Differ Equ Dyn Syst (April 2021) 29(2):339–351
https://doi.org/10.1007/s12591-019-00460-4

ORIGINAL RESEARCH

Existence and Regularity for Some Degenerate Parabolic Equation

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Published online: 1 March 2019
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Abstract
By making use of the approximation method, we obtain the existence and regularity of the viscosity solutions for the generalized mean curvature flow. The asymptotic behavior of the flow is also considered. In particular, the Dirichlet problem of the degenerate elliptic equation

$$-|\nabla v|\left(\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right)+\nu\right)=0$$

is solvable in viscosity sense, which is the main new ingredient of this paper.

Keywords Generalized mean curvature flow · Viscosity solutions · Maximum principle · Schauder estimates

Mathematics Subject Classification 35K55 · 35K65

Introduction

In this paper, we will study the global properties of solutions of the generalized mean curvature flow equations

$$u_t-|\nabla u|\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)+\nu\right)=0,\quad (x,t)\in D\times(0,+\infty), \tag{1.1}$$

where ν is a constant. The Eq. (1.1) has a geometric significance because γ-level surface $\Gamma(t)$ of u moves by its mean curvature and an external force field provided ∇u does not vanish on $\Gamma(t)$ (cf. [1,2]). When $\nu=0$ such a motion of surfaces has been presented by many authors under various conditions (cf. [3–7]). However, the uniformly gradient estimates for

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solutions of (1.1) are little known and crucial for studying the global properties of viscosity solutions. Our motivation for studying Eq. (1.1) comes from providing a parabolic approach to prescribe the boundary value problems for some degenerate elliptic equation.

Firstly we introduce some relevant works according to the above equation. Let $n < 6$, and $D \subset \mathbb{R}^{n+1}$ a bounded domain with a C^2 boundary of mean curvature $H \geq 0$ with respect to its outer unit normal. For $\Omega = \mathbb{R}^{n+1} \setminus D$ and a nonnegative function $f \in C^{0,1}(\Omega)$, Bernhard Hein considered the viscosity solutions of the inverse mean curvature flow (cf. [8])

$$\begin{cases} u_t - \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| = 0, & (x,t) \in \Omega \times (0, +\infty), \\ u = 0, & (x,t) \in \partial D \times (0, +\infty), \\ u = f(x), & (x,t) \in \Omega \times \{0\}. \end{cases} \tag{1.2}$$

Here $u_t = \frac{\partial u}{\partial t}$, $\nabla u = \operatorname{grad} u$, div is the divergence operator in $\mathbb{R}^{n+1}$. He proved that there exists an unique nonnegative weak solution which satisfies (1.2). And there is a positive constant $C = C(n, D, f)$ such that for $x \in \Omega$ and all $t > 0$,

$$|\nabla u| \leq C, \qquad -\frac{\sqrt{(n+1)C}}{\sqrt{t}} - 2C \leq \frac{\partial u}{\partial t} \leq \frac{\sqrt{(n+1)C}}{\sqrt{t}} + C.$$

Y.Giga, M.Ohuma and M.Sato studied the following Neumann problem (cf. [9])

$$\begin{cases} u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0, & (x,t) \in D \times (0, +\infty), \\ \frac{\partial u}{\partial \gamma} = 0, & (x,t) \in \partial D \times (0, +\infty), \\ u = f(x), & x \in D \times \{0\}, \end{cases} \tag{1.3}$$

where γ is the outer unit normal of ∂D and $f(x) \in C^2(\overline{D})$. They discovered some interesting properties of the solution $u(x,t)$ (see Theorem 1.1 in [9]) which satisfies (1.3) in viscosity sense.

In the present paper we consider the initial and boundary value problem

$$\begin{cases} u_t - |\nabla u| \left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nu\right) = 0, & (x,t) \in D \times (0, +\infty), \\ u = h(x), & (x,t) \in \partial D \times [0, +\infty), \\ u = g(x), & x \in D \times \{0\}. \end{cases} \tag{1.4}$$

Here $h(x)$ and $g(x)$ are the given functions on $\overline{D}$.

Our main purposes are to show the existence and regularity of the viscosity solutions for (1.4), to study their asymptotic behavior, and to prove that $u(x,t)$ converges to a solution of the Dirichlet problem of degenerate elliptic equation

$$\begin{cases} -|\nabla v| \left(\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) + \nu\right) = 0, & x \in D, \\ v = h(x), & x \in \partial D, \end{cases} \tag{1.5}$$

as $t \to +\infty$. The solvability of (1.5) does not seem to be easily found in the literature as far as we know.

Quite naturally, we always use the following notations

$$\varphi_i = \frac{\partial \varphi}{\partial x_i}, \quad \varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}.$$

Throughout the paper the following Einstein's convention of summation over repeated indices will be adopted. Firstly we introduce the definition of viscosity solutions from [2].

