Nonlinear Analysis 75 (2012) 3629–3640

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Existence for translating solutions of Gauss curvature flow on exterior domains

Hongjie Ju [a,*], Jiguang Bao [a], Huaiyu Jian [b]

[a] School of Mathematical Sciences, Beijing Normal University, Beijing 100875, PR China
[b] Department of Mathematics, Tsinghua University, Beijing 100084, PR China

ARTICLE INFO

Article history:
Received 27 October 2011
Accepted 19 January 2012
Communicated by Enzo Mitidieri

Keywords:
Dirichlet problem
Gauss curvature
Monge–Ampère
Exterior domain
Asymptotic behavior

ABSTRACT

In this paper, we use the Perron method to prove the existence of viscosity solutions to a class of Monge–Ampère equations on exterior domains in $\mathbb{R}^n (n \geq 2)$ with prescribed asymptotic behavior at infinity. This problem comes from the study of Gauss curvature flow and its generalization, the flow by powers of Gauss curvature.

1. Introduction and main results

Let $M_t,\ t \in [0, T)$, be a family of hypersurfaces given by smooth immersions $X_t = X(\cdot, t) : M \to \mathbb{R}^{n+1}$, where M is a given n-dimensional manifold. The hypersurfaces M_t are said to move by K^α-flow for some $\alpha > 0$ if

$$\frac{\partial X}{\partial t}(p, t) = -K^\alpha(p, t)\vec{n}(p, t), \quad \forall\, (p, t) \in M \times (0, T), \tag{1.1}$$

where $K(\cdot, t)$ is the Gauss curvature of M_t and $\vec{n}(\cdot, t)$ is the unit normal vector field of M_t. We use the conventions that for a complete nonplanar convex hypersurface, $\vec{n}$ points out of the convex region defined by the hypersurface, and the second fundamental form of such a hypersurface is nonnegative. The flow (1.1) was studied by Firey in [1] and Tso in [2] for $\alpha = 1$, Chow in [3] for $\alpha = \frac{1}{n}$, and Andrews in [4] for general $\alpha > 0$.

Locally, the K^α-flow of hypersurfaces in $\mathbb{R}^{n+1}$ can be described by the nonlinear parabolic equation,

$$\frac{\partial V}{\partial t} = \sqrt{1 + |DV|^2}\left[\frac{\det(D^2V)}{(1 + |DV|^2)^{\frac{n+2}{2}}}\right]^\alpha. \tag{1.2}$$

A function $v = v(y)$ is called a translating solution to the K^α-flow if the function $V(y, t) = v(y) + \lambda t$ solves (1.2), where λ is a positive constant that represents the translating velocity. Equivalently, $v(y)$ is an initial hypersurface satisfying

$$\det(D^2v) = \lambda^{\frac{1}{\alpha}}(1 + |Dv|^2)^{\frac{n+2-\frac{1}{\alpha}}{2}}. \tag{1.3}$$

* Corresponding author. Tel.: +86 10 58808868.
E-mail address: hjju@bnu.edu.cn (H. Ju).

0362-546X/$ – see front matter
doi:10.1016/j.na.2012.01.020

In [5], Urbas proved that for any $\alpha \in (0, \frac{1}{2}]$ there is a convex radially symmetric solution $v \in C^\infty(\mathbb{R}^n)$ to (1.3). See Theorem 3 in [5]. To obtain this result, the author first used the Legendre transform and reduced Eq. (1.3) to the equation

$$\det(D^2u) = \lambda^{-\beta}(1+|x|^2)^{-\gamma}, \quad x \in \mathbb{R}^n, \tag{1.4}$$

where $\beta = \frac{1}{\alpha}$, $\gamma = \frac{n+2-\beta}{2}$ and $u = v^*$ is the Legendre transform of v, i.e., for $v : \Omega \to \mathbb{R}$ $(\Omega \subseteq \mathbb{R}^n)$, $v^* : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$v^*(x) = \sup_{y\in\Omega}(y \cdot x - v(y)).$$

Then he found an entire convex radially symmetric solution to (1.4).

For the special case $\lambda = 1$, $\gamma = 0$ (which is equivalent to $\alpha = \frac{1}{n+2}$), (1.4) is reduced to the equation

$$\det(D^2u) = 1, \quad x \in \mathbb{R}^n, \tag{1.5}$$

which is well understood. The results of Jörgens [6] for $n = 2$, Calabi [7] for $n \le 5$, Pogorelov [8] and Cheng and Yau [9] for all dimensions respectively, assert that any convex solution of (1.5) must be a quadratic polynomial.

In [10], Caffarelli and Li investigated the asymptotic behavior of solutions to the equation

$$\det(D^2u) = g, \tag{1.6}$$

where $g \in C^0(\mathbb{R}^n)$ is bounded and $\mathrm{supp}(g-1)$ is bounded, and proved the existence of solutions to (1.5) in exterior domains of $\mathbb{R}^n$ $(n \ge 3)$ with prescribed asymptotic behavior at infinity. In dimension two, similar Dirichlet problem of (1.5) on exterior domains was studied by Ferrer et al. in [11,12] using complex variable methods. Recently, Wang and Bao [13] also considered the exterior problem of (1.5) for $n = 2$, with an appropriate asymptotic behavior at infinity. They obtained the necessary and sufficient conditions on existence and convexity of radial solutions.

Chou and Wang [14] constructed infinitely many entire solutions to (1.6) under the assumption that $g(x)$ is between two positive constants, which does not hold for Eq. (1.4) since the right side hand of (1.4) with $\gamma \ne 0$ does not have any positive lower bound or positive upper bound. Recently, Jian and Wang [15] constructed infinitely many entire solutions to (1.6) under the *doubling condition*:

$$\int_E g(x)dx \le b\int_{\frac{E}{2}} g(x)dx$$

for any ellipsoid E centered at the origin and some b independent of E. By this result, they proved that for any $\alpha \in (0, \frac{1}{2})$, there exist infinitely many smooth, nonrotationally symmetric solutions to (1.4).

In this paper, we re-find the radially symmetric solution to (1.4) by ODE method, obtain the asymptotic behavior of the solutions and then prove the existence of viscosity solutions to (1.4) on the domain $\mathbb{R}^n \setminus D$ with prescribed asymptotic behavior at infinity for any smooth bounded strictly convex domain D.

For the reader's convenience, we recall the definition of viscosity solutions to the equation

$$\det(D^2u) = g \quad \text{in } \Omega, \tag{1.7}$$

see [16,17] and the references therein.

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^n$, $g \in C^0(\Omega)$ a positive function, and $u \in C^0(\Omega)$ a locally convex function.

