proposition 4.2 with $\lambda^2 = |x-y_\lambda||x-y|$, we have

$$\left(\frac{\lambda}{|x-y|}\right)^2 \le \frac{|y_\lambda|}{|y|},$$

which implies that

$$\left(\frac{\lambda}{|x-y|}\right)^{p^\sharp} \le \left(\frac{|y|}{|y_\lambda|}\right)^\tau,$$

due to $-2\tau \le p^\sharp$ and $\frac{\lambda}{|x-y|} \le 1$. Therefore,

$$\left(\frac{\lambda}{|y-x|}\right)^{p^\sharp}|y_\lambda|^\tau \le |y|^\tau, \tag{24}$$

and

$$\int\limits_{\mathcal{B}^+_\mu(X)\backslash\mathcal{B}^+_\lambda(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2 \le \int\limits_{B_\mu(x)\backslash B_\lambda(x)} |y|^\tau(u^p_{x,\lambda}(y)-u^p(y))(u_{x,\lambda}-u)^+.$$

For any $y\in B_\mu(x)\backslash B_\lambda(x)$,

$$u_{x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2\sigma} u(y_\lambda) \le u(y_\lambda),$$

where $y_\lambda := x + \frac{\lambda^2(y-x)}{|y-x|^2}$. Combining $y_\lambda \in B_\lambda(x) \subset \overline{B}_{\frac{|x|}{2}}(x)$ with $u\in C^2(B_1\setminus\{0\})$, we deduce that there exists a positive constant $C$ depending on $x$, such that for any $y\in B_\mu(x)\backslash B_\lambda(x)$,

$$u_{x,\lambda}(y) \le C(x). \tag{25}$$

With the help of mean value theorem, $\lambda < \frac{1}{2}|x|$ and the Hölder inequality, we obtain that

$$\begin{aligned}
&\int\limits_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2 \\
\leq &\int\limits_{B_\mu(x)\setminus B_\lambda(x)} 2^{-\tau}|x|^\tau p u_{x,\lambda}^{p-1}(y)[(u_{x,\lambda}-u)^+]^2 \\
\leq &2^{-\tau}|x|^\tau p\Big(\int\limits_{B_\mu(x)\setminus B_\lambda(x)} (u_{x,\lambda})^{\frac{n(p-1)}{2\sigma}}\Big)^{\frac{2\sigma}{n}} \left(\int\limits_{B_\mu(x)\setminus B_\lambda(x)} [(u_{x,\lambda}-u)^+]^{\frac{2n}{n-2\sigma}}\right)^{\frac{n-2\sigma}{n}}.
\end{aligned}$$

It follows from (25) and the trace inequality (Proposition 4.3) that

$$\begin{aligned}
&\int\limits_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2 \\
\leq &2^{-\tau}|x|^\tau pC(C(x))^{\frac{n(p-1)}{2\sigma}}\Big(\int\limits_{B_\mu(x)} 1\Big)^{\frac{2\sigma}{n}} \left(\int\limits_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2\right) \\
= &C|B_\mu(x)|^{\frac{2\sigma}{n}} \left(\int\limits_{\mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X)} t^{1-2\sigma}|\nabla(U_{X,\lambda}-U)^+|^2\right),
\end{aligned}$$

where $C$ is a positive constant depending only on $n$, $p$, $\sigma$, $\tau$ and $x$.

We can fix $\mu$ sufficiently small such that

$$C|B_\mu(x)|^{\frac{2\sigma}{n}} \leq \frac{1}{2}.$$

Then

$$\nabla(U_{X,\lambda}(\xi)-U(\xi))^+ = 0 \qquad \text{in} \quad \mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X).$$

Since (22), we deduce that

$$(U_{X,\lambda}(\xi)-U(\xi))^+ = 0 \qquad \text{in} \quad \mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X).$$

Therefore,

$$U_{X,\lambda}(\xi) \leq U(\xi) \qquad \text{in} \quad \mathcal{B}_\mu^+(X)\setminus\mathcal{B}_\lambda^+(X).$$

Next, we are going to prove (21).