Definition 1.1 Suppose that $u(x,t)$ is a function in $C(\overline{D}\times[0,+\infty))$ and satisfies the initial and boundary conditions of (1.4). If $\varphi \in C^\infty(D\times(0,+\infty))$, $(x,t)\in\Theta\subset D\times(0,+\infty)$ and Θ is a bounded open set, satisfy

$$(u-\varphi)(x,t)=\max_{\overline{\Theta}}(u-\varphi),$$

and at (x,t) such that

$$\varphi_t \le \left(\delta_{ij}-\frac{\varphi_i\varphi_j}{|\nabla\varphi|^2}\right)\varphi_{ij}+\nu|\nabla\varphi|, \quad |\nabla\varphi|\neq 0.$$

Or there exists $\eta=(\eta_1,\eta_2,\ldots,\eta_{n+1})$ with $|\eta|\le 1$ at (x,t) such that

$$\varphi_t \le (\delta_{ij}-\eta_i\eta_j)\varphi_{ij}, \quad |\nabla\varphi|=0.$$

Then $u(x,t)$ is viscosity sub-solution of (1.4).

Definition 1.2 Suppose that $u(x,t)$ is a function in $C(\overline{D}\times[0,+\infty))$ and satisfies the initial and boundary conditions of (1.4). If $\varphi \in C^\infty(D\times(0,+\infty))$, $(x,t)\in\Theta\subset D\times(0,+\infty)$ and Θ is a bounded open set, satisfy

$$(u-\varphi)(x,t)=\min_{\overline{\Theta}}(u-\varphi),$$

and at (x,t) such that

$$\varphi_t \ge \left(\delta_{ij}-\frac{\varphi_i\varphi_j}{|\nabla\varphi|^2}\right)\varphi_{ij}+\nu|\nabla\varphi|, \quad |\nabla\varphi|\neq 0.$$

Or there exists $\eta=(\eta_1,\eta_2,\ldots,\eta_{n+1})$ with $|\eta|\le 1$ at (x,t) such that

$$\varphi_t \ge (\delta_{ij}-\eta_i\eta_j)\varphi_{ij}, \quad |\nabla\varphi|=0.$$

Then $u(x,t)$ is viscosity super-solution of (1.4).

Definition 1.3 If $u(x,t)$ is a viscosity sub-solution and also is a viscosity super-solution of (1.4), then $u(x,t)$ is a viscosity solution of (1.4).

Let us fix $h(x)=g(x)$ on ∂D, $h(x)\in C^2(\partial D)$, $g(x)\in C^2(\overline{D})$. One of the main results in this paper is the existence and regularity of viscosity solutions of (1.4).

Theorem 1.4 *Suppose that D is a smooth strictly convex bounded domain in $\mathbb{R}^{n+1}$, and $|\nu|<\frac{nH_0}{n+1}$, where H_0 is the positive lower bound of the mean curvature of ∂D. Then there exists an unique function $u(x,t)$ which satisfies (1.4) in viscosity sense and*

$$u\in C(\overline{D}\times[0,+\infty)), \quad u_t\in L^\infty(D\times[0,+\infty)), \quad \nabla u\in L^\infty(D\times[0,+\infty)), \tag{1.6}$$

$$\|u\|_{L^\infty(D\times[0,+\infty))}+\|\nabla u\|_{L^\infty(D\times[0,+\infty))}+\|u_t\|_{L^\infty(D\times[0,+\infty))}\le C, \tag{1.7}$$

$$\int_0^{+\infty}\int_D |u_t|^2dxdt\le C, \tag{1.8}$$

where the constant C depends only on n, ν, D, $\|h\|_{C^2(\partial D)}$ and $\|g\|_{C^2(\overline{D})}$.

As an application of Theorem 1.4, we have

Corollary 1.5 *Suppose that the domain D satisfies the conditions in the above theorem and $u(x,t)$ is the viscosity solution of (1.4). Then there exists a function $v(x)$ which satisfies $v(x) \in C(\overline{D})$, $\nabla v \in L^\infty(D)$, such that*

$$\lim_{t\to+\infty} u(x,t) = v(x), \quad in \quad C(\overline{D}), \tag{1.9}$$

and $v(x)$ satisfies (1.5) in viscosity sense.

Remark 1.6 By Corollary 1.5 the Dirichlet problem (1.5) is solvable. But we do not know whether a viscosity solution of (1.5) is unique.

The second result of this paper is the Liouville-type property of the viscosity solutions. Suppose that D' is a smooth convex bounded domain in $\mathbb{R}^n$, such that

$$(A): \quad D \cap \{(x', x_{n+1}) \in \mathbb{R}^{n+1} \mid |x_{n+1}| < m+1\} = D' \times (-m-1, m+1),$$

where $x' = (x_1, \dots, x_n)$ and m is a positive constant.

Theorem 1.7 *Let $\nu \geq 0$ and a domain D satisfying the condition (A). Suppose that $g(x', x_{n+1})$ is a non-decreasing function of x_{n+1} which satisfies*

$$g(x', x_{n+1}) = \lambda, \quad (x', x_{n+1}) \in D \cap \{x | x_{n+1} \geq m\}, \tag{1.10}$$

where λ is a constant. Then the viscosity solution $u(x,t)$ in $C(\overline{D}\times[0,+\infty))$ of (1.4) satisfies

$$u(x', x_{n+1}, t) = \lambda, \quad (x', x_{n+1}) \in D \cap \{x | x_{n+1} \geq m\}. \tag{1.11}$$

In the next section we construct approximated problem for (1.4) and establish uniform estimates for its classical solutions. In the last section we present the proof of the main results.

