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Research Article

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Entire Solutions of Cauchy Problem for Parabolic Monge–Ampère Equations

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Abstract: In this paper, we study the Cauchy problem of the parabolic Monge–Ampère equation

$$-u_t \det D^2 u = f(x, t)$$

and obtain the existence and uniqueness of viscosity solutions with asymptotic behavior by using the Perron method.

Keywords: Parabolic Monge–Ampère Equations, Cauchy Problem, Asymptotic Behavior, Perron Method

MSC 2010: 35K96, 35D40

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1 Introduction

The parabolic Monge–Ampère equation

$$-u_t \det D^2 u = f(x, t) \tag{1.1}$$

is an important class of fully nonlinear parabolic equations. This type of parabolic differential operator was first considered by Krylov [11] with other two parabolic versions of the elliptic Monge–Ampère operator. This operator is relevant in the study of deformation of surfaces by Gauss–Kronecker curvature [13], and in a maximum principle for parabolic equations [14]. There are many results for (1.1), see [7–9, 15, 16, 18, 19] and the references therein. Especially, the interior Dirichlet problem

$$\begin{aligned} -u_t \det D^2 u &= f(x, t) \quad \text{in } Q, \\ u &= \varphi(x, t) \quad \text{on } \partial_p Q, \end{aligned}$$

was studied in [15, 16], where $Q = \Omega \times (0, T]$ is a cylinder in $\mathbb{R}^{n+1}$, $\Omega \subset \mathbb{R}^n$ is a bounded and strictly convex domain, T is a positive constant, and $\partial_p Q = (\partial\Omega \times (0, T)) \cup (\overline{\Omega} \times \{0\})$ is the parabolic boundary of Q, see [12] for a complete description of Q. The existence and uniqueness of viscosity solution of the interior Dirichlet problem were obtained in [15, 16]. The first author [7] considered the exterior Dirichlet problem of (1.1).

In this paper, we will study the viscosity solution of the Cauchy problem

$$-u_t \det D^2 u = f(x, t), \quad (x, t) \in \mathbb{R}^n \times (0, T], \tag{1.2}$$

$$u = \phi(x), \quad (x, t) \in \mathbb{R}^n \times \{t = 0\}, \tag{1.3}$$

where f and ϕ are continuous functions.

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For a domain $D \subset \mathbb{R}^{n+1}$, we say a function $u \in C^{2k,k}(D)$ which means that u is $2k$-th continuous differentiable with spatial variables $x \in \mathbb{R}^n$ and k-th continuous differentiable with time variable t for $(x,t) \in D$. Let USC(D) and LSC(D) denote respectively the set of upper and lower semicontinuous real-valued functions on D. A function $u \in$ USC(D) (or LSC(D)) is called parabolically convex if u is convex in x and nonincreasing in t. The following is the definition of viscosity solutions, see [16].

Definition 1.1. Let $u \in \mathrm{USC}(\mathbb{R}^n \times (0,T])$ (or $\mathrm{LSC}(\mathbb{R}^n \times (0,T])$) be parabolically convex. Then the function u is called a viscosity subsolution (or supersolution) of (1.2) if for any function $h \in C^{2,1}(Q_r(\bar{x},\bar{t}))$ (with some $Q_r(\bar{x},\bar{t}) := \{(x,t) : |x-\bar{x}| < r,\ \bar{t} - r^2 < t \le \bar{t}\} \subset \mathbb{R}^n \times (0,T]$), whenever

$$u(x,t) - h(x,t) \le (\text{or } \ge)\ u(\bar{x},\bar{t}) - h(\bar{x},\bar{t}) \quad \text{for any } (x,t) \in Q_r(\bar{x},\bar{t})$$

we must have

$$-h_t(\bar{x},\bar{t}) \det D^2 h(\bar{x},\bar{t}) \ge (\text{or } \le)\ f(\bar{x},\bar{t}).$$

For the supersolution, we also require that $D^2 h(\bar{x},\bar{t}) > 0$ in the matrix sense.

A function $u \in C^0(\mathbb{R}^n \times (0,T])$ is called a viscosity solution of (1.2) if it is both a viscosity subsolution and supersolution of (1.2).

Definition 1.2. A function $u \in \mathrm{USC}(\mathbb{R}^n \times [0,T])$ (or $(\mathrm{LSC}(\mathbb{R}^n \times [0,T]))$ is called a viscosity subsolution (or supersolution) of problem (1.2) and (1.3) if u is a viscosity subsolution (or supersolution) of (1.2), and $u \le (\text{or } \ge)\ \phi(x)$ for $(x,t) \in \mathbb{R}^n \times \{t=0\}$. Then $u \in C^0(\mathbb{R}^n \times [0,T])$ is called a viscosity solution of (1.2) and (1.3) if it is a viscosity solution of (1.2) and $u = \phi(x)$ for $(x,t) \in \mathbb{R}^n \times \{t=0\}$.

We obtain the existence of the Cauchy problem for parabolic Monge–Ampère equations.