(i) u is a viscosity subsolution of (1.7) if for every $\bar{x} \in \Omega$ and every function $\varphi \in C^2(\Omega)$ satisfying

$$\varphi \ge u \quad \text{on } \Omega \quad \text{and} \quad \varphi(\bar{x}) = u(\bar{x}),$$

we have

$$\det(D^2\varphi(\bar{x})) \ge g(\bar{x}).$$

(ii) u is called a viscosity supersolution of (1.7) if for every $\bar{x} \in \Omega$ and every convex function $\varphi \in C^2(\Omega)$ satisfying

$$\varphi \le u \quad \text{on } \Omega \quad \text{and} \quad \varphi(\bar{x}) = u(\bar{x}),$$

we have

$$\det(D^2\varphi(\bar{x})) \le g(\bar{x}).$$

(iii) u is called a viscosity solution of (1.7), if u is both a viscosity subsolution and a viscosity supersolution of (1.7).

Remark 1.1. In the definition of viscosity subsolution, φ is not required to be convex. But for the Monge–Ampère equation (1.7), Urbas proved the definition in which φ is required to be convex is equivalent to the one in which φ is not required to be convex, see the remarks (ii) in [18].

The main result of this paper is the following theorem.

Theorem 1.1. *Let D be a smooth, bounded, strictly convex open subset of $\mathbb{R}^n$ $(n \ge 2)$, $\phi \in C^2(\partial D)$. Assume $\lambda > 0$ and $-\infty < \gamma < \frac{n(n-2)}{2(n-1)}$. Then for any given $b \in \mathbb{R}^n$, there exists some constant c^*, depending only on n, b, λ, γ, D and ϕ, such* *that for every $c > c^*$ there exists a locally convex viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ to the Dirichlet problem*

$$\begin{cases} \det(D^2u) = \lambda^{-\beta}(1+|x|^2)^{-\gamma}, & \text{in } \mathbb{R}^n \setminus \bar{D}, \\ u = \phi, & \text{on } \partial D. \end{cases} \tag{1.8}$$

Moreover, u satisfies

$$u(x) \le f_0(|x|) + b \cdot x + c \quad \text{in } \mathbb{R}^n \setminus D \tag{1.9}$$

$$\text{and} \quad \liminf_{|x|\to\infty} |x|^{n-2-2\gamma+\frac{2\gamma}{n}} [u(x) - f_0(|x|) - b \cdot x - c] \quad \text{exists and is finite}, \tag{1.10}$$

where $f_0(|x|)$ is the radially symmetric solution of (1.4) in $\mathbb{R}^n$ with $f_0(0) = f_0'(0) = 0$, given explicitly by (2.3).

Remark 1.2. We can obtain $u \in C^0(\mathbb{R}^n \setminus D) \cap C^\infty(\mathbb{R}^n \setminus \bar{D})$ by the regularity theory of the Monge–Ampère equation, see [17,19].

Remark 1.3. It is necessary that c has lower bound and $\gamma < \frac{n(n-2)}{2(n-1)}$ by the counterexamples in the last section.

When $\gamma = 0$, $\lambda = 1$, the result is compatible with Theorem 1.5 in [10].

The paper is organized as follows. In Section 2, we re-find the radially symmetric solution to (1.4) by the ODE method, and show the asymptotic behavior of the solution at infinity for $\alpha \in (0, \frac{1}{2})$. In Section 3, we prove Theorem 1.1 by the Perron method. Finally, in Section 4, we use an example to compute $c*$ explicitly and show that the large c is necessary in Theorem 1.1. Then we show that it is necessary for γ having a upper bound by discussing radially symmetric solutions to the exterior Dirichlet problem in $\mathbb{R}^2$ with $\frac{n(n-2)}{2(n-1)} \le \gamma < \frac{n}{2}$, i.e., $0 \le \gamma < 1$.

2. Radially symmetric solutions of (1.4)

Let $u(x) = f(|x|)$, then

$$\det(D^2u(x)) = \left(\frac{f'(r)}{r}\right)^{n-1} f''(r), \quad r = |x|.$$

See (3.1) in [20]. We rewrite (1.4) as

$$(f'(r))^{n-1} f''(r) = \lambda^{-\beta} r^{n-1}(1+r^2)^{-\gamma}. \tag{2.1}$$

(2.1) is equivalent to

$$((f'(r))^n)' = n\lambda^{-\beta} r^{n-1}(1+r^2)^{-\gamma}.$$

Integrating the above equation on $[0, r]$ for $r > 0$, we obtain

$$(f'(r))^n = n\lambda^{-\beta} \int_0^r s^{n-1}(1+s^2)^{-\gamma} ds + (f'(0))^n.$$

Assume $f'(0) = 0, f'(r) \ge 0$. Then

$$f'(r) = (n\lambda^{-\beta})^{\frac{1}{n}} \left[\int_0^r s^{n-1}(1+s^2)^{-\gamma} ds\right]^{\frac{1}{n}},$$

which, together with (2.1), implies

$$f''(r) = n^{\frac{1-n}{n}} \lambda^{-\frac{\beta}{n}} (1+r^2)^{-\gamma} \left[\frac{r^n}{\int_0^r s^{n-1}(1+s^2)^{-\gamma} ds}\right]^{\frac{n-1}{n}}, \quad r > 0.$$

Since

$$\lim_{r\to 0} \frac{r^n}{\int_0^r s^{n-1}(1+s^2)^{-\gamma} ds} = \lim_{r\to 0} \frac{nr^{n-1}}{r^{n-1}(1+r^2)^{-\gamma}} = n,$$

we have

$$\lim_{r\to 0} f''(r) = n^{\frac{1-n}{n}} \lambda^{-\frac{\beta}{n}} \lim_{r\to 0}(1+r^2)^{-\gamma} \cdot \lim_{r\to 0} \left[\frac{r^n}{\int_0^r s^{n-1}(1+s^2)^{-\gamma} ds}\right]^{\frac{n-1}{n}} = \lambda^{-\frac{\beta}{n}}$$

and

$$\lim_{r\to 0}\frac{f'(r)}{r} = \lim_{r\to 0} f''(r) = \lambda^{-\frac{\beta}{n}} \tag{2.2}$$

by L'Hospital's Rule.

If we define $f''(0) = \lambda^{-\frac{\beta}{n}}$, then $f \in C^2([0, +\infty))$ and $f''(r) > 0$ for $r \in [0, \infty)$, which together with (2.2) and $\frac{f'(r)}{r} > 0$ for $r > 0$, implies that $u \in C^2(\mathbb{R}^n)$ is convex. By the regularity theory of Monge–Ampère equations, see [19], and the standard Schauder theory of linear elliptic equations, see [21], we have $u \in C^\infty(\mathbb{R}^n)$.