By Proposition 4.1, we have

$$U(\xi) \geq \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial'' \mathcal{B}^+_\mu(X)} U \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\mu(X). \tag{26}$$

Then for all $\xi \in \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\mu(X)$ and $\lambda \in (0, \lambda_0) \subset (0, \mu)$, it follows that

$$\begin{aligned} U_{X,\lambda}(\xi) &= \left(\frac{\lambda}{|\xi - X|}\right)^{n-2\sigma} U(X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2}) \leq \left(\frac{\lambda_0}{|\xi - X|}\right)^{n-2\sigma} \sup_{\mathcal{B}^+_\mu(X)} U \\ &= \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial'' \mathcal{B}^+_\mu(X)} U \leq U(\xi), \end{aligned}$$

where (26) is used in the last inequality. Then Claim 1 is proved. □

**Proof of Claim 2.** By Claim 1, $\bar{\lambda}(x)$ is well defined, and we also know that for $x \neq 0$, $\bar{\lambda}(x) \leq |x|$. From the definition of $\bar{\lambda}(x)$, it is obvious to see that for $0 < \lambda \leq \bar{\lambda}(x)$, we have

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X). \tag{27}$$

We prove Claim 2 by contradiction. That is, suppose $\bar{\lambda}(x) < |x|$ for some $x \neq 0$. We want to show that there exists a positive constant $\varepsilon \in (0, \frac{|x| - \bar{\lambda}(x)}{2})$ such that for any $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon)$,

$$U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\lambda(X), \tag{28}$$

which contradicts with the definition of $\bar{\lambda}(x)$, then we obtain $\bar{\lambda}(x) = |x|$.

W divide the region into two parts,

$$\begin{aligned} K_1 &= \left\{\xi \in \mathbb{R}^{n+1}_+ \mid |\xi - X| \geq \bar{\lambda}(x) + \delta_2\right\}, \\ K_2 &= \left\{\xi \in \mathbb{R}^{n+1}_+ \mid \lambda \leq |\xi - X| \leq \bar{\lambda}(x) + \delta_2\right\}, \end{aligned}$$

where $\delta_2$ will be fixed later. Then in order to obtain (28) it suffices to prove that it established on $K_1$, $K_2$.

By (27), we have

$$U_{X,\bar{\lambda}(x)}(\xi) \leq U(\xi) \quad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_{\bar{\lambda}(x)}(X).$$

On the other hand,

$$\begin{aligned} \lim_{\xi \to 0} U_{X,\bar{\lambda}}(\xi) &= \lim_{\xi \to 0} \left(\frac{\bar{\lambda}}{|\xi - X|}\right)^{n-2\sigma} U(X + \frac{\bar{\lambda}^2(\xi - X)}{|\xi - X|^2}) \\ &= \left(\frac{\bar{\lambda}}{|X|}\right)^{n-2\sigma} U(X - \frac{\bar{\lambda}^2 X}{|X|^2}) < \infty. \end{aligned} \tag{29}$$

Hence, by (29) and the strong maximum principle, we conclude that

$$U_{X,\bar{\lambda}(x)}(\xi) < U(\xi) \quad \text{in } \mathbb{R}^{n+1}_{+} \backslash \overline{\mathcal{B}^{+}_{\bar{\lambda}(x)}(X)}. \tag{30}$$

By calculation, it follows that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U - U_{X,\bar{\lambda}})) = 0 & \text{in } \mathbb{R}^{n+1}_{+} \backslash \mathcal{B}^{+}_{\bar{\lambda}(x)+\delta_2}(X), \\ \frac{\partial}{\partial \nu^{\sigma}}(U - U_{X,\bar{\lambda}}) = |y|^{\tau} u^{p}(y) - \left(\frac{\lambda}{|y-x|}\right)^{p^{\sharp}} \left|y_{\lambda}\right|^{\tau} u^{p}_{x,\lambda}(y) & \text{on } \mathbb{R}^{n} \backslash (B_{\bar{\lambda}(x)+\delta_2}(X) \cup \{0\}). \end{cases} \tag{31}$$

Using Proposition 4.2 and by the same argument as (24) in Claim 1 for (31), we obtain that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla(U - U_{X,\bar{\lambda}})) = 0 & \text{in } \mathbb{R}^{n+1}_{+} \backslash \mathcal{B}^{+}_{\bar{\lambda}(x)+\delta_2}(X), \\ \frac{\partial}{\partial \nu^{\sigma}}(U - U_{X,\bar{\lambda}}) \geq 0 & \text{on } \mathbb{R}^{n} \backslash (B_{\bar{\lambda}(x)+\delta_2}(X) \cup \{0\}). \end{cases} \tag{32}$$