Theorem 1.1. *Assume that f and ϕ are continuous, f has positive upper bound and positive lower bound in $\mathbb{R}^n \times [0,T]$, $D^2\phi$ is positive definite and $\det D^2\phi$ has positive upper bound and positive lower bound in the viscosity sense in $\mathbb{R}^n$. Then the Cauchy problem* (1.2) *and* (1.3) *has a viscosity solution $u \in C^0(\mathbb{R}^n \times [0,T])$.*

To obtain the uniqueness of solutions to (1.2) and (1.3), we suppose that f and ϕ satisfy the following assumptions:

Assumption (F). The function $f \in C^0(\mathbb{R}^n \times [0,T])$ is positive satisfying that for the constant $\beta > 0$,

$$f(x,t) = f_0(|x|) + O(|x|^{-\beta}) \quad \text{uniformly for } t, \quad |x| \to \infty,$$

where $f_0 \in C^0([0,+\infty))$ is positive,

$$f_0(r) = O(r^\alpha), \quad r \to +\infty,$$

with constant $\alpha \ge -\beta$ and

$$\frac{-n(\min\{\beta,n\}-2)}{n-1} < \alpha < \infty. \tag{1.4}$$

Assumption (Φ). Let $\phi \in C^0(\mathbb{R}^n)$. Suppose that there is a constant $\tau > 0$ such that in the viscosity sense

$$\tau \det D^2\phi = f(x,0) \quad \text{and} \quad D^2\phi > 0, \quad x \in \mathbb{R}^n, \tag{1.5}$$

and for some $b \in \mathbb{R}^n$ and some constant c, $\phi(x)$ satisfies

$$\limsup_{|x|\to\infty} |x|^{\min\{\beta,n\}-2+\alpha-\frac{\alpha}{n}} |\phi(x) - (u_0(|x|) + b \cdot x + c)| < \infty, \tag{1.6}$$

where

$$u_0(|x|) = \left(\frac{n}{\tau}\right)^{\frac{1}{n}} \int_0^{|x|} \left(\int_0^s z^{n-1} f_0(z)\,dz \right)^{\frac{1}{n}} ds \tag{1.7}$$

is the solution of

$$\det D^2 u_0 = \frac{f_0(|x|)}{\tau}$$

with $u_0(0) = 0$, $u_0'(0) = 0$.

We obtain the existence and uniqueness of the Cauchy problem for parabolic Monge–Ampère equations.

Theorem 1.2. *Let $n \geq 2$. Assume that f satisfies* (F) *and ϕ satisfies* (Φ). *Then for the $b \in \mathbb{R}^n$ and the constant c in* (1.6)*, the Cauchy problem* (1.2) *and* (1.3) *has a unique viscosity solution $u \in C^0(\mathbb{R}^n \times [0, T])$ with the asymptotic behavior*

$$\limsup_{|x|\to\infty} (|x|^{\min\{\beta,n\}-2+\alpha-\frac{\alpha}{n}} |u(x,t) - (-\tau t + u_0(|x|) + b \cdot x + c)|) < \infty, \quad t \in [0, T]. \tag{1.8}$$

Remark 1.1. In [10], a counterexample is given to show the necessity of (1.4) for the elliptic Monge–Ampère equations. Then (1.4) is needed for (1.6) and so it is necessary for the parabolic Monge–Ampère equations.

Remark 1.2. If $\tau \det D^2\phi = f(x,t)$, $(x,t) \in \mathbb{R}^n \times [0, T]$, then the unique solution of problem (1.2) and (1.3) is $u(x,t) = -\tau t + \phi(x)$.

If $f_0(|x|) \equiv 1$, $x \in \mathbb{R}^n$, then

$$u_0(|x|) = \frac{1}{2\sqrt[n]{\tau}}|x|^2.$$

Corollary 1.1. *Let $n \geq 2$, $f = 1 + O(|x|^{-\beta})$ with $\beta > 2$ and ϕ satisfy* (1.5) *and* (1.6). *Then for the $b \in \mathbb{R}^n$ and the constant c in* (1.6)*, the Cauchy problem* (1.2) *and* (1.3) *has a unique viscosity solution $u \in C^0(\mathbb{R}^n \times [0, T])$ with the asymptotic behavior*

$$\limsup_{|x|\to\infty} \left(|x|^{\min\{\beta,n\}-2}\left|u(x,t) - \left(-\tau t + \frac{1}{2\sqrt[n]{\tau}}|x|^2 + b \cdot x + c\right)\right|\right) < \infty, \quad t \in [0, T].$$

Suppose that Ω is a smooth, bounded and strictly convex open subset in $\mathbb{R}^n$. Let Σ, diffeomorphic to an $(n-1)$-disc, be the intersection of Ω and a hyperplane in $\mathbb{R}^n$, and let Γ be the boundary of Σ. Caffarelli and Li [1] introduced the domain Γ when they investigated the multi-valued solutions of elliptic Monge–Ampère equations $\det D^2 u = f(x)$ in $(\tilde{\Omega} \setminus \Gamma) \times \mathbb{Z}$, where $\Omega \subset \tilde{\Omega}$ and $\tilde{\Omega}$ is bounded strictly convex. They obtained the existence and uniqueness of multi-valued solutions with prescribed value on Γ. For a detailed description, see [1]. Xiong and Bao [17] studied the isolated singularity of parabolic Monge–Ampère equations $-u_t \det D^2 u = 1$ in $\mathbb{R}^{n+1}_- \setminus X_0$ with $\mathbb{R}^{n+1}_- = \mathbb{R}^n \times (-\infty, 0)$ and $X_0 = (x_0, t_0)$. For more results about the singular solutions, we can refer to [2–4].