Denote $f_0(r)$ is the solution of (2.1) with $f_0(0) = 0,\ f_0'(0) = 0$, then

$$f_0(r) = (n\lambda^{-\beta})^{\frac{1}{n}} \int_0^r \left[\int_0^\tau s^{n-1}(1+s^2)^{-\gamma}\,ds\right]^{\frac{1}{n}} d\tau. \tag{2.3}$$

Next, we assume $-\infty < \gamma < \frac{n}{2}$ and study the asymptotic behavior of $f_0(r)$ at infinity. It follows from L'Hospital's Rule that

$$\begin{aligned}\lim_{\tau\to\infty}\frac{\int_0^\tau s^{n-1}(1+s^2)^{-\gamma}\,ds}{\tau^{n-2\gamma}} &= \lim_{\tau\to\infty}\frac{\tau^{n-1}(1+\tau^2)^{-\gamma}}{(n-2\gamma)\tau^{n-1-2\gamma}}\\ &= \frac{1}{n-2\gamma}\lim_{\tau\to\infty}\frac{\tau^{2\gamma}}{(1+\tau^2)^\gamma}\\ &= \frac{1}{n-2\gamma},\end{aligned}$$

i.e.,

$$\int_0^\tau s^{n-1}(1+s^2)^{-\gamma}\,ds = \frac{1}{n-2\gamma}\tau^{n-2\gamma} + o(\tau^{n-2\gamma}), \quad \text{as } \tau \to \infty. \tag{2.4}$$

This, together with (2.3) implies

$$f_0(r) = \left(\frac{n}{n-2\gamma}\right)^{\frac{1}{n}} \frac{n}{2n-2\gamma}\lambda^{-\frac{\beta}{n}} r^{2-\frac{2\gamma}{n}} + o\left(r^{2-\frac{2\gamma}{n}}\right), \quad \text{as } r \to \infty. \tag{2.5}$$

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. By subtracting a linear function from u, we need only to prove the theorem for $b = 0$. The proof will be completed by several lemmas. The following comparison principle is well known, see [5, Proposition 2.1].

Lemma 3.1 ([10]). *Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$, and let $g \in C^0(\Omega)$ be a positive function. Assume that $w \in C^0(\bar{\Omega})$ is a locally convex viscosity subsolution (supersolution) of*

$$\det D^2 w = g, \quad \text{in } \Omega,$$

and $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ is a locally convex supersolution (subsolution) of

$$\det D^2 v = g, \quad \text{in } \Omega.$$

Assume also that

$$w \leq v\ (w \geq v) \quad \text{on } \partial\Omega.$$

Then

$$w \leq v\ (w \geq v) \quad \text{on } \bar{\Omega}.$$

Next, we prove the existence of the Dirichlet problem on bounded convex domain, which is necessary to the proof of Theorem 1.1. The method of proof follows from Lemma A.3 in [10].

Lemma 3.2. *Let Ω be a smooth, bounded, strictly convex subset of $\mathbb{R}^n (n \geq 2)$. Assume that $\underline{u} \in C^0(\bar{\Omega})$ is a convex viscosity subsolution to $\det(D^2\underline{u}) \geq \lambda^{-\beta}(1+|x|^2)^{-\gamma}$.*

Then the Dirichlet problem

$$\begin{cases} \det(D^2u) = \lambda^{-\beta}(1+|x|^2)^{-\gamma}, & \text{in } \Omega, \\ u = \underline{u}, & \text{on } \partial\Omega \end{cases}$$

has a unique convex viscosity solution $u \in C^0(\bar{\Omega}) \cap C^\infty(\Omega)$.

Proof. Uniqueness follows from the comparison principle. Let $\varphi_i \in C^\infty(\partial\Omega)$ satisfy

$$\underline{u} < \varphi_i \leq \underline{u} + \frac{1}{i} \quad \text{on } \partial\Omega \quad \text{and} \quad \varphi_i \to \underline{u} \quad \text{in } C^0(\bar{\Omega}).$$

It follows from [22] that there exists a unique, strictly convex solution $u_i \in C^\infty(\bar{\Omega})$ of

$$\begin{cases} \det(D^2u_i) = \lambda^{-\beta}(1+|x|^2)^{-\gamma}, & \text{in } \Omega, \\ u = \varphi_i, & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle, we have

$$\underline{u} \leq u_i \leq h_i \quad \text{on } \bar{\Omega},$$

where h_i is the harmonic function on Ω with boundary value φ_i. We can see that $\{u_i\}$ are uniformly bounded. This, together with the convexity of u_i, implies that $|\nabla u_i|$ is bounded on compact subsets of Ω. So, after passing to a subsequence, u_i uniformly converges on compact subsets of Ω to some convex function $u \in C^0(\Omega)$. Consequently, u is a viscosity solution to $\det(D^2u) = \lambda^{-\beta}(1+|x|^2)^{-\gamma}$.

On the other hand, it is obvious that $\underline{u} \leq u \leq h$, where h is the harmonic function on Ω with boundary value $h = \underline{u}$. It follows that u can be extended as a continuous function on $\bar{\Omega}$ with $u = \underline{u}$ on $\partial\Omega$. By the regularity theory of Monge–Ampère equations, see [19], and the standard Schauder theory of linear elliptic equations, see [21], we obtain $u \in C^\infty(\Omega)$. The lemma is established. □

Lemma 3.3. *Let Ω be a domain in $\mathbb{R}^n$ and $g \in C^0(\mathbb{R}^n)$ be a nonnegative function. Suppose that convex functions $v \in C^0(\bar{\Omega}),\ u \in C^0(\mathbb{R}^n)$ satisfy*

$$\det D^2 v \geq g(x), \quad x \in \Omega,$$
$$\det D^2 u \geq g(x), \quad x \in \mathbb{R}^n$$

in the viscosity sense, respectively, and

$$u \leq v, \quad x \in \Omega, \tag{3.1}$$
$$u = v, \quad x \in \partial\Omega. \tag{3.2}$$

Set

$$w(x) = \begin{cases} v(x), & x \in \Omega, \\ u(x), & x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then, $w \in C^0(\mathbb{R}^n)$ is a convex function and satisfies

$$\det D^2 w \geq g(x), \quad x \in \mathbb{R}^n$$

in the viscosity sense.

Proof. Let $\bar{x} \in \mathbb{R}^n,\ \varphi \in C^2(\mathbb{R}^n)$ satisfy $w(\bar{x}) = \varphi(\bar{x})$,

$$w(x) \leq \varphi(x), \quad x \in \mathbb{R}^n. \tag{3.3}$$

If $\bar{x} \in \Omega$, we have

$$v(\bar{x}) = w(\bar{x}) = \varphi(\bar{x}), \qquad v(x) = w(x) \leq \varphi(x), \quad x \in \Omega.$$

Therefore,

$$\det(D^2\varphi(\bar{x})) \geq g(\bar{x}).$$

If $\bar{x} \in \mathbb{R}^n \setminus \Omega$, we have

$$u(\bar{x}) = w(\bar{x}) = \varphi(\bar{x}), \qquad u(x) = w(x) \leq \varphi(x), \quad x \in \mathbb{R}^n \setminus \Omega.$$

By (3.1)–(3.3),

$$u(x) \le \varphi(x), \quad x \in \mathbb{R}^n.$$

Therefore,

$$\det(D^2\varphi(\bar{x})) \ge g(\bar{x}).$$

The lemma is completed. □

The following lemma can be found in [10].