Preliminary Estimates

Consider the approximated problem of (1.4) with $\epsilon \in (0,1)$

$$\begin{cases} u_t - \sqrt{\epsilon^2 + |\nabla u|^2} \cdot \left(\operatorname{div}\left(\frac{\nabla u}{\sqrt{\epsilon^2+|\nabla u|^2}}\right) + \nu\right) = 0, & (x,t) \in D \times (0,+\infty), \\ u = h(x), & (x,t) \in \partial D \times [0,+\infty), \\ u = g(x), & (x,t) \in D \times \{0\}. \end{cases} \tag{2.1}$$

where D is a smooth strictly convex bounded domain in $\mathbb{R}^{n+1}$. We want to use the continuity method to prove the solvability of (2.1) and then obtain the estimates similarly to (1.7) and (1.8).

In order to solve (2.1) we use the following form of fixed point theorem (cf. [10]).

Lemma 2.1 (Leray–Schauder) *Suppose that $\mathfrak{B}$ is a Banach space, $\chi(b,\sigma)$ is a map from $\mathfrak{B} \times [0,1]$ to $\mathfrak{B}$. If χ satisfies*

(1) *χ is continuous and compact.*
(2) $\chi(b, 0) = 0$, $\forall b \in \mathfrak{B}$.
(3) *There exists constant $C > 0$ such that*

$$\|b_0\|_{\mathfrak{B}} \leq C, \quad \forall b_0 \in \{b \in \mathfrak{B} | \exists \sigma \in [0,1], \quad b = \chi(b,\sigma)\}.$$

Then there exists $b_0 \in \mathfrak{B}$ such that $\chi(b_0, 1) = b_0$.

For any $T>0$, if we define

$$\mathfrak{B}=\{u|u\in C(\overline{D}\times[0,T)),\quad \nabla u\in C(\overline{D}\times[0,T))\},\quad D_T=D\times[0,T),$$

with the norm $\|u\|=\sup_{\overline{D}\times[0,T)}|u|+\sup_{\overline{D}\times[0,T)}|\nabla u|$, then it is easily to see that $\mathfrak{B}$ must be a Banach space. From the theory on linear parabolic equations (cf. [11]) and for any $\tilde{u}\in\mathfrak{B}$, $\sigma\in[0,1]$, there exists an unique function u, where $u\in\mathfrak{B}$, $u\in W_p^{2,1}(D_T)$ with any $p>0$ such that u satisfies

$$\begin{cases} u_t-\left(\delta_{ij}-\sigma^2\frac{\tilde{u}_i\tilde{u}_j}{\epsilon^2+\sigma^2|\nabla\tilde{u}|^2}\right)u_{ij}=\sigma\nu\sqrt{\epsilon^2+\sigma^2|\nabla\tilde{u}|^2}, & (x,t)\in D\times(0,T),\\ u=\sigma h(x), & (x,t)\in\partial D\times[0,T),\\ u=\sigma g(x), & (x,t)\in D\times\{0\}. \end{cases} \tag{2.2}$$

By (2.2) we can define a map from $\mathfrak{B}\times[0,1]$ to $\mathfrak{B}$ and denote $u=\chi(\tilde{u},\sigma)$. The main step in our argument is to validate the three conditions of Lemma 2.1 one by one.

It is obvious that $\chi(\tilde{u},0)=0$ for every $\tilde{u}\in\mathfrak{B}$ by the uniqueness of the initial and boundary value problem (2.2). We see that the map χ is compact by Schauder estimates and Sobolev embedding theorem (cf. [11]). Consequently we claim that χ is continuous. Indeed, this fact follows from the compactness of χ and the uniqueness of the mapping $\chi(\tilde{u},\sigma)$.

So it remains to verify the third condition for applying Lemma 2.1 to the problem (2.2). Suppose $\chi(u,\sigma)=u$. It follows from (2.2) that u satisfies

$$\begin{cases} u_t-\sqrt{\epsilon^2+\sigma^2|\nabla u|^2}\cdot\left(\mathrm{div}\left(\frac{\nabla u}{\sqrt{\epsilon^2+\sigma^2|\nabla u|^2}}\right)+\sigma\nu\right)=0, & (x,t)\in D\times(0,T),\\ u=\sigma h(x), & (x,t)\in\partial D\times[0,T),\\ u=\sigma g(x), & (x,t)\in D\times\{0\}. \end{cases} \tag{2.3}$$

By using regularity theory, $u\in C^\infty(D_T)\cap C^{2,1}(\overline{D}_T)$. Then the condition (3) in Lemma 2.1 is equivalent to the boundness of u and ∇u in the L^∞ norm which is independence of σ if $u\in C^\infty(D_T)\cap C^{2,1}(\overline{D}_T)$ and u satisfies (2.3).

In this section we derive $W^{1,\infty}$ estimates for the classical solutions of (2.3) in which the bound is not only independent of σ, but also independent of ϵ and T.

Let

$$L_\sigma u=u_t-\sqrt{\epsilon^2+\sigma^2|\nabla u|^2}\cdot\left(\mathrm{div}\left(\frac{\nabla u}{\sqrt{\epsilon^2+\sigma^2|\nabla u|^2}}\right)+\sigma\nu\right), \tag{2.4}$$

and

$$\partial_p D_T=(\partial D\times[0,T))\cup(D\times\{t=0\}).$$

The estimates follow from the next three lemmas. The following comparison principle is by Theorem 14.1 in [11].

