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EXISTENCE OF ROTATING STARS WITH PRESCRIBED ANGULAR VELOCITY LAW

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Abstract. The existence of solutions of the equations for a self-gravitating fluid with prescribed angular velocity law is proved. The conditions on the angular velocity are nearly optimal. The system is formulated as a variational problem and concentration-compactness methods are used to prove the existence of minimizers of the energy functional.

1. Introduction

Since Newton's time, the relative equilibrium figures of self-gravitating rotating fluids, including the incompressible and compressible cases, have been received extensive attention. These models drive their primary interest from astrophysics, where they may be used to study the figures of stars and planets. Later on, many distinguished mathematicians and physicists such as Maclaurin, Jacobi, Liouville, Dirichlet, Lyapunov and Poincaré made great contributions. In the last century, Milne, Chandrasekhar and Tassoul produced an impressive amount of work. The interested readers can see [6][21] for more details.

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Our problem here considered is to find an equilibrium configuration of a mass of compressible fluid that is rotating about a fixed axis (say x_3-axis) under the influence of self-gravitation. We are interested mainly in the existence and properties of the steady state solution, which is called by Luo & Smoller rotating star solution. So far, on the aspect of existence, there are many results obtained by using different methods. Specifically, Auchmuty & Beals [3] (compressible case), Auchmuty [1] (incompressible case), and Li [14] (uniformly rotating case) all first restricted to functions with support in a ball to consider a variational problem and obtain the existence of a constrained minimizer. Then they showed an a priori bound on the support of this minimizer, independent of the radius of the ball, which implied that for a sufficiently large radius, the minimizer above is just a solution of the original problem. By means of a series asymptotic estimates, Friedman & Turkington [11] proved that there exist solutions for "white dwarf stars". Deng et al [8] applied the mountain pass theorem on bounded domain and established the existence of positive solution of the Euler-Poisson equations. McCann [19] proved an existence for rotating binary stars. Particularly, in the latest paper [18], Luo & Smoller, using the concentration-compactness principle, proved the existence of rotating star solution of compressible fluid with given angular momentum. On the aspect of the properties of solution, Friedman & Turkington ([9][10]) obtained the asymptotic estimates on the diameter of fluids with given angular momentum, and Caffarelli & Friedman [5] studied the regularity of the stars' boundary. Chanillo & Li [7] gave an a priori bound on diameters and the number of connected components of white dwarfs.

What's significant is that the above recent papers may be separated two cases, one with given angular momentum ([1][2][3][4][5][9][10][11][17][18]), and the other with given angular velocity ([3][7][14]), which seems comparatively less. However, the ancient mathematicians and physicists (e.g. Maclaurin, Jacobi, Dirichlet, etc.) actually paid more attention to the latter case. Naturally, we here are also interested in the case with prescribed angular velocity.

In this paper, we focus on the rotating stars with given angular velocity and total mass, which is distinguished with the case with given angular momentum studied by Luo & Smoller. Firstly we formulate this problem as a variational problem of finding a minimizer of an energy functional. And then under the nearly optimal assumptions on the angular velocity we show the existence of rotating star solution by using the concentration-compactness principle, due to P.L. Lions [15]. We remark that the existence and stability of rotating star solution with given angular momentum was discussed in [18].

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2. Formulation of the Problem and Main Results

Our starting point here is to formulate this problem. For $x = (x_1, x_2, x_3) \in \mathbf{R^3}$, write

$$r(x) = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad z(x) = x_3.$$

Let ρ, p and $\omega(r)$ denote the density, pressure and angular velocity of the fluid rotating about z-axis, respectively. Recall the Euler equation in three dimensions space which models the motion of a compressible fluid with self-gravitation (see [19]):

$$\nabla p(\rho) = \rho\{\nabla B\rho + \omega^2(r) r\mathbf{e}_r\}, \tag{1}$$

where ∇ is the spatial gradient,

$$B\rho(x) = \int \frac{\rho(y)}{|x-y|} dy, \qquad \mathbf{e}_r = \left(\frac{x_1}{r(x)}, \frac{x_2}{r(x)}, 0\right).$$