With the help of Proposition 4.1, we have

$$(U - U_{X,\bar{\lambda}})(\xi) \geq \left(\frac{\bar{\lambda}(x) + \delta_2}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial'' \mathcal{B}^{+}_{\bar{\lambda}(x)+\delta_2}(X)} (U - U_{X,\bar{\lambda}}) \quad \text{in } K_1. \tag{33}$$

From (30), it is easy to see that

$$\inf_{\partial'' \mathcal{B}^{+}_{\bar{\lambda}(x)+\delta_2}(X)} (U - U_{X,\bar{\lambda}(x)}) > 0.$$

By the uniform continuity of $U$ on compact sets, there exists a positive constant $\varepsilon_1 (< \frac{|x| - \bar{\lambda}(x)}{2})$ sufficient small such that for any $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \varepsilon_1)$,

$$|U_{X,\bar{\lambda}(x)}(\xi) - U_{X,\lambda}(\xi)| \leq \frac{1}{2}\left(\frac{\bar{\lambda}(x) + \delta_2}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial'' \mathcal{B}^{+}_{\bar{\lambda}(x)+\delta_2}(X)} (U - U_{X,\bar{\lambda}(x)}) \quad \text{in } K_1. \tag{34}$$

Indeed, notice that $\xi_{\bar{\lambda}(x)} := X + \frac{\bar{\lambda}^2(x)(\xi - X)}{|\xi - X|^2}$, $\xi_{\lambda} := X + \frac{\lambda^2(\xi - X)}{|\xi - X|^2} \in \overline{\mathcal{B}^{+}_{\frac{|x| + \bar{\lambda}(x)}{2}}(X)}$ and

$$
\begin{aligned}
|U_{X,\bar\lambda(x)}(\xi) - U_{X,\lambda}(\xi)| &= \left|\left(\frac{\bar\lambda(x)}{|\xi - X|}\right)^{n-2\sigma} U(\xi_{\bar\lambda(x)}) - \left(\frac{\lambda}{|\xi - X|}\right)^{n-2\sigma} U(\xi_\lambda)\right| \\
&\leq \left(\frac{\bar\lambda(x)}{|\xi - X|}\right)^{n-2\sigma} \left|U(\xi_{\bar\lambda(x)}) - U(\xi_\lambda)\right| + \frac{|\bar\lambda^{n-2\sigma}(x) - \lambda^{n-2\sigma}|}{|\xi - X|^{n-2\sigma}} U(\xi_\lambda) \\
&= \left(\frac{\bar\lambda(x) + \delta_2}{|\xi - X|}\right)^{n-2\sigma} \left(\frac{\bar\lambda(x)}{\bar\lambda(x) + \delta_2}\right)^{n-2\sigma} \left|U(\xi_{\bar\lambda(x)}) - U(\xi_\lambda)\right| \\
&\quad + \left(\frac{\bar\lambda(x) + \delta_2}{|\xi - X|}\right)^{n-2\sigma} \frac{|\bar\lambda^{n-2\sigma}(x) - \lambda^{n-2\sigma}|}{(\bar\lambda(x) + \delta_2)^{n-2\sigma}} U(\xi_\lambda).
\end{aligned}
$$

From the fact that $U$ is the uniform continuity on compact sets, we can choose $\varepsilon_1$ sufficient small, such that for all $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon_1)$,

$$
\left(\frac{\bar\lambda(x)}{\bar\lambda(x) + \delta_2}\right)^{n-2\sigma} \left|U(\xi_{\bar\lambda(x)}) - U(\xi_\lambda)\right| \leq \frac{1}{4} \inf_{\partial'' \mathcal{B}^+_{\bar\lambda(x)+\delta_2}(X)} (U - U_{X,\bar\lambda(x)}),
$$

and

$$
\frac{|\bar\lambda^{n-2\sigma}(x) - \lambda^{n-2\sigma}|}{(\bar\lambda(x) + \delta_2)^{n-2\sigma}} U(\xi_\lambda) \leq \frac{1}{4} \inf_{\partial'' \mathcal{B}^+_{\bar\lambda(x)+\delta_2}(X)} (U - U_{X,\bar\lambda(x)}).
$$

Therefore, (34) holds.