Lemma 3.4. *Let $D \subset \mathbb{R}^n$ be a bounded strictly convex domain, $\partial D \in C^2$, $\varphi \in C^2(\partial D)$. Then there exists a constant C, depending only on n, φ and D, such that, for every $\xi \in \partial D$, there exists $\bar{x}_\xi \in \mathbb{R}^n$ satisfying*

$$|\bar{x}_\xi| \le C, \quad w_\xi(\xi) = \varphi(\xi) \quad \text{and} \quad w_\xi < \varphi \quad \text{on } \partial D \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2}|x - \bar{x}_\xi|^2 - \frac{1}{2}|\xi - \bar{x}_\xi|^2, \quad x \in \mathbb{R}^n.$$

Definition 3.5. The subfunction class S_c for some constant c is defined as follows: a function v is in S_c if and only if

(1) $v \in C^0(\mathbb{R}^n \setminus D)$ and $v \le \phi$ on ∂D;
(2) v is a locally convex viscosity subsolution of (1.4) in $\mathbb{R}^n \setminus \bar{D}$;
(3) $v(x) \le f_0(|x|) + c, \ \forall x \in \mathbb{R}^n \setminus D$.

Lemma 3.6. *There exists some constant c^*, depending only on n, γ, λ and D, such that, for any $c > c^*$, $S_c \ne \emptyset$.*

Proof. Fix $R_2 > R_1 > 1$ such that $D \subset\subset B_{R_1}$ and $R_2 > 3R_1$. Let

$$C = \max_{x \in \overline{B_{R_2+1}}} \lambda^{-\beta}(1 + |x|^2)^{-\gamma} > 0.$$

By Lemma 3.4, we know that

$$v_\xi(x) := C^{\frac{1}{n}} w_\xi(x) = C^{\frac{1}{n}}\varphi(\xi) + \frac{1}{2}C^{\frac{1}{n}}|x - \bar{x}_\xi|^2 - \frac{1}{2}C^{\frac{1}{n}}|\xi - \bar{x}_\xi|^2$$

satisfies the equation $\det(D^2u) = C$ in $\mathbb{R}^n$ and

$$v_\xi(\xi) = \phi(\xi), \qquad v_\xi < \phi \quad \text{on } \partial D \setminus \{\xi\},$$

where $\phi = C^{\frac{1}{n}}\varphi$. In particular, v_ξ is a convex smooth subsolution of (1.4) on B_{R_2+1}. Hence,

$$V(x) := \sup_{\xi \in \partial D} v_\xi(x), \quad x \in B_{R_2+1}$$

is a convex viscosity subsolution of (1.4) in B_{R_2+1} and satisfies

$$V(\xi) \le \phi(\xi), \quad \xi \in \partial D.$$

By the definition of V, for any $\xi \in \partial D$,

$$V(\xi) \ge v_\xi(\xi) = \phi(\xi).$$

Therefore,

$$V(\xi) = \phi(\xi) \quad \text{on } \partial D.$$

On the other hand, for $a \ge 0$, define

$$w_a(x) := \inf_{x \in B_{R_1}} V(x) + \int_{2R_1}^{|x|} [g(\tau) + a]^{\frac{1}{n}} d\tau,$$

where

$$g(\tau) = n\lambda^{-\beta} \int_0^\tau s^{n-1}(1 + s^2)^{-\gamma} ds. \tag{3.4}$$

From Section 2 we can see that $w_a \in C^0(\mathbb{R}^n)$ is a convex viscosity solution of (1.4) in $\mathbb{R}^n$. Obviously,

$$w_a(x) \leq V(x), \quad |x| \leq R_1.$$

Since $R_2 > 3R_1$, we choose $a_1 > 0$ large enough such that for $a \geq a_1$,

$$w_a(x) \geq \inf_{x\in B_{R_1}} V(x) + \int_{2R_1}^{3R_1} [g(\tau) + a]^{\frac{1}{n}} d\tau \geq 1 + V(x), \quad |x| = R_2. \tag{3.5}$$

By the definition of w_a,

$$\begin{aligned} w_a(x) &= f_0(|x|) + \inf_{x\in B_{R_1}} V(x) + \int_{2R_1}^{|x|} [g(\tau) + a]^{\frac{1}{n}} d\tau - \int_0^{|x|} [g(\tau)]^{\frac{1}{n}} d\tau, \\ &= f_0(|x|) + \inf_{x\in B_{R_1}} V(x) - f_0(2R_1) + \int_{2R_1}^{\infty} (g(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{a}{g(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau \\ &\quad - \int_{|x|}^{\infty} (g(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{a}{g(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau. \end{aligned} \tag{3.6}$$

It follows from (2.4) and $-\infty < \gamma < \frac{n(n-2)}{2(n-1)}$ that

$$\int_{2R_1}^{\infty} (g(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{a}{g(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau = \left(\frac{n}{n-2\gamma} \right)^{\frac{1-n}{n}} \lambda^{\frac{\beta(n-1)}{n}} \frac{a}{n} \int_{2R_1}^{\infty} \left[\tau^{1-n+2\gamma-\frac{2\gamma}{n}} + o\left(\tau^{1-n+2\gamma-\frac{2\gamma}{n}} \right) \right] d\tau < +\infty$$

and

$$\begin{aligned} \int_{|x|}^{\infty} (g(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{a}{g(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau &= \left(\frac{n}{n-2\gamma} \right)^{\frac{1-n}{n}} \lambda^{\frac{\beta(n-1)}{n}} \frac{a}{n} \int_{|x|}^{\infty} \tau^{1-n+2\gamma-\frac{2\gamma}{n}} + o\left(\tau^{1-n+2\gamma-\frac{2\gamma}{n}} \right) d\tau \\ = -\left(\frac{n}{n-2\gamma} \right)^{\frac{1-n}{n}} \lambda^{\frac{\beta(n-1)}{n}} \frac{a}{2n-2\gamma-n^2+2n\gamma} |x|^{2-n+2\gamma-\frac{2\gamma}{n}} &+ o\left(|x|^{2-n+2\gamma-\frac{2\gamma}{n}} \right), \end{aligned}$$

as $|x| \to \infty$.

