Calc. Var. (2019) 58:193
https://doi.org/10.1007/s00526-019-1639-4

Calculus of Variations

Existence of entire solutions of Monge–Ampère equations with prescribed asymptotic behavior

Jiguang Bao[1] · Jingang Xiong[1] · Ziwei Zhou[1]

Received: 18 September 2018 / Accepted: 30 September 2019

Abstract
We prove the existence of entire solutions of the Monge–Ampère equations with prescribed asymptotic behavior at infinity of the plane, which was left unsolved by Caffarelli–Li in 2003. The special difficulty of the problem in dimension two is due to the *global logarithmic term* in the asymptotic expansion of solutions at infinity. Furthermore, we give a PDE proof of the characterization of the space of solutions of the Monge–Ampère equation $\det \nabla^2 u = 1$ with $k \geq 2$ singular points, which was established by Gálvez–Martínez–Mira in 2005. We also obtain the existence of solutions in higher dimensional cases with general right hand sides.

Mathematics Subject Classification 35J96 · 35B40

1 Introduction

In 1954, K. Jörgens [12] proved that, modulo the unimodular affine equivalence, $\frac{1}{2}|x|^2$ is the unique convex smooth solution of

$$\det \nabla^2 u = 1 \quad \text{in } \mathbb{R}^2.$$

Jörgens' theorem was extended to smooth convex solutions in higher dimensions by Calabi [5] for dimensions less than or equal to 5 and by Pogorelov [20] for all dimensions. Different

Communicated by O. Savin.

All authors are supported in part by the key project NSFC 11631002, and J. Xiong is also supported in part by NSFC 11501034 and 11571019.

✉ Jingang Xiong
jx@bnu.edu.cn

Jiguang Bao
jgbao@bnu.edu.cn

Ziwei Zhou
zhouziwei@mail.bnu.edu.cn

1 School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

Published online: 25 October 2019

proofs were given by Cheng–Yau [6], Caffarelli [2] and Jost–Xin [14]. In dimension two, elementary and simpler proofs were found by Nitsche [19] and Jin–Xiong [15].

In [3], Caffarelli and Li established a quantitative version of the theorem of Jörgens–Calabi–Pogorelov. They considered

$$\det \nabla^2 u = f \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $f \in C^0(\mathbb{R}^n)$ satisfies that

$$0 < \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < \infty \text{ and } \operatorname{supp}(f-1) \text{ is bounded.} \tag{2}$$

Denote

$$\mathcal{A} := \{A : \ A \text{ is a symmetric, positive definite } n \times n \text{ matrix and } \det A = 1\}.$$

Theorem 1 (Caffarelli–Li [3]) *Let u be a convex viscosity (Alexandrov) solution of (1) with f satisfying (2). Then $u \in C^\infty(\mathbb{R}^n \backslash supp(f-1))$, and we have the following:*

- *For $n \geq 3$, there exist a linear function $\ell(x)$ and $A \in \mathcal{A}$ such that*

$$\limsup_{x\to\infty} |x|^{n-2} \left| u(x) - \left(\frac{1}{2}x^t Ax + \ell(x)\right)\right| < \infty. \tag{3}$$

- *For $n = 2$, there exist a linear function $\ell(x)$ and $A \in \mathcal{A}$ such that*

$$\limsup_{x\to\infty} |x| \left| u(x) - \left(\frac{1}{2}x^t Ax + d\ln\sqrt{x^t Ax} + \ell(x)\right)\right| < \infty, \tag{4}$$

where

$$d = \frac{1}{2\pi}\int_{\mathbb{R}^2} (f-1)\,\mathrm{d}x. \tag{5}$$

The asymptotic behavior in exterior domains of dimension two had been established by Ferrer–Martínez–Milán [8].

In addition, Caffarelli–Li [3] proved that (1) with the condition (3) admits a unique viscosity solution when $n \geq 3$; see Theorem 1.7 of [3]. However, it was not known whether (1) with the condition (4) has a unique solutions in the plane. The difficulty stems from the global constant d in (4), which makes it hard to construct sub- and super- solutions with quadratic growth. In this paper, we solve the problem positively.

In fact, we can relax the assumptions on f. Let ν be a locally finite Borel measure defined in $\mathbb{R}^2$ and $\mathrm{d}\nu = f\,\mathrm{d}x$ in $\mathbb{R}^n \backslash \Omega$, where Ω is a bounded open set and $f \in C^3(\mathbb{R}^n\backslash\Omega)$ is a positive function satisfying

$$\limsup_{|x|\to\infty} |x|^{\beta+j} |\nabla^j (f(x)-1)| < \infty, \quad j = 0, 1, 2, 3, \tag{6}$$

for some $\beta > 2$. Extending [3], Bao–Li–Zhang [1] proved that for every Alexandrov solution of