Combining (33) with (34), we deduce that for any $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon_1)$,

$$
U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{in } K_1. \tag{35}
$$

Now let us focus on the region $K_2$. From (35), it follows that

$$
U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{on } \partial'' K_2.
$$

Using the narrow domain technique as before (see the proof of (20) in Claim 1), we can choose $\delta_2$ small (notice that we can choose $\varepsilon$ as small as we want less than $\varepsilon_1$ such that for $\lambda \in (\bar\lambda(x), \bar\lambda(x) + \varepsilon)$, we have

$$
U_{X,\lambda}(\xi) \leq U(\xi) \quad \text{in } K_2. \tag{36}
$$

Combining (35) with (36), we can see that the moving sphere procedure may continue beyond $\bar\lambda(x)$ where we reach a contradiction. □

## 3. Singularity and decay estimates

First, we recall the doubling property [29, Lemma 5.1] and denote $B_R(x)$ as the ball in $\mathbb{R}^n$ with radius $R$ and center $x$. For convenience, we write $B_R(0)$ as $B_R$ for short.

**Proposition 3.1.** *Suppose that $\emptyset \neq D \subset \Sigma \subset \mathbb{R}^n$, $\Sigma$ is closed and $\Gamma = \Sigma \setminus D$. Let $M : D \to (0, \infty)$ be bounded on compact subset of $D$. If for a fixed positive constant $k$, there exists $y \in D$ satisfying*

$$M(y)\mathrm{dist}(y, \Gamma) > 2k,$$

*then there exists $x \in D$ such that*

$$M(x) \geq M(y), \qquad M(x)\mathrm{dist}(x, \Gamma) > 2k,$$

*and for all $z \in D \cap B_{kM^{-1}(x)}(x)$,*

$$M(z) \leq 2M(x).$$

The second one is called the interior Schauder estimates. See [19, Theorem 2.11] for the proof. Many regularity properties can be founded in Cabre–Sire [5], Cabre–Tan [6].

**Proposition 3.2.** *Suppose that $g \in C^\gamma(B_R)$, $\gamma > 0$ and $u$ is a nonnegative solution of*

$$(-\Delta)^\sigma u = g(x) \qquad \text{in } \; B_R.$$

*If $2\sigma + \gamma \leq 1$, then $u \in C^{0,2\sigma+\gamma}(B_{R/2})$. Moreover,*

$$\|u\|_{C^{0,2\sigma+\gamma}(B_{R/2})} \leq C\left(\|u\|_{L^\infty(B_{3R/4})} + \|g\|_{C^\gamma(B_{3R/4})}\right),$$

*where $C$ is a positive constant depending on $n$, $\sigma$, $\gamma$, $R$.*

*If $2\sigma + \gamma > 1$, then $u \in C^{1,2\sigma+\gamma-1}(B_{R/2})$. Moreover,*

$$\|u\|_{C^{1,2\sigma+\gamma-1}(B_{R/2})} \leq C\left(\|u\|_{L^\infty(B_{3R/4})} + \|g\|_{C^\gamma(B_{3R/4})}\right),$$

*where $C$ is a positive constant depending on $n$, $\sigma$, $\gamma$, $R$.*

Next, in order to prove Theorem 1.2, we start with the following lemma.

### *3.1. Proof of the Lemma 3.3*

**Lemma 3.3.** *Let $1 < p < \frac{n+2\sigma}{n-2\sigma}$, $0 < \alpha \leq 1$ and $c(x) \in C^{2,\alpha}(\overline{B_1})$ satisfy*

$$\|c\|_{C^{2,\alpha}(\overline{B_1})} \leq C_1, \quad c(x) \geq C_2 \quad \text{in } \; \overline{B_1} \tag{37}$$

*for some positive constants $C_1$, $C_2$. Suppose that $u \in C^2(B_1) \cap L_\sigma(\mathbb{R}^n)$ is a nonnegative solution of*

$$(-\triangle)^\sigma u = c(x)u^p \qquad \text{in } \; B_1, \tag{38}$$

*then there exists a positive constant $C$ depending only on $n$, $\sigma$, $p$, $C_1$, $C_2$ such that*

$$|u(x)|^{\frac{p-1}{2\sigma}} + |\nabla u(x)|^{\frac{p-1}{p+2\sigma-1}} \le C[\mathrm{dist}(x, \partial B_1)]^{-1} \qquad \text{in} \quad B_1.$$

**Proof.** Arguing by contradiction, for $k = 1, 2, \cdots$, we assume that there exist nonnegative functions $u_k$ satisfying (38) and points $y_k \in B_1$ such that

$$|u_k(y_k)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(y_k)|^{\frac{p-1}{p+2\sigma-1}} > 2k[\mathrm{dist}(y_k, \partial B_1)]^{-1}. \tag{39}$$