A rotating star solution $(\tilde{\rho}, \tilde{\mathbf{v}})(r, z)$ is an axi-symmetric solution of equation (1), which models a star rotating about the z-axis. For convenience, throughout this paper, we use $\int$ and $\|\cdot\|_q$ to denote $\int_{\mathbf{R^3}}$ and $\|\cdot\|_{L^q(\mathbf{R^3})}$, respectively. We consider the problem of minimizing the functional, which corresponds to (1) (see Theorem 2.1(c))

$$\begin{aligned} E(\rho) &= \int \left(A(\rho(x)) + \frac{1}{2}\rho(x)|\mathbf{v}(r(x))|^2 - \frac{1}{2}\rho(x) \cdot B\rho(x) \right) dx \\ &:= E_{int}(\rho) + E_{kin}(\rho) + E_{pot}(\rho) \end{aligned} \tag{2}$$

over the set

$$\begin{aligned} \Gamma_M = \{\rho \in L^1(\mathbf{R^3}) \mid \rho &\text{ is axisymmetric about the } z\text{-axis}, \ \rho \geq 0, \\ &\int \rho dx = M, \ \int \left(A(\rho) + \frac{1}{2}\rho|\mathbf{v}|^2 \right) dx < \infty\}. \end{aligned}$$

Here $A(\rho)$ is determined by the pressure function $p(\rho)$, which is continuous, with the relationship

$$A(s) = s\int_0^s p(t)t^{-2} dt.$$

Suppose that the angular velocity $\omega(r)(\geq 0)$ is prescribed, then the velocity field

$$\mathbf{v}(x) = (-x_2\omega(r), x_1\omega(r), 0) \quad \text{and} \quad |\mathbf{v}(x)| = \omega(r)r.$$

Suppose

$$J(r) = \int_r^\infty s\omega^2(s)ds,$$

it is well known that Auchmuty and Beals [3] assumed that $J(r)$ satisfies

($J1$)**:** $J(r)$ is absolutely continuous and bounded on $[0,\infty)$,

($J2$)**:** $rJ(r) \to 0$, as $r \to \infty$

(i.e. $J(r)$ satisfies $(P_4^{'})$ and (23) in [3]), and then proved the existence of a minimizer of the functional in the class $\Gamma_{M,S} = \Gamma_M \cap \Gamma_S$, where

$$\Gamma_S = \{\rho \in L^1(\mathbf{R^3}) \mid \rho(x_1,x_2,x_3) = \rho(x_1,x_2,-x_3), (x_1,x_2,x_3) \in \mathbf{R^3}\}.$$

In this present paper, our interest is focused on finding a minimizer of functional $E(\rho)$ in Γ_M, and considering only the case that the state equation of the fluid is polytropic, say

$$p(\rho) = \rho^\gamma, \ \ \gamma > 4/3,$$

and

$$A(\rho) = \frac{\rho^\gamma}{\gamma - 1}.$$

In order to show the existence, as to the angular velocity ω, we assume that

($\omega1$)**:** $r\omega^2(r) \in L^1([0,\infty))$,

($\omega2$)**:** $\omega(r)/r^\alpha$ is nonincreasing in $(0,\infty)$, for some constant $\alpha \geq 0$.

Remark. Note that ($\omega1$) is equivalent to ($J1$) but ($\omega2$) is weaker than ($J2$). Consider the case $w(r) = r^{-p}$ for $p > 1$ and $r > r_0 > 0$. By a simple calculation, we see that in order to make $\omega(r)$ satisfy the condition ($J2$),

$$rJ(r) = r\int_r^\infty sw^2(s)ds = r\int_r^\infty \frac{ds}{s^{2p-1}} = \frac{r^{3-2p}}{2p-2} \to 0, \text{ as } r \to \infty,$$

it has to require

$$p > \frac{3}{2},$$

while $p > 1$ suffices to make our assumption ($\omega1$) and ($\omega2$) hold.

Actually, the angular velocity is monotone is common in astrophysics and is related to some stability criteria. So in this sense our assumption on $\omega(r)$ is nearly optimal.

By the argument as in [3], it is not difficult to prove the following regularity theorem of the minimizer.

Theorem 2.1. *Under the assumption* $(\omega 1)$, *if* $\tilde{\rho}$ *is a minimizer of the energy functional* $E(\rho)$ *in* Γ_M *and let*

$$G = \{x \in \mathbf{R^3} \mid \tilde{\rho}(x) > 0\},$$

then

(a) $\tilde{\rho} \in C(\mathbf{R^3}) \cap C^1(G)$.