In this paper, we will also consider the Cauchy problem

$$-u_t \det D^2 u = f(x,t), \quad (x,t) \in (\mathbb{R}^n \setminus \Gamma) \times (0, T], \tag{1.9}$$

$$u = \phi(x), \quad (x,t) \in (\mathbb{R}^n \setminus \Gamma) \times \{t = 0\}. \tag{1.10}$$

Assumption (Φ'). Let ϕ satisfy (Φ) with $b = 0$ and there exists some constant γ^* such that for any $\gamma > \gamma^*$,

$$\phi = -\gamma \quad \text{on } \Gamma. \tag{1.11}$$

Assumption (H). For some positive constant h_1, $h(t) \in C^1[0, T]$ satisfies $h(0) = 0$ and

$$h'(t) \leq -h_1 < 0. \tag{1.12}$$

Theorem 1.3. *Let $n \geq 2$,* (F), (Φ') *and* (H) *hold. Then for the constant c in* (1.6) *and the constant γ in* (1.11)*, there exists a unique viscosity solution u of* (1.9) *and* (1.10) *which satisfies* (1.8) *with $b = 0$ and*

$$u = h(t) - \gamma, \quad (x,t) \in \Gamma \times [0, T]. \tag{1.13}$$

This paper is arranged as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2 and Corollary 1.1. Section 4 is devoted to providing the proof of Theorem 1.3. In Section 5, we give some basic lemmas.

2 Proof of Theorem 1.1

Choose positive constants τ_1, τ_2 such that $0 < \tau_1 \leq 1 \leq \tau_2$ and

$$\tau_2 \det D^2\phi \geq f(x,t), \ \tau_1 \det D^2\phi \leq f(x,t), (x,t) \in \mathbb{R}^n \times (0, T].$$

Let

$$A(x,t) = -\tau_2 t + \phi(x), \quad (x,t) \in \mathbb{R}^n \times [0,T],$$
$$B(x,t) = -\tau_1 t + \phi(x), \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Then

$$-A_t \det D^2 A = \tau_2 \det D^2 \phi \geq f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T],$$
$$-B_t \det D^2 B = \tau_1 \det D^2 \phi \leq f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T].$$

Clearly,

$$A(x,t) \leq B(x,t)$$

and

$$A(x,0) = B(x,0) = \phi(x).$$

So $A(x,t)$ and $B(x,t)$ are respectively viscosity subsolution and supersolution of (1.2) and (1.3).

Let $\mathcal{S}$ denote the set of parabolically convex functions v which are viscosity subsolutions of (1.2) and (1.3) satisfying

$$v(x,t) \leq B(x,t).$$

Then $A \in \mathcal{S}$. So $\mathcal{S} \neq \emptyset$. Define

$$u(x,t) = \sup\{v(x,t) : v \in \mathcal{S}\}, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Therefore

$$A(x,t) \leq u(x,t) \leq B(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

As a result, $u(x,0) = \phi(x)$, $x \in \mathbb{R}^n$.

As in [5, Step 4 of the proof of Theorem 1.1], we can prove that u is a viscosity solution of (1.2).

3 Proof of Theorem 1.2

By an affine transformation in x-space and subtracting a linear function to u, we only need to prove the case $b = 0$. We divide the proof into six steps.

Step 1: Construct a viscosity subsolution of (1.2)–(1.3). Let $\bar{f}(|x|)$, $\underline{f}(|x|)$ be two positive continuous functions such that

$$\tau \bar{f}(|x|) \geq f(x,t) \geq \tau \underline{f}(|x|),$$
$$\bar{f}(|x|) \geq \det D^2 \phi \geq \underline{f}(|x|),$$

with

$$\tau \underline{f}(|x|) = f_0(|x|) - c_1 |x|^{-\beta}, \quad |x| \to \infty,$$
$$\tau \bar{f}(|x|) = f_0(|x|) + c_2 |x|^{-\beta}, \quad |x| \to \infty,$$

and c_1, c_2 being positive constants. For $a > 0$, define functions

$$u_1(x,t) = -\tau t + \int_1^{|x|} \left(\int_1^s n z^{n-1} \bar{f}(z)\,dz + a \right)^{\frac{1}{n}} ds, \quad (x,t) \in \mathbb{R}^n \times [0,T],$$
$$u_2(x,t) = -\tau t + \int_1^{|x|} \left(\int_1^s n z^{n-1} \underline{f}(z)\,dz + a \right)^{\frac{1}{n}} ds, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Then u_1 and u_2 are parabolically convex, and

$$-(u_1)_t \det D^2 u_1 = \tau \bar{f} \geq f, \quad (x,t) \in \mathbb{R}^n \times (0,T], \tag{3.1}$$
$$-(u_2)_t \det D^2 u_2 = \tau \underline{f} \leq f, \quad (x,t) \in \mathbb{R}^n \times (0,T], \tag{3.2}$$
$$\det(D^2 u_1(x,t)) = \bar{f}, \quad (x,t) \in \mathbb{R}^n \times [0,T], \tag{3.3}$$
$$\det(D^2 u_2(x,t)) = \underline{f}, \quad (x,t) \in \mathbb{R}^n \times [0,T]. \tag{3.4}$$

Furthermore, we find that for $(x,t)\in\mathbb{R}^n\times[0,T]$,

$$u_1(x,t)=-\tau t+u_0(|x|)+\mu_1(a)-\int_{|x|}^{\infty}\bigg[\bigg(\int_1^s nz^{n-1}\bar f(z)\,dz+a\bigg)^{\frac1n}-\bigg(\int_0^s \frac{n}{\tau}z^{n-1}f_0(z)\,dz\bigg)^{\frac1n}\bigg]\,ds,$$

where $u_0(|x|)$ is the same as (1.7), and

$$\mu_1(a)=\int_1^{\infty}\bigg[\bigg(\int_1^s nz^{n-1}\bar f(z)\,dz+a\bigg)^{\frac1n}-\bigg(\int_0^s \frac{n}{\tau}z^{n-1}f_0(z)\,dz\bigg)^{\frac1n}\bigg]\,ds-u_0(1).$$