Lemma 2.2 *Suppose that $u_1,u_2\in C^{2,1}(D\times(0,T))\cap C(\overline{D}\times[0,T))$. If*

$$L_\sigma u_1|_{D_T}\geq L_\sigma u_2|_{D_T},\quad u_1|_{\partial_p D_T}\geq u_2|_{\partial_p D_T},$$

then

$$u_1|_{D_T}\geq u_2|_{D_T}.$$

The estimate for the maximum norm for the solutions of (2.3) is the following:

Lemma 2.3 *If $u \in C^{\infty}(D \times (0,T)) \cap C(\overline{D} \times [0,T))$ is a solution of (2.3) with $|\nu| < \frac{nH_0}{n+1}$. Then*

$$\|u\|_{L^{\infty}(D\times[0,T))} \le C, \tag{2.5}$$

where C is depending only on $\|h\|_{C(\partial D)}$, $\|g\|_{C(\overline{D})}$, and D.

Proof Step 1. By $|\nu| < \frac{nH_0}{n+1}$ and Theorem 16.10 in [10], there exists $\alpha > 0$, $v^{\epsilon} \in C^{2+\alpha}(\overline{D})$, such that

$$\begin{cases} -\sqrt{\epsilon^2 + \sigma^2 |\nabla v^{\epsilon}|^2} \cdot \left(\operatorname{div}\left(\frac{\nabla v^{\epsilon}}{\sqrt{\epsilon^2+\sigma^2|\nabla v^{\epsilon}|^2}} \right) + \sigma \nu \right) = 0, & x \in D, \\ v^{\epsilon} = 1, & x \in \partial D. \end{cases}$$

Set $w = \dfrac{\sigma}{\epsilon} v^{\epsilon}$. Then w is a classical solution of the following Dirichlet problem:

$$\begin{cases} \operatorname{div}\left(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}} \right) + \sigma^2 \nu = 0, & x \in D, \\ w = \frac{\sigma}{\epsilon}, & x \in \partial D. \end{cases}$$

It follows from Theorem 6.1 in [12] that there exists a constant C depending only on n and $diam D$ such that

$$\max_{\overline{D}} |w| \le \frac{\sigma}{\epsilon} + C\sigma^2 \nu.$$

So

$$\max_{\overline{D}} |v^{\epsilon}| \le 1 + \epsilon\sigma\nu C \le C. \tag{2.6}$$

Step 2. Suppose that κ is a positive constant which will be determined later. Let $v_1^{\epsilon} = v^{\epsilon} + \kappa$ then v_1^{ϵ} satisfies

$$\begin{cases} -\sqrt{\epsilon^2 + \sigma^2 |\nabla v_1^{\epsilon}|^2} \cdot \left(\operatorname{div}\left(\frac{\nabla v_1^{\epsilon}}{\sqrt{\epsilon^2+\sigma^2|\nabla v_1^{\epsilon}|^2}} \right) + \sigma \nu \right) = 0, & x \in D, \\ v_1^{\epsilon} = 1 + \kappa, & x \in \partial D. \end{cases}$$

By (2.6) we can choose κ depending only on $\|h\|_{C(\partial D)}$, $\|g\|_{C(\overline{D})}$, and D such that

$$v_1^{\epsilon}(x) \ge g(x), \quad v_1^{\epsilon}(x) \ge h(x), \quad x \in \overline{D}.$$

By applying Lemma 2.2 we arrive at

$$u(x,t) \le v_1^{\epsilon}(x) \le C + \kappa \le C, \quad (x,t) \in \overline{D} \times [0,T).$$

For the same reason we obtain

$$u(x,t) \ge -C, \quad (x,t) \in \overline{D} \times [0,T).$$

This yields the desired results. □

The following is the gradient estimate for a solutions of (2.3).

Lemma 2.4 *If $u \in C^{\infty}(D\times(0,T)) \cap C(\overline{D}\times[0,T))$ and is a solution of (2.3) with $|\nu| < \frac{nH_0}{n+1}$. Then*

$$\|\nabla u\|_{L^{\infty}(D\times[0,T))} \le C. \tag{2.7}$$

where C is depending only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$ and D.

Proof Step 1. We derive the gradient estimates of u at the boundary using the methods from [3]. Set $w = u - h$. Then by (2.3) w satisfies the following equations on $D \times (0, T)$:

$$\mathfrak{L}w \triangleq w_t - \left(\delta_{ij} - \sigma^2 \frac{(w_i + h_i)(w_j + h_j)}{\epsilon^2 + \sigma^2|\nabla w + \nabla h|^2}\right)(w_{ij} + h_{ij}) - \sigma\nu\sqrt{\epsilon^2 + \sigma^2|\nabla w + \nabla h|^2} = 0.$$