Let

$$\mu(a) := \inf_{x\in B_{R_1}} V(x) - f_0(2R_1) + \int_{2R_1}^{\infty} (g(\tau))^{\frac{1}{n}} \left\{ \left[1 + \frac{a}{g(\tau)} \right]^{\frac{1}{n}} - 1 \right\} d\tau.$$

It is clear that $\mu(a)$ is continuous, monotonic increasing for a, and $\mu(a) \to \infty$ as $a \to \infty$. Also,

$$w_a(x) \leq f_0(|x|) + \mu(a), \quad a \geq a_1,\ x \in \mathbb{R}^n \setminus D. \tag{3.7}$$

Moreover,

$$w_a(x) = f_0(|x|) + \mu(a) + O\left(|x|^{2-\frac{2\gamma}{n}-n+2\gamma} \right), \quad \text{as } |x| \to \infty. \tag{3.8}$$

We choose $a_2 > 0$ large enough such that for $a > a_2$,

$$V(x) \leq f_0(|x|) + \mu(a), \quad |x| \leq R_2. \tag{3.9}$$

Set $a^* = \max\{a_1, a_2\}$, then for any $a > a^*$, (3.5), (3.7) and (3.9) hold.

Define

$$\underline{u}_a(x) = \begin{cases} \max\{V(x), w_a(x)\}, & |x| \leq R_2, \\ w_a(x), & |x| \geq R_2. \end{cases} \tag{3.10}$$

Then

$$\underline{u}_a(x) = V(x) = \phi(x), \quad x \in \partial D. \tag{3.11}$$

By Lemma 3.3, we know that $\underline{u}_a$ is a convex viscosity subsolution of (1.4) in $\mathbb{R}^n$. Since $\mu(a)$ is continuous and monotonic increasing for a and $\mu(a) \to \infty$ as $a \to \infty$, then for $c > c^* := \mu(a^*)$, there is a number $a > a^*$, such that $c = \mu(a)$.

By (3.7) and (3.9), we have, for $c > c^*$, $a = \mu^{-1}(c)$,

$$\underline{u}_a(x) \le f_0(|x|) + c, \quad \forall x \in \mathbb{R}^n.$$

Moreover, by (3.8) we have

$$\underline{u}_a(x) = w_a(x) = f_0(|x|) + c + O\left(|x|^{2-\frac{2\gamma}{n}-n+2\gamma}\right), \quad \text{as } |x| \to \infty. \tag{3.12}$$

Therefore, for $c > c^*$, $S_c \neq \emptyset$. □

Define

$$u_c(x) = \sup\{v(x) : v \in S_c\}, \quad x \in \mathbb{R}^n \setminus \bar{D},\ c > c^*.$$

Lemma 3.7. *We have*

(i) $u_c(x) \le f_0(|x|) + c,\ x \in \mathbb{R}^n \setminus \bar{D}$;
(ii) u_c *is a locally convex viscosity subsolution of* (1.4) *in* $\mathbb{R}^n \setminus \bar{D}$;
(iii) u_c *can be extended to a continuous function on* $\mathbb{R}^n \setminus D$ *with* $u_c = \phi$ *on* ∂D;
(iv) u_c *is a viscosity solution of* (1.4) *in* $\mathbb{R}^n \setminus \bar{D}$.

Proof. (i) follows from the definition of u_c, since $v(x) \le f_0(|x|) + c$ for all $v \in S_c$. (ii) holds since u_c locally is the sup over a family of convex viscosity subsolutions.

Next, we prove (iii). For $\xi_0 \in \partial D$. By Lemma 3.6, we know that for $c > c^*$, $\underline{u}_a \in S_c$ with $a = \mu^{-1}(c)$. Therefore,

$$u_c(x) \ge \underline{u}_a(x) \quad \text{in } \mathbb{R}^n \setminus \bar{D},$$

which, together with (3.11) and the continuity of $\underline{u}_a$ in $\mathbb{R}^n$, implies

$$\liminf_{x \to \xi_0} u_c(x) \ge \underline{u}_a(\xi_0) = \phi(\xi_0). \tag{3.13}$$

On the other hand, we claim that $\limsup_{x\to\xi_0} u_c(x) \le \phi(\xi_0)$. Indeed, for any $v \in S_c$, v is a viscosity subsolution of (1.4) in $\mathbb{R}^n \setminus \bar{D}$, i.e., for every $\bar{x} \in \mathbb{R}^n \setminus \bar{D}$ and every function $\varphi \in C^2(\mathbb{R}^n \setminus \bar{D})$ satisfying

$$\varphi \ge v \quad \text{on } \mathbb{R}^n \setminus \bar{D}, \qquad \varphi(\bar{x}) = v(\bar{x}),$$

we have $\det(D^2\varphi(\bar{x})) \ge \lambda^{-\beta}(1 + |\bar{x}|^2)^{-\gamma}$. By Remark 1.3.2 in [17], we obtain $D^2\varphi(\bar{x}) \ge 0$. Thus,

$$\Delta\varphi(\bar{x}) \ge n[\det(D^2\varphi(\bar{x}))]^{\frac{1}{n}} \ge n\lambda^{-\frac{\beta}{n}}(1 + |\bar{x}|^2)^{-\frac{\gamma}{n}}.$$

Therefore, v is a viscosity subsolution of $\Delta v = n\lambda^{-\frac{\beta}{n}}(1 + |x|^2)^{-\frac{\gamma}{n}}$ in $\mathbb{R}^n \setminus \bar{D}$ and $v \le \phi$ on ∂D, $v \le u_c$ in $\mathbb{R}^n \setminus \bar{D}$.

Choose a ball $B_R(0)$, such that $D \subset\subset B_R(0)$. It is well known that the following Dirichlet problem

$$\begin{cases} \Delta v^+ = n\lambda^{-\frac{\beta}{n}}(1 + |x|^2)^{-\frac{\gamma}{n}}, & \text{in } B_R(0) \setminus \bar{D}, \\ v^+ = \phi, & \text{on } \partial D, \\ v^+ = u_c, & \text{on } \partial B_R(0) \end{cases} \tag{3.14}$$

has a unique classical solution $v^+ \in C^2(B_R(0) \setminus \bar{D}) \cap C^0(\overline{B_R(0) \setminus D})$, see [21]. By a comparison principle, we obtain, for any $v \in S_c$,

$$v \le v^+ \quad \text{in } \overline{B_R(0) \setminus D}.$$

Hence, $u_c \le v^+$ in $B_R(0) \setminus \bar{D}$ and

$$\limsup_{x \to \xi_0} u_c(x) \le v^+(\xi_0) = \phi(\xi_0).$$

This, together with (3.13), implies (iii).

Finally, we prove (iv). For $x_0 \in \mathbb{R}^n \setminus \bar{D}$, choose an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset \mathbb{R}^n \setminus \bar{D}$. By Lemma 3.2, there is a unique convex viscosity solution $\tilde{u} \in C^0(\overline{B_\varepsilon(x_0)}) \cap C^\infty(B_\varepsilon(x_0))$ to

$$\begin{cases} \det(D^2\tilde{u}) = \lambda^{-\beta}(1 + |x|^2)^{-\gamma} & \text{in } B_\varepsilon(x_0), \\ \tilde{u} = u_c & \text{on } \partial B_\varepsilon(x_0). \end{cases}$$

We also know that $f_0(|x|) + c$ is a convex smooth solution to

$$\begin{cases} \det(D^2(f_0 + c)) = \lambda^{-\beta}(1 + |x|^2)^{-\gamma} & \text{in } B_\varepsilon(x_0), \\ f_0 + c \ge u_c & \text{on } \partial B_\varepsilon(x_0). \end{cases}$$

By the comparison principle, Lemma 3.1, $\tilde{u} \ge u_c$ and $\tilde{u} \le f_0 + c$ on $\overline{B_\varepsilon(x_0)}$.