Define

$$M_k(x) := |u_k(x)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x)|^{\frac{p-1}{p+2\sigma-1}}.$$

Via Proposition 3.1, for $D = B_1$, $\Gamma = \partial B_1$, there exist $x_k \in B_1$ such that

$$M_k(x_k) \ge M_k(y_k), \quad M_k(x_k) > 2k[\mathrm{dist}(x_k, \partial B_1)]^{-1} \ge 2k, \tag{40}$$

and for any $z \in B_1$ and $|z - x_k| \le kM_k^{-1}(x_k)$,

$$M_k(z) \le 2M_k(x_k). \tag{41}$$

It follows from (40) that

$$\lambda_k := M_k^{-1}(x_k) \to 0 \qquad \text{as } k \to \infty, \tag{42}$$

$$\mathrm{dist}(x_k, \partial B_1) > 2k\lambda_k, \qquad \text{for } k = 1, 2, \cdots. \tag{43}$$

Consider

$$w_k(y) := \lambda_k^{\frac{2\sigma}{p-1}} u_k(x_k + \lambda_k y) \qquad \text{in} \ B_k.$$

Together with (43), we obtain that for any $y \in B_k$,

$$|x_k + \lambda_k y - x_k| \le \lambda_k |y| \le \lambda_k k < \frac{1}{2}\mathrm{dist}(x_k, \partial B_1),$$

that is,

$$x_k + \lambda_k y \in B_{\frac{1}{2}\mathrm{dist}(x_k, \partial B_1)}(x_k) \subset B_1.$$

Therefore, $w_k$ is well defined in $B_k$ and

$$|w_k(y)|^{\frac{p-1}{2\sigma}} = \lambda_k |u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma}},$$

$$|\nabla w_k(y)|^{\frac{p-1}{2\sigma+p-1}} = \lambda_k |\nabla u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma+p-1}}.$$

From (41), we find that for all $y \in B_k$,

$$|u_k(x_k+\lambda_k y)|^{\frac{p-1}{2\sigma}}+|\nabla u_k(x_k+\lambda_k y)|^{\frac{p-1}{2\sigma+p-1}}\le 2\left(|u_k(x_k)|^{\frac{p-1}{2\sigma}}+|\nabla u_k(x_k)|^{\frac{p-1}{p+2\sigma-1}}\right).$$

That is,

$$|w_k(y)|^{\frac{p-1}{2\sigma}}+|\nabla w_k(y)|^{\frac{p-1}{2\sigma+p-1}}\le 2\lambda_k M_k(x_k)=2. \tag{44}$$

Moreover, $w_k$ satisfies

$$(-\triangle)^\sigma w_k=c_k(y)w_k^p \qquad \text{in } \ B_k, \tag{45}$$

and

$$|w_k(0)|^{\frac{p-1}{2\sigma}}+|\nabla w_k(0)|^{\frac{p-1}{2\sigma+p-1}}=1,$$

where $c_k(y):=c(x_k+\lambda_k y)$.

By condition (37), we obtain that $\{c_k\}$ is uniformly bounded in $\mathbb{R}^n$. For each $R>0$, and for all $y,z\in B_R$, we have

$$|D^\beta c_k(y)-D^\beta c_k(z)|\le C_1\lambda_k^{|\beta|}|\lambda_k(y-z)|^\alpha\le C_1|y-z|^\alpha, \quad |\beta|=0,1,2,$$

for $k$ is large enough. Therefore, by Arzela–Ascoli's Theorem, there exists a function $c\in C^2(\mathbb{R}^n)$, after extracting a subsequence, $c_k\to c$ in $C^2_{loc}(\mathbb{R}^n)$. Moreover, by (42), we obtain

$$|c_k(y)-c_k(z)|\to 0 \qquad \text{as } \ k\to\infty. \tag{46}$$

This implies that the function $c$ actually is a constant $C$. By (37) again, $c_k\ge C_2>0$, we conclude that $C$ is a positive constant.