(b) *there exists a constant* $\mu < 0$ *such that*

$$\begin{cases} A'(\tilde{\rho}(x)) + J(r(x)) - B\tilde{\rho}(x) = \mu, & x \in G, \\ J(r(x)) - B\tilde{\rho}(x) \geq \mu, & x \in \mathbf{R^3} - G. \end{cases}$$

(c) $\tilde{\rho}$ *is a solution of (1).*

Our main result is the following existence theorem.

Theorem 2.2. *Suppose the angular velocity satisfies the conditions* $(\omega 1)$ *and* $(\omega 2)$*. Then*

(a) *the functional* $E(\rho)$ *is bounded from below on* Γ_M *and*

$$e_M := \inf_{\Gamma_M} E(\rho) < 0,$$

(b) *there exists* $\tilde{\rho}$ *in* $\Gamma_{M,S}$, *which is non-increasing in* $z \geq 0$, *such that*

$$E(\tilde{\rho}) = \min_{\rho \in \Gamma_M} E(\rho) = \min_{\rho \in \Gamma_{M,S}} E(\rho).$$

Thus $\tilde{\rho}$ *is a rotating star solution with total mass* M *and angular velocity* ω.

In the rest of this paper we give some elementary results in Section 3 and establish the existence of a rotating star solution in Section 4.

3. Preliminary Results

We notice that the integrability of the potential term in (2) can be implied from that of the density function ρ. We begin with the convexity inequality, a consequence of Young's inequality (see page 145 in [12]).

Lemma 3.1. *If* $f \in L^s(\mathbf{R}^3)$, $p \leq q \leq s$, *then*

$$\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_s^{1-\lambda}, \qquad 1/q = \lambda/p + (1-\lambda)/s.$$

The following lemma is proved in [3] (see p259).

Lemma 3.2. *Suppose $f \in L^1(\mathbf{R}^3) \cap L^q(\mathbf{R}^3)$. If $1 < q \le 3/2$, then $Bf \in L^r(\mathbf{R}^N)$ for $3 < r < 3q/(3-2q)$, and*

$$\|Bf\|_r \le C(\|f\|_1^{\eta}\|f\|_q^{1-\eta} + \|f\|_1^{\theta}\|f\|_q^{1-\theta}),$$

where $C > 0$ and $0 < \eta, \theta < 1$. If $q > 3/2$, then Bf is bounded continuous function and satisfies the above inequality with $r = \infty$.

Lemma 3.3. *Let $f \in L^1(\mathbf{R}^3) \cap L^\gamma(\mathbf{R}^3)$. if $\gamma \ge 4/3$, then $\nabla Bf \in L^2(\mathbf{R}^3)$, and*

$$\|\nabla Bf\|_2^2 \le C\|f\|_1^{\frac{5\gamma-6}{3(\gamma-1)}}\|f\|_\gamma^{\frac{\gamma}{3(\gamma-1)}},$$

for some constant C.

Indeed, it follows from Hölder's inequality, Lemma 3.2 and 3.1 that

$$\begin{aligned}
\frac{1}{4\pi}\|\nabla Bf\|_2^2 &= \frac{1}{4\pi}\int (\triangle Bf)(Bf)dx \\
&= \int f(x)\cdot Bf(x)dx \\
&\le C\left(\int |f|^{6/5}dx\right)^{5/6}\left(\int |Bf|^6 dx\right)^{1/6} \\
&\le C\|f\|_{6/5}^2 \\
&\le C\|f\|_1^{\frac{5\gamma-6}{3(\gamma-1)}}\|f\|_\gamma^{\frac{\gamma}{3(\gamma-1)}}.
\end{aligned}$$

4. The Existence Proof

We now establish the existence of rotating star solution with given angular velocity. Since our proof follows closely the method detailed in [18] or [20], we shall make our discussion brief.

Split our argument into a series of lemmas.

Lemma 4.1. *There exists a positive constant C depending only on M such that*

$$\int \left(A(\rho(x)) + \rho(x)|\mathbf{v}(r(x))|^2\right)dx \le 2E(\rho) + C, \quad \rho \in \Gamma_M. \tag{3}$$

This implies $e_M > -\infty$.

