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Existence and nonexistence theorem for entire subsolutions of k-Yamabe type equations

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ABSTRACT

In this paper, we consider fully nonlinear elliptic equations related to Yamabe problem. We improve a result of Jin, Li and Xu and establish the existence and nonexistence for positive entire subsolutions with a generalized Keller–Osserman condition.

1. Introduction and main results

For $1 \leqslant k \leqslant n$, let

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}$$

denote the k-th elementary symmetric function of n variations, $\lambda_1, \ldots, \lambda_n$, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone

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$$\Gamma_n = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0,\ i = 1, \ldots, n\}.$$

It is well known (see, e.g. [6]) that Γ_k is a convex cone in $\mathbb{R}^n$ with its vertex at the origin, and

$$\Gamma_n \subseteq \cdots \subseteq \Gamma_2 \subseteq \Gamma_1 = \{\lambda \in \mathbb{R}^n \mid \lambda_1 + \cdots + \lambda_n > 0\}. \tag{1}$$

Let $\lambda(D^2u)$ denote the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the Hessian matrix of u. Fully nonlinear elliptic equations involving $\sigma_k(\lambda(D^2u))$, as well as for more general f instead of σ_k, have been investigated in the classical and pioneering paper of Caffarelli, Nirenberg and Spruck [6]. For extensive studies and outstanding results on such equations, see, for example, Guan and Spruck [11], Trudinger [28], Trudinger and Wang [29], and the references therein.

Let (M, g) be an n-dimensional, smooth Riemannian manifold without boundary. For $n \geqslant 3$, the well-known Yamabe conjecture states that there exist metrics which are pointwise conformal to g and have constant scalar curvature. The Yamabe conjecture is proved through the work of Yamabe [32], Trudinger [27], Aubin [1] and Schoen [25]. The Yamabe and related problems have attracted much attention in the last 40 years or so, see, e.g., [26,2] and the references therein. The Schouten tensor of g is defined as

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g\right),$$

where Ric_g and R_g denote, respectively, the Ricci tensor and the scalar curvature associated with g. We use $\lambda(A_g) = (\lambda_1(A_g), \ldots, \lambda_n(A_g))$ to denote the eigenvalues of A_g. Clearly,

$$\sigma_1(\lambda(A_g)) = \frac{1}{2(n-1)} R_g.$$

Let

$$V_1 = \left\{\lambda \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \lambda_i > 1\right\},$$

and let

$$\Gamma(V_1) = \{s\lambda \mid s > 0,\ \lambda \in V_1\}$$

be the cone with vertex at the origin generated by V_1, i.e. $\Gamma(V_1) = \Gamma_1$. Then the Yamabe problem in the positive case can be formulated as follows: Assuming $\lambda(A_g) \in \Gamma(V_1)$, then there exists a Riemannian metric $\hat{g}$ which is pointwise conformal to g and satisfies $\lambda(A_{\hat{g}}) \in \partial V_1$ on M, i.e. $\sigma_1(\lambda(A_{\hat{g}})) = 1$ on M. A fully nonlinear version of the Yamabe problem on locally conformally flat manifolds was studied in [19,20] and the references therein.

Viaclovsky [30,31] introduced and systematically studied equations

$$\sigma_k^{\frac{1}{k}}(\lambda(A_g)) = \psi(x, u). \tag{2}$$

On 4-dimensional general Riemannian manifolds, remarkable results on (2) for $k = 2$ were obtained by Chang, Gursky, and Yang in [7,8], which include Liouville-type theorems, existence and compactness of solutions, as well as applications to topology.

Let $g_1 = u^{4/(n-2)} g_0$ be a conformal change of metrics; then (see, e.g. [30])

$$A_{g_1} = -\frac{2}{n-2} u^{-1} \nabla^2_{g_0} u + \frac{2n}{(n-2)^2} u^{-2} \nabla_{g_0} u \otimes \nabla_{g_0} u - \frac{2}{(n-2)^2} u^{-2} |\nabla_{g_0} u|^2_{g_0} g_0 + A_{g_0}.$$

Let $g = u^{4/(n-2)} g_{\text{flat}}$, where g_{flat} denotes the Euclidean metric on $\mathbb{R}^n$. Then by the above transformation formula,

$$A_g = u^{\frac{4}{n-2}} A^u_{ij}\, dx^i\, dx^j,$$

where A^u is given by

$$A^u = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I, \tag{3}$$

and I is the $n \times n$ identity matrix. In this case,

$$\lambda(A_g) = \lambda\big(A^u\big),$$

where $\lambda(A^u)$ denotes the eigenvalues of the symmetric $n \times n$ matrix A^u.

For $n \geqslant 3$, consider equations

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(A^u\big)\big) = u^{p-\frac{n+2}{n-2}}, \quad \lambda\big(A^u\big) \in \Gamma_k, \text{ in } \mathbb{R}^n. \tag{4}$$

For the case $p = \frac{n+2}{n-2}$, Li and Li [20] extended the celebrated Liouville-type theorem of Caffarelli, Gidas and Spruck [5] to all σ_k, $1 \leqslant k \leqslant n$. They also showed that for $-\infty < p < \frac{n+2}{n-2}$ problem (4) has no positive solution $u \in C^2(\mathbb{R}^n)$. For $k = 1$ and $p = \frac{n+2}{n-2}$, Eq. (4) takes the form

$$-\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u = \sum_{i=1}^{n} A^u_{ii} = \operatorname{trace}\big(A^u\big) = 1, \quad \text{in } \mathbb{R}^n. \tag{5}$$

That is,

$$-\Delta u = \frac{1}{2}(n-2) u^{\frac{n+2}{n-2}}, \quad \text{in } \mathbb{R}^n.$$

A related result of Gidas and Spruck in [10] states that there is no positive solution to the equation $-\Delta u = u^p$ in $\mathbb{R}^n$ when $1 \leqslant p < \frac{n+2}{n-2}$.

There have been also many works on the nonlinear partial differential equation

$$\Delta u = u^p, \quad p > 0, \text{ in } \mathbb{R}^n. \tag{6}$$

It is well known that Eq. (6) has no positive solution if $p > 1$, see, for example Keller [16], Osserman [23], Loewner and Nirenberg [17] and Brezis [4]. It is worthwhile to point out that Osserman [23] considered the necessary and sufficient condition under which the following equation

$$\Delta u = f(u), \quad \text{in } \mathbb{R}^n \tag{7}$$

has a subsolution, where f is a positive, continuous, monotone non-decreasing function defined on $\mathbb{R}$. The following growth condition on f at infinity,

$$\int^{\infty} \left(\int_0^t f(s)\, ds \right)^{-\frac{1}{2}} dt = \infty \tag{8}$$

is well known as Keller–Osserman condition, where and later we omit the lower limit of integral to admit an arbitrary positive number. This condition and its generalized forms are crucial in the investigation of existence of blow-up solutions. See [18,22,33,34,21] and the references therein.