Then by the fact that $\bar f(z)=\frac{f_0(z)}{\tau}+\frac{c_2}{\tau}z^{-\beta}$, $f_0(z)=O(z^{\alpha})$, we know that

$$\bigg(\int_1^s nz^{n-1}\bar f(z)\,dz+a\bigg)^{\frac1n}-\bigg(\int_0^s \frac{n}{\tau}z^{n-1}f_0(z)\,dz\bigg)^{\frac1n}=O(s^{1-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}),\quad s\to+\infty.$$

So

$$\begin{aligned}\int_{|x|}^{\infty}\bigg[\bigg(\int_1^s nz^{n-1}\bar f(z)\,dz+a\bigg)^{\frac1n}-\bigg(\int_0^s \frac{n}{\tau}z^{n-1}f_0(z)\,dz\bigg)^{\frac1n}\bigg]\,ds&=\int_{|x|}^{\infty}O(s^{1-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}})\,ds\\&=O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}),\end{aligned}\tag{3.5}$$

where $2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}<0$ by (1.4). In addition, $\mu_1(a)$ is strictly increasing in $(0,+\infty)$ and

$$\lim_{a\to+\infty}\mu_1(a)=+\infty.$$

So

$$u_1(x,t)=-\tau t+u_0(|x|)+\mu_1(a)+O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}})\quad\text{as }|x|\to\infty.$$

Similarly, we have that

$$u_2(x,t)=-\tau t+u_0(|x|)+\mu_2(a)-\int_{|x|}^{\infty}\bigg[\bigg(\int_1^s nz^{n-1}\underline{f}(z)\,dz+a\bigg)^{\frac1n}-\bigg(\int_0^s \frac{n}{\tau}z^{n-1}f_0(z)\,dz\bigg)^{\frac1n}\bigg]\,ds,$$

where

$$\mu_2(a)=\int_1^{\infty}\bigg[\bigg(\int_1^s nz^{n-1}\underline{f}(z)\,dz+a\bigg)^{\frac1n}-\bigg(\int_0^s \frac{n}{\tau}z^{n-1}f_0(z)\,dz\bigg)^{\frac1n}\bigg]\,ds-u_0(1).$$

Then $\mu_2(a)$ is also strictly increasing in $(0,+\infty)$ and

$$\lim_{a\to+\infty}\mu_2(a)=+\infty.$$

So as $|x|\to\infty$, we also have

$$u_2(x,t)=-\tau t+u_0(|x|)+\mu_2(a)+O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

For the sufficiently large constant c in (1.6), there exist $a_1(c)$ and $a_2(c)$ satisfying $\mu_1(a_1(c))=\mu_2(a_2(c))=c$. Therefore as $|x|\to\infty$, $0\le t\le T$, we have

$$u_1(x,t)=-\tau t+u_0(|x|)+c+O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}})\tag{3.6}$$

and

$$u_2(x,t)=-\tau t+u_0(|x|)+c+O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

Thus

$$\lim_{|x|\to\infty}(u_1(x,t)-u_2(x,t))=0,\quad 0\le t\le T.\tag{3.7}$$

In virtue of (3.3), (3.4), (3.7) and the comparison principle, we get that

$$u_1(x,0) \le u_2(x,0), \quad x \in \mathbb{R}^n. \tag{3.8}$$

By (3.1), (3.2), (3.7), (3.8) and the comparison principle, we have

$$u_1(x,t) \le u_2(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

By (1.5), (1.6), (3.3), (3.6) and the comparison principle, we get

$$u_1(x,0) \le \phi(x), \quad x \in \mathbb{R}^n.$$

Step 2: Define the Perron solution of (1.2). Let $\mathcal{S}$ denote the set of parabolically convex functions v which are viscosity subsolutions of (1.2) and (1.3) satisfying

$$v(x,t) \le u_2(x,t).$$

Then $u_1 \in \mathcal{S}$. So $\mathcal{S} \neq \emptyset$. Define

$$u_c(x,t) = \sup\{v(x,t) : v \in \mathcal{S}\}, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Step 3: We prove that u_c has the asymptotic behavior at infinity. Firstly, by the definition of u_c, we have

$$u_c(x,t) \le u_2(x,t).$$

Then as $|x| \to \infty$,

$$u_c(x,t) + \tau t - u_0(|x|) - c \le O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

On the other hand, since $u_1 \in \mathcal{S}$, by (3.6), as $|x| \to \infty$, we have

$$u_c(x,t) + \tau t - u_0(|x|) - c \ge O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

Thus,

$$\limsup_{|x|\to\infty} \big(|x|^{\min\{\beta,n\}+\alpha-\frac{\alpha}{n}-2}|u_c(x,t) - (-\tau t + u_0(|x|) + c)|\big) < \infty.$$

Step 4: We prove that $u_c(x,0) = \phi(x)$, $x \in \mathbb{R}^n$. Choose positive constants τ_1, τ_2 such that $0 < \tau_1 \le \tau \le \tau_2$ and

$$\tau_2 \det D^2\phi \ge f(x,t), \quad \tau_1 \det D^2\phi \le f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T].$$

Let

$$A(x,t) = -\tau_2 t + \phi(x), (x,t) \in \mathbb{R}^n \times [0,T],$$

$$B(x,t) = -\tau_1 t + \phi(x), (x,t) \in \mathbb{R}^n \times [0,T].$$

Then

$$-A_t \det D^2 A = \tau_2 \det D^2\phi \ge f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T],$$
$$-B_t \det D^2 B = \tau_1 \det D^2\phi \le f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T].$$

As $|x| \to \infty$,

$$\lim_{|x|\to\infty} (A(x,t) - u_c(x,t)) \le 0,$$

$$\lim_{|x|\to\infty} (B(x,t) - u_c(x,t)) \ge 0.$$

Clearly, for $x \in \mathbb{R}^n$,

$$A(x,0) = B(x,0) = \phi(x).$$

So $A(x,t)$ and $B(x,t)$ are respectively viscosity subsolution and supersolution of (1.2)–(1.3). Then $A \in \mathcal{S}$ and for any $v \in \mathcal{S}$, we have $v(x,t) \le B(x,t)$. Therefore

$$A(x,t) \le u_c(x,t) \le B(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

As a result, $u_c(x,0) = \phi(x)$, $x \in \mathbb{R}^n$.