In the neighborhood Θ of $\partial D \times [0, T)$ we will construct the functions $\psi^{\pm}$ which are independent of t and satisfy

$$\pm\mathfrak{L}\psi^{\pm} \geq 0, \quad (x, t) \in \Theta \cap (D \times (0, T)), \tag{2.8}$$

$$\psi^{\pm} = w = 0, \quad (x, t) \in \Theta \cap (\partial D \times [0, T)), \tag{2.9}$$

$$\psi^{-} \leq w \leq \psi^{+}, \quad (x, t) \in (\partial\Theta \cap (D \times [0, T))) \cup (\Theta \cap (D \times \{0\})). \tag{2.10}$$

Consequently by Lemma 2.2 we have

$$\psi^{-} \leq w \leq \psi^{+}, \quad (x, t) \in \bar{\Theta} \cap (D \times [0, T)). \tag{2.11}$$

For $(x, t) \in \partial D \times [0, T)$, if $\mathbf{a}$ is the normal vector of ∂D such that

$$x + s\mathbf{a} \in \bar{\Theta} \cap (D \times [0, T)), \qquad \text{when} \quad 0 < s \leq 1.$$

Then by (2.9) and (2.11) we obtain

$$\frac{\psi^{-}(x + s\mathbf{a}) - \psi^{-}(x)}{s} \leq \frac{w(x + s\mathbf{a}, t) - w(x, t)}{s} \leq \frac{\psi^{+}(x + s\mathbf{a}) - \psi^{+}(x)}{s}.$$

Letting $s \to 0$, we have

$$\frac{\partial\psi^{-}}{\partial\mathbf{a}}(x) \leq \frac{\partial w}{\partial\mathbf{a}}(x, t) \leq \frac{\partial\psi^{+}}{\partial\mathbf{a}}(x).$$

A direct calculation yields on $\Theta \cap (\partial D \times [0, T))$

$$\begin{aligned} |\nabla u| &\leq \|\nabla w\|_{C(\Theta\cap(\partial D\times[0,T)))} + \|\nabla h\|_{C(\Theta\cap(\partial D\times[0,T)))} \\ &\leq \|\nabla\psi^{+}\|_{C(\Theta\cap(\partial D\times[0,T)))} + \|\nabla\psi^{-}\|_{C(\Theta\cap(\partial D\times[0,T)))} + \|\nabla h\|_{C(\Theta\cap(\partial D\times[0,T)))} \leq C. \end{aligned} \tag{2.12}$$

In the following we define ψ^{+} and ψ^{-} which satisfy (2.8)–(2.10) in detail. Firstly set

$$\psi^{+}(x) = \lambda d(x), \ x \in \overline{D}, \qquad N = \{x \in D | d(x) < \rho\},$$

where $d(x)$ is the distance from x to ∂D, ρ and λ are positive constants which will be determined later. Selecting the positive constant ρ to be small enough such that $d(x)$ satisfies

(a) $d(x) \in C^2(N)$.

(b) In N, $|\nabla d| = 1$, and

$$\sum_{i=1}^{n+1} d_i d_{ij} = 0, \quad j = 1, 2, \ldots, n+1.$$

(c) If $x \in N$, then there exists $x_0 \in \partial D$ such that $d(x) = |x - x_0|$. By Lemma 14.17 in [10] we have the formula

$$-\Delta d(x) = \sum_{i=1}^{n} \frac{k_i}{1 - k_i d(x)}, \tag{2.13}$$

where $k_1, k_2, \ldots, k_n$ are the principle curvature of ∂D at x_0.

Because ∂D is strictly convex, the mean curvature of ∂D have a positive lower bound and we denote it by H_0. Choosing $\rho < \frac{1}{H_0}$ and by (2.13) we have

$$\triangle d(x) \leq -nH_0, \quad x \in N. \tag{2.14}$$

Now we verifies ψ^+ (2.8)–(2.10).

(1) By the definition of $d(x)$, ψ^+ satisfies (2.9).
(2) If $x \in N$, then we can choose $x_0 \in \partial D$ such that $d(x) = |x - x_0|$. And by $w(x_0, 0) = 0$ we obtain

$$w(x, 0) = g(x) - h(x) - [g(x_0) - h(x_0)] \leq \beta|x - x_0| = \beta d(x),$$

where β is depending only on $\|h\|_{C^1(\partial D)}$ and $\|g\|_{C^1(\overline{D})}$. On the other hand, if $x \in \partial N \cap D$, then $d(x) = \rho$. So we can select a positive constant λ, such that ψ^+ satisfies (2.10).
(3) ψ^+ satisfies (2.8). In fact,

$$\mathfrak{L}\psi^+ = -\lambda\Delta d - \Delta h + \sigma^2 \left(\frac{\lambda^2 d_i d_j + \lambda d_i h_j + \lambda h_i d_j + h_i h_j}{\epsilon^2 + \sigma^2|\lambda\nabla d + \nabla h|^2}\right)(\lambda d_{ij} + h_{ij})$$
$$-\sigma\nu\sqrt{\epsilon^2 + \sigma^2|\nabla h|^2 + 2\lambda\sigma^2\nabla d \cdot \nabla h + \sigma^2\lambda^2}.$$