Consider the following fully nonlinear partial differential equations

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(D^2u\big)\big) = f(u), \quad 1 \leqslant k \leqslant n, \ \text{in } \mathbb{R}^n, \tag{9}$$

where f is a positive, monotone non-decreasing, continuous function on $\mathbb{R}$. A function $u \in C^2(\mathbb{R}^n)$ is said to be a subsolution of (9), if $\lambda(D^2u) \in \Gamma_k$ and

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(D^2u\big)\big) \geqslant f(u), \quad \text{in } \mathbb{R}^n.$$

We remark that for Eq. (9) with $k = n$ and $f(u) \equiv 1$ some Bernstein-type theorems have been established. The well-known theorem of Jörgens [15], Calabi [9] and Pogorelov [24] says that any convex solution of $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. For $1 \leqslant k \leqslant n$, Bao, Chen, Guan and Ji [3] showed that any convex solution of $\sigma_k(\lambda(D^2u)) = 1$ in $\mathbb{R}^n$ satisfying a quadratic growth condition is a quadratic polynomial. In [14], Jin, Li and Xu proved that $\sigma_k^{\frac{1}{k}}(\lambda(D^2u)) = u^p$ in $\mathbb{R}^n$ has no positive subsolution for any $p > 1$. Combining these facts it seems interesting to study the existence of positive subsolution of $\sigma_k^{\frac{1}{k}}(\lambda(D^2u)) = u^p$ in $\mathbb{R}^n$ for $0 < p \leqslant 1$.

Ji and Bao [13] extended the Keller–Osserman condition (8) to the fully nonlinear partial differential equation (9) and established a necessary and sufficient condition for the solvability of Hessian equations (9).

Theorem 1. *(See [13, Theorem 1.1].) If $f(\cdot)$ is a continuous function defined on $\mathbb{R}$ and satisfies*

$$\begin{cases} f(t) > 0, \ \text{and is monotonically non-decreasing in } (0, +\infty), \\ f(t) = 0 \quad \text{on } (-\infty, 0], \end{cases} \tag{10}$$

then Eq. (9) has a positive subsolution $u \in C^2(\mathbb{R}^n)$ if and only if

$$\int^{\infty} \left(\int_0^t f^k(s)\, ds \right)^{-\frac{1}{k+1}} dt = \infty. \tag{11}$$

Remark 1. Especially, it is easily to see that when $k = 1$, (11) is exactly (8).

Remark 2. For the case $f(u) = u^p$, Theorem 1 shows that (9) has a positive subsolution if and only if $0 < p \leqslant 1$.

In this paper, we consider the following fully nonlinear elliptic equations

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(-A^u\big)\big) = u^{-\frac{n+2}{n-2}} f(u), \quad \lambda\big(-A^u\big) \in \Gamma_k, \ \text{in } \mathbb{R}^n, \tag{12}$$

where $n \geqslant 3$, $f(\cdot)$ is a continuous function in $\mathbb{R}$. For the case $f(u) = u^p$, Jin, Li and Xu [14] proved that

Theorem 2. *(See [14, Theorem 1].) Let $n \geqslant 3$, $f(u) = u^p$. Then the problem (12) has no positive continuous viscosity subsolution, if*

$$p > 1 + \max\left\{0, \frac{2(2k-n)}{(n-2)k}\right\}.$$

The definition of viscosity subsolutions can be referred to [14]. In this paper, we restrict us to classical subsolution. A function $u \in C^2(\mathbb{R}^n)$ is said to be a (classical) subsolution of (12), if $\lambda(-A^u) \in \Gamma_k$ and

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(-A^u\big)\big) \geqslant u^{-\frac{n+2}{n-2}} f(u), \quad \text{in } \mathbb{R}^n.$$

Remark 3. By the definitions of viscosity subsolutions in [14], it is straightforward to show that if $u \in C^2(\mathbb{R}^n)$ is a positive function satisfying $\lambda(-A^u) \in \Gamma_k$ in $\mathbb{R}^n$, then u is a viscosity subsolution of (12) if and only if u is a classical subsolution of (12).

For the general case that $f(\cdot)$ is a continuous function in $\mathbb{R}$, we establish the existence and nonexistence for positive subsolutions of (12) in $\mathbb{R}^n$. Our main theorem is:

Theorem 3. *For $n \geqslant 3$, suppose that $f(\cdot)$ is a continuous function in $\mathbb{R}$. Then*

(a) *for $k > \frac{n}{2}$, if f satisfies (10) and*

$$\frac{f(t)}{t} \quad \text{is strictly increasing in } (0, \infty), \tag{13}$$

then (12) has no positive entire subsolution $u \in C^2(\mathbb{R}^n)$;

(b) *for $k \leqslant \frac{n}{2}$, if f satisfies (10) and*

$$\int^{\infty} t^{\frac{n(1-k)}{(n-2)k}} \left(\int^{t} s^{\frac{(n+2)(1-k)}{n-2}} f^k(s)\, ds \right)^{-\frac{1}{2k}} dt = \infty, \tag{14}$$

then (12) has a positive entire subsolution $u \in C^2(\mathbb{R}^n)$.

By the main theorem, we can easily get the corollary below, which extends Theorem 2, one result of Jin, Li and Xu in [14].

Corollary 4. *Let $f(u) = u^p$, $p > 0$. Then*

(a) *for $k > \frac{n}{2}$, (12) has no radial positive entire solution $u \in C^2(\mathbb{R}^n)$. In particular, (12) has no positive entire subsolution if $p > 1$;*
(b) *for $k \leqslant \frac{n}{2}$, (12) has a positive subsolution $u \in C^2(\mathbb{R}^n)$ if and only if $p \leqslant 1$.*

Remark 4. Although we cannot obtain a necessary and sufficient condition similar as (11) under which Eq. (12) has a positive subsolution, for the case $f(u) = u^p$ we essentially improve the range of p such that the problem (12) has a positive solution for $k > \frac{n}{2}$. Namely, Corollary 4 shows that for $k > \frac{n}{2}$, (12) has no positive entire solution $u \in C^2(\mathbb{R}^n)$ if $p > 1$, while the result in [14] is $p > 1 + \frac{2(2k-n)}{(n-2)k}$.

Remark 5. Corollary 4 also shows that for $k \leqslant \frac{n}{2}$, (12) has a positive subsolution $u \in C^2(\mathbb{R}^n)$ if $p \leqslant 1$. It is still open for us whether (12) has a positive subsolution $u \in C^2(\mathbb{R}^n)$ for $k > \frac{n}{2}$ and $p \leqslant 1$. To solve this question, it needs new methods involved.

If n is even, and $k = \frac{n}{2}$, then we further have

Corollary 5. *For $n \geqslant 3$, suppose that n is even, $k = \frac{n}{2}$, and $f(\cdot)$ is a continuous function in $\mathbb{R}$. Then*

(a) *if f satisfies (10), (13) and*

$$\int^{\infty} t^{-1}\left(\int^{t} s^{-\frac{n+2}{2}} f^{\frac{n}{2}}(s)\,ds\right)^{-\frac{1}{n}} dt < \infty, \tag{15}$$

then (12) has no positive entire subsolution $u \in C^2(\mathbb{R}^n)$;

(b) *if f satisfies (10), and*

$$\int^{\infty} t^{-1}\left(\int_{a}^{t} s^{-\frac{n+2}{2}} f^{\frac{n}{2}}(s)\,ds\right)^{-\frac{1}{n}} dt = \infty, \tag{16}$$

then (12) has a positive entire subsolution $u \in C^2(\mathbb{R}^n)$.

In Section 2 we will introduce some results on radial solutions as preliminaries. The proof of the main theorems will be given in Section 3.

2. Preliminary results on radial solutions

In this section, we study some properties of radial solutions.