$$\det \nabla^2 u = \nu \quad \text{in } \mathbb{R}^n, \tag{7}$$

there exist a linear function $\ell(x)$ and $A \in \mathcal{A}$ such that

$$\begin{cases} \limsup_{x\to\infty} |x|^{\min\{\beta,n\}-2+j} \left|\nabla^j (u(x) - \left(\frac{1}{2}x^t Ax + \ell(x)\right))\right| < \infty, & \text{if } \beta \neq n, \\ \limsup_{x\to\infty} |x|^{n-2+j} (\ln|x|)^{-1} \left|\nabla^j (u(x) - \left(\frac{1}{2}x^t Ax + \ell(x)\right))\right| < \infty, & \text{if } \beta = n \end{cases} \tag{8}$$

when $n \geq 3$, and

$$\limsup_{x\to\infty} |x|^{\sigma+j} \left|\nabla^j \left(u(x) - \left(\frac{1}{2}x^t Ax + d \ln \sqrt{x^t Ax} + \ell(x)\right)\right)\right| < \infty \tag{9}$$

when $n = 2$, where $j = 0, 1, 2, 3, 4$, $\sigma \in (0, \min\{\beta - 2, 2\})$, and the constant d will be

$$d = \frac{1}{2\pi} \lim_{R\to\infty} \left(\int_{B_R} \mathrm{d}\nu - \pi R^2\right) \tag{10}$$

by following the proof of (1.9) in [3] (when $n \geq 3$, the case "$\beta = n$" was missed in the original paper [1]). See also recent paper Li–Lu [18] for related exterior problems.

The main result of this paper is

Theorem 2 *Let $n = 2$ and ν be as above. For any linear function ℓ and $A \in \mathcal{A}$, the Monge–Ampère equation (7) has a unique Alexandrov solution satisfying (9) with d given by (10).*

Theorem 2 confirms the Conjecture 1 of [1] particularly. Our proof is different from the one in [3] for $n \geq 3$. It contains three new ingredients: (i) constructing upper barrier functions which, however, are not supper solutions; (ii) modifying approximation solutions; (iii) using the upper barrier and asymptotics of [1,3] to conclude that the Hessian of the limiting solution converges to the identity matrix at the infinity.

For $n \geq 3$, the proof is more like the one in [3]. Together with some properties of Alexandrov solutions, we have the following theorem.

Theorem 3 *Let $n \geq 3$ and ν be as above. For any linear function ℓ and $A \in \mathcal{A}$, the Monge–Ampère equation (7) has a unique Alexandrov solution satisfying (8).*

Remark 4 The condition $\beta > 2$ is necessary for the asymptotic behavior (8) and (9). Indeed, let f be a radial, smooth, positive function satisfying $f(r) \equiv 1$ for $r \in [0, 1]$ and $f(r) = 1 + r^{-2}$ for $r > 2$. Then

$$u(x) = \int_0^{|x|} \left(\int_0^s nt^{n-1} f(t)\,\mathrm{d}t\right)^{\frac{1}{n}} \mathrm{d}s$$

is a solution of (7) with $\mathrm{d}\nu = f\,\mathrm{d}x$ in $\mathbb{R}^n$. But, as $|x| \to \infty$,

$$u(x) = \begin{cases} \frac{1}{2}|x|^2 + O((\log|x|)^2) & \text{for } n = 2, \\ \frac{1}{2}|x|^2 + O(\log|x|) & \text{for } n \geq 3. \end{cases}$$

In 1955, Jörgens [13] further proved that, modulo the unimodular affine equivalence, every smooth locally convex solution of

$$\det \nabla^2 u = 1 \quad \text{in } \mathbb{R}^2 \backslash \{0\}$$

has to be

$$\int_0^{|x|} (c + t^2)^{1/2}\,\mathrm{d}t, \quad c \geq 0. \tag{11}$$

In 2016, Jin–Xiong [16] extended Jörgens theorem to all dimensions, i.e,

$$\int_0^{|x|} (c + t^n)^{1/n}\,\mathrm{d}t, \quad c \geq 0$$

is the unique solution of $\det \nabla^2 u = 1$ in $\mathbb{R}^n \backslash \{0\}$, $n \geq 3$, modulo the unimodular affine equivalence. Furthermore, they identified the set of locally convex entire solutions with $k \geq 1$ singular points to an orbifold of dimension $d(n, k)$, where

$$d(n,k) = \begin{cases} k - 1 + \frac{(k-1)k}{2}, & \text{if } k - 1 \leq n, \\ k - 1 + \frac{n(n+1)}{2} + (k - 1 - n)n, & \text{if } k - 1 > n \end{cases}$$

when $n \geq 3$. The later result in dimension two was obtained by Gálvez–Martínez–Mira [10], using a complex analysis method. Jin-Xiong's proof is based on the result which they proved: If u is a locally convex solution of

$$\det \nabla^2 u = 1 \ \text{ in } \mathbb{R}^n \backslash \{P_1, \ldots, P_k\},$$

then u is convex in $\mathbb{R}^n$ and there exist nonnegative constants c_i such that

$$\det \nabla^2 u = 1 + \sum_{i=1}^{k} c_i \delta_{P_i}$$

in the Alexandrov sense, where P_i, $i = 1, \ldots, k$, are distinct points, and δ_{P_i} is the Dirac measure centered at P_i. This result holds for all $n \geq 2$. Together with the asymptotic behavior at infinity, we have all the parameters to determinate the dimensions of the orbifolds. When $n \geq 3$, the existence of solutions to such equations was proved in [16]. Theorem 2 applies here to obtain existence in dimension two.