On the other hand, applying Proposition 3.2 a finite number of times to (44) and (45), there exists some positive $\gamma\in(0,1)$ such that for every $R\in(1,k)$,

$$\|w_k\|_{C^{2,\gamma}(\overline{B_{R/2}})}\le C\left(\|w_k\|_{L^\infty(B_{3R/4})}+\|c_k w_k^p\|_{C^\gamma(B_{3R/4})}\right)\le C(R), \tag{47}$$

where $C(R)$ is a positive constant independent of $k$. Thus, after passing to a subsequence, we have, for some nonnegative function $w\in C^2_{\text{loc}}(\mathbb{R}^n)$,

$$w_k\to w \qquad \text{in } \ C^2_{\text{loc}}(\mathbb{R}^n),$$

and

$$|w(0)|^{\frac{p-1}{2\sigma}}+|\nabla w(0)|^{\frac{p-1}{2\sigma+p-1}}=1.$$

Since $p<\frac{n+2\sigma}{n-2\sigma}$, this contradicts the Liouville-type result ([19, Remark 1.9]) that

$$w=0 \qquad \text{in } \ \mathbb{R}^n. \tag{48}$$

The proof of (48) will be provided in Appendix. Then we conclude the lemma. □

*3.2. Proof of Theorem 1.2*

We now turn to prove Theorem 1.2.

**Proof.** For $x_0 \in \mathbb{R}^n \setminus B_2$, we denote $R := \frac{1}{2}|x_0|$. Then for any $y \in B_1$, we have $\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}$, and deduce that $x_0 + Ry \in \mathbb{R}^n \setminus B_1$. Define

$$w(y) := R^{\frac{2\sigma+\tau}{p-1}} u(x_0 + Ry).$$

Therefore, we obtain that

$$(-\Delta)^\sigma w = c(y) w^p \qquad \text{in } \ B_1,$$

where $c(y) := |y + \frac{x_0}{R}|^\tau$. Notice that

$$1 < |y + \frac{x_0}{R}| < 3 \quad \text{in } \ \overline{B_1}.$$

Moreover,

$$\|c\|_{C^3(\overline{B_1})} \leq C, \quad c(y) \geq 3^{-2\sigma} \quad \text{in } \ \overline{B_1}.$$

Applying Lemma 3.3, we obtain that

$$|w(0)|^{\frac{p-1}{2\sigma}} + |\nabla w(0)|^{\frac{p-1}{p+2\sigma-1}} \leq C.$$

That is,

$$(R^{\frac{2\sigma+\tau}{p-1}} u(x_0))^{\frac{p-1}{2\sigma}} + (R^{\frac{2\sigma+\tau}{p-1}+1} |\nabla u(x_0)|)^{\frac{p-1}{p+2\sigma-1}} \leq C.$$

Hence,

$$u(x_0) \leq C R^{-\frac{2\sigma+\tau}{p-1}} \leq C|x_0|^{-\frac{2\sigma+\tau}{p-1}},$$
$$|\nabla u(x_0)| \leq C R^{-\frac{2\sigma+\tau+p-1}{p-1}} \leq C|x_0|^{-\frac{2\sigma+\tau+p-1}{p-1}}.$$

Since $x_0 \in \mathbb{R}^n \setminus B_2$ is arbitrary, Theorem 1.2 is proved. □

## 4. Appendix

*4.1. Proof of the (48)*

Before that, we introduce the definition of Weight Sobolev Space and weak solutions. Let $D$ be an open set in $\mathbb{R}^{n+1}_+$. Denote by $L^2(t^{1-2\sigma}, D)$ the Banach space of all measurable functions $U$ defined on $D$, for which

$$\|U\|_{L^2(t^{1-2\sigma},D)} := \left( \int_D t^{1-2\sigma} |U|^2 dX \right)^{\frac{1}{2}} < \infty,$$

and $X := (x, t) \in \mathbb{R}^n \times \mathbb{R}_+$. We say that $U \in W^{1,2}(t^{1-2\sigma}, D)$ if $U \in L^2(t^{1-2\sigma}, D)$, and its weak derivatives $\nabla U$ exist and belong to $L^2(t^{1-2\sigma}, D)$. The norm of $U$ in $W^{1,2}(t^{1-2\sigma}, D)$ is given by

$$\|U\|_{W^{1,2}(t^{1-2\sigma},D)} := \left( \int_D t^{1-2\sigma} |U|^2 dX + \int_D t^{1-2\sigma} |\nabla U|^2 dX \right)^{\frac{1}{2}}.$$

See Fabes–Jerison–Kenig [15], Heinonen–Kilpeläinen–Martio [17] for more details about the Weight Sobolev Space.