Proof. Since $\rho \in \Gamma_M$, we have

$$\int \rho^\gamma dx < \infty, \quad \int \rho dx = M.$$

By the Appendix and the Young's inequality, we obtain

$$\begin{aligned}-E_{pot}(\rho) &= \frac{1}{2}\int \rho(x)\cdot B\rho(x)dx \\ &\le C\|\rho\|_1^{\frac{5\gamma-6}{3(\gamma-1)}}\|\rho\|_\gamma^{\frac{\gamma}{3(\gamma-1)}} \\ &= CM^{\frac{5\gamma-6}{3(\gamma-1)}}\left(\int \rho^\gamma(x)dx\right)^{\frac{\gamma}{3(\gamma-1)}} \\ &\le \frac{1}{2}\int A(\rho(x))dx + C.\end{aligned}$$

This implies

$$\begin{aligned}E(\rho) &= \int A(\rho(x))dx + \frac{1}{2}\int \rho(x)|\mathbf{v}(r(x))|^2dx + E_{pot}(\rho) \\ &\ge \frac{1}{2}\int A(\rho(x))dx + \frac{1}{2}\int \rho(x)|\mathbf{v}(r(x))|^2dx - C.\end{aligned}$$

Hence (3) holds. □

Lemma 4.2. *Under* ($\omega 2$), *we have*
(*a*) $e_M < 0$, *and*
(*b*) $e_M/M^{(5+2\alpha)/3}$ *is non-increasing for* $M > 0$.

Proof. (a) If ρ is a minimizer of $E(\rho)$ in Γ_M, by Theorem 2.1, we know that ρ is continuous and satisfies the equation (1). Moreover, Caffarelli and Friedman proved in [5] that the boundary ∂G is smooth enough to apply the Gauss-Green formula. Noting that $\rho|_{\partial G} = 0$, we obtain

$$\int_G x\cdot\nabla p(\rho(x))dx = -3\int p(\rho(x))dx.$$

By the argument in [18] (see page 9), we have

$$\int_G x\cdot\rho(x)\nabla(B\rho(x))dx = -\frac{1}{2}\int \rho(x)B\rho(x)dx.$$

Next since $x\cdot\mathbf{e}_r = r(x)$, it follows that

$$\int_G x\cdot\rho(x)\omega^2(r)r(x)\mathbf{e}_r dx = \int_G \rho(x)\omega^2(r)r^2(x)dx = \int_G \rho(x)|\mathbf{v}(x)|^2dx.$$

Therefore, by (1)

$$\frac{1}{2}\int \rho(x)B\rho(x)dx = \int \rho(x)|\mathbf{v}(x)|^2dx + 3\int p(\rho(x))dx.$$

So that

$$\begin{aligned} E(\rho) &= \int A(\rho(x))dx + \frac{1}{2}\int \rho(x)|\mathbf{v}(x)|^2dx - \frac{1}{2}\int \rho(x)B\rho(x)dx \\ &= \frac{4-3\gamma}{\gamma-1}\int \rho^\gamma(x)dx - \frac{1}{2}\int \rho(x)|\mathbf{v}(x)|^2dx \\ &< 0 \end{aligned}$$

since $\gamma > 4/3$.

(b) For $a, b > 0$, by a scaling argument as in [20] (see page 905), we define $\bar{\rho}(x) = a\rho(bx)$. Then it is not difficult to show that

$$\int \bar{\rho}dx = ab^{-3}\int \rho dx, \qquad \int A(\bar{\rho})dx = b^{-3}\int A(a\rho)dx,$$
$$E_{pot}(\bar{\rho}) = a^2b^{-5}E_{pot}(\rho).$$

By the condition on the angular velocity ($\omega 2$), for $b > 1$, we have

$$\begin{aligned} \int \bar{\rho}|\mathbf{v}(x)|^2dx &= a\int \rho(bx)\omega^2(r(x))r^2(x)dx \\ &\geq ab^{-2\alpha}\int \rho(bx)\omega^2(r(bx))r^2(x)dx \\ &= ab^{-(5+2\alpha)}\int \rho(x)|\mathbf{v}(x)|^2dx. \end{aligned}$$