Lemma 6. *Suppose $\varphi(r) \in C^2[0, R)$ and $\varphi'(0) = 0$. If $v(x) = \varphi(r)$, where $r = |x| < R$, then $v(x) \in C^2(B_R(0))$, and*

$$\lambda\big(-A^v\big) := \tilde{\lambda}(r) = \big(\tilde{\lambda}_1(r), \tilde{\lambda}_2(r), \ldots, \tilde{\lambda}_n(r)\big), \tag{17}$$

where

$$\begin{cases} \tilde{\lambda}_1(r) = \dfrac{2}{n-2}\varphi^{-\frac{2n}{n-2}}(r)\lambda_1(r), \\ \tilde{\lambda}_2(r) = \cdots = \tilde{\lambda}_n(r) = \dfrac{2}{n-2}\varphi^{-\frac{2n}{n-2}}(r)\lambda_2(r), \end{cases} \tag{18}$$

and

$$\lambda_1(r) := \begin{cases} \varphi(r)\varphi''(r) - \frac{n-1}{n-2}(\varphi'(r))^2, & r > 0, \\ \varphi(0)\varphi''(0), & r = 0, \end{cases}$$

$$\lambda_2(r) := \begin{cases} \frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}(\varphi'(r))^2, & r > 0, \\ \varphi(0)\varphi''(0), & r = 0. \end{cases} \tag{19}$$

So that

$$\sigma_k\big(\lambda\big(-A^v\big)\big) = \frac{\varphi^{-\frac{2nk}{n-2}}(r)}{C_0}\left(\lambda_1(r)\lambda_2^{k-1}(r) + \frac{n-k}{k}\lambda_2^k(r)\right), \quad r \in [0, R), \tag{20}$$

where $C_0 = (\frac{n-2}{2})^k (C_{n-1}^{k-1})^{-1}$, and $C_n^k = \frac{n!}{(n-k)!k!}$.

Proof. In fact, for $x \neq 0$, $1 \leqslant i, j \leqslant n$, we have

$$\frac{\partial v}{\partial x_i}(x) = \varphi'(r)\frac{x_i}{r},$$
$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = \varphi''(r)\frac{x_i x_j}{r^2} - \varphi'(r)\frac{x_i x_j}{r^3} + \frac{\varphi'(r)}{r}\delta_{ij}.$$

By $\varphi'(0) = 0$, we have

$$\lim_{x\to 0}\frac{\partial v}{\partial x_i}(x) = \lim_{x\to 0}\left(\frac{\varphi'(r) - \varphi'(0)}{r-0}\right)x_i = \varphi''(0)\cdot 0 = 0,$$
$$\lim_{x\to 0}\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = \lim_{x\to 0}\left(\left(\varphi''(r) - \frac{\varphi'(r)}{r}\right)\frac{x_i x_j}{r^2} + \left(\frac{\varphi'(r)}{r}\right)\delta_{ij}\right) = \lim_{x\to 0}\left(\frac{\varphi'(r)}{r}\right)\delta_{ij} = \varphi''(0)\delta_{ij}.$$

Define

$$\frac{\partial v}{\partial x_i}(0) = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x_i \partial x_j}(0) = \varphi''(0)\delta_{ij}.$$

Then $v(x) \in C^2(B_R(0))$. For convenience, throughout this paper, we denote $\frac{\varphi'(r)}{r}|_{r=0} = \varphi''(0)$. Then we have

$$\nabla^2 v(x) = \left(\frac{\varphi''(r)}{r^2} - \frac{\varphi'(r)}{r^3}\right)x^T x + \frac{\varphi'(r)}{r}I,$$
$$\nabla v \otimes \nabla v = \left(\frac{\varphi'(r)}{r}\right)^2 x^T x, \quad |\nabla v|^2 = \left(\varphi'(r)\right)^2.$$

Substituting them to (3), we have

$$\begin{aligned}
-A^v &= \frac{2}{n-2}v^{-\frac{n+2}{n-2}}\nabla^2 v - \frac{2n}{(n-2)^2}v^{-\frac{2n}{n-2}}\nabla v \otimes \nabla v + \frac{2}{(n-2)^2}v^{-\frac{2n}{n-2}}|\nabla v|^2 I \\
&= \frac{2}{n-2}\varphi^{-\frac{n+2}{n-2}}(r)\left[\left(\frac{\varphi''(r)}{r^2} - \frac{\varphi'(r)}{r^3}\right)x^T x + \left(\frac{\varphi'(r)}{r}\right)I\right] \\
&\quad - \frac{2n}{(n-2)^2}\varphi^{-\frac{2n}{n-2}}(r)\left(\frac{\varphi'(r)}{r}\right)^2 x^T x + \frac{2}{(n-2)^2}\varphi^{-\frac{2n}{n-2}}(r)\left(\varphi'(r)\right)^2 I \\
&= \frac{2}{n-2}\varphi^{-\frac{2n}{n-2}}(r)\left[\left(\frac{\varphi(r)\varphi''(r)}{r^2} - \frac{\varphi(r)\varphi'(r)}{r^3} - \frac{n}{n-2}\left(\frac{\varphi'(r)}{r}\right)^2\right)x^T x \right. \\
&\quad \left. + \left(\frac{\varphi(r)\varphi'(r)}{r} + \frac{(\varphi'(r))^2}{n-2}\right)I\right].
\end{aligned}$$

Denote

$$\begin{cases} a = \dfrac{2}{n-2}\varphi^{-\frac{2n}{n-2}}(r)\left[\dfrac{\varphi(r)\varphi''(r)}{r^2} - \dfrac{\varphi(r)\varphi'(r)}{r^3} - \dfrac{n}{n-2}\left(\dfrac{\varphi'(r)}{r}\right)^2\right], \\ b = \dfrac{2}{n-2}\varphi^{-\frac{2n}{n-2}}(r)\left[\dfrac{\varphi(r)\varphi'(r)}{r} + \dfrac{1}{n-2}\left(\varphi'(r)\right)^2\right]. \end{cases} \tag{21}$$

By linear algebra, we know that the eigenvalues of the symmetric matrix of the form $ax^Tx + bI$ are $(a|x|^2 + b, b, \dots, b)$. So we obtain the $\tilde{\lambda}$ defined by (18).

By definition of σ_k, we have

$$\sigma_k\big(\lambda(-A^v)\big) = C_{n-1}^{k-1}\tilde{\lambda}_1(r)\tilde{\lambda}_2^{k-1}(r) + C_{n-1}^{k}\tilde{\lambda}_2^{k}(r)$$
$$= C_{n-1}^{k-1}\left(\frac{2}{n-2}\right)^k \varphi^{-\frac{2nk}{n-2}}(r)\left(\lambda_1(r)\lambda_2^{k-1}(r) + \frac{n-k}{k}\lambda_2^k(r)\right).$$

This is (20). □

By (20), we know that the radial solution $\varphi(r)$ of (12) satisfies

$$\lambda_1(r)\lambda_2^{k-1}(r) + \frac{n-k}{k}\lambda_2^k(r) = C_0\varphi^k(r) f^k\big(\varphi(r)\big), \quad r \in [0, R). \tag{22}$$

Next, we will prove that there exists a local solution of the ordinary differential equation (22) in a neighborhood of the origin, with initial conditions $\varphi(0) > 0$ and $\varphi'(0) = 0$. In order to show this, we need the following lemmas.

Lemma 7. *Let $f(t)$ be a continuous function defined on $\mathbb{R}$, and satisfy (10). For any positive constant a, there exists a positive constant R such that the Cauchy problem*

$$\begin{cases} \varphi'(r) = C\left(r^\alpha\varphi^\beta(r)\displaystyle\int_0^r s^{\alpha+k-1}\varphi^{-\beta}(s) f^k\big(\varphi(s)\big)\,ds\right)^{1/k}, & r > 0, \\ \varphi(0) = a \end{cases}$$

has a solution in $[0, R]$.

