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EULER-POISSON EQUATIONS RELATED TO GENERAL COMPRESSIBLE ROTATING FLUIDS

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Abstract. This paper is mainly concerned with Euler-Poisson equations modeling Newtonian stars. We establish the existence of rotating star solutions for general compressible fluids with prescribed angular velocity law, which is the main point distinguished with the case with prescribed angular momentum per unit mass. The compactness of any minimizing sequence is established, which is important from the stability point of view.

1. **Introduction.** Since Newton's time, many distinguished mathematicians and physical scientists such as Maclaurin, Clairaut, Liouville, Lyapunov and Poincaré have made great contributions on the relative equilibrium figures of self-gravitating rotating fluids, including incompressible and compressible cases. In 1902, Jeans proposed the first serious theory of galaxy formation. He supposed that the universe is filled with a non-relativistic fluids, governed by the Euler-Poisson equations (see p562 in [26])

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{g} \\ \nabla \times \mathbf{g} = 0 \\ \nabla \cdot \mathbf{g} = -4\pi\rho. \end{cases} \tag{1}$$

Here $t \in \mathbb{R}^+$, $x \in \mathbb{R}^3$, ∇ is the spatial gradient, $\mathbf{g}$ is the gravitational field. ρ, $\mathbf{v} = (v_1, v_2, v_3)$, and P are the density, velocity field, and pressure, respectively. A classical problem is to investigate the stability of fluids in equilibrium. The fact shows that the stability of a fluid depends mainly on two elements: the equation of state $P(\rho)$ and the rotating velocity $\mathbf{v}$.

When $\mathbf{v} = 0$, the existence and property of the self-gravitating non-rotating star is classical, which shows that the star is exactly a ball. However when the star rotates with a fixed axis, its configuration will be no longer radial symmetric. So it is more challenging and significant in astrophysics and mathematics to study

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rotating stars. Luo and Smoller [21] call an axi-symmetry time-independent solution of system (1) as a rotating star solution.

Here it is worthwhile in astrophysics to mention the well-known white dwarfs. By virtue of the uncertainty principle and Pauli's exclusion principle, the stellar material occupies a ground state pressure P, which depends on the local density ρ. The well-known fact from quantum statistics (see Chapter 10 in [6]) shows that $P(\rho)$ obeys the asymptotic relations:

$$\begin{cases} P(\rho) = c_1\rho^{5/3} - c_2\rho^{7/3} + O(\rho^3), & \rho \to 0, \\ P(\rho) = d_1\rho^{4/3} - d_2\rho^{2/3} + \cdots, & \rho \to \infty, \end{cases}$$

for the envelope and the core, respectively, where c_1, c_2, d_1, d_2 are positive constants. In the last century, Milne and Chandrasekhar produced an impressive amount of work on slowly rotating stars. The interested readers can see [6, 7, 25] for more details.

In 1971, a rigorous mathematical theory for rotating stars of compressible fluids with angular velocity was initiated by Auchmuty and Beals [3, 4]. They proved the existence of the rotating star solution if the angular velocity satisfies certain decay conditions, to be precisely given in the next section. Subsequently, the stars with uniform rotation were studied. Li [16] proved there exists a rotating star solution if the angular velocity is smaller than a certain constant. Chanillo and Li [8] gave an a-priori bound on diameters and the number of connected components of white dwarfs with small uniform rotation.

There is another model of this problem, in which the angular momentum per unit mass, rather than the angular velocity, is prescribed. There are many results, such as Auchnuty and Beals [3], Auchmuty [1, 2], Lions [17], Friedman and Turkington [9, 10], Caffarelli and Friedman [5] and McCann [22]. Recently, Luo and Smoller [21, 20] proved the existence and nonlinear stability of rotating star solutions for compressible isentropic fluids and applied to rotating white dwarfs and high density supermassive stars. The method developed in [21, 20] is very important to deal with the fluids with angular velocity law.