Finally, we would like to mention a further extension of the theorem of Jörgens–Calabi–Pogorelov. In another paper [4], Caffarelli–Li classified entire solutions of Monge–Ampère equations with periodic functions on the right hand side. See also the recent work of Teixeira–Zhang [21].

The paper is organized as follows. Theorem 2 is proved in the next section. Using the arguments of [3,16], we give a Proof of Theorem 3 in Sect. 3.

2 Proof of Theorem 2

For convenience, we recall the definition of Alexandrov solutions, see e.g., Gutierrez [11] and Figalli [9]. Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a locally convex function. The normal mapping of u, or subdifferential of u, at $x_0 \in \Omega$ is the set-valued function $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$\partial u(x_0) = \{p : u(x) \geq u(x_0) + p \cdot (x - x_0), \ \text{forall } x \in \Omega\},$$

where $\mathcal{P}(\mathbb{R}^n)$ denotes the class of all subsets of $\mathbb{R}^n$. Given $E \subset \Omega$, define $\partial u(E) = \cup_{x \in E} \partial u(x)$. One can show that the class

$$S = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable}\}$$

is a Borel σ-algebra. The set function $Mu : S \to \overline{\mathbb{R}}$ defined by

$$Mu(E) = |\partial u(E)|$$

is called the Monge–Ampère measure associated with the function u, where $|\cdot|$ is the n-dimensional Lebesgue measure. For a Borel measure ν in Ω, we say a locally convex function u is an Alexandrov solution of the Monge–Ampère equation

$$\det \nabla^2 u = \nu$$

if $Mu = \nu$.

Now we start to prove Theorem 2.

Proof of Theorem 2 We only need to prove the existence part as the uniqueness part follows from the comparison principle. By the affine invariance, we can assume that A is the identity matrix I and $\ell = 0$.

Take $\rho > 0$ such that $\Omega \subset B_\rho$. Let

$$\underline{f}(r) = \begin{cases} 0, & r < \rho, \\ \min_{x \in \partial B_r} f(x), & r \geq \rho, \end{cases}$$
$$\underline{d} = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\underline{f} - 1)\,\mathrm{d}x = \int_0^\infty r(\underline{f}(r) - 1)\,\mathrm{d}r,$$

and

$$w_c(r) = \int_0^r \left(\int_0^s 2t\underline{f}(t)\,\mathrm{d}t + 2c \right)^{1/2} \mathrm{d}s,$$

where $c \geq 0$. It is easy to check that w_c is a convex solution of

$$\det \nabla^2 w_c = |\partial w_c(0)|\delta_0 + \underline{f} = 2\pi c \delta_0 + \underline{f} \quad \text{in } \mathbb{R}^2. \tag{12}$$

Using the condition (6) on f, by a direct calculation we have

$$\begin{aligned} &\left(\int_0^s 2t\underline{f}(t)\,\mathrm{d}t + 2c \right)^{1/2} \\ &= \left(s^2 + 2c + \int_0^\infty 2t(\underline{f}(t) - 1)\,\mathrm{d}t - \int_s^\infty 2t(\underline{f}(t) - 1)\,\mathrm{d}t \right)^{1/2} \\ &= \left(s^2 + 2(c + \underline{d}) + O(s^{2-\beta}) \right)^{1/2} \\ &= s\left(1 + 2(\underline{d} + c)s^{-2} + O(s^{-\beta}) \right)^{1/2} \\ &= s\left(1 + (\underline{d} + c)s^{-2} + O(s^{-\min\{\beta,4\}}) \right) \\ &= s + (\underline{d} + c)s^{-1} + O(s^{-\min\{\beta-1,3\}}) \quad \text{as } s \to \infty. \end{aligned}$$

Thus,

$$h(s) := \left(\int_0^s 2t\underline{f}(t)\,\mathrm{d}t + 2c \right)^{1/2} - s - (\underline{d} + c)s^{-1} = O(s^{-\min\{\beta-1,3\}})$$

as $s \to \infty$. It follows that

$$\begin{aligned} w_c(r) =& \int_1^r \left(\int_0^s 2t\underline{f}(t)\,\mathrm{d}t + 2c \right)^{1/2} \mathrm{d}s + \int_0^1 \left(\int_0^s 2t\underline{f}(t)\,\mathrm{d}t + 2c \right)^{1/2} \mathrm{d}s \\ =& \frac{1}{2}r^2 + (\underline{d} + c)\ln r - \frac{1}{2} + \int_1^\infty h(s)\,\mathrm{d}s - \int_r^\infty h(s)\,\mathrm{d}s \\ &+ \int_0^1 \left(\int_0^s 2t\underline{f}(t)\,\mathrm{d}t + 2c \right)^{1/2} \mathrm{d}s \\ =& \frac{1}{2}r^2 + (\underline{d} + c)\ln r + O(1) \quad \text{as } r \to \infty. \end{aligned} \tag{13}$$