Letting $C = (\frac{k!(n-k)!}{(n-1)!})^{1/k}$, $\alpha = n - k$ and $\beta = 0$, this is Lemma 2.3 in [13]. So the proof is very similar as in [13] and we omit the proof here. The interested readers could refer to [13].

Lemma 8. *(See [12, Theorem 1.2.3].) Let $T > 0$ be a constant. Suppose that $f(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $v_0(t)$ and $w_0(t)$ are subsolution and supersolution of*

$$u' = f(t, u), \qquad u(0) = x_0, \tag{23}$$

respectively, and satisfy

$$v_0(t) \leqslant w_0(t), \quad \forall t \in [0, T].$$

Then (23) has at least one solution in $D = \{u \in C[0, T]:\ v_0 \leqslant u \leqslant w_0\}$. Here, we call $v(t) \in C^1[0, T]$ is a subsolution (or supersolution) of (23), if

$$v'(t) \leqslant (\geqslant)\, f\big(t, v(t)\big), \quad \forall t \in [0, T]; \quad \textit{and} \quad v(0) \leqslant (\geqslant)\, x_0. \tag{24}$$

Remark 6. Notice that this definition of subsolution (or supersolution) is different from that for partial differential equations of second order. The proof could be found in [12].

By this definition of subsolution and supersolution of ODEs, we have

Lemma 9. *If f is a continuous function in $\mathbb{R}$, and satisfies (10), then for any positive constant a, there exists a positive constant R, such that the Cauchy problem*

$$\begin{cases} \lambda_1\lambda_2^{k-1} + \dfrac{n-k}{k}\lambda_2^k = C_0\big(\varphi(r)\big)^k f^k\big(\varphi(r)\big), \quad r \in (0,R), \\ \varphi'(0) = 0, \\ \varphi(0) = a \end{cases} \tag{25}$$

has a solution $\varphi \in C^2[0,R]$.

Proof. First, we claim that the Cauchy problem (25) has subsolution and supersolution near $r = 0$, with the same initial values.

In fact, Eq. (22) could be written as

$$\left(\lambda_1(r) + \frac{n-k}{k}\lambda_2(r)\right)\lambda_2^{k-1}(r) = C_0\varphi^k(r) f^k\big(\varphi(r)\big). \tag{26}$$

Under the initial value conditions

$$\varphi(0) = a > 0 \quad \text{and} \quad \varphi(0)' = 0,$$

denoting

$$\varphi''(0) := \lim_{r\to 0}\frac{\varphi'(r)}{r},$$

then

$$\begin{aligned} &\lim_{r\to 0}\left[\lambda_1(r) + \frac{n-k}{k}\lambda_2(r)\right] \\ &\quad = \lim_{r\to 0}\left[\varphi(r)\varphi''(r) - \frac{n-1}{n-2}\big(\varphi'(r)\big)^2 + \frac{n-k}{k}\left(\frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2\right)\right] \\ &\quad = \frac{n}{k}\big(a\varphi''(0)\big)^k. \end{aligned}$$

So sending $r \to 0$ on both sides of (26), we have

$$\frac{n}{k}\big(a\varphi''(0)\big)^k = C_0\big(af(a)\big)^k.$$

By (10), we have

$$\varphi''(0) = \left(\frac{k}{n}C_0\right)^{\frac{1}{k}} f(a) > 0$$

and

$$\lim_{r\to 0}\lambda_2(r) = \lim_{r\to 0}\left(\frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2\right) = a\varphi''(0) > 0.$$

Then there exists $\bar{r} > 0$, such that

$$\lambda_2(r) > 0, \qquad \lambda_1(r) + \frac{n-k}{k}\lambda_2(r) > 0, \quad \text{on } [0, \bar{r}]. \tag{27}$$

For $1 < k < n$, we write it as a total differential form; we have

$$\begin{aligned}
&\lambda_1(r) + \frac{n-k}{k}\lambda_2(r) \\
&\quad = \varphi(r)\varphi''(r) - \frac{n-1}{n-2}\big(\varphi'(r)\big)^2 + \frac{n-k}{k}\left(\frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2\right) \\
&\quad = \varphi(r)\varphi''(r) + \frac{n-k}{k}\frac{\varphi(r)\varphi'(r)}{r} + \left(\frac{n-k}{k(n-2)} - \frac{n-1}{n-2}\right)\big(\varphi'(r)\big)^2 \\
&\quad = \varphi(r)\varphi''(r) + \frac{l\varphi(r)\varphi'(r)}{r} + m\big(\varphi'(r)\big)^2 \\
&\quad = r^{-l}\varphi^{1-m}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)',
\end{aligned} \tag{28}$$

where

$$l = \frac{n-k}{k} > 0, \qquad m = \frac{n(1-k)}{(n-2)k} < 0, \quad 1 < k < n. \tag{29}$$

Since

$$\lambda_2(r) = \frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2 \geqslant \frac{\varphi(r)\varphi'(r)}{r}, \quad r \in [0, \bar{r}],$$

it follows from (26) that for $r \in [0, \bar{r}]$,

$$r^{-l}\varphi^{1-m}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)'\left(\frac{\varphi(r)\varphi'(r)}{r}\right)^{k-1} \leqslant C_0\varphi^k(r)f^k\big(\varphi(r)\big),$$

that is,

$$r^{-l-(k-1)(l+1)}\varphi^{1-m+(k-1)(1-m)}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)'\big(r^l\varphi^m(r)\varphi'(r)\big)^{k-1} \leqslant C_0\varphi^k(r)f^k\big(\varphi(r)\big),$$
$$\big(r^l\varphi^m(r)\varphi'(r)\big)'\big(r^l\varphi^m(r)\varphi'(r)\big)^{k-1} \leqslant C_0 r^{kl+k-1}\varphi^{km}(r)f^k\big(\varphi(r)\big).$$

Integrating on both sides, we have

$$\big(r^l\varphi^m(r)\varphi'(r)\big)^k \leqslant kC_0\int_0^r s^{kl+k-1}\varphi^{km}(s)f^k\big(\varphi(s)\big)\,ds,$$

then

$$\varphi'(r) \leqslant (kC_0)^{\frac{1}{k}} r^{-l}\varphi^{-m}(r)\left(\int_0^r s^{kl+k-1}\varphi^{km}(s)f^k\big(\varphi(s)\big)\,ds\right)^{\frac{1}{k}}.$$

On the other hand,

$$\lim_{r\to 0}\frac{\varphi(r)\varphi'(r)}{r}=a\varphi''(0)>0\quad\text{and}\quad\frac{1}{n-2}\big(\varphi'(0)\big)^2=0,$$

then, without loss of generality, we can assume that for the above $\bar{r}>0$,

$$\frac{\varphi(r)\varphi'(r)}{r}>\frac{1}{n-2}\big(\varphi'(r)\big)^2,\quad\text{on } [0,\bar{r}]. \tag{30}$$

Then

$$\lambda_2(r)=\frac{\varphi(r)\varphi'(r)}{r}+\frac{1}{n-2}\big(\varphi'(r)\big)^2\leqslant\frac{2\varphi(r)\varphi'(r)}{r},$$

by (28) and (26), we have

$$r^{-l}\varphi^{1-m}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)'\left(2\frac{\varphi(r)\varphi'(r)}{r}\right)^{k-1}>C_0\varphi^k(r)f^k\big(\varphi(r)\big). \tag{31}$$

Similarly, we have

$$\varphi'(r)>\big(2^{1-k}kC_0\big)^{\frac{1}{k}}r^{-l}\varphi^{-m}(r)\left(\int_0^r s^{kl+k-1}\varphi^{km}(s)f^k\big(\varphi(s)\big)\,ds\right)^{\frac{1}{k}}. \tag{32}$$