Therefore, we choose $a = 1$ and $b = (M/\bar{M})^{1/3} > 1$, it follows that

$$\begin{aligned} E(\bar{\rho}) &= \int A(\bar{\rho}(x))dx + \frac{1}{2}\int \bar{\rho}(x)|\mathbf{v}(x)|^2dx + E_{pot}(\bar{\rho}) \\ &\geq b^{-3}\int A(\rho(x))dx + \frac{b^{-(5+2\alpha)}}{2}\int \rho(x)|\mathbf{v}(x)|^2dx + b^{-5}E_{pot}(\rho) \\ &\geq b^{-(5+2\alpha)}\left(\int A(\rho(x))dx + \frac{1}{2}\int \rho(x)|\mathbf{v}(x)|^2dx + E_{pot}(\rho)\right) \\ &= (\bar{M}/M)^{(5+2\alpha)/3}E(\rho). \end{aligned}$$

Since the map $\rho \to \bar{\rho}$ is one-to-one and onto between Γ_M and $\Gamma_{\bar{M}}$, thus we prove Part (b). □

We next consider the minimizing sequence $\{\rho_i\} \subset \Gamma_M$ of the energy functional $E(\rho)$. Recalling the Steiner symmetrization [13], we denote $\hat{\rho}_i$ as the symmetric rearrangement of ρ_i with respect to z. By the properties of the rearrangement, the integrals

$$\int A(\rho(x))dx, \qquad \int \rho(x)|\mathbf{v}(x)|^2dx$$

are not changed, while the potential term

$$\iint \frac{\rho(x)\rho(y)}{|x-y|}dxdy$$

is increased. Thus,

$$E(\hat{\rho}_i) \le E(\rho_i).$$

Therefore, in the following the minimizing sequences are assumed being rearranged, so that every ρ_i is even with respect to $z = 0$ and decreasing in $z \ge 0$.

Lemma 4.3. *Under ($\omega 2$), any minimizing sequence of $E(\rho)$ in Γ_M is bounded in $L^\gamma(\mathbf{R^3})$ and therefore has a subsequence which converges weakly in $L^\gamma(\mathbf{R^3})$.*

Proof. Let $\{\rho_i\} \subset \Gamma_M$ be a minimizing sequence of $E(\rho)$ in Γ_M. By Lemma 4.1 and Lemma 4.2, for i large enough,

$$\int A(\rho_i)dx \le 2E(\rho_i) + C \le e_M + C \le C.$$

So $\{\rho_i\}$ is bounded in $L^\gamma(\mathbf{R^3})$, the assertion is true by the self-flexibility. □

To prove the result of Theorem 2.2, we need the following lemma due to Rein (see Lemma 3.4 in [20]). It is included here for easier reference.

Lemma 4.4. *Let $\rho \in \Gamma_M$. Then for $R > 1$,*

$$\sup_{a\in\mathbf{R^3}} \int_{a+B_R} \rho(x)dx \ge \frac{1}{RM}\left(-2E_{pot}(\rho) - \frac{M^2}{R} - \frac{C\|\rho\|_\gamma^2}{R^{5-6/\gamma}}\right),$$

where $B_R = \{y \in \mathbf{R^3} \big| \; |y| < R\}$.

Lemma 4.5. *Under ($\omega 2$), let $\{\rho_i\} \subset \Gamma_M$ be a minimizing sequence of $E(\rho)$ in Γ_M. Then there exist positive constants $\delta_0 = \delta_0(M)$, $R_0 = R_0(M)$, and $i_0 \in \mathbf{N}$ such that*

$$\int_{B_R} \rho_i(x)dx \ge \delta_0, \quad \textit{for } i > i_0, \;\; R > R_0.$$

Proof. It follows by Lemma 4.3 that $\{\rho_i\}$ is bounded in $L^\gamma(\mathbf{R^3})$. And by Lemma 4.2(a), $e_M < 0$, we have for $i \ge i_0$,

$$-E_{pot}(\rho_i) = \frac{1}{2}\int \rho_i B\rho_i dx \ge -E(\rho_i) \ge -\frac{e_M}{2} > 0.$$

Therefore, by Lemma 3.4, there exist $\delta_0 = \delta_0(M) > 0$, $R' = R'(M) > 0$ and a sequence of shift vectors $\{a_i\} \subset \mathbf{R^3}$ such that

$$\int_{a_i+B_R} \rho_i(x)dx \ge \delta_0, \quad \text{for } i \ge i_0, \;\; R > R'.$$

By the property of Steiner symmetrization, ρ_i is even in z and decrease in $z > 0$, and a_i can be chosen on the plane $z = 0$.