Step 5: We prove that u_c is a viscosity solution of (1.2). As in [5, Step 4 of the proof of Theorem 1.1], we can prove that u_c is a viscosity solution of (1.2).

Step 6: We prove the uniqueness. Suppose that u and v all satisfy (1.2)–(1.3) and (1.6). Then

$$\lim_{x\to\infty}(u(x,t)-v(x,t))=0.$$

By the comparison principle, $u\equiv v$, $(x,t)\in\mathbb{R}^n\times[0,T]$.

Theorem 1.2 is proved.

Proof of Corollary 1.1. In Step 1 of the proof of Theorem 1.2, we let $f_0(|x|)\equiv 1$, then

$$\int_{|x|}^{\infty}\bigg[\bigg(\int_1^s nz^{n-1}\bar{f}(z)\,dz+a\bigg)^{\frac{1}{n}}-\bigg(\int_0^s\frac{n}{\tau}z^{n-1}\bigg)^{\frac{1}{n}}\bigg]\,ds=O(|x|^{2-\min\{\beta,n\}}).$$

So

$$u_1(x,t)=-\tau t+\frac{1}{2\sqrt[n]{\tau}}|x|^2+c+O(|x|^{2-\min\{\beta,n\}}),$$
$$u_2(x,t)=-\tau t+\frac{1}{2\sqrt[n]{\tau}}|x|^2+c+O(|x|^{2-\min\{\beta,n\}}).$$

The remainder of the proof is the same as Theorem 1.2. □

4 Proof of Theorem 1.3

Let $B_2(0)\subset\subset\Omega$ and $R_1=\operatorname{diam}(\Omega)$; then $\Omega\subset\subset B_{R_1}(0)$. Choose $R_2=3R_1$.

To prove the theorem, let $\tilde{g}\in C^\infty(\overline{\Omega})$ satisfy

$$\begin{cases}\det D^2\tilde{g}=1 & \text{in }\Omega,\\ \tilde{g}=0 & \text{on }\partial\Omega.\end{cases}$$

Set $\Psi(x,t)=h(t)+\tilde{c}\tilde{g}(x)\in C^{2,1}(\overline{\Omega}\times[0,T])$. Then $\Psi|_{\partial\Omega}=h(t)$ and $\Psi_t(x,t)|_{\partial\Omega}=h'(t)$. By Lemma 5.4, for any $\xi\in\partial\Omega$,

$$w_\xi(x,t)=\Psi(\xi,t)+\frac{c_*}{2}[|x-\bar{x}(\xi,t)|^2-|\xi-\bar{x}(\xi,t)|^2],\quad (x,t)\in\mathbb{R}^n\times[0,T],$$

satisfying

$$w_\xi(x,t)<\Psi(x,t)\quad\text{on }(\overline{\Omega}\setminus\{\xi\})\times[0,T].$$

In virtue of Remark 5.1, we know that $(w_\xi)_t=\Psi_t(\xi,t)=h'(t)$. Then, by (H), we can choose c_* and $\tilde{c}$ large enough such that

$$\begin{aligned}-(w_\xi)_t\det D^2w_\xi&\ge f(x,t), && (x,t)\in B_{R_2}(0)\times(0,T],\\ \det D^2w_\xi(x,0)&\ge\det D^2\phi(x), && x\in B_{R_2}(0),\\ -\Psi_t\det D^2\Psi&\ge f(x,t), && (x,t)\in\Omega\times(0,T],\\ \det D^2\Psi(x,0)&\ge\det D^2\phi(x), && x\in\Omega.\end{aligned}$$

Define

$$w(x,t)=\begin{cases}\Psi(x,t), & (x,t)\in\Omega\times[0,T],\\ \sup_{\xi\in\partial\Omega}w_\xi(x,t), & (x,t)\in(\mathbb{R}^n\setminus\Omega)\times[0,T].\end{cases}$$

Then

$$w(x,t)=\Psi(x,t)=h(t),\quad (x,t)\in\Gamma\times[0,T],$$

and by Lemma 5.1 and Lemma 5.2,

$$\begin{aligned}-w_t\det D^2w&\ge f, && (x,t)\in B_{R_2}(0)\times(0,T],\\ \det D^2w(x,0)&\ge\det D^2\phi(x), && x\in B_{R_2}(0).\end{aligned}$$

Similar to the proof of Theorem 1.2, we choose two functions $\bar{f}(|x|)$ and $\underline{f}(|x|)$. For $a > 0$, we construct two functions

$$v_1(x,t) = -\tau t + \inf_{B_{R_1}\times[0,T]} w + \int_{2R_1}^{|x|} \bigg(\int_1^s nz^{n-1}\bar{f}(z)\,dz + a \bigg)^{\frac{1}{n}} ds, \quad (x,t) \in \mathbb{R}^n \times [0,T],$$

$$v_2(x,t) = -\tau t + \sup_{B_{R_1}\times[0,T]} w + \int_{2}^{|x|} \bigg(\int_1^s nz^{n-1}\underline{f}(z)\,dz + a \bigg)^{\frac{1}{n}} ds, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Then v_1 and v_2 are parabolically convex, and

$$\begin{aligned}
-(v_1)_t \det D^2 v_1 &= \tau\bar{f} \geq f, && (x,t) \in \mathbb{R}^n \times (0,T],\\
-(v_2)_t \det D^2 v_2 &= \tau\underline{f} \leq f, && (x,t) \in \mathbb{R}^n \times (0,T],\\
\det(D^2 v_1(x,t)) &= \bar{f}, && (x,t) \in \mathbb{R}^n \times [0,T],\\
\det(D^2 v_2(x,t)) &= \underline{f}, && (x,t) \in \mathbb{R}^n \times [0,T].
\end{aligned}$$