Then by (2.14) and $\Sigma_{i=1}^{n+1} d_{ij} d_i = 0$ we have

$$\mathfrak{L}\psi^+ \geq n\lambda H_0 - \|h\|_{C^2(\overline{D})} + \sigma^2 \left(\frac{\lambda^2 d_i d_j h_{ij} + 2\lambda h_i d_j h_{ij} + h_i h_j h_{ij} + \lambda d_{ij} h_i h_j}{\epsilon^2 + \sigma^2\lambda^2 + 2\sigma^2\lambda\nabla d \cdot \nabla h + \sigma^2|\nabla h|^2}\right)$$
$$-\sigma\nu\sqrt{\epsilon^2 + \sigma^2|\nabla h|^2 + 2\sigma^2\lambda\nabla d \cdot \nabla h + \sigma^2\lambda^2}.$$

By Lemma 14.17 in [10], $|d_{ij}|$ have an upper bound depending only on ∂D. Let the positive constant λ to be large enough then we obtain

$$\mathfrak{L}\psi^+ \geq n\lambda H_0 - \lambda\sigma^2|\nu| - C \geq n\lambda H_0 - \lambda|\nu| - C, \tag{2.15}$$

where C is depending only on ∂D, $\|h\|_{C^2(\partial D)}$. From (2.15) and $|\nu| < nH_0$ let λ to be large enough which is depending only on ∂D and $\|h\|_{C^2(\partial D)}$ then we have

$$\mathfrak{L}\psi^+ \geq 0.$$

For the same reason we can construct ψ^- which satisfies (2.8)–(2.10). So we have obtained the desired results of step 1 by (2.12).

Step 2. For $i \in \{1, 2, \ldots, n+1\}$, let $\varpi = u_i$. Differentiating (2.3) with respect to x_i we get

$$\varpi_t - a^{kl}\varpi_{kl} - b^l\varpi_l = 0, \quad (x, t) \in D \times (0, T),$$

where

$$a^{kl} = \delta_{kl} - \frac{\sigma^2 u_k u_l}{\epsilon^2 + \sigma^2|\nabla u|^2},$$
$$b^l = \frac{2\sigma^4 u_k^\epsilon u_m^\epsilon u_{km}^\epsilon u_l}{(\epsilon^2 + \sigma^2|\nabla u|^2)^2} - \frac{2\sigma^2 u_k u_{kl}}{\epsilon^2 + \sigma^2|\nabla u|^2} - \frac{\nu\sigma^3 u_l}{\sqrt{\epsilon^2 + \sigma^2|\nabla u|^2}}.$$

By the maximum principle for linear parabolic equation (cf. [11]) and (2.12) we obtain (2.7). □

From Lemma 2.1–2.4 and the Schauder estimates we conclude that

Theorem 2.5 *For any $\epsilon > 0$ and $|\nu| < \frac{nH_0}{n+1}$, there exists u^ϵ which satisfies*

$$u^\epsilon \in C^\infty(D \times (0, +\infty)), \quad u^\epsilon \in C(\overline{D} \times [0, +\infty)), \quad \nabla u^\epsilon \in C(\overline{D} \times [0, +\infty)),$$

and u^ϵ is a classical solution of (2.1). And there holds

$$\|u^\epsilon\|_{L^\infty(D\times[0,+\infty))} \leq C, \quad \|\nabla u^\epsilon\|_{L^\infty(D\times[0,+\infty))} \leq C,$$

where C depends only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$, H_0 and D.

Corollary 2.6 *Suppose that u^ϵ is a classical solution of (2.1) with $|\nu| < \frac{nH_0}{n+1}$. Then there holds*

$$\int_0^{+\infty} \int_D |u_t^\epsilon|^2 dxdt \leq C, \tag{2.16}$$

where C depends only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$, H_0 and D.

Proof Set

$$J(t) = \int_D \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} dx.$$

Then

$$J'(t) = \int_D \frac{\nabla u^\epsilon \cdot \nabla u_t^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dx = -\int_D \operatorname{div} \frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} u_t^\epsilon dx. \tag{2.17}$$

From (2.1) we see that

$$\operatorname{div} \frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} = \frac{u_t^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} - \nu. \tag{2.18}$$

Substituting (2.18) into (2.17) we obtain

$$J'(t) + \int_D \frac{|u_t^\epsilon|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dx = \nu \int_D u_t^\epsilon dx. \tag{2.19}$$

For (2.19) integrating from 0 to T and using (2.5) we have

$$\begin{aligned}\int_0^T \int_D \frac{|u_t^\epsilon|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dxdt &= J(0) - J(t) + \nu \int_D u^\epsilon|_{t=T} dx - \nu \int_D u^\epsilon|_{t=0} dx \\ &\leq J(0) + C,\end{aligned}$$

where C is a constant which is independent of ϵ. Letting $T \to +\infty$ we get

$$\int_0^{+\infty} \int_D \frac{|u_t^\epsilon|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dxdt \leq C. \tag{2.20}$$

Combining (2.7) with (2.20) we arrive at

$$\begin{aligned}\int_0^{+\infty} \int_D |u_t^\epsilon|^2 dxdt &= \int_0^{+\infty} \int_D \frac{|u_t^\epsilon|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} dxdt \\ &\leq (\|\nabla u^\epsilon\|_{L^\infty(D\times[0,+\infty))} + \epsilon) \int_0^{+\infty} \int_D \frac{|u_t^\epsilon|^2}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} dxdt \\ &\leq C,\end{aligned}$$

where C depends only on $\|h\|_{C^2(\partial D)}$, $\|g\|_{C^1(\overline{D})}$, H_0 and D. □

Corollary 2.7 *Suppose u^ϵ is a classical solution of* (2.1). *Then there holds*

$$\|u_t^\epsilon\|_{L^\infty(D\times[0,+\infty))} \le C. \tag{2.21}$$

where C depends only on $\|g\|_{C^2(\overline{D})}$.