Letting $\alpha=kl$, $\beta=-km$, by Lemma 7, we know that

$$\varphi'(r)=Cr^{-l}\varphi^{-m}(r)\left(\int_0^r s^{kl+k-1}\varphi^{km}(s)f^k\big(\varphi(s)\big)\,ds\right)^{\frac{1}{k}} \tag{33}$$

has a local solution near $r=0$ under the initial value condition $\varphi(0)=a>0$. Let $\bar{\varphi}$ and $\underline{\varphi}(r)$ be solutions of the Cauchy problems

$$\begin{cases}\varphi'(r)=(kC_0)^{\frac{1}{k}}r^{-l}\varphi^{-m}(r)\left(\displaystyle\int_0^r s^{kl+k-1}\varphi^{km}(s)f^k\big(\varphi(s)\big)\,ds\right)^{\frac{1}{k}},\quad 0<r<\bar{r},\\ \varphi(0)=a,\end{cases} \tag{34}$$

and

$$\begin{cases}\varphi'(r)=\big(2^{1-k}kC_0\big)^{\frac{1}{k}}r^{-l}\varphi^{-m}(r)\left(\displaystyle\int_0^r s^{kl+k-1}\varphi^{km}(s)f^k\big(\varphi(s)\big)\,ds\right)^{\frac{1}{k}},\quad 0<r<\bar{r},\\ \varphi(0)=a,\end{cases} \tag{35}$$

respectively. Denote their common interval as $[0,R]\subset[0,\bar{r})$. Then at the origin, we have

$$\overline{\varphi}''(0) = \left(\frac{k}{n}C_0\right)^{\frac{1}{k}} f(a) > \left(2^{1-k}\frac{k}{n}C_0\right)^{\frac{1}{k}} f(a) = \underline{\varphi}''(0),$$
$$\overline{\varphi}'(0) = \underline{\varphi}'(0) = 0,$$
$$\overline{\varphi}(0) = \underline{\varphi}(0) = a.$$

Without loss of generality, we assume that

$$\overline{\varphi} > \underline{\varphi}, \quad \text{in } (0, R].$$

Next, we will prove the existence of local solution of the Cauchy problem (25). Let $\phi(r) = \varphi'(r)$, then (25) is equivalent with the following Cauchy problem for systems of ordinary differential equations

$$\begin{cases} \begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix}' = \begin{pmatrix} \dfrac{C_0\varphi^{k-1}(r) f^k(\varphi(r))}{(\frac{\varphi(r)\phi(r)}{r} + \frac{1}{n-2}\phi^2(r))^{k-1}} - lr^{-1}\phi(r) - m\varphi^{-1}(r)\phi^2(r) \\ \phi(r) \end{pmatrix} \\ \qquad := F\big(r, \varphi(r), \phi(r)\big), \quad r > 0, \\ \begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix}\Big|_{r=0} = \begin{pmatrix} 0 \\ a \end{pmatrix}. \end{cases} \tag{36}$$

Then there exist a supersolution and a subsolution on $[0, R]$ of (36)

$$\begin{pmatrix} \overline{\phi}(r) \\ \overline{\varphi}(r) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \underline{\phi}(r) \\ \underline{\varphi}(r) \end{pmatrix}.$$

Defining

$$G\big(r, \varphi(r), \phi(r)\big) := \max\left\{\begin{pmatrix} \underline{\phi}(r) \\ \underline{\varphi}(r) \end{pmatrix}, \min\left\{\begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix}, \begin{pmatrix} \overline{\phi}(r) \\ \overline{\varphi}(r) \end{pmatrix}\right\}\right\},$$

we construct a new system

$$\begin{cases} \begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix}' = F\big(r, G\big(r, \varphi(r), \phi(r)\big)\big), \quad r > 0, \\ \begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix}\Big|_{r=0} = \begin{pmatrix} 0 \\ a \end{pmatrix}. \end{cases} \tag{37}$$

Denote

$$D := \left\{\begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix} : \begin{matrix} \varphi \in C^1[0, R], \\ \phi \in C^1[0, R], \end{matrix} \begin{pmatrix} \underline{\phi}(r) \\ \underline{\varphi}(r) \end{pmatrix} \leqslant \begin{pmatrix} \phi(r) \\ \varphi(r) \end{pmatrix} \leqslant \begin{pmatrix} \overline{\phi}(r) \\ \overline{\varphi}(r) \end{pmatrix}\right\}.$$

It is clear that

$$F\big(r, G\big(r, \varphi(r), \phi(r)\big)\big) = F\big(r, \varphi(r), \phi(r)\big) \quad \text{on } [0, R] \times D. \tag{38}$$

By Lemma 8, the Cauchy problem (37) has a solution $(\hat{\phi}(r), \hat{\varphi}(r))$, and $(\hat{\phi}(r), \hat{\varphi}(r)) \in D$ on $[0, R]$. Then by (38), $(\hat{\phi}(r), \hat{\varphi}(r))$ is exactly a local solution of the Cauchy problem (36) on $[0, R]$. By $\hat{\varphi}' = \hat{\phi} \in C^1[0, R]$, we have $\hat{\varphi} \in C^2[0, R]$ is a local solution of the Cauchy problem (25). □

Next, we also need the following fact.

Lemma 10. *Let $f(\cdot)$ be a continuous function in $\mathbb{R}$ and satisfy (10). For any positive constant a, if $\varphi \in C^2[0,R)$ is a solution of Cauchy problem (25), where $[0,R)$ is the maximal existence interval, then for $0 \leqslant r < R$, we have*

$$\frac{n-2}{2}\varphi^{\frac{2n}{n-2}}(r)\tilde{\lambda}(r) =: \lambda(r) = \big(\lambda_1(r), \lambda_2(r), \ldots, \lambda_2(r)\big) \in \Gamma_k, \tag{39}$$

where $\lambda_1(r)$ and $\lambda_2(r)$ are defined by (19). And if $k > \frac{n}{2}$, then (25) has no positive entire subsolution.

Proof. It follows from (32) that

$$\varphi'(r) > 0, \quad \text{for } 0 < r \leqslant \bar{r}.$$

On the other hand, if there exists $\hat{r} \in (\bar{r}, R)$ satisfying $\varphi'(\hat{r}) < 0$, then by the continuity of the derivative, there exists $\hat{\hat{r}} \in (\bar{r}, \hat{r}) \subset [0, R)$, such that $\varphi'(\hat{\hat{r}}) = 0$, and

$$\lambda_2(\hat{\hat{r}}) = \frac{\varphi(\hat{\hat{r}})\varphi'(\hat{\hat{r}})}{\hat{\hat{r}}} + \frac{1}{n-2}\big(\varphi'(\hat{\hat{r}})\big)^2 = 0,$$

$$\sigma_k\big(\lambda(\hat{\hat{r}})\big) = \binom{n-1}{k-1}\lambda_1(\hat{\hat{r}})\lambda_2^{k-1}(\hat{\hat{r}}) + \binom{n-1}{k}\lambda_2^k(\hat{\hat{r}}) = 0,$$

which is contradiction with $\lambda(\hat{\hat{r}}) \in \Gamma_k$. So we have

$$\varphi'(r) > 0, \quad r \in (0, R). \tag{40}$$

Therefore, by (26), we have for $r \in [0, R)$,

$$\lambda_2(r) = \frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2 > 0 \quad \text{and} \quad \lambda_1(r) + \frac{n-k}{k}\lambda_2(r) > 0.$$