Let $\partial' D \neq \emptyset$, and $a \in L^\infty(\partial' D)$. For $1 < p \leq \frac{n+2\sigma}{n-2\sigma}$, we say $U \in W^{1,2}(t^{1-2\sigma}, D)$ is a weak solution of

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in} \quad D, \\ \dfrac{\partial U}{\partial \nu^\sigma} = a U^p & \text{on} \ \ \partial' D, \end{cases}$$

if for every nonnegative $\Phi \in C_c^\infty(D \cup \partial' D)$,

$$\int_D t^{1-2\sigma} \nabla U \nabla \Phi dX = \int_{\partial' D} a U^p \Phi dx.$$

Now, we shall prove (48). As before, define

$$W_k(x,t) := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x - y, t) w_k(y) dy.$$

Together with (45), we have

$$\begin{cases} \operatorname{div}(t^{1-2\sigma} \nabla W_k) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial W_k}{\partial \nu^\sigma} = c_k(y) w_k^p & \text{on } \partial' \mathcal{B}_k^+. \end{cases}$$

Since $W_k \in C^2(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$, it follows that $W_k \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_k^+)$. Thus, by Proposition 4.5, there exist some $\nu \in (0, 1)$ such that for every $R > 1$

$$\|W_k\|_{W^{1,2}(t^{1-2\sigma},\mathcal{B}_R^+)} + \|W_k\|_{C^\nu(\overline{\mathcal{B}_R^+})} \leq C(R),$$

where $C(R)$ is independent of $k$. Combining with (47), after passing to a subsequence, we have, for some nonnegative function $W \in W^{1,2}_{\text{loc}}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+}) \cap C^\nu_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$

$$\begin{cases} W_k \rightharpoonup W & \text{weakly in } W^{1,2}_{loc}(t^{1-2\sigma}, \overline{\mathbb{R}^{n+1}_+}), \\ W_k \to W & \text{in } C^{\nu/2}_{loc}(\overline{\mathbb{R}^{n+1}_+}), \\ w_k \to w & \text{in } C^2_{loc}(\mathbb{R}^n), \end{cases}$$

where $w(y) = W(y, 0)$. Moreover, $W$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla W) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \dfrac{\partial W}{\partial \nu^\sigma} = C w^p & \text{on } \partial'\mathbb{R}^{n+1}_+. \end{cases} \tag{49}$$

Since $p < \frac{n+2\sigma}{n-2\sigma}$, by the Liouville-type result ([19, Remark 1.9]) that $w = 0$. Hence, we derive (48).

### 4.2. *Some useful propositions*

Then, we prove some propositions which plays a vital role in our proof.

**Proposition 4.1.** *Let $x \in \mathbb{R}^n$, $\mu > 0$, $x_0 \in \mathbb{R}^n \backslash B_\mu(x)$ and $X = (x, 0)$. Let $U \in C^2(\mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\mu(X))$ and $C^1$ up to the boundary $\mathbb{R}^n \backslash (B_\mu(x) \cup \{x_0\})$. Suppose that $U$ is a nonnegative solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) \le 0 & \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\mu(X), \\ \dfrac{\partial U}{\partial \nu^\sigma} \ge 0 & \text{on } \mathbb{R}^n \backslash (B_\mu(x) \cup \{x_0\}), \end{cases}$$

*then*

$$U(\xi) \ge \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U \qquad \text{in } \mathbb{R}^{n+1}_+ \backslash \mathcal{B}^+_\mu(X).$$

**Proof.** For an arbitrary positive constant $\varepsilon$, define

$$U_\varepsilon(\xi) := U(\xi) - \phi(\xi) + \varepsilon|\xi - X_0|^{2\sigma - n},$$

where

$$\phi(\xi) := \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U.$$

By a direct calculation, we obtain that

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U_\varepsilon) \le 0 & \text{in } \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X), \\ \dfrac{\partial U_\varepsilon}{\partial \nu^\sigma} \ge 0 & \text{on } \mathbb{R}^n \backslash (B_\mu(x) \cup \{x_0\}). \end{cases}$$

It follows that

$$U_\varepsilon(\xi) = U(\xi) - \inf_{\partial''\mathcal{B}^+_\mu(X)} U + \varepsilon|\xi - X_0|^{2\sigma-n} \geq 0 \qquad \text{on } \partial''\mathcal{B}^+_\mu(X). \tag{50}$$

$$\liminf_{\xi\to\infty} U_\varepsilon(\xi) = \liminf_{\xi\to\infty} U(\xi) \geq 0. \tag{51}$$