Define

$$\tilde{w}(x) = \begin{cases} \tilde{u}(x), & x \in B_\varepsilon(x_0), \\ u_c(x), & x \in \mathbb{R}^n \setminus (D \cup B_\varepsilon(x_0)). \end{cases}$$

Clearly, $\tilde{w} \in S_c$. So, by the definition of u_c, $u_c \geq \tilde{w}$ on $B_\varepsilon(x_0)$. It follows that $u_c \equiv \tilde{u}$ on $B_\varepsilon(x_0)$. In this way, we have proved (iv). □

Proof of Theorem 1.1. It follows from Lemma 3.7 that for any $c > c^*$, there exists a viscosity solution $u_c \in C^0(\overline{\mathbb{R}^n \setminus D})$ to

$$\det(D^2 u_c) = \lambda^{-\beta}(1 + |x|^2)^{-\gamma}, \quad \text{in } \mathbb{R}^n \setminus \bar{D} \tag{3.15}$$

with $u_c = \phi$ on ∂D. We need only to prove (1.10). By the definition of u_c and Lemma 3.7, we have

$$\underline{u}_a \leq u_c \leq f_0(|x|) + c, \quad \text{in } \mathbb{R}^n \setminus D,$$

where $a = \mu^{-1}(c)$. Then the asymptotic behavior (1.10) follows from (3.12). The theorem is completed. □

4. Examples

Fix a ball $B_R(0) \subset \mathbb{R}^n (n \geq 2)$ and a constant d. We consider the existence of the radially symmetric locally convex solution of the exterior Dirichlet problem

$$\begin{cases} \det(D^2 u) = \lambda^{-\beta}(1 + |x|^2)^{-\gamma}, & \text{in } \mathbb{R}^n \setminus \overline{B_R(0)}, \\ u = d, & \text{on } \partial B_R(0), \end{cases} \tag{4.1}$$

with some appropriate asymptotic behavior at infinity. We will discuss the problem in the following cases.

(i) $n \geq 2$ and $-\infty < \gamma < \frac{n(n-2)}{2(n-1)}$;
(ii) $n = 2$ and $\frac{2k-2}{2k-1} < \gamma < \frac{2k}{2k+1}$ for some positive integer k;
(iii) $n = 2$ and $\gamma = \frac{2k-2}{2k-1}$ for some positive integer k.

Theorem 4.1 will tell us that it is necessary for c^* having lower bound in Theorem 1.1, and Theorems 4.2 and 4.3 will show the necessity of $\gamma < \frac{n(n-2)}{2(n-1)}$ in Theorem 1.1.

Theorem 4.1. *Assume $-\infty < \gamma < \frac{n(n-2)}{2(n-1)}$. The exterior problem (4.1) has a radially symmetric locally convex solution $u(x) = f(|x|) \in C^0(\mathbb{R}^n \setminus B_R(0)) \cap C^2(\mathbb{R}^n \setminus \overline{B_R(0)})$ satisfying*

$$u(x) \leq f_0(|x|) + C \quad \text{in } \mathbb{R}^n \setminus B_R(0) \tag{4.2}$$

and

$$\liminf_{|x|\to\infty} |x|^{n-2-2\gamma+\frac{2\gamma}{n}}[u(x) - f_0(|x|) - C] \quad \text{exists and is finite} \tag{4.3}$$

for some C if and only if $C \in [C_0, \infty)$, where $f_0(|x|)$ is the radially symmetric locally convex solution of (1.4) in $\mathbb{R}^n$ with $f_0(0) = f_0'(0) = 0$ and $C_0 := d - f_0(R)$.

Proof. If $u(x) = f(|x|)$ and $u \in C(\mathbb{R}^n \setminus B_R(0)) \cap C^2(\mathbb{R}^n \setminus \overline{B_R(0)})$ is a radially symmetric locally convex solution of (4.1), then $f''(r) > 0$, $\frac{f'(r)}{r} > 0$ for $r > R$, $r = |x|$ and

$$(f'(r))^{n-1} f''(r) = \lambda^{-\beta} r^{n-1}(1 + r^2)^{-\gamma},$$

which is equivalent to

$$((f'(r))^n)' = n\lambda^{-\beta} r^{n-1}(1 + r^2)^{-\gamma}.$$

Integrating the above equation on $[R, r]$ for $r > R$, we obtain

$$f'(r) = \left[n\lambda^{-\beta} \int_R^r s^{n-1}(1 + s^2)^{-\gamma} ds + b \right]^{\frac{1}{n}},$$

where $b = (f'(R))^n \geq 0$. Then we have by recalling the definition of g and f_0,

$$\begin{aligned}
f(|x|) &= \int_R^{|x|}\left[n\lambda^{-\beta}\int_R^{\tau} s^{n-1}(1+s^2)^{-\gamma}ds+b\right]^{\frac{1}{n}}d\tau+f(R)\\
&= \int_R^{|x|}[g(\tau)-g(R)+b]^{\frac{1}{n}}d\tau+d\\
&= \int_R^{|x|}[g(\tau)]^{\frac{1}{n}}+\int_R^{|x|}\left\{[g(\tau)-g(R)+b]^{\frac{1}{n}}-[g(\tau)]^{\frac{1}{n}}\right\}d\tau+d\\
&= f_0(|x|)-f_0(R)+\int_R^{|x|}(g(\tau))^{\frac{1}{n}}\left\{\left[1+\frac{b-g(R)}{g(\tau)}\right]^{\frac{1}{n}}-1\right\}d\tau+d\\
&= f_0(|x|)+C(b)-\int_{|x|}^{\infty}(g(\tau))^{\frac{1}{n}}\left\{\left[1+\frac{b-g(R)}{g(\tau)}\right]^{\frac{1}{n}}-1\right\}d\tau\\
&= f_0(|x|)+C(b)+O\left(|x|^{2-n+2\gamma-\frac{2\gamma}{n}}\right),\quad \text{as } |x|\to\infty,
\end{aligned}\tag{4.4}$$

where

$$C(b):=d-f_0(R)+\int_R^{\infty}(g(\tau))^{\frac{1}{n}}\left\{\left[1+\frac{b-g(R)}{g(\tau)}\right]^{\frac{1}{n}}-1\right\}d\tau.$$

If $u(x)=f(|x|)$ satisfies (4.2), (4.3) for a constant C, then $C=C(b)$ for some b by (4.4). Hence, we have

$$f(|x|)\le f_0(|x|)+C(b),\quad \forall\, |x|\ge R$$

and

$$f(|x|)=f_0(|x|)+C(b)+O\left(|x|^{2-n+2\gamma-\frac{2\gamma}{n}}\right),\quad \text{as } |x|\to\infty.$$

Again by (4.4) we see that $b\ge g(R)$. It is obvious that $C(t)$ is continuous, monotonic increasing for t, and $C(t)\to\infty$ as $t\to\infty$. Thus, $C=C(b)\in[C(g(R)),\ \infty)=[C_0,\ \infty)$.