Let

$$\bar{c} = d - \underline{d}.$$

By (12),

$$\det \nabla^2 w_{\bar{c}} = 2\pi \bar{c}\delta_0 + \underline{f} \quad \text{in } \mathbb{R}^2.$$

Since

$$d = \frac{1}{2\pi}\left(\int_\Omega \mathrm{d}\nu - \int_\Omega \mathrm{d}x + \int_{\mathbb{R}^2\setminus\Omega} (f-1)\,\mathrm{d}x\right)$$

and

$$\underline{d} = \frac{1}{2\pi}\left(-\int_\Omega \mathrm{d}x + \int_{\mathbb{R}^2\setminus\Omega} (\underline{f}-1)\,\mathrm{d}x\right),$$

we have

$$\bar{c} = \frac{1}{2\pi}\left(\int_\Omega \mathrm{d}\nu + \int_{\mathbb{R}^2\setminus\Omega} (f-\underline{f})\,\mathrm{d}x\right).$$

For any large $R > \rho$, choose $\lambda_{\bar{c}}(R)$ such that

$$w_{\bar{c}}(R) + \lambda_{\bar{c}}(R) = \frac{R^2}{2} + d\ln R.$$

By (13), $\lambda_{\bar{c}}(R)$ is uniformly bounded in $R > \rho$.

Let $u_R \in C(\bar{B}_R)$ be the unique Alexandrov solution of

$$\begin{cases} \det \nabla^2 u_R = \nu & \text{in } B_R, \\ u_R = \frac{R^2}{2} + d\ln R & \text{on } \partial B_R; \end{cases} \tag{14}$$

see Theorem 1.6.2 in [11].

We claim that $u_R(0) \geq \lambda_{\bar{c}}(R)$.

Indeed, for any large R and any $c > \bar{c}$, let $\lambda_c(R) \in \mathbb{R}$ such that

$$w_c(R) + \lambda_c(R) = \frac{R^2}{2} + d\ln R.$$

If $u_R(0) \leq \lambda_c(R)$, then, considering that for any Borel set $E \subset B_R\setminus\{0\}$,

$$|\partial u_R(E)| = \int_E \mathrm{d}\nu \geq \int_E \underline{f}\,\mathrm{d}x = |\partial(w_c + \lambda_c(R))(E)|,$$

it follows from the comparison principle that $u_R(x) \leq w_c(|x|) + \lambda_c(R)$ for all $x \in B_R$. By Lemma 1.4.1 in [11], we have $\partial(w_c + \lambda_c(R))(B_R) \subset \partial u_R(B_R)$. However, note that when $c > \bar{c}$,

$$\begin{aligned} |\partial(w_c + \lambda_c(R))(B_R)| &= \int_{B_R} \underline{f}\,\mathrm{d}x + 2\pi c \\ &> \int_\Omega \mathrm{d}\nu + \int_{B_R\setminus\Omega} f\,\mathrm{d}x + \int_{\mathbb{R}^2\setminus B_R} (f-\underline{f})\,\mathrm{d}x \\ &\geq \int_{B_R} \mathrm{d}\nu = |\partial u_R(B_R)|. \end{aligned}$$

Hence, we have derived a contradiction. It follows that $u_R(0) > \lambda_c(R)$. For any fixed R, since $w_c(R)$ is continuous with respect to c, $\lambda_c(R)$ is continuous with respect to c. Sending $c \to \bar{c}$, we have

$$u_R(0) \geq \lambda_{\bar{c}}(R) \tag{15}$$

for any large R.

Let $v_R(x) = w_{\bar{c}}(|x|) + u_R(0)$. By (15), we have

$$v_R \geq w_{\bar{c}}(R) + \lambda_{\bar{c}}(R) = u_R \quad \text{on } \partial B_R,$$

and $v_R(0) = u_R(0)$. By the comparison principle, we have

$$v_R \geq u_R \quad \text{in } B_R. \tag{16}$$

Since u_R is a convex function, there exists a vector $p_R(0)$ such that

$$u_R(x) \geq p_R(0)x + u_R(0) \quad \text{for all } x \in B_R.$$

By (16), we have

$$p_R(0)x \leq w_{\bar{c}}(|x|) + u_R(0) - u_R(0) \leq w_{\bar{c}}(|x|).$$

It follows that $|p_R(0)| \leq C$ for some constant C independent of R.