In this paper we prove the existence of the rotating star solutions for general compressible fluids with prescribed angular velocity law. The compactness of any minimizing sequence is established, which is important from the stability point of view. We first reformulate this problem as a variational problem to find a minimizer of the energy functional. In order to further discuss the stability problem, we weaken the symmetry of the admissible functions. Then we use the concentration-compactness principle, due to P. L. Lions [17] and a more specific setting recovered by Rein [23], to show that any minimizing sequence of the energy functional must be compact, which leads to the existence and compactness of rotating star solution. The proof is quite different from that in [3].

In particular, for polytropic perfect gas, $P(\rho) = \rho^\gamma$, $\gamma > 1$. We remark that when $\gamma = \frac{6}{5}$, the nonlinear instability was discussed by Jang [14]. when $\gamma > \frac{4}{3}$, we studied the existence of rotating stars with prescribed angular velocity law in [15]. In this paper we will improve our result to more general fluids.

The rest of this paper runs as follows. In the next section we reformulate Euler-Poisson equations as a variational problem. Some basic inequalities were given in section 3. The main result is proved in section 4. Once the energy is proved to be negative, the existence theorem will be established by a standard technique. More details could be found in [15].

2. **Reformulation and main result.** For $x = (x_1, x_2, x_3) \in \mathbf{R^3}$, we denote

$$r(x) = \sqrt{x_1^2 + x_2^2} \quad \text{and} \quad z(x) = x_3.$$

A rotating star solution $(\rho, \mathbf{v})(r, z)$ is an axi-symmetric time-independent solution of system (1), which models a star rotating about z-axis [21]. If the angular velocity $\omega(r)(\geq 0)$ is prescribed, then the velocity field

$$\mathbf{v}(x) = (-x_2\omega(r), x_1\omega(r), 0),$$

and $|\mathbf{v}(x)| = \omega(r)r$. The Euler-Poisson Equations (1) can be written as

$$\nabla P(\rho) = \rho \left\{ \nabla B\rho + \omega^2(r) r \mathbf{e}_r \right\}, \tag{2}$$

where the gravitational field $\mathbf{g} = \nabla B\rho$,

$$B\rho(x) = \int \frac{\rho(y)}{|x-y|} dy, \qquad \mathbf{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right),$$

the function P are prescribed.

For convenience, we will use $\int$ and $\|\cdot\|_q$ to denote $\int_{\mathbf{R^3}}$ and $\|\cdot\|_{L^q(\mathbf{R^3})}$, respectively, throughout this paper. We consider the problem of minimizing the energy functional, which corresponds to (2) (see Theorem 4.1 (c)),

$$\begin{aligned} E(\rho) &= \int \left(A(\rho(x)) + \rho(x) J(r(x)) - \frac{1}{2}\rho(x) \cdot B\rho(x) \right) dx \\ &:= E_{int}(\rho) + E_{kin}(\rho) + E_{pot}(\rho) \end{aligned} \tag{3}$$

over the set

$$\Gamma_M = \Big\{ \rho \in L^1(\mathbf{R^3}) : \rho \text{ is axisymmetric about the } z\text{-axis}, \ \rho \geq 0,$$
$$\int \rho dx = M, \ \int (A(\rho) + \rho\, J(r))\, dx < \infty \Big\}.$$

Here $E_{int}, E_{kin}, E_{pot}$ represent, respectively, the internal energy, the rotating kinetic energy, and the gravitational potential energy of the fluids. $A(\rho)$ is determined by the pressure function $P(\rho)$, which is continuous, with the relationship

$$A(\rho) = \rho \int_0^\rho \frac{P(t)}{t^2} dt.$$

And

$$J(r) = \int_r^\infty s\omega^2(s) ds.$$

We studied in [15] the case that $P(\rho)$ is polytropic, say

$$P(\rho) = \rho^\gamma, \ \ \gamma > \frac{4}{3}.$$

In this paper we generalize our result [15] to more general compressible fluids, including rotating white dwarfs. Suppose that $P(\rho)$ satisfies

$(P1)$ $P(0) = 0$, and $P'(\rho) > 0$, for $\rho > 0$,

$(P2)$ $\rho^{1+1/n} \leq c^0 P(\rho)$, for $\rho > \rho^0$,

$(P3)$ $P(\rho) \leq c_0 \rho^{1+1/m}$, for $\rho < \rho_0$,

where $c_0, c^0, \rho_0 < \rho^0$ are positive constants and $0 < m, n \leq 3$. Assume that $\omega(r)$ satisfies