Proof Set $\omega = u_t^\epsilon$. Differentiating (2.1) with respect to t we get

$$\omega_t - a^{kl}\omega_{kl} - b^l\omega_l = 0, \quad (x,t)\in D\times(0,+\infty),$$

where

$$a^{kl} = \delta_{kl} - \frac{u_k^\epsilon u_l^\epsilon}{\epsilon^2 + |\nabla u^\epsilon|^2},$$
$$b^l = \frac{2u_k^\epsilon u_m^\epsilon u_{km}^\epsilon u_l^\epsilon}{(\epsilon^2+|\nabla u^\epsilon|^2)^2} - \frac{2u_k^\epsilon u_{kl}^\epsilon}{\epsilon^2+|\nabla u^\epsilon|^2} - \frac{\nu u_l^\epsilon}{\sqrt{\epsilon^2+|\nabla u^\epsilon|^2}}.$$

From $u^\epsilon|_{\partial D\times[0,T)} = h(x)$, there holds

$$\omega = 0, \quad (x,t)\in\partial D\times[0,+\infty).$$

From (2.1) we see that

$$\omega = \sqrt{\epsilon^2+|\nabla g|^2}\cdot\left(\operatorname{div}\left(\frac{\nabla g}{\sqrt{\epsilon^2+|\nabla g|^2}}\right)+\nu\right), \quad (x,t)\in D\times\{0\}.$$

This yields (2.21) by using maximum principle . □

The Proof of Main Results

In the third section, we give the proof of Theorem 1.4, Corollary 1.5 and Theorem 1.7.

Proof of Theorem 1.4 Consider the classical solution of the approximated problem (2.1). From Theorem 2.5 and Corollary 2.7 we see that there exists $\{\epsilon_i\}|_{i=1}^{+\infty}$ satisfying $\lim_{i\to+\infty}\epsilon_i = 0$ such that there holds

$$\begin{aligned} u^{\epsilon_i} &\to u, && \text{in} \quad C(\overline{D}\times[0,+\infty)),\\ \nabla u^{\epsilon_i} &\rightharpoonup \nabla u && \text{in} \quad L^\infty(D\times[0,+\infty)),\\ u_t^{\epsilon_i} &\rightharpoonup u_t && \text{in} \quad L^\infty(D\times[0,+\infty)). \end{aligned}$$

Combining Corollary 2.6 with Fatou's Lemma we verify that u satisfies (1.6)–(1.8). On the other hand, by the stability theorem of viscosity solutions (cf. Theorem 2.4 in [2]) u is a viscosity solution of (1.4). This completes the proof of Theorem 1.4. □

Proof of Corollary 1.5 The main idea comes from Y.Giga, M.Ohnuma and M.Sato (cf. [9]).

Consider the viscosity solution u of (1.4). For $(x,t)\in\overline{D_1}\triangleq\overline{D}\times[0,1]$, set

$$u_k(x,t) = u(x,k+t), \quad k=1,2,\ldots.$$

From (1.7) and the Ascoli–Arzela's Theorem, there exists a subsequence of $\{u_k\}$ (still denote the subsequence by $\{u_k\}$) and the function $v(x,t)$, such that

$$\lim_{k\to+\infty} u_k(x,t) = v(x,t), \quad \text{in} \quad C(\overline{D_1}). \tag{3.1}$$

By (1.8) we obtain

$$\lim_{k\to+\infty}\int_0^1\int_D |u_{kt}|^2 dxdt = \lim_{k\to+\infty}\int_k^{k+1}\int_D |u_t|^2 dxdt = 0.$$

Letting $k \to +\infty$ we have

$$u_{kt} \rightharpoonup 0, \quad \text{in} \quad L^2(D_1). \tag{3.2}$$

It follows from (3.1) and (3.2) that for any $\phi \in C_0^\infty(D)$ and $\chi \in C_0^\infty(0,1)$,

$$\int_D\int_0^1 v\phi\chi_t dtdx = 0.$$

Then

$$v_t = 0, \quad (x,t) \in D_1. \tag{3.3}$$

By (1.4) u_k satisfies the following equation in viscosity sense

$$\begin{cases} u_{kt} - |\nabla u_k|\left(\operatorname{div}\left(\frac{\nabla u_k}{|\nabla u_k|}\right) + \nu\right) = 0, & (x,t) \in D \times (0,1), \\ u_k = h(x), & (x,t) \in \partial D \times [0,1]. \end{cases} \tag{3.4}$$