Let $\tilde{u}_R(x) = u_R(x) - (p_R(0)x + u_R(0))$. Note that

$$\begin{aligned} 0 \leq \tilde{u}_R(x) &\leq v_R(x) - (p_R(0)x + u_R(0)) \\ &= w_{\bar{c}}(|x|) + u_R(0) - p_R(0)x - u_R(0) \\ &\leq w_{\bar{c}}(|x|) + C|x| \end{aligned}$$

and

$$\det \nabla^2 \tilde{u}_R = \nu \quad \text{in } B_R.$$

By the Lipschitz estimates for convex functions (see, e.g., Theorem 6.7 in [7]), for any $K \subset\subset B_{R/2}$,

$$||\tilde{u}_R||_{C^{0,1}(K)} \leq C(K),$$

where $C(K)$ is a constant independent of R. Then after passing to a subsequence, denoted by $\tilde{u}_{R_i}$, we have

$$\tilde{u}_{R_i} \to u_\infty \quad \text{in } C^\alpha_{loc}(\mathbb{R}^2)$$

where $\alpha \in (0, 1)$ for some convex function u_∞ satisfying

$$0 \leq u_\infty(x) \leq w_{\bar{c}}(|x|) + C|x| = \frac{1}{2}|x|^2 + C|x| + d \ln |x| + O(1) \tag{17}$$

and

$$\det \nabla^2 u_\infty = \nu \quad \text{in } \mathbb{R}^2$$

in the Alexandrov sense. It follows from Corollary 1.1 in [1] that there exist $A \in \mathcal{A}$, $D \in \mathbb{R}$ and a linear function $\ell(x)$ such that

$$\limsup_{|x|\to\infty} |x|^{j+\sigma} \left| \nabla^j \left(u_\infty(x) - \left(\frac{1}{2} x' A x + D \ln \sqrt{x' A x} + \ell(x) \right) \right) \right| < \infty, \tag{18}$$

for $j = 0, 1, 2, 3, 4$, and $\sigma \in (0, \min\{\beta - 2, 2\})$. By (17), A can not have one eigenvalue greater than 1. This forces that all the eigenvalues equal 1 and thus $A = I$.

To prove $D = d$, we use the argument of proving (1.9) in [3]. Let $u = u_\infty - \ell$. We first assume that $u \in C^3(\mathbb{R}^2)$ and write

$$E(x) = u(x) - \left(\frac{1}{2}|x|^2 + D \ln |x|\right),$$

and

$$\det \nabla^2 u = \partial_1(u_1 u_{22}) - \partial_2(u_1 u_{12}).$$

By (18), as $|x| \to \infty$,

$$|E(x)| = O(|x|^{-\sigma}),\ |DE(x)| = O(|x|^{-\sigma-1}),\ |D^2 E(x)| = O(|x|^{-\sigma-2}).$$

Integrating the equation of u on B_R and integrating by parts, we have, as $R \to \infty$,

$$\begin{aligned}
\int_{B_R} \mathrm{d}\nu &= \int_{B_R} \partial_1(u_1 u_{22}) - \partial_2(u_1 u_{12})\, \mathrm{d}x \\
&= \int_{|x|=R} \left[u_1 u_{22} \frac{x_1}{|x|} - u_1 u_{12} \frac{x_2}{|x|}\right] \mathrm{d}x \\
&= \int_{|x|=R} \left[\left(x_1 + \frac{Dx_1}{|x|^2} + E_1\right)\left(1 + D\frac{|x|^2 - 2x_2^2}{|x|^4} + E_{22}\right)\frac{x_1}{|x|}\right. \\
&\quad \left. - \left(x_1 + \frac{Dx_1}{|x|^2} + E_1\right)\left(-2D\frac{x_1 x_2}{|x|^4} + E_{12}\right)\frac{x_2}{|x|}\right] \mathrm{d}x \\
&= \int_{|x|=R} \left(x_1 + \frac{Dx_1}{|x|^2}\right)\left(\frac{x_1}{|x|} + \frac{Dx_1}{|x|^3}\right) \mathrm{d}x + O(R^{-\sigma}) \\
&= \int_{|x|=R} \left(\frac{x_1^2}{|x|} + \frac{2Dx_1^2}{|x|^3}\right) \mathrm{d}x + O(R^{-\sigma}) \\
&= \pi R^2 + 2\pi D + O(R^{-\sigma}),
\end{aligned}$$

where $E_i = \partial_i E$ and $E_{ij} = \partial_{ij}^2 E$ for $i, j = 1, 2$. Sending R to infinity, we have $D = d$.