On the other hand, there exists a positive constant $C$ such that

$$\liminf_{\xi\to X_0} (U - \phi)(\xi) \geq -\left(\frac{\mu}{|X_0 - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U \geq -C,$$

where the nonnegative of $U$ gives the first inequality. Since

$$\lim_{\xi\to X_0} \varepsilon|\xi - X_0|^{2\sigma-n} = +\infty,$$

we deduce that there exists a positive constant $\delta$ such that

$$U_\varepsilon(\xi) \geq 0 \qquad \text{in } \mathcal{B}^+_\delta(X_0). \tag{52}$$

Combining (50), (51), (52) and the standard maximum principle argument, we conclude that

$$U_\varepsilon(\xi) \geq 0 \qquad \text{in } \mathbb{R}^{n+1}_+ \setminus (\mathcal{B}^+_\mu(X) \cup \mathcal{B}^+_\delta(X_0)).$$

Using (52), we obtain that

$$U_\varepsilon(\xi) \geq 0 \qquad \text{in } \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X).$$

Sending $\varepsilon \to 0$, we conclude that

$$U(\xi) - \phi(\xi) \geq 0 \qquad \text{in } \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X).$$

That is

$$U(\xi) \geq \left(\frac{\mu}{|\xi - X|}\right)^{n-2\sigma} \inf_{\partial''\mathcal{B}^+_\mu(X)} U \qquad \text{in } \mathbb{R}^{n+1}_+ \setminus \mathcal{B}^+_\mu(X).$$

This finish the proof of this proposition. □

**Proposition 4.2.** [25, Corollary 1.4.] *For $x \in \mathbb{R}^n \setminus \{0\}$, $0 < \lambda < \min\{|x|, |y - x|\}$, we have*

$$|x - y_\lambda||y| \leq |x - y||y_\lambda|, \tag{53}$$

*where $y_\lambda = x + \frac{\lambda^2(y-x)}{|y-x|^2}$.*

The following one is called the trace inequality.

**Proposition 4.3.** [19, Proposition 2.1] *If $U \in C_c^2(\mathbb{R}_+^{n+1})$, then there exists a positive constant $C$ depending only on $n$ and $\sigma$ such that*

$$\left(\int_{\mathbb{R}^n} |U(\cdot,0)|^{\frac{2n}{n-2\sigma}} dx\right)^{\frac{n-2\sigma}{2n}} \le C\left(\int_{\mathbb{R}_+^{n+1}} t^{1-2\sigma}|\nabla U|^2 dxdt\right)^{\frac{1}{2}} \tag{54}$$

We also recall the standard maximum principle.

**Proposition 4.4.** [19, Lemma 2.5] *Suppose that $U \in C^2(D) \cap C^1(\overline{D})$ is a solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) \le 0 & \text{in } D, \\ \dfrac{\partial U}{\partial \nu^\sigma} \ge 0 & \text{on } \partial' D, \end{cases}$$

*where $D \subset \mathbb{R}_+^{n+1}$ is an open domain. If $U \ge 0$ on $\partial'' D$, then $U \ge 0$ in $D$.*

Last, we also introduce a result about the regularity.

**Proposition 4.5.** [19, Corollary 2.10] *Suppose that $a \in L^\infty(\partial' \mathcal{B}_R^+)$, $U \in W^{1,2}(t^{1-2\sigma}, \mathcal{B}_R^+)$ is a nonnegative solution of*

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla U) = 0 & \text{in } \mathcal{B}_R^+, \\ \dfrac{\partial U}{\partial \nu^\sigma} = a(x)U^p & \text{on } \partial' \mathcal{B}_R^+. \end{cases}$$

*Then*

*(i) $U \in L^\infty_{loc}(\mathcal{B}_R^+ \cup \partial' \mathcal{B}_R^+)$ and hence $U(\cdot,0) \in L^\infty_{loc}(\partial' \mathcal{B}_R^+)$.*

*(ii) There exist $C > 0$ and $\nu \in (0,1)$ depending only on $n$, $\sigma$, $p$, $R$, $\|U(\cdot,0)\|_{L^\infty(\partial' \mathcal{B}_{3R/4}^+)}$ and $\|a\|_{L^\infty(\partial' \mathcal{B}_{3R/4}^+)}$ such that $U \in C^\nu(\overline{\mathcal{B}_{R/2}^+})$ and*

$$\|U\|_{W^{1,2}(t^{1-2\sigma}, \mathcal{B}_{R/2}^+)} + \|U\|_{C^\nu(\overline{\mathcal{B}}_{R/2}^+)} \le C.$$

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