On the other hand, by the properties of $C(b)$, for any $C\in[C_0,\ \infty)$, there exists a number $b\in[g(R),\ \infty)$ such that $C=C(b)$. Then we consider the function

$$u(x):=\int_R^{|x|}\left[n\lambda^{-\beta}\int_R^{\tau}s^{n-1}(1+s^2)^{-\gamma}ds+b\right]^{\frac{1}{n}}d\tau+d.$$

It is easy to see that $u\in C^0(\mathbb{R}^n\setminus B_R(0))\cap C^2(\mathbb{R}^n\setminus\overline{B_R(0)})$ satisfies (4.1)–(4.3) for the constant C. □

Theorem 4.2. *Assume $n=2$, $\frac{2k-2}{2k-1}<\gamma<\frac{2k}{2k+1}$ for some positive integer k, the exterior problem (4.1) has a radially symmetric locally convex solution $u(x)=f(|x|)\in C^0(\mathbb{R}^2\setminus B_R(0))\cap C^2(\mathbb{R}^2\setminus\overline{B_R(0)})$ satisfying*

$$\limsup_{|x|\to\infty}|x|^{-\theta_{k+1}}\left|u(x)-f_0(|x|)-c_1a|x|^{\theta_1}-c_2a^2|x|^{\theta_2}-\cdots-c_ka^k|x|^{\theta_k}-c_0\right|<\infty,\tag{4.5}$$

if and only if $a\ge -g(R)$, where

$$c_m=\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdot\cdots\cdot\left(\frac{1}{2}-m+1\right)\lambda^{-\beta\left(\frac{1}{2}-m\right)}}{m![(1-\gamma)(1-2m)+1](1-\gamma)^{\frac{1}{2}-m}},\qquad \theta_m=(1-\gamma)(1-2m)+1\in(0,1)\tag{4.6}$$

for $m=1,2,\ldots,k$, $\theta_{k+1}=(\gamma-1)(2k+1)+1<0$, and c_0 depends only on d, R, γ, λ and a.

Proof. As above, if $u(x)=f(|x|)$ and $u\in C^0(\mathbb{R}^2\setminus B_R(0))\cap C^2(\mathbb{R}^2\setminus\overline{B_R(0)})$ is a radially symmetric locally convex solution of (4.1), then $f''(r)>0$, $\frac{f'(r)}{r}>0$ for $r>R$, $r=|x|$ and

$$(f'(r))^2=2\lambda^{-\beta}\int_R^r s(1+s^2)^{-\gamma}ds+b=g(r)-g(R)+b,$$

where $g(\tau) = 2\lambda^{-\beta}\int_0^\tau s(1+s^2)^{-\gamma}ds$, $b = (f'(R))^2$. Clearly, the exterior problem (4.1) has a locally convex solution $u \in C^0(\mathbb{R}^2 \setminus B_R(0)) \cap C^2(\mathbb{R}^2 \setminus \overline{B_R(0)})$ if and only if $b \geq 0$, i.e., $a := b - g(R) \geq -g(R)$. By recalling the definition of f_0, we have

$$f(|x|) = \int_R^{|x|}\left[g(\tau)+a\right]^{\frac{1}{2}} d\tau + f(R)$$
$$= f_0(|x|) - f_0(R) + \int_R^{|x|}(g(\tau))^{\frac{1}{2}}\left\{\left[1+\frac{a}{g(\tau)}\right]^{\frac{1}{2}} - 1\right\}d\tau + d.$$

From Taylor's expansion, we have

$$g(\tau) = 2\lambda^{-\beta}\int_0^\tau s(1+s^2)^{-\gamma}ds = \frac{\lambda^{-\beta}}{1-\gamma}\left[(1+\tau^2)^{1-\gamma} - 1\right]$$
$$= \frac{\lambda^{-\beta}}{1-\gamma}\tau^{2-2\gamma} + O(1), \quad \text{as } \tau \to \infty. \tag{4.7}$$

Let $\theta_m = (1-\gamma)(1-2m)+1$ for $m = 1, 2, \ldots, k+1$. Notice that $\frac{2k-2}{2k-1} < \gamma < \frac{2k}{2k+1}$, then $0 < \gamma < 1$, $0 < \theta_m < 1$ for $m = 1, 2, \ldots, k$ and $\theta_{k+1} < 0$. By (4.7) and Taylor's expansion, we obtain

$$\int_R^{|x|}(g(\tau))^{\frac{1}{2}}\left\{\left[1+\frac{a}{g(\tau)}\right]^{\frac{1}{2}} - 1\right\}d\tau = c_1 a|x|^{(1-\gamma)(1-2)+1} + c_2 a^2|x|^{(1-\gamma)(1-4)+1} + \cdots$$
$$+ c_k a^k|x|^{(1-\gamma)(1-2k)+1} + \hat{c}_0 + O(|x|^{(\gamma-1)(2k+1)+1})$$
$$= c_1 a|x|^{\theta_1} + c_2 a^2|x|^{\theta_2} + \cdots + c_k a^k|x|^{\theta_k} + \hat{c}_0 + O(|x|^{\theta_{k+1}})$$

as $|x| \to \infty$, where c_m defined by (4.6) for $m = 1, 2, \ldots, k$, $\hat{c}_0$ depends only on R, γ, λ and a. Let $c_0 = \hat{c}_0 - f_0(R) + d$, we obtain (4.5).

On the other hand, if $a \geq -g(R)$, then we consider the function

$$u(x) := \int_R^{|x|}\left[2\lambda^{-\beta}\int_0^\tau s(1+s^2)^{-\gamma}ds + a\right]^{\frac{1}{2}} d\tau + d.$$

It is easy to see that $u \in C^0(\mathbb{R}^2 \setminus B_R(0)) \cap C^2(\mathbb{R}^2 \setminus \overline{B_R(0)})$ satisfies (4.1) and (4.5). The theorem is completed. □

Theorem 4.3. *Assume $n = 2$, $\gamma = \frac{2k-2}{2k-1}$ for some positive integer k, the exterior problem (4.1) has a radially symmetric locally convex solution $u(x) = f(|x|) \in C^0(\mathbb{R}^2 \setminus B_R(0)) \cap C^2(\mathbb{R}^2 \setminus \overline{B_R(0)})$ satisfying*

$$\limsup_{|x|\to\infty} |x|^{-\theta_{k+1}}\left|u(x) - f_0(|x|) - c_1 a|x|^{\theta_1} - \cdots - c_{k-1}a^{k-1}|x|^{\theta_{k-1}} - \tilde{c}_k a^k \ln|x| - \tilde{c}_0\right| < \infty \tag{4.8}$$

if and only if $a \geq -g(R)$, where c_m and θ_m defined by (4.6) for $m = 1, 2, \ldots, k-1$,

$$\tilde{c}_k = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdot\cdots\cdot\left(\frac{1}{2}-k+1\right)\lambda^{-\beta\left(\frac{1}{2}-k\right)}}{k!(1-\gamma)^{\frac{1}{2}-k}}, \qquad \theta_{k+1} = (\gamma-1)(2k+1)+1 < 0,$$

and $\tilde{c}_0$ depends only on d, R, γ, λ and a.