For $u \in C(\mathbb{R}^2)$, by (18) we know that u is of C^4 near ∂B_R for large R. Let $u_\epsilon \in C^\infty(\mathbb{R}^2)$ be a family of convex functions such that $u_\epsilon \to u$ in $C^0_{loc}(\mathbb{R}^2)$, and $u_\epsilon \to u$ in C^4 near ∂B_R as $\epsilon \to 0$. Let η be a continuous cutoff function satisfying $\eta = 1$ in B_R, and $\eta = 0$ in $\mathbb{R}^2 \backslash B_{R+1}$. By Lemma 1.2.3 in [11],

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^2} \eta \det \nabla^2 u_\epsilon \,\mathrm{d}x = \int_{\mathbb{R}^2} \eta \,\mathrm{d}\nu.$$

Note that

$$\lim_{\epsilon \to 0} \int_{B_{R+1} \backslash B_R} \eta \det \nabla^2 u_\epsilon \,\mathrm{d}x = \int_{B_{R+1} \backslash B_R} \eta \,\mathrm{d}\nu.$$

Subtracting the two equalities above, we have

$$\lim_{\epsilon \to 0} \int_{B_R} \det \nabla^2 u_\epsilon \,\mathrm{d}x = \int_{B_R} \mathrm{d}\nu.$$

As shown above,

$$\int_{B_R} \det \nabla^2 u_\epsilon \, \mathrm{d}x = \int_{|x|=R} \left[u_{\epsilon 1} u_{\epsilon 22} \frac{x_1}{|x|} - u_{\epsilon 1} u_{\epsilon 12} \frac{x_2}{|x|} \right] \mathrm{d}x.$$

Sending $\epsilon \to 0$, we have

$$\begin{aligned}\int_{B_R} \mathrm{d}\nu &= \int_{|x|=R} \left[u_1 u_{22} \frac{x_1}{|x|} - u_1 u_{12} \frac{x_2}{|x|} \right] \mathrm{d}x \\ &= \pi R^2 + 2\pi D + O(R^{-\sigma}).\end{aligned}$$

Sending R to infinity, again we have $D = d$. Then u is the solution we want.

Therefore, Theorem 2 is proved. □

3 Proof of Theorem 3

When $f \equiv 1$ outside Ω, Theorem 3 was proved by [16].

We only show the existence part as the uniqueness part follows from the comparison principle. Due to the affine invariance, we assume that $A = I$, $\ell = 0$ and $\Omega \subset B_{\frac{1}{2}}$. We assume $\nu = f \mathrm{d}x$ in $\mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$ is positive and satisfies (6). The bounds we will obtain are independent of the smoothness and the lower bound of f in $B_{1/2}$. By an approximation argument, Theorem 3 will follow.

Next we are going to construct sub- and super- solutions by following the arguments in [3,16].

Let η be a nonnegative smooth function supported in $B_{\frac{1}{4}}$ satisfying $\int_{B_1} \eta \, \mathrm{d}x = 1$, and v_1 be the smooth solution of

$$\begin{cases} \det \nabla^2 v_1 = f + a\eta & \text{in } B_1, \\ v_1 = 0 & \text{on } \partial B_1, \end{cases}$$

where $a > 0$ will be chosen later. It follows from Alexandrov's maximum principle (see, e.g., Theorem 1.4.2 in [11]) that

$$v_1 \geq -c(n) |\partial v_1(B_1)|^{\frac{1}{n}} = -c(n) \left(\int_{B_1} f(x) \, \mathrm{d}x + a \right)^{\frac{1}{n}} =: -c_0 \quad \text{in } B_{\frac{1}{2}},$$

where $c(n)$ is a constant depending only on the dimension n.

Let $r = |x|$ and define

$$\bar{f}(r) = \max_{|x|=r} f(x), \quad r \geq \frac{1}{2}.$$

Let $c_1 = \int_{\frac{1}{2}}^1 (\int_1^s nt^{n-1} \bar{f}(t) \, \mathrm{d}t)^{\frac{1}{n}} \, \mathrm{d}s$, $K = \frac{c_0}{c_1}$,

$$v_2(r) = \begin{cases} K \int_1^r \left(\int_1^s nt^{n-1} \bar{f}(t) \, \mathrm{d}t \right)^{\frac{1}{n}} \, \mathrm{d}s, & r \geq \frac{1}{2}, \\ -c_0, & 0 \leq r < \frac{1}{2}. \end{cases}$$

First of all, $v_1 \geq v_2$ in $\bar{B}_{\frac{1}{2}}$. Secondly, by choosing a large such that $c_0 \geq c_1$, we have

$$\det \nabla^2 v_2 = K^n \bar{f} \geq f = \det \nabla^2 v_1 \quad \text{in } B_1 \backslash \bar{B}_{\frac{1}{2}},$$

and $v_1 = v_2 = 0$ on ∂B_1. By the comparison principle, we have $v_1 \geq v_2$ in $B_1 \backslash \bar{B}_{\frac{1}{2}}$. So $v_1 \geq v_2$ in B_1.