For $1 \leqslant l \leqslant k$,

$$\begin{aligned}
\sigma_l\big(\lambda(r)\big) &= \binom{n-1}{l-1}\lambda_1(r)\lambda_2^{l-1}(r) + \binom{n-1}{l}\lambda_2^l(r) \\
&= \binom{n-1}{l-1}\left(\lambda_1(r) + \frac{n-l}{l}\lambda_2(r)\right)\lambda_2^{l-1}(r) \\
&\geqslant \binom{n-1}{l-1}\left(\lambda_1(r) + \frac{n-k}{k}\lambda_2(r)\right)\lambda_2^{l-1}(r) \\
&> 0.
\end{aligned}$$

That is, for $r \in [0, R)$,

$$\lambda(r) = \big(\lambda_1(r), \lambda_2(r), \ldots, \lambda_2(r)\big) \in \Gamma_k.$$

Letting $\psi(r) := r^l\varphi^m(r)\varphi'(r)$, by (27) and (28), we have $\psi'(r) > 0$ for $r \in (0, R)$. Since

$$\psi(r) = r^{\frac{n}{k}}\varphi^m(r) \cdot \frac{\varphi'(r)}{r} \to 0, \quad \text{as } r \to 0,$$

it follows that

$$\psi(r) > 0, \quad r \in (0, R).$$

Fix $r_1 \in (0, R)$, and choose $C_1 = \psi(r_1) > 0$. By the monotonicity of ψ, we have

$$\varphi^m(r)\varphi'(r) > r^{-l}C_1, \quad \text{for } r > r_1.$$

Case 1. $m = -1$. We have $n = 2k$, $l = 1$, and

$$\varphi^{-1}(r)\varphi'(r) > C_1 r^{-1},$$
$$\ln\varphi(r) > C_1 \ln r - C_1 \ln r_1 + \ln\varphi(r_1),$$
$$\varphi > C_2 r^{C_1}. \tag{41}$$

Case 2. $m \neq -1$. We have $n \neq 2k$, $l \neq 1$,

$$\frac{\varphi^{m+1}(r)}{m+1} > \frac{C_1}{1-l} r^{1-l} + \left(\frac{\varphi(r_1)^{m+1}}{m+1} - \frac{C_1}{1-l} r_1^{1-l}\right). \tag{42}$$

In fact, by (29)

$$m + 1 = \frac{n - nk}{(n-2)k} + 1 = \frac{n - 2k}{(n-2)k}$$

and

$$1 - l = 1 - \frac{n-k}{k} = \frac{2k - n}{k}$$

have different signs. If $k < \frac{n}{2}$, then $m + 1 > 0$, $1 - l < 0$, that is, $m > -1$ and $l > 1$. If $k > \frac{n}{2}$, then $m + 1 < 0$, $1 - l > 0$. Letting $r \to +\infty$, (42) does not hold any longer. This is a contradiction. Therefore, the maximal existence interval of the solution is finite, in another word, Cauchy problem (25) has no entire solution. □

The following lemma and its proof could be found in [14].

Lemma 11. *(See [14, Lemma 1].) Let Ω be a bounded open set in $\mathbb{R}^n$, $B(\cdot,\cdot,\cdot) : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a map, $h(x, t)$ be a positive function defined in $\Omega \times \mathbb{R}_+$, and let $t \mapsto t^{-1}h(x, t)$ be strictly increasing on $(0, \infty)$ for each $x \in \Omega$. Suppose that $u \in C^2(\overline{\Omega})$ is a positive subsolution of*

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(D^2u + B(x, u, Du)\big)\big) = h(x, u), \quad \lambda\big(D^2u + B(x, u, Du)\big) \in \Gamma_k, \ \forall x \in \Omega, \tag{43}$$

and $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is a positive supersolution of (43) with $\lambda(D^2v + t^{-1}B(x, tv, tDv)) \in \Gamma_k$ for each $t \geqslant 1$. Suppose also that for each $x \in \Omega$ and $\xi, \mathbf{p} \in \mathbb{R}^n$, the function

$$t \mapsto t^{-1}\big\langle B(x, t, t\mathbf{p})\xi, \xi\big\rangle \tag{44}$$

is non-increasing on $(0, \infty)$. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ on $\overline{\Omega}$.

Remark 7. For the convenience, we change f in original lemma to $\sigma_k^{\frac{1}{k}}$, and change the viscosity subsolution to the classical subsolution.

By Lemma 11, we obtain the following comparison principle.

Lemma 12. *Let $f(t)$ be a monotonically non-decreasing function in $\mathbb{R}$, and satisfy (10) and (13). Suppose that $\varphi(r) \in C^2[0, R)$ satisfies (22) and (39) for $r \in [0, R)$, and $\varphi'(0) = 0$, $\lim_{r\to R} \varphi(r) = \infty$. If $u(x) \in C^2(\mathbb{R}^n)$ is a positive subsolution of (12), then we have $u(x) \leqslant \varphi(|x|)$ in B_R.*

Proof. Let

$$B(x,t,\mathbf{p}) = t^{-1}\left(-\frac{n}{n-2}\mathbf{p}\otimes\mathbf{p} + \frac{1}{n-2}|\mathbf{p}|^2 I\right), \qquad h(x,t) = \frac{n-2}{2} f(t),$$

where $x, \mathbf{p} \in \mathbb{R}^n$, $t \in \mathbb{R}$. It is clear that

$$\begin{aligned} t^{-1}\langle B(x,t,t\mathbf{p})\xi,\xi\rangle &= t^{-1}\left\langle t^{-1}\left(-\frac{n}{n-2}(t\mathbf{p})\otimes(t\mathbf{p}) + \frac{1}{n-2}|t\mathbf{p}|^2 I\right)\xi,\xi\right\rangle \\ &= \left\langle\left(-\frac{n}{n-2}\mathbf{p}\otimes\mathbf{p} + \frac{1}{n-2}|\mathbf{p}|^2 I\right)\xi,\xi\right\rangle \end{aligned}$$

is non-increasing in $(0,\infty)$ with respect to t. By (13), we have

$$t^{-1}h(x,t) = \frac{n-2}{2}\left(\frac{f(t)}{t}\right)$$

is strictly monotone in $(0,\infty)$. Then (12) could be written as

$$\sigma_k^{\frac{1}{k}}\big(\lambda\big(D^2u + B(x,u,Du)\big)\big) = h(x,u), \quad \lambda\big(D^2u + B(x,u,Du)\big) \in \Gamma_k,\ x \in B_R. \tag{45}$$

Denote $v(x) = \varphi(r)$, where $r = |x|$. By Lemma 6, we have $v(x)$ satisfies (12) in B_R. $v(x)$ is naturally a positive solution of (45) in B_R, and

$$\begin{aligned} \lambda\big(D^2v + t^{-1}B(x,tv,tDv)\big) &= \lambda\left(D^2v + t^{-2}v^{-1}\left(-\frac{n}{n-2}(tDv)\otimes(tDv) + \frac{1}{n-2}|tDv|^2 I\right)\right) \\ &= \lambda\big(D^2v + B(x,v,Dv)\big) \in \Gamma_k. \end{aligned}$$

We notice that a positive subsolution of (12), $u \in C^2(\mathbb{R}^n)$, is naturally a positive subsolution of (45) in $\mathbb{R}^n$. Since $\lim_{x\to\partial B_R} v(x) = \infty$, it follows from Lemma 11 that $u(x) \leqslant v(x)$ in B_R. □

3. The proof of main results

Lemma 13. *Let $f(t)$ be a continuous function in $\mathbb{R}$ and satisfy (10) and (13). Then (12) has a positive entire subsolution $u \in C^2(\mathbb{R}^n)$ if and only if (22) has a positive solution $\varphi(r) \in C^2[0,\infty)$ and satisfies the initial value condition $\varphi'(0) = 0$.*

Proof. Firstly, the sufficiency is obvious. If there exists such a solution φ of (22), then, letting $v(x) = \varphi(|x|)$, by Lemmas 6 and 10, $v(x)$ is exactly a positive solution of (12) in $\mathbb{R}^n$.