$(\omega 1)$ $r\omega^2(r) \in L^1([0, \infty))$,

($\omega 2$) $rJ(r) \to 0$, as $r \to \infty$
(i.e. $J(r)$ satisfies (P_4') and (23) in [3]). It is well known that Auchmuty and Beals [3] have proved the existence of a minimizer of the functional in the class $\Gamma_{M,S}$, where $\Gamma_{M,S} = \Gamma_M \cap \Gamma_S$,

$$\Gamma_S = \left\{\rho \in L^1(\mathbf{R^3}) : \rho(x_1, x_2, x_3) = \rho(x_1, x_2, -x_3)\right\}.$$

In order to further discuss the stability problem, we weaken the symmetry of the admissible functions. Here we just require them belong to Γ_M.

Our main result is as follows.

Theorem 2.1. *Suppose ($\omega 1$)($\omega 2$) hold. Then there exists $M_c > 0$, depending only on n and c^0 (if $n = 3$ then $M_c < +\infty$, if $n < 3$ then $M_c = +\infty$), such that if $M < M_c$, then*

(a) *the functional $E(\rho)$ is bounded from below on Γ_M and*

$$e_M := \inf_{\Gamma_M} E(\rho) < 0,$$

(b) *there exists $\tilde{\rho}$ in $\Gamma_{M,S}$, which is non-increasing in $z \geq 0$, such that*

$$E(\tilde{\rho}) = \min_{\rho \in \Gamma_M} E(\rho) = \min_{\rho \in \Gamma_{M,S}} E(\rho).$$

Thus $\tilde{\rho}$ is a rotating star solution with total mass M and angular velocity ω.

Remark 1. It is essentially important for the fluids in equilibrium that its energy is negative. This shows that its gravitational binding energy exceeds the sum of its internal energy and its kinetic energy, which guarantees that the gravitation forces can pull together the stellar matter.

Remark 2. When $n = 3$, the pressure $P(\rho)$ of the fluids includes the white dwarf star case. Theorem 2.1 shows that the existence of the rotating white dwarf star solution requires the total mass $M < M_c$, "a critical mass", which is also called "Chandrasekhar limit" (see [6]).

3. **Preliminaries.**

For convenience to read, we give several elementary results, beginning with a convexity inequality, which is a consequence of Young's inequality (p145 in [12]).

Lemma 3.1. *If $f \in L^s(\mathbf{R}^3)$, $p \leq q \leq s$, then*

$$\|f\|_q \leq \|f\|_p^\lambda \|f\|_s^{1-\lambda}, \qquad 1/q = \lambda/p + (1-\lambda)/s.$$

The following lemma is proved in [3].

Lemma 3.2. *Suppose $f \in L^1(\mathbf{R}^3) \cap L^q(\mathbf{R}^3)$. If $1 < q \leq 3/2$, then $Bf \in L^r(\mathbf{R}^3)$ for $3 < r < 3q/(3-2q)$, and*

$$\|Bf\|_r \leq C(\|f\|_1^\eta \|f\|_q^{1-\eta} + \|f\|_1^\theta \|f\|_q^{1-\theta}),$$

where $C > 0$ and $1 < \eta, \theta < 1$. If $q > 3/2$, then Bf is bounded continuous function and satisfies the above inequality with $r = \infty$.

Lemma 3.3. *Let $f \in L^1(\mathbf{R}^3) \cap L^\gamma(\mathbf{R}^3)$. if $\gamma \geq 4/3$, then $\nabla Bf \in L^2(\mathbf{R}^3)$, and*

$$\|\nabla Bf\|_2^2 \leq C \|f\|_1^{\frac{5\gamma-6}{3(\gamma-1)}} \|f\|_\gamma^{\frac{\gamma}{3(\gamma-1)}},$$

for some constant $C = C(\gamma)$.