Let

$$\underline{u}(x) = \begin{cases} \int_1^r \left(\int_1^s nt^{n-1} \bar{f}(t)\, \mathrm{d}t + K \right)^{\frac{1}{n}} \mathrm{d}s, & r \geq 1, \\ v_1, & 0 \leq r < 1. \end{cases}$$

Then $\underline{u} \in C^0(\mathbb{R}^n) \cap C^\infty(B_1) \cap C^\infty(\mathbb{R}^n \backslash \bar{B}_1)$, $\underline{u}$ is locally convex in $\mathbb{R}^n \backslash B_1$,

$$\det \nabla^2 \underline{u} = \bar{f} \quad \text{in } \mathbb{R}^n \backslash \overline{B_1},$$
$$\det \nabla^2 \underline{u} \geq f \quad \text{in } B_1.$$

Moreover, we have $\underline{u} \geq v_2$ in B_1, and $\underline{u} = v_2$ on ∂B_1, then

$$\lim_{r \to 1^-} \partial_r \underline{u} \leq \lim_{r \to 1^-} \partial_r v_2.$$

Since

$$\lim_{r \to 1^-} \partial_r v_2 = 0 < (K)^{\frac{1}{n}} = \lim_{r \to 1^+} \partial_r \underline{u},$$

we have

$$\lim_{r \to 1^-} \partial_r \underline{u} < \lim_{r \to 1^+} \partial_r \underline{u}. \tag{19}$$

It follows that $\underline{u}$ is convex in $\mathbb{R}^n$. By a simple computation,

$$\sup_{\mathbb{R}^n} \left| \underline{u}(x) - \frac{1}{2}|x|^2 \right| \leq C$$

for some $C > 0$ depending only on n, $\int_{B_1} f(x)\, \mathrm{d}x$ and $\bar{f}$ outside $B_{1/2}$.

Define

$$\underline{f}(r) = \min_{|x|=r} f(x), \quad r \geq \frac{1}{2},$$

and

$$\bar{u}(x) = \begin{cases} \int_1^{|x|} \left(\int_1^s nt^{n-1} \underline{f}(t)\, \mathrm{d}s \right)^{\frac{1}{n}} \mathrm{d}s, & |x| > 1, \\ 0, & |x| \leq 1. \end{cases}$$

It follows that

$$\lim_{r \to 1^-} \partial_r \bar{u} = \lim_{r \to 1^+} \partial_r \bar{u} = 0, \tag{20}$$

and

$$\sup_{\mathbb{R}^n} \left| \bar{u}(x) - \frac{1}{2}|x|^2 \right| < +\infty.$$

By the above construction, we have

$$\beta_+ := \sup_{\mathbb{R}^n} \left(\frac{|x|^2}{2} - \bar{u}(x) \right) < +\infty \text{ and } \beta_- := \inf_{\mathbb{R}^n} \left(\frac{|x|^2}{2} - \underline{u}(x) \right) > -\infty.$$

Moreover, β_+ and β_- depend only on n, $\int_{B_1} f(x)\, \mathrm{d}x$ and f outside $B_{1/2}$.

For $R > 1$, let u_R be the unique convex smooth solution of

$$\begin{cases} \det \nabla^2 u_R = f & \text{in } B_R, \\ u_R = \frac{R^2}{2} & \text{on } \partial B_R. \end{cases}$$

We claim that

$$\underline{u}(x) + \beta_- \le u_R(x) \le \bar{u}(x) + \beta_+, \quad x \in B_R. \tag{21}$$

To establish the first inequality, let $\underline{x}$ be a maximum point of the function

$$\underline{h}(x) := \underline{u}(x) + \beta_- - u_R(x)$$

in $\bar{B}_R$. Since

$$\det \nabla^2 \underline{u} \ge \det \nabla^2 u_R \quad \text{in } B_R \backslash \bar{B}_1$$

and

$$\det \nabla^2 \underline{u} \ge \det \nabla^2 u_R \ \text{ in } B_1,$$

we have, by the strong maximum principle, $\underline{x} \in \partial B_R$ or $\underline{x} \in \partial B_1$. If $\underline{x} \in \partial B_R$, then by the definition of β_-,

$$\underline{h}(x) \le \underline{u}(\underline{x}) + \beta_- - u_R(\underline{x}) \le \frac{|\underline{x}|^2}{2} - \frac{R^2}{2} = 0 \ \text{ in } \bar{B}_R$$

and the inequality holds. If $\underline{x} \in \partial B_1$, then considering the smoothness of u_R, it contradicts to the condition (19). Hence, the first inequality of (21) holds. For the second inequality, let $\bar{x}$ be a minimum point of the function

$$\bar{h}(x) := \bar{u}(x) + \beta_+ - u_R(x)$$

in $\overline{B_R}$. Similar to the above, $\bar{x} \in \partial B_R$ or $\bar{x} \in \partial B_1$. If $\bar{x} \in \partial B_R$, then by the definition of β_+,

$$\bar{h}(x) \ge \bar{u}(\bar{x}) + \beta_+ - u_R(\bar{x}) \ge \frac{|\bar{x}|^2}{2} - \frac{R^2}{2} = 0 \ \text{ in } \bar{B}_R$$

and the inequality holds. If $\bar{x} \in \partial B_1$, in view of (20) and the equation u_R satisfies, this is impossible. Therefore, the inequality (21) is proved.