Next, we will prove the necessity. On the contrary, suppose that there is no such function $\varphi(r)$ in whole space, and suppose that (12) has a positive subsolution u. By Lemma 9, for any constant

$a > 0$, Cauchy problem (25) has a positive solution $\varphi(r)$ in some finite interval. Hence we assume that $[0, R)$ is the maximal interval in which the solution exists. Since for $r > 0$, $\varphi'(r) > 0$, it follows that $\varphi(r) \to \infty$ as $r \to R$. Then by Lemma 10, we know that $\varphi(|x|)$ satisfies (22) and (39) in B_R. Hence by Lemma 12, for any positive subsolution of (12), $u(x)$, we have $u(x) \leqslant \varphi(|x|)$ for each $x \in B_R$. Especially, we have $u(0) \leqslant \varphi(0) = a$. But, by the arbitrariness of a, if we take $a = \frac{u(0)}{2}$, then we obtain a contradiction, which means the necessary condition holds. □

Proof of Theorem 3. By Lemmas 9, 10 and 13, part (a) of Theorem 3 holds.

In the following, we assume that $k \leqslant \frac{n}{2}$. For part (b), we need only to show that if f satisfies (10) and (14), then (22) has a positive entire solution.

First, by (22) and (28), we have

$$C_0\varphi^k(r)f^k\big(\varphi(r)\big) = \left(\lambda_1(r) + \frac{n-k}{k}\lambda_2(r)\right)\lambda_2^{k-1}(r)$$
$$= r^{-l}\varphi^{1-m}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)'\left(\frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2\right)^{k-1}. \tag{46}$$

On the other hand, we have

$$r^{-l}\varphi^{1-m}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)'\left(\frac{1}{n-2}\big(\varphi'(r)\big)^2\right)^{k-1} < C_0\varphi^k(r)f^k\big(\varphi(r)\big),$$

that is,

$$r^{-l}\varphi^{1-m}(r)\big(r^l\varphi^m(r)\varphi'(r)\big)'\big(r^l\varphi^m(r)\varphi'(r)\big)^{2k-1} < C\big(r^l\varphi^m(r)\big)^{2k-1}\varphi'(r)\varphi^k(r)f^k\big(\varphi(r)\big).$$

Then

$$\Big(\big(r^l\varphi^m(r)\varphi'(r)\big)^{2k}\Big)' < Cr^{2kl}\varphi^{2km-1+k}(r)f^k\big(\varphi(r)\big)\varphi'(r). \tag{47}$$

Integrating from 0 to r on both sides, we have

$$\big(r^l\varphi^m(r)\varphi'(r)\big)^{2k} < C\int_0^r s^{2kl}\varphi^{2km-1+k}(s)f^k\big(\varphi(s)\big)\varphi'(s)\,ds$$
$$< Cr^{2kl}\int_a^{\varphi(r)} t^{2km-1+k}f^k(t)\,dt.$$

Then

$$\varphi^m(r)\frac{d\varphi(r)}{dr} < C\left(\int_a^{\varphi(r)} t^{2km-1+k}f^k(t)\,dt\right)^{\frac{1}{2k}}, \tag{48}$$

that is,

$$\varphi^m(r)\left(\int_a^{\varphi(r)} t^{2km-1+k}f^k(t)\,dt\right)^{-\frac{1}{2k}} d\varphi(r) < C\,dr. \tag{49}$$

Substituting $m = \frac{n(1-k)}{(n-2)k}$, we have

$$\varphi^{\frac{n(1-k)}{(n-2)k}}(r)\left(\int_a^{\varphi(r)} t^{\frac{(n+2)(1-k)}{n-2}} f^k(t)\,dt\right)^{-\frac{1}{2k}} \varphi'(r)\,dr < C\,dr. \tag{50}$$

By Lemma 9, (22) has an entire solution. If it was not an entire solution, then its existence interval is finite, denoting as $[0, R)$. Then by $\varphi' > 0$, it is easy to know that φ blows up on the boundary, that is,

$$\varphi(r) \to \infty, \quad \text{as } r \to R.$$

Integrating from 0 to R on both sides of (50), we have

$$\int_a^{\infty} \varphi^{\frac{n(1-k)}{(n-2)k}}\left(\int_a^{\varphi(r)} t^{\frac{(n+2)(1-k)}{n-2}} f^k(t)\,dt\right)^{-\frac{1}{2k}} d\varphi < CR < \infty. \tag{51}$$

This is a contradiction. □

Proof of Corollary 4. Letting $k \leqslant \frac{n}{2}$, $f(t) = t^p$, $p > 0$, for $t > 0$, we have

$$\begin{aligned} t^{\frac{n(1-k)}{(n-2)k}}\left(\int_0^t s^{\frac{(n+2)(1-k)}{n-2}+kp}\,ds\right)^{-\frac{1}{2k}} &= t^m\left(\int_0^t s^{2km-1+k+kp}\,ds\right)^{-\frac{1}{2k}} \\ &= t^m\left(\frac{2km+k+kp}{t^{2km+k+kp}}\right)^{\frac{1}{2k}} \\ &= (2km+k+kp)^{\frac{1}{2k}} t^{-\frac{p+1}{2}}. \end{aligned}$$

For $0 < p \leqslant 1$, it satisfies (14). It follows from the proof of Theorem 3 that Corollary 4 holds. □

Especially, if n is even, and $k = \frac{n}{2}$, we have

Lemma 14. *Let $k = \frac{n}{2}$, and let f be a continuous function in $\mathbb{R}$. If f satisfies (10), (15), and*

$$L := \inf_{t \in [1,\infty)} \frac{f(t)}{2t} > 0, \tag{52}$$

then (22) has no positive entire solution $\varphi \in C^2[0, \infty)$.

Proof. Suppose (22) has a positive entire solution $\varphi \in C^2[0, \infty)$. Clearly, $\varphi' > 0$.

First, we will show that $\varphi(r) \to \infty$ as $r \to \infty$. On the contrary, suppose that there exists a constant $M > 0$, such that

$$\varphi(r) < M, \quad \text{on } [0, \infty).$$

By (48) and $m = -1$, $l = 1$, we have

$$\begin{aligned}\varphi'(r) &< C\varphi(r)\left(\int_a^{\varphi(r)} s^{-\frac{n}{2}-1} f^{\frac{n}{2}}(s)\,ds\right)^{\frac{1}{n}} \\ &< CM\left(\int_a^{M} s^{-\frac{n}{2}-1} f^{\frac{n}{2}}(s)\,ds\right)^{\frac{1}{n}} \\ &< Ca^{-\frac{1}{2}} M f^{\frac{1}{2}}(M) \\ &:= M_1 < \infty.\end{aligned}$$

Then choosing $r_1 > 0$ in (41), for $r > r_1$,

$$\lambda_2(r) = \frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2 < \frac{MM_1}{r_1} + \frac{1}{n-2}(M_1)^2 := M_2 < \infty. \tag{53}$$

Substituting it to (46), we have

$$\psi'(r) = \big(r\varphi^{-1}(r)\varphi'(r)\big)' > C_0 r M^{-2} M_2^{1-\frac{n}{2}} \varphi(r_1)^{\frac{n}{2}} f^{\frac{n}{2}}\big(\varphi(r_1)\big) = Cr. \tag{54}$$

Integrating on both sides, we have

$$r\varphi^{-1}(r)\varphi'(r) > Cr^2,$$

i.e.

$$\varphi^{-1}(r)\varphi'(r) > Cr.$$

Integrating again, we have

$$\ln\varphi(r) > Cr^2 + C'.$$

Clearly, this contradicts with the assumption that $\varphi(r)$ is bounded on $[0,\infty)$. Therefore, there exists $r_2 > r_1$, such that $\varphi(r) > 1$ for $r > r_2$.