From (3.4) taking $k \to +\infty$ and using (1.7), (3.1), (3.3) and applying Theorem 2.4 in [2] we deduce that v satisfies

$$\begin{cases} -|\nabla v|\left(\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) + \nu\right) = 0, & x \in D, \\ v = h(x), & x \in \partial D, \end{cases}$$

in viscosity sense. This completes the proof of Corollary 1.5. □

Proof of Theorem 1.7 Firstly taking positive constant δ to be small enough we can construct a pair of non-decreasing C^2 functions $g^+(\tau)$ and $g^-(\tau)$ such that

$$\begin{cases} g^-(x_{n+1}) = g^+(x_{n+1}) = \lambda, & x_{n+1} \geq m+\delta, \\ g^-(x_{n+1}) \leq \max_{x'\in\overline{D}} g(x', x_{n+1}) \leq g^+(x_{n+1}), & x_{n+1} \leq m+\delta. \end{cases} \tag{3.5}$$

In fact, by the hypothesis of g we can choose $g^+(\tau) \equiv \lambda$. Set

$$g_\varepsilon(\tau) = \begin{cases} \lambda, & \text{if} \quad \tau \geq m+\delta, \\ \frac{\lambda}{\varepsilon}(\tau - m - \delta + \varepsilon), & \text{if} \quad \tau \leq m+\delta. \end{cases}$$

By smoothing the point $(m+\delta, \lambda)$ and letting $\epsilon = \epsilon(\delta)$ to be small enough we obtain $g^-(\tau)$ which satisfies (3.5).

Let

$$u^+(x', x_{n+1}, t) = g^+(x_{n+1} + \nu t),$$
$$u^-(x', x_{n+1}, t) = g^-(x_{n+1}).$$

We claim that $u^+(x', x_{n+1}, t)$ and $u^-(x', x_{n+1}, t)$ are viscosity sub-solution and viscosity super-solution of (1.4) respectively. Then using Theorem 1.7 in [13], i.e, the comparison principle for the viscosity solution of (1.4) which is likely to Lemma 2.2, we get

$$u^-(x', x_{n+1}, t) \leq u(x', x_{n+1}, t) \leq u^+(x', x_{n+1}, t), \quad (x', x_{n+1}, t) \in \overline{D} \times [0, +\infty). \tag{3.6}$$

In particular, if $x_{n+1} \geq m+\delta$ for any $\delta > 0$, then $u^+(x', x_{n+1}, t) = u^-(x', x_{n+1}, t) \equiv \lambda$ by making use of (3,5). Taking $\delta \to 0$ we obtain (1.11).

Now we prove that $u^+(x', x_{n+1}, t)$ is viscosity super-solution of (1.4). In a similar way we can prove that $u^-(x', x_{n+1}, t)$ is a viscosity sub-solution of (1.4).

In fact, for any $(x,t) \in D \times [0,+\infty)$, if $\varphi \in C^\infty(D \times [0,+\infty))$ and there exists a neighborhood Θ of (x,t) in $D \times [0,+\infty)$ such that

$$(u^+ - \varphi)(x,t) = \min_{\overline{\Theta}}(u^+ - \varphi).$$

Then at (x,t) we have

$$u_t^+ - \varphi_t \leq 0, \quad \nabla u^+ = \nabla\varphi, \quad D^2u^+ \geq D^2\varphi. \tag{3.7}$$

If $\nabla\varphi = 0$. Then by taking $\eta = (\eta_1, \eta_2, \ldots, \eta_n, \eta_{n+1}) = (0,0,\ldots,0,1)$ and using (3.7) we obtain

$$(\delta_{ij} - \eta_i\eta_j)\varphi_{ij} \leq (\delta_{ij} - \eta_i\eta_j)u_{ij}^+ = (1 - \eta_{n+1}\eta_{n+1})u_{n+1,n+1}^+ = 0. \tag{3.8}$$

By (3.7) and (3.8) there holds

$$\varphi_t \geq u_t^+ = \nu(g^+)' = \nu\varphi_{n+1} = 0 \geq (\delta_{ij} - \eta_i\eta_j)\varphi_{ij}.$$

On the other hand, if $\nabla\varphi \neq 0$. Then by (3.7) we get

$$\varphi_i = u_i^+ = 0, \quad u_{ii}^+ = 0, \quad i = 1,2,\ldots,n, \quad \varphi_{n+1} = u_{n+1}^+ \neq 0. \tag{3.9}$$

Combining (3.7) with (3.9), we obtain

$$\left(\delta_{ij} - \frac{\varphi_i\varphi_i}{|\nabla\varphi|^2}\right)\varphi_{ij} = \sum_{i=1}^n \varphi_{ii} \leq \sum_{i=1}^n u_{ii}^+ = 0, \tag{3.10}$$

$$\varphi_t \geq u_t^+ = \nu\varphi_{n+1}. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\left(\delta_{ij} - \frac{\varphi_i\varphi_i}{|\nabla\varphi|^2}\right)\varphi_{ij} + \nu|\nabla\varphi| \leq \varphi_t.$$

So we conclude that $u^+(x', x_{n+1}, t)$ is viscosity super-solution of (1.4). This completes the proof of Theorem 1.7. □

Acknowledgements The first author was supported by National Nature Science Foundation of China (No.11771103) and Guangxi Nature Science Foundation(2017GXNSFFA198013). The second author was supported by National Nature Science Foundation of China (No.11631002) and National Nature Science Foundation of China (No.11371060). They also would like to thank referees for useful comments, which improved the paper.

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