Indeed, it follows from Hölder's inequality, Lemma 3.2 and Lemma 3.1 that

$$\begin{aligned}\frac{1}{4\pi}\|\nabla Bf\|_2^2 &= -\frac{1}{4\pi}\int(\triangle Bf)(Bf)dx\\ &= \int f(x)\cdot Bf(x)dx\\ &\le \left(\int |f|^{6/5}dx\right)^{5/6}\left(\int |Bf|^6 dx\right)^{1/6}\\ &\le C\|f\|_{6/5}^2\\ &\le C\|f\|_1^{\frac{5\gamma-6}{3(\gamma-1)}}\|f\|_\gamma^{\frac{\gamma}{3(\gamma-1)}}.\end{aligned}$$

4. **Proof of the main Theorem.** Our goal is to establish the existence of rotating star solution of (2). Before proving our existence theorem, we first present the following theorem for the minimizer by the argument used in [3].

Theorem 4.1. *Under the assumption* $(\omega 1)$*, if* $\tilde\rho$ *is a minimizer of the energy functional* $E(\rho)$ *in* Γ_M *and let*

$$G = \{x \in \mathbf{R^3} \mid \tilde\rho(x) > 0\},$$

then

(*a*) $\tilde\rho \in C(\mathbf{R^3}) \cap C^1(G)$.

(*b*) *there exists a constant* $\mu < 0$ *such that*

$$\begin{cases} A'(\tilde\rho(x)) + J(r(x)) - B\tilde\rho(x) = \mu, & x \in G,\\ J(r(x)) - B\tilde\rho(x) \ge \mu, & x \in \mathbf{R^3} - G.\end{cases}$$

(*c*) $\tilde\rho$ *is a solution of (2).*

Therefore, by the approach used in [15], in order to establish the existence of rotating star solution with given angular velocity, it suffice now to show

$$e_M > -\infty, \quad \text{and} \quad e_M < 0.$$

Since the rest is a standard argument, we list some Lemmas and outline the proof of our main result. More details can be found in [21] or [15].

It is easy to check that the condition $(P2)$ and $(P3)$ imply that

$$n(\rho^{1+1/n} - (\rho^0)^{1/n}\rho) \le c^0 A(\rho), \quad \text{for} \quad \rho > \rho^0, \tag{4}$$

and

$$A(\rho) \le mc_0\rho^{1+1/m}, \quad \text{for} \quad \rho < \rho_0. \tag{5}$$

Then we have

Lemma 4.2. *Under (4), there exists a critical constant* M_c*, depending only on* n *and* c^0*, (if* $n = 3$*, then* $M_c < +\infty$*, if* $n < 3$*, then* $M_c = +\infty$*), such that if* $M < M_c$*, then*

$$\int (A(\rho(x)) + 2\rho(x)J(r(x)))\,dx \le 2E(\rho) + C, \quad \rho \in \Gamma_M. \tag{6}$$

where C *is a positive constant, depending only on* ρ^0, c^0, n *and* M*. This implies* $e_M > -\infty$.

Proof. Since $\rho \in \Gamma_M$, we have

$$\int A(\rho)dx < \infty, \quad \int \rho dx = M.$$

It follows from (4) that

$$\begin{aligned}
&\int \rho^{1+1/n}(x)dx \\
&= \left(\int_{0<\rho\leq\rho^0} + \int_{\rho>\rho^0} \right) \rho^{1+1/n}(x)dx \\
&\leq (\rho^0)^{1/n} \int_{\rho\leq\rho^0} \rho(x)dx + (\rho^0)^{1/n} \int_{\rho>\rho^0} \rho(x)dx + \frac{c^0}{n} \int_{\rho>\rho^0} A(\rho(x))dx \\
&\leq (\rho^0)^{1/n} M + \frac{c^0}{n} \int_{\rho>\rho^0} A(\rho(x))dx. \qquad (7)
\end{aligned}$$