Next, we prove $r(a_i)$ is uniformly less than a positive constant r_0. Indeed, Since ρ_i is symmetric about z-axis and has mass at least δ_0 in the ball $a_i + B_R$, it has mass more than $Cr(a_i)\delta_0$ in the torus obtained by revolving this ball around z-axis. Hence $r(a_i) \leq M/C\delta_0$, denoted by r_0. Then letting $R_0 = r_0 + R'$, the proof is completed. $\square$

The following result indicates the well-known compactness property, which is also due to Rein (see Lemma 3.7 in [20]).

Lemma 4.6. *Let* $\{\rho_i\} \subset L^\gamma(\mathbf{R^3})$ *be bounded and*

$$\rho_i \rightharpoonup \tilde{\rho} \text{ weakly in } L^\gamma(\mathbf{R^3}).$$

(a) *For any domain* $\Omega \subset \mathbf{R^3}$,

$$\nabla B(\chi_\Omega \rho_i) \to \nabla B(\chi_\Omega \tilde{\rho}) \text{ strongly in } L^2(\mathbf{R^3}),$$

where χ_Ω *is the indicator function on set* Ω.

(b) *If in addition* $\{\rho_i\}$ *is bounded in* $L^1(\mathbf{R^3})$, $\tilde{\rho} \in L^1(\mathbf{R^3})$, *and for any* $\epsilon > 0$ *there exist* $R > 0$ *and* $i_0 \in \mathbf{N}$ *such that*

$$\int_{|x| \geq R} \rho_i(x)dx < \epsilon, \qquad i \geq i_0,$$

then

$$\nabla B\rho_i \to \nabla B\tilde{\rho} \text{ strongly in } L^2(\mathbf{R^3}).$$

Next, we prove the main result Theorem 2.2.

Proof of Theorem 2.2. Fix $M > 0$. By lemma 4.1 and 4.2(a), we have proved Theorem 2.2(a). It is sufficient to give the proof of Theorem 2.2(b).

For any $0 < R_1 < R_2$ (to be determined), we denote

$$B^{(1)} = \{x \in \mathbf{R^3} | \ |x| \leq R_1\},$$

$$B^{(2)} = \{x \in \mathbf{R^3} | \ R_1 < |x| \leq R_2\},$$

$$B^{(3)} = \{x \in \mathbf{R^3} | \ |x| > R_2\},$$

and split $\rho \in \Gamma_M$ into three different parts as in [20],

$$\rho = \rho\chi_{_{B^{(1)}}} + \rho\chi_{_{B^{(2)}}} + \rho\chi_{_{B^{(3)}}} =: \rho^{(1)} + \rho^{(2)} + \rho^{(3)},$$

where $\chi_{_{B^{(j)}}}$ is the indicator function on sets $B^{(j)}$, $j = 1, 2, 3$. Thus we have

$$E(\rho) = E(\rho^{(1)}) + E(\rho^{(2)}) + E(\rho^{(3)}) - P_{12} - P_{13} - P_{23},$$

where

$$P_{kl} = \int\int \frac{\rho^{(k)}(x)\rho^{(l)}(y)}{|x-y|}dxdy, \qquad 1 \le k < l \le 3.$$

If we choose $R_2 > 2R_1$, then (see [20])

$$P_{13} \le \frac{C}{R_2}, \qquad P_{12} + P_{23} \le C\|\rho\|_\gamma^{\gamma/6(\gamma-1)}\|\nabla B\rho^{(2)}\|_2.$$

Denote

$$M^{(j)} = \int \rho^{(j)}(x)dx, \qquad j = 1, 2, 3.$$

Noticing that $e_M < 0$ and using Lemma 4.2(b), we find

$$\begin{aligned}
& e_M - E(\rho^{(1)}) - E(\rho^{(2)}) - E(\rho^{(3)}) \\
& \le e_M \left(1 - \left(\frac{M^{(1)}}{M}\right)^{\frac{5+2\alpha}{3}} - \left(\frac{M^{(2)}}{M}\right)^{\frac{5+2\alpha}{3}} - \left(\frac{M^{(3)}}{M}\right)^{\frac{5+2\alpha}{3}}\right) \\
& \le e_M \left(1 - \left(\frac{M^{(1)}}{M}\right)^{\frac{5}{3}} - \left(\frac{M^{(2)}}{M}\right)^{\frac{5}{3}} - \left(\frac{M^{(3)}}{M}\right)^{\frac{5}{3}}\right) \\
& \le \frac{10e_M}{9M^2}\left(M^{(1)}M^{(3)} + M^{(1)}M^{(2)} + M^{(2)}M^{(3)}\right) \\
& \le \frac{10e_M}{9M^2} \cdot M^{(1)}M^{(3)}.
\end{aligned}$$