And

$$v_1(x,t) \leq w(x,t) \quad \text{for } (x,t) \in \overline{B_{R_1}(0)} \times [0,T].$$

Choose $a_0 > 0$ such that for $a \geq a_0$,

$$\begin{aligned}
v_1(x,t) &\geq w(x,t) && \text{for } (x,t) \in \partial B_{R_2}(0) \times [0,T],\\
v_2(x,t) &\geq w(x,t) && \text{for } (x,t) \in \partial B_{R_2}(0) \times [0,T],\\
v_2(x,t) &\geq h(t) && \text{for } (x,t) \in \Gamma \times [0,T].
\end{aligned}$$

Furthermore, we find that for $(x,t) \in \mathbb{R}^n \times [0,T]$,

$$v_1(x,t) = -\tau t + u_0(|x|) + c + \nu_1(a) - \int_{|x|}^{\infty} \bigg[\bigg(\int_1^s nz^{n-1}\bar{f}(z)\,dz + a \bigg)^{\frac{1}{n}} - \bigg(\int_0^s \frac{n}{\tau} z^{n-1} f_0(z)\,dz \bigg)^{\frac{1}{n}} \bigg] ds,$$

where $u_0(|x|)$ is the same as (1.7), and

$$\nu_1(a) = \int_{2R_1}^{\infty} \bigg[\bigg(\int_1^s nz^{n-1}\bar{f}(z)\,dz + a \bigg)^{\frac{1}{n}} - \bigg(\int_0^s \frac{n}{\tau} z^{n-1} f_0(z)\,dz \bigg)^{\frac{1}{n}} \bigg] ds - u_0(2R_1) + \inf_{B_{R_1}\times[0,T]} w - c.$$

Then $\nu_1(a)$ is strictly increasing in $(0,+\infty)$ and

$$\lim_{a\to+\infty} \nu_1(a) = +\infty.$$

By (3.5),

$$v_1(x,t) = -\tau t + u_0(|x|) + c + \nu_1(a) + O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}) \quad \text{as } |x| \to \infty.$$

Similarly, we have that

$$v_2(x,t) = -\tau t + u_0(|x|) + c + \nu_2(a) - \int_{|x|}^{\infty} \bigg[\bigg(\int_1^s nz^{n-1}\underline{f}(z)\,dz + a \bigg)^{\frac{1}{n}} - \bigg(\int_0^s \frac{n}{\tau} z^{n-1} f_0(z)\,dz \bigg)^{\frac{1}{n}} \bigg] ds,$$

where

$$\nu_2(a) = \int_{2}^{\infty} \bigg[\bigg(\int_1^s nz^{n-1}\underline{f}(z)\,dz + a \bigg)^{\frac{1}{n}} - \bigg(\int_0^s \frac{n}{\tau} z^{n-1} f_0(z)\,dz \bigg)^{\frac{1}{n}} \bigg] ds - u_0(2) + \sup_{B_{R_1}\times[0,T]} w - c.$$

Then $\nu_2(a)$ is also strictly increasing in $(0,+\infty)$ and

$$\lim_{a\to+\infty} \nu_2(a) = +\infty.$$

So as $|x| \to \infty$,

$$v_2(x,t) = -\tau t + u_0(|x|) + c + v_2(a) + O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

For the γ in (Φ'), there exist $a_1(\gamma)$ and $a_2(\gamma)$ such that

$$v_1(a_1(\gamma)) = v_2(a_2(\gamma)) = \gamma.$$

Then as $|x| \to \infty$,

$$\begin{aligned} v_1(x,t) - \gamma &= -\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}),\\ v_2(x,t) - \gamma &= -\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}). \end{aligned}$$

Define

$$\underline{u}_a(x,t) = \begin{cases} \max\{w(x,t), v_1(x,t)\} - \gamma, & (x,t) \in \overline{B_{R_2}(0)} \times [0,T],\\ v_1(x,t) - \gamma, & (x,t) \in (\mathbb{R}^n \setminus B_{R_2}(0)) \times [0,T]. \end{cases}$$

Then $\underline{u}_a \in C^0(\mathbb{R}^n \times [0,T])$. By Lemma 5.2, $\underline{u}_a$ satisfies in the viscosity sense

$$-(\underline{u}_a)_t \det D^2 \underline{u}_a \geq f, \quad (x,t) \in \mathbb{R}^n \times (0,T],$$

and

$$\det D^2 \underline{u}_a(x,0) \geq \det D^2 \phi(x), \quad x \in \mathbb{R}^n.$$

As $|x| \to \infty$,

$$\underline{u}_a(x,t) = -\tau t + u_0(|x|) + c + O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}). \tag{4.1}$$

So

$$\limsup_{|x|\to\infty} (\underline{u}_a(x,0) - \phi(x)) = 0.$$

Thus from the comparison principle, we know that

$$\underline{u}_a(x,0) \leq \phi(x), \quad x \in \mathbb{R}^n.$$

In addition,

$$\underline{u}_a(x,t) = w(x,t) - \gamma = h(t) - \gamma, \quad (x,t) \in \Gamma \times [0,T]. \tag{4.2}$$