By (21) and the Lipschitz estimate for convex functions (see Theorem 6.7 in [7]), we have, along a subsequence $R_i \to \infty$,

$$u_{R_i} \to u_\infty \quad \text{in } C^\alpha_{loc}(\mathbb{R}^n),$$

where $0 < \alpha < 1$, u_∞ satisfies $\det \nabla^2 u_\infty = f$ in $\mathbb{R}^n$ in the Alexandrov sense and

$$\underline{u}(x) + \beta_- \le u_\infty(x) \le \bar{u}(x) + \beta_+ \quad \text{in } \mathbb{R}^n,$$

which particularly implies that

$$\sup_{\mathbb{R}^n} \left| u_\infty(x) - \frac{1}{2}|x|^2 \right| \le C \tag{22}$$

for some $C > 0$ depending only on n, $\int_{B_1} f(x)\,\mathrm{d}x$ and f outside $B_{1/2}$. By Bao–Li–Zhang [1], there exist $A \in \mathcal{A}$ and a linear function $\ell(x)$ such that (8) holds for $j = 0, 1, 2, 3, 4$. Considering (22), we have $A = I$ and $\ell = \tilde{c}$ for some constant $\tilde{c}$. Then

$$u = u_\infty - \tilde{c}$$

is the solution we want.

Therefore, we complete the proof of Theorem 3.

References

1. Bao, J., Li, H., Zhang, L.: Monge–Ampère equation on exterior domains. Calc. Var. Partial Differ. Equ. **52**, 39–63 (2015)
2. Caffarelli, L.A.: Topics in PDEs: The Monge–Ampère equation. Graduate course. Courant Institute, New York University (1995)
3. Caffarelli, L.A., Li, Y.Y.: An extension to a theorem of Jörgens, Calabi, and Pogorelov. Commun. Pure Appl. Math. **56**, 549–583 (2003)
4. Caffarelli, L.A., Li, Y.Y.: A Liouville theorem for solutions of the Monge–Ampère equation with periodic data. Ann. Inst. H. Poincar Anal. Non Linaire **21**, 97–120 (2004)
5. Calabi, E.: Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. Mich. Math. J. **5**, 105–126 (1958)
6. Cheng, S.Y., Yau, S.T.: Complete affine hypersurfaces. I. The completeness of affine metrics. Commun. Pure Appl. Math. **39**(6), 839–866 (1986)
7. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton (1992)
8. Ferrer, L., Martínez, A., Milán, F.: The space of parabolic affine spheres with fixed compact boundary. Monatsh. Math. **130**(1), 19–27 (2000)
9. Figalli, A.: The Monge–Ampère Equation and its Applications. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2017)
10. Gálvez, J.A., Martínez, A., Mira, P.: The space of solutions to the Hessian one equation in the finitely punctured plane. J. Math. Pures Appl. (9) **84**(12), 1744–1757 (2005)
11. Gutierrez, C.E.: The Monge–Ampère Equation. Progress in Nonlinear Differential Equations and Applications, vol. 44. Birkhauser Boston Inc., Boston (2001)
12. Jörgens, K.: Über die Lösungen der Differentialgleichung $rt - s^2 = 1$. Math. Ann. **127**, 130–134 (1954)
13. Jörgens, K.: Harmonische Abbildungen und die Differentialgleichung $rt - s^2 = 1$. Math. Ann. **129**, 330–344 (1955)
14. Jost, J., Xin, Y.L.: Some aspects of the global geometry of entire space-like submanifolds. Results Math. **40**, 233–245 (2001)
15. Jin, T., Xiong, J.: A Liouville theorem for solutions of degenerate Monge–Ampère equations. Commun. Partial Differ. Equ. **39**, 306–320 (2014)
16. Jin, T., Xiong, J.: Solutions of some Monge–Ampère equations with isolated and line singularities. Adv. Math. **289**, 114–141 (2016)
17. Li, Y.Y.: Some existence results of fully nonlinear elliptic equations of Monge–Ampère type. Commun. Pure Appl. Math. **43**, 233–271 (1990)
18. Li, Y.Y., Lu, S.: Existence and nonexistence to exterior Dirichlet problem for Monge–Ampère equation. To appear in Calculus of Variatoins and PDEs
19. Nitsche, J.C.C.: Elementary proof of Bernsteins theorem on minimal surfaces. Ann. Math. **66**, 543–544 (1957)
20. Pogorelov, A.V.: On the improper convex affine hyperspheres. Geom. Dedicata **1**, 33–46 (1972)
21. Teixeira, E.V., Zhang, L.: Global Monge–Ampère equation with asymptotically periodic data. Indiana Univ. Math. J. **65**, 399–422 (2016)