Here we have used the elementary formula

$$1 - \left(a^{5/3} + b^{5/3} + c^{5/3}\right) \ge \frac{10}{9}(ab + bc + ac)$$

for $0 < a, b, c < 1$, $a + b + c = 1$. From these estimates above, it follows

$$\begin{aligned}
e_M - E(\rho) &= e_M - E(\rho^{(1)}) - E(\rho^{(2)}) - E(\rho^{(3)}) + P_{12} + P_{13} + P_{23} \\
&\le Ce_M M^{(1)}M^{(3)} + C\left(R_2^{-1} + \|\rho\|_\gamma^{\gamma/6(\gamma-1)}\|\nabla B\rho^{(2)}\|_2\right). \qquad (4)
\end{aligned}$$

Let $\{\rho_i\} \subset \Gamma_M$ be a minimizing sequence of $E(\rho)$. By Lemma 4.3, the sequence $\{\rho_i\}$ is bounded in $L^\gamma(\mathbf{R^3})$ so there exists a subsequence, still denoted by $\{\rho_i\}$, such that

$$\rho_i \rightharpoonup \tilde{\rho}, \quad \text{weakly in } L^\gamma(\mathbf{R^3}).$$

We now choose $R_1 > R_0$ by Lemma 4.5, such that $M_i^{(j)} \ge \delta_0$ for i large. By (4),

$$\begin{aligned}
-Ce_M\delta_0 M_i^{(3)} &\le \frac{C}{R_2} + C\|\nabla B\rho_i^{(2)}\|_2 + E(\rho_i) - e_M \\
&\le \frac{C}{R_2} + C\|\nabla B\tilde{\rho}^{(2)}\|_2 + C\|\nabla B\rho_i^{(2)} - \nabla B\tilde{\rho}^{(2)}\|_2 + E(\rho_i) - e_M. \qquad (5)
\end{aligned}$$

where $\rho_i^{(j)} = \rho_i \chi_{B^{(j)}}$ and $M_i^{(j)} = \int \rho_i^{(j)}(x)dx$, $i = 1, 2, \cdots$, $j = 1, 2, 3$, referring to the splitting.

Given any $\epsilon > 0$, by the same argument as [20], we can increase $R_1 > R_0$ such that the mass of $\tilde{\rho}$ mainly concentrates in the ball $B^{(1)}$, so the potential energy generated by $\tilde{\rho}_2$, the second term on the right-hand side of (5), can be sufficiently small, say less than $\epsilon/4$. Next, we take $R_2 > 2R_1$ such that the first term of (5) is also small and less than $\epsilon/4$. Now, R_1 and R_2 are fixed, so the third term in (5) converges to zero as $i \to \infty$, by Lemma 4.6(a). And the remainder $|E(\rho_i) - e_M|$ can be small if i is large. Therefore, for i sufficiently large, we can make

$$M_i^{(3)} = \int_{B^{(3)}} \rho_i(x)dx < \epsilon,$$

$$\int_{B_{R_2}} \rho_i(x)dx = M - M_i^{(3)} \geq M - \epsilon.$$

By Lemma 4.6(b), we obtain

$$\|\nabla B\rho_i - \nabla B\tilde{\rho}\|_2 \to 0, \quad \text{as} \quad i \to \infty.$$

Obviously, $\tilde{\rho} \geq 0$ a.e. By the weak convergence we have that for any $\epsilon > 0$,

$$M \geq \int_{B_R} \tilde{\rho}(x)dx \geq M - \epsilon, \quad \text{if } R \text{ large enough},$$

which in particular implies that $\tilde{\rho} \in L^1(\mathbf{R^3})$ with $\int \tilde{\rho}dx = M$. The functional $\rho \mapsto \int A(\rho)dx$ is convex, so by a standard argument, we have

$$E(\tilde{\rho}) = e_M.$$

Therefore, $\tilde{\rho}$ is a minimizer of $E(\rho)$. □

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