Then

$$\underline{u}_a(x,0) = w(x,0) - \gamma = h(0) - \gamma = -\gamma = \phi(x), \quad x \in \Gamma.$$

By the comparison principle, we also have

$$\underline{u}_a(x,t) \leq v_2(x,t) - \gamma, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Step 2: Define the Perron solution of (1.9). Let $\mathcal{S}$ denote the set of locally parabolically convex functions v which are viscosity subsolutions of (1.9) satisfying

$$\begin{aligned} v(x,t) &\leq v_2(x,t) - \gamma, & (x,t) &\in \mathbb{R}^n \times [0,T],\\ v(x,t) &= h(t) - \gamma, & (x,t) &\in \Gamma \times [0,T],\\ v(x,0) &\leq \phi(x), & x &\in \mathbb{R}^n \setminus \Gamma. \end{aligned}$$

Then $\underline{u}_a \in \mathcal{S}$. So $\mathcal{S} \neq \emptyset$. Define

$$u_c(x,t) = \sup\{v(x,t) : v \in \mathcal{S}\}, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Step 3: We prove that u_c has the asymptotic behavior at infinity. Firstly, by the definition of u_c, we have

$$u_c(x,t) \leq v_2(x,t) - \gamma, \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

Then as $|x| \to \infty$,

$$u_c(x,t) + \tau t - u_0(|x|) - c \leq O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

On the other hand, since $\underline{u}_a \in \mathcal{S}$, then by (4.1), as $|x| \to \infty$, we have

$$u_c(x,t) + \tau t - u_0(|x|) - c \geq O(|x|^{2-\alpha+\frac{\alpha}{n}-\min\{\beta,n\}}).$$

Thus,

$$\limsup_{|x|\to\infty} \big(|x|^{\min\{\beta,n\}+\alpha-\frac{\alpha}{n}-2} |u_c(x,t) - (-\tau t + u_0(|x|) + c)|\big) < \infty.$$

Step 4: We prove that $u_c(x,t) = h(t) - \gamma$, $(x,t) \in \Gamma \times [0,T]$, and $u_c(x,0) = \phi(x)$, $x \in \mathbb{R}^n \setminus \Gamma$. We first prove that $u_c(x,0) = \phi(x)$, $x \in \mathbb{R}^n \setminus \Gamma$. Since $h \in C^1[0,T]$, by (1.12) there exists some positive constant $h_2 \geq h_1$ such that $h'(t) \geq -h_2$. Choose positive constants τ_1, τ_2 such that $\tau_1 < 1 < \tau_2$, $\tau_1 h_1 \leq \tau \leq \tau_2 h_2$ and

$$\tau_2 h_2 \det D^2\phi(x) \geq f(x,t), \quad \tau_1 h_1 \det D^2\phi(x) \leq f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T].$$

Let

$$\begin{aligned} A(x,t) &= -\tau_2 h_2 t + \phi(x), \quad (x,t) \in \mathbb{R}^n \times [0,T], \\ B(x,t) &= -\tau_1 h_1 t + \phi(x), \quad (x,t) \in \mathbb{R}^n \times [0,T]. \end{aligned}$$

Then

$$\begin{aligned} -A_t \det D^2 A &= \tau_2 h_2 \det D^2\phi \geq f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T], \\ -B_t \det D^2 B &= \tau_1 h_1 \det D^2\phi \leq f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0,T]. \end{aligned}$$

In addition, on $\Gamma \times [0,T]$,

$$\begin{aligned} A(x,t) &= -\tau_2 h_2 t + \phi(x) \leq -h_2 t + \phi(x) = -h_2 t - \gamma \leq h(t) - \gamma, \\ B(x,t) &= -\tau_1 h_1 t + \phi(x) \geq -h_1 t + \phi(x) = -h_1 t - \gamma \geq h(t) - \gamma. \end{aligned}$$

As $|x| \to \infty$,

$$\lim_{|x|\to\infty} (A(x,t) - u_c(x,t)) \leq 0$$

and

$$\lim_{|x|\to\infty} (B(x,t) - u_c(x,t)) \geq 0.$$

Clearly, as $x \in \mathbb{R}^n \setminus \Gamma$,

$$A(x,0) = B(x,0) = \phi(x).$$

So $A(x,t)$ and $B(x,t)$ are respectively viscosity subsolution and viscosity supersolution of (1.9), (1.10) and (1.13). So, $A \in \mathcal{S}$ and for any $v \in \mathcal{S}$, we have $v(x,t) \leq B(x,t)$. Therefore,

$$A(x,t) \leq u_c \leq B(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T].$$

As a result, $u_c(x,0) = \phi(x)$, $x \in \mathbb{R}^n \setminus \Gamma$.