By (52), we have $f(t) \geqslant 2Lt > Lt$, for $t > 1$. Then

$$f\big(\varphi(r)\big) > L\varphi(r), \quad \text{for } r > r_2. \tag{55}$$

Next, we will show that $\psi(r) > 1$, for r sufficiently large. On the contrary, suppose that there exists $r_2 > 0$ such that

$$\psi(r) \leqslant 1, \quad r \in [r_2, \infty). \tag{56}$$

By (46), we have

$$\begin{aligned}C_0\varphi^{\frac{n}{2}}(r) f^{\frac{n}{2}}\big(\varphi(r)\big) &= r^{-1}\varphi^2(r)\big(r\varphi^{-1}(r)\varphi'(r)\big)'\left(\frac{\varphi(r)\varphi'(r)}{r} + \frac{1}{n-2}\big(\varphi'(r)\big)^2\right)^{\frac{n}{2}-1} \\ &= r^{-1}\varphi^2(r)\psi'(r)\left(r^{-2}\varphi^2(r)\psi(r) + \frac{1}{n-2} r^{-2}\varphi^2(r)\psi^2(r)\right)^{\frac{n}{2}-1} \\ &= r^{1-n}\varphi^n(r)\psi'(r)\left(\psi(r) + \frac{1}{n-2}\psi^2(r)\right)^{\frac{n}{2}-1}.\end{aligned} \tag{57}$$

Combining with $\psi(r) \leqslant 1$, we have

$$\psi'(r) \geqslant \frac{C_0\varphi^{\frac{n}{2}}(r) f^{\frac{n}{2}}(\varphi(r))}{(\frac{n-1}{n-2})^{\frac{n}{2}-1} r^{1-n}\varphi^n(r)} = \frac{Cr^{n-1} f^{\frac{n}{2}}(\varphi(r))}{\varphi^{\frac{n}{2}}(r)} \to \infty, \quad \text{as } r \to \infty.$$

This contradicts with the assumption (56). Therefore, there exists $r_3 > r_2$, such that

$$\psi(r) > 1, \quad \text{for } r > r_3.$$

Then, by (57),

$$r^{1-n}\varphi^n(r)\psi'(r)\left(\frac{n-1}{n-2}\psi^2(r)\right)^{\frac{n}{2}-1} > C_0\varphi^{\frac{n}{2}}(r) f^{\frac{n}{2}}\big(\varphi(r)\big),$$

that is,

$$\begin{aligned} \psi'(r)\psi^{2(\frac{n}{2}-1)}(r) &> Cr^{n-1}\varphi^{-\frac{n}{2}}(r) f^{\frac{n}{2}}\big(\varphi(r)\big) \\ &> Cr^{n-1}. \end{aligned} \tag{58}$$

Integrating from r_3 to r, we have

$$\psi^{n-1}(r) - \psi^{n-1}(r_3) > C\big(r^n - r_3^n\big).$$

Thus

$$r\varphi^{-1}(r)\varphi'(r) = \psi(r) > Cr^{\frac{n}{n-1}},$$

that is,

$$\varphi^{-1}(r)\varphi'(r) > Cr^{\frac{1}{n-1}}. \tag{59}$$

Multiplying ψ on both sides of the first line in (58),

$$\psi'(r)\psi^{n-1}(r) > Cr^n\varphi^{-\frac{n}{2}-1}(r) f^{\frac{n}{2}}\big(\varphi(r)\big)\varphi'(r),$$

and integrating from r_3 to r, we have

$$\psi^n(r) - \psi^n(r_3) > C\int_{r_3}^{r} s^n\varphi^{-\frac{n}{2}-1}(s) f^{\frac{n}{2}}\big(\varphi(s)\big)\varphi'(s)\,ds. \tag{60}$$

In the following we will show that

$$\int_{r_3}^{r} s^n\varphi^{-\frac{n}{2}-1}(s) f^{\frac{n}{2}}\big(\varphi(s)\big)\varphi'(s)\,ds > \left(\frac{r}{2}\right)^n \int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt. \tag{61}$$

Denote

$$F(r) := \int_{r_3}^{r} s^n \varphi^{-\frac{n}{2}-1}(s) f^{\frac{n}{2}}\big(\varphi(s)\big)\varphi'(s)\,ds - \left(\frac{r}{2}\right)^n \int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt,$$

then $F(r_3) = 0$. For $r > r_3$, by (59), we have

$$\begin{aligned} F'(r) &= \left(r^n - \left(\frac{r}{2}\right)^n\right)\varphi^{-\frac{n}{2}-1}(r) f^{\frac{n}{2}}\big(\varphi(r)\big)\varphi'(r) - \frac{n}{2}\left(\frac{r}{2}\right)^{n-1} r \int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt \\ &> C\left(1 - \frac{1}{2^n}\right) r^n \varphi^{-\frac{n}{2}}(r) f^{\frac{n}{2}}\big(\varphi(r)\big) r^{\frac{1}{n-1}} + \left(\frac{r}{2}\right)^{n-1} r\big(\varphi^{-\frac{n}{2}}(r_3) - \varphi^{-\frac{n}{2}}(r)\big) f^{\frac{n}{2}}\big(\varphi(r)\big) \\ &> \left[C\left(1 - \frac{1}{2^n}\right) r^{\frac{1}{n-1}} - \frac{1}{2^{n-1}}\right] r^n \varphi^{-\frac{n}{2}}(r) f^{\frac{n}{2}}\big(\varphi(r)\big). \end{aligned}$$

Clearly, if r_3 is sufficiently large, then $F'(r) > 0$ for $r \in (r_3, \infty)$. Therefore, (61) holds. Substituting it to (60), we have

$$\psi^n(r) > C r^n \int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt.$$

Extracting n times, and substituting $\psi(r) = r\varphi^{-1}(r)\varphi'(r)$, we have

$$r\varphi^{-1}(r)\varphi'(r) > Cr\left(\int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt\right)^{\frac{1}{n}}.$$

That is,

$$\varphi^{-1}(r)\left(\int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt\right)^{-\frac{1}{n}} d\varphi(r) > C\,dr.$$

Integrating from r_3 to ∞, we have

$$\int_{\varphi(r_3)}^{\infty} \varphi^{-1}(r)\left(\int_{\varphi(r_3)}^{\varphi(r)} t^{-\frac{n}{2}-1} f^{\frac{n}{2}}(t)\,dt\right)^{-\frac{1}{n}} d\varphi(r) > C\int_{r_3}^{\infty} dr = \infty.$$

This contradicts with (15). □

Proof of Corollary 5. From Lemmas 13 and 14, part (a) of Corollary 5 holds. Let $k = \frac{n}{2}$ in (14), we know part (b) holds from Theorem 3. □

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