Proof. The proof is similar to the proof of Theorem 4.2. We need only to establish (4.8). As above, we have

$$f(|x|) = f_0(|x|) - f_0(R) + \int_R^{|x|}(g(\tau))^{\frac{1}{2}}\left\{\left[1+\frac{a}{g(\tau)}\right]^{\frac{1}{2}} - 1\right\}d\tau + d.$$

In view of $\gamma = \frac{2k-2}{2k-1}$, i.e., $(1-\gamma)(1-2k) = -1$, we can obtain by (4.7) and Taylor's expansion,

$$\int_R^{|x|}(g(\tau))^{\frac{1}{2}}\left\{\left[1+\frac{a}{g(\tau)}\right]^{\frac{1}{2}} - 1\right\}d\tau = c_1 a|x|^{(1-\gamma)(1-2)+1} + \cdots + c_{k-1}a^{k-1}|x|^{(1-\gamma)(3-2k)+1}$$
$$+ \tilde{c}_k a^k \ln|x| + \bar{c}_0 + O(|x|^{(\gamma-1)(2k+1)+1})$$
$$= c_1 a|x|^{\theta_1} + \cdots + c_{k-1}a^{k-1}|x|^{\theta_{k-1}} + \tilde{c}_k \ln|x| + \bar{c}_0 + O(|x|^{\theta_{k+1}})$$

as $|x| \to \infty$, where c_m and θ_m defined by (4.6) for $m = 1, 2, \ldots, k-1$, $\tilde{c}_k = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)\lambda^{-\beta(\frac{1}{2}-k)}}{k!(1-\gamma)^{\frac{1}{2}-k}}$, $\theta_{k+1} = (\gamma-1)(2k+1)+1$ and $\bar{c}_0$ depends only on R, γ, λ and a.

In view of $\gamma = \frac{2k-2}{2k-1}$, we have $0 < \theta_m < 1$ for $m = 1, 2, \ldots, k-1$, and $\theta_{k+1} < 0$. Thus, we obtain (4.8). Let $\tilde{c}_0 = \bar{c}_0 - f_0(R) + d$, the theorem is established. □

Remark 4.1. For $n = 2$, $\gamma = 0$, Theorem 4.3 is compatible with Theorem 3 in [13].

Acknowledgments

The authors would like to thank the referee of this paper for very helpful comments and suggestions.

The first author was partially supported by China Postdoctoral Science Foundation (20110490311). The work of the second author was partially supported by the Program for Changjiang Scholars and Innovative Research Teams at University in China (IRT0908), NSFC (11071020), and Doctoral Program Foundation of Institution of Higher Education of China (20100003110003). The third author was partially supported by NSFC (11131005) and Doctoral Program Foundation of Institution of Higher Education of China (20110002110064).

References

[1] W.J. Firey, On the shapes of worn stones, Mathematika 21 (1974) 1–11.
[2] K.S. Chou, Kaiseng Tso, Deforming a hypersurface by its Gauss–Kronecker curvature, Comm. Pure Appl. Math. 38 (1985) 867–882.
[3] B. Chow, Deforming convex hypersurfaces by the nth root of the Gaussian curvature, J. Differential Geom. 23 (1985) 117–138.
[4] B. Andrews, Gauss curvature flow: the fate of the rolling stones, Invent. Math. 138 (1999) 151–161.
[5] J. Urbas, Complete noncompact self-similar solutions of Gauss curvature flows I. Positive powers, Math. Ann. 311 (1998) 251–274.
[6] K. Jörgens, Über die Lösungen der differentialgleichung $rt - s^2 = 1$, Math. Ann. 127 (1954) 130–134.
[7] E. Calabi, Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, Michigan Math. J. 5 (1958) 105–126.
[8] A.V. Pogorelov, On the improper affine hyperspheres, Geom. Dedicata 1 (1972) 33–46.
[9] S.Y. Cheng, S.T. Yau, Complete affine hypersurfaces, Part I. The completeness of affine metrics, Comm. Pure Appl. Math. 39 (1986) 839–866.
[10] L. Caffarelli, Y.Y. Li, An extension to a theorem of Jörgens, Calabi and Pogorelov, Comm. Pure Appl. Math 56 (2003) 0549–0583.
[11] L. Ferrer, A. Martínez, F. Milán, An extension of a theorem by K. Jörgens and a maximum principle at infinity for parabolic affine spheres, Math. Z. 230 (3) (1999) 471–486.
[12] L. Ferrer, A. Martínez, F. Milán, The space of parabolic affine spheres with fixed compact boundary, Monatsh. Math. 130 (1) (2000) 19–27.
[13] C. Wang, J.G. Bao, Necessary and sufficient conditions on existence and convexity of solutions for dirichlet problems of Hessian equations on exterior domains, Preprint.
[14] K.S. Chou, X.J. Wang, Entire solutions of the Monge–Ampère equation, Comm. Pure Appl. Math. 49 (1996) 529–539.
[15] H.Y. Jian, X.J. Wang, Entire solutions to Monge–Ampère equations and translating solutions to Gauss curvature flow, 2010, Preprint.
[16] L. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, in: American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.
[17] C.E. Gutiérrez, The Monge–Ampère equation, in: Progress in Nonlinear Differential Equations and their Applications, vol. 44, Birkhäuser, Boston, 2001.
[18] J. Urbas, On the existence of nonclassical solutions for two class of fully nonlinear elliptic equations, Indiana Univ. Math. J. 39 (1990) 355–382.
[19] L. Caffarelli, Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation, Ann. of Math. (2) 131 (1) (1990) 135–150.
[20] B. Guan, H.Y. Jian, The Monge–Ampère equation with infinite boundary value, Pacific J. Math. 216 (2004) 77–94.
[21] D. Gilbarg, N. Trudinger, Second Order Elliptic Partial Differential Equations, second ed., Springer-Verlag, 1983.
[22] L. Caffarelli, L. Nirenberg, J. Spruck, The dirichlet problem for nonlinear second-order elliptic equations. I. Monge–Ampère equation, Comm. Pure Appl. Math. 37 (3) (1984) 369–402.