Next we prove that $u_c(x,t) = h(t) - \gamma$, $(x,t) \in \Gamma \times [0,T]$. For any $\bar{\xi} \in \Gamma$, $0 \leq \bar{\tau} \leq T$, on one hand, since $\underline{u}_a \in \mathcal{S}$, then by (4.2),

$$\liminf_{(x,t)\to(\bar{\xi},\bar{\tau})} u_c(x,t) \geq \lim_{(x,t)\to(\bar{\xi},\bar{\tau})} \underline{u}_a(x,t) = h(\bar{\xi}) - \gamma.$$

On the other hand, we have

$$\limsup_{(x,t)\to(\bar{\xi},\bar{\tau})} u_c(x,t) \leq h(\bar{\xi}) - \gamma.$$

Indeed, choose $B_R = \{x : |x| \leq R\}$ such that $\mathbb{R}^n \supset\supset B_R \supset\supset \Omega$. Let

$$Q_R^T = (B_R \setminus \Gamma) \times (0,T]$$

and

$$\partial_p Q_R^T = (\partial B_R \times [0,T]) \cup ((B_R \setminus \Gamma) \times \{t = 0\}).$$

For every $v \in \mathcal{S}$, we have

$$\begin{cases} -v_t + \Delta v \geq 0, & (x,t) \in Q_R^T, \\ v \leq h(t) - \gamma, & (x,t) \in \Gamma \times [0,T], \\ v \leq B, & (x,t) \in \partial_p Q_R^T. \end{cases}$$

Let w^+ (see [12, Theorem 5.14]) satisfy

$$\begin{cases} -w_t^+ + \Delta w^+ = 0, & (x,t) \in Q_R^T, \\ w^+ = h(t) - \gamma, & (x,t) \in \Gamma \times [0,T], \\ w^+ = B, & (x,t) \in \partial_p Q_R^T. \end{cases}$$

By the comparison principle, $v \le w^+$, $(x,t) \in \overline{Q_R^T}$. So $u_c \le w^+$, $(x,t) \in \overline{Q_R^T}$ and

$$\limsup_{(x,t)\to(\bar\xi,\bar\tau)} u_c(x,t) \le \lim_{(x,t)\to(\bar\xi,\bar\tau)} w^+(x,t) = h(\bar\xi) - \gamma.$$

Step 5: We prove that u_c is a viscosity solution of (1.9). As in [5, Step 4 of the proof of Theorem 1.1], we can prove that u_c is a viscosity solution of (1.9).

Step 6: We prove the uniqueness. Suppose that u and v all satisfy (1.9), (1.10), (1.13) and (1.8). Then

$$\lim_{x\to\infty}(u(x,t) - v(x,t)) = 0.$$

By the comparison principle, $u \equiv v$, $(x,t) \in \mathbb{R}^n \times [0,T]$.

Theorem 1.3 is proved.

5 Appendix

In this section, we give some basic results.

Lemma 5.1 ([6]). *Let $\mathcal{S}$ denote the nonempty set of viscosity subsolutions of*

$$-v_t \det D^2 v = f \quad \text{in } Q = \Omega \times (0,T]. \tag{5.1}$$

Set

$$w(x,t) = \sup\{v(x,t) \mid v \in \mathcal{S}\} \quad \text{for } (x,t) \in Q.$$

Then w is a viscosity subsolution of (5.1).

Similar to [7, Lemma 2.3], we have the following:

Lemma 5.2. *Let $\Omega \subset \Omega_1$ be two bounded open strictly convex subsets with smooth boundaries in $\mathbb{R}^n$ and $Q = \Omega \times (0,T]$, $U = \Omega_1 \times (0,T]$. Suppose that v and u are parabolically convex and satisfy respectively*

$$\begin{aligned} -v_t \det D^2 v &\ge f \quad \text{in } Q, \\ -u_t \det D^2 u &\ge f \quad \text{in } U. \end{aligned}$$

Furthermore,

$$\begin{cases} u \le v & \text{in } Q, \\ u = v & \text{on } \partial\Omega \times [0,T]. \end{cases}$$

Let

$$w(x,t) = \begin{cases} v(x,t), & (x,t) \in Q, \\ u(x,t), & (x,t) \in U. \end{cases}$$

Then w is parabolically convex and satisfies, in the viscosity sense,

$$-w_t \det D^2 w \ge f \quad \text{on } U.$$

In [15, 16], a comparison principle in $V = U \times (0,T]$ with U being a convex domain is proved, see [16, Proposition 2.2] or [15, Proposition 2.1]. We find that if we adopt the definition of viscosity solution (Definition 1.2), the comparison principle still holds for any bounded domain.

Lemma 5.3 (Comparison Principle). *Let Ω be a bounded domain in $\mathbb{R}^n$, let $Q = \Omega \times (0, T]$ and let the $f, g \in C(\bar{Q})$ be positive functions. Suppose that u and v are locally parabolically convex viscosity solution of the equation*

$$-u_t \det D^2 u = f(x, t) \quad \text{in } Q$$

and the equation

$$-v_t \det D^2 v = g(x, t) \quad \text{in } Q,$$

respectively. If

$$f(x, t) \geq g(x, t) \quad \text{on } \bar{Q}.$$

Then

$$\sup_Q (u - v) \leq \sup_{\partial_p Q} (u - v).$$

Lemma 5.4 ([7]). *Let $\Psi(x, t) \in C^{2,1}(\overline{\Omega} \times [0, T])$. Then there exists some constant C_0, depending only on n, Ψ, Ω, T, such that, for any $\xi \in \partial\Omega$, there exists $\bar{x}(\xi, t) \in \mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi, t)| \leq C_0$$

and

$$w_\xi(x, t) < \Psi(x, t) \quad \text{on } (\overline{\Omega} \setminus \{\xi\}) \times [0, T],$$

where

$$w_\xi(x, t) = \Psi(\xi, t) + \frac{c_*}{2}[|x - \bar{x}(\xi, t)|^2 - |\xi - \bar{x}(\xi, t)|^2], \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

and c_* is any bounded positive constant.

Remark 5.1. In [7], from the proof of Lemma 5.4, we can see that if $\Psi_{x_i,t}(x, t) = 0$, $x \in \partial\Omega$, then we have $(w_\xi)_t = \Psi_t(\xi, t)$, $\xi \in \partial\Omega$.

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