By Lemma 3.3, taking $\gamma = 1 + \frac{1}{n}$, and combining with (7), we obtain

$$\begin{aligned}
-E_{pot}(\rho) &= \frac{1}{2} \int \rho(x) \cdot B\rho(x)dx \leq \|\rho\|_{6/5}^2 \\
&\leq C\|\rho\|_1^{(5-n)/3} \|\rho\|_{1+1/n}^{(n+1)/3} \\
&\leq CM^{(5-n)/3} \left((\rho^0)^{1/n} M + \frac{c^0}{n} \int_{\rho>\rho^0} A(\rho(x))dx \right)^{n/3} \\
&\leq CM^{5/3}(\rho^0)^{1/3} + CM^{(5-n)/3} \left(\frac{c^0}{n} \right)^{n/3} \left(\int A(\rho(x))dx \right)^{n/3}, \qquad (8)
\end{aligned}$$

here C depends only on n, and in the last inequality we used $(a+b)^p \leq a^p + b^p$ for $a > 0, b > 0$ and $p \leq 1$.

If $n = 3$, then there exists a constant $M_c = \left(\frac{3}{2Cc^0} \right)^{3/2}$, such that if $M < M_c$, we have

$$CM^{2/3} \cdot \frac{c^0}{3} \int A(\rho(x))dx \leq \frac{1}{2} \int A(\rho(x))dx.$$

Combining with (8), we obtain

$$-E_{pot}(\rho) \leq \frac{1}{2} \int A(\rho(x))dx + C, \qquad (9)$$

where C depends only on c^0, ρ^0 and M. If $n < 3$, then we can obtain inequality (9) by Young's inequality, where C depends only on c^0, ρ^0, n and M.

It is clear that inequality (9) implies that

$$\begin{aligned}
E(\rho) &= \int A(\rho(x))dx + \int \rho(x)J(r(x))dx + E_{pot}(\rho) \\
&\geq \frac{1}{2} \int A(\rho(x))dx + \int \rho(x)J(r(x))dx - C
\end{aligned}$$

where C depends only on c^0, ρ^0, n and M. Hence (6) holds. □

Next we prove another important lemma.

Lemma 4.3. *For $M < M_c$ (determined in Lemma 4.2),*

(*a*) *Under $(\omega 1)(\omega 2)$, we have $e_M < 0$.*

(*b*) *And under a weaker condition*

($\omega 3$) $\omega(r)/r^{\alpha}$ *is non-increasing in* $(0,\infty)$, *for some constant* $\alpha \geq 0$,

we have, $e_M/M^{(5+2\alpha)/3}$ *is non-increasing for* $0 < M < M_c$.

Proof. Let $\hat{\rho} \in \Gamma_{M,S} \subset \Gamma_M$ $(M < M_c)$ be a minimizer of $E(\rho)$ in $\Gamma_{M,S}$ with

$$G = \{x \in \mathbb{R}^3 : \hat{\rho}(x) > 0\}$$

being compact set in $\mathbb{R}^3$, and $\hat{\rho} \in C^1(G)$ (the existence of such a minimizer is proved in [3]). Moreover, Caffarelli and Friedman proved in [5] that the boundary ∂G is smooth enough to apply the Gauss-Green formula. Noting that $\rho|_{\partial G} = 0$, we obtain

$$\int_G x \cdot \nabla P(\rho(x))dx = -3 \int P(\rho(x))dx.$$

By the argument in [21] (see page 455), we have

$$\int_G x \cdot \rho(x) \nabla (B\rho(x))dx = -\frac{1}{2} \int \rho(x) B\rho(x)dx.$$

Next since $x \cdot \mathbf{e}_r = r(x)$, it follows that

$$\int_G x \cdot \rho(x)\omega^2(r)r(x)\mathbf{e}_r dx = -\int_G \rho(x) r J'(r)dx = \int_G \rho(x)J(r(x))dx.$$

Therefore, by (2),

$$\frac{1}{2}\int \rho(x)B\rho(x)dx = \int \rho(x)J(r(x))dx + 3\int P(\rho(x))dx. \tag{10}$$

For $a, b > 0$, by a scaling argument as in [23], we define $\bar{\rho}(x) = a\rho(bx)$. Then it is not difficult to show that

$$\int \bar{\rho}dx = ab^{-3}\int \rho dx, \qquad E_{pot}(\bar{\rho}) = a^2 b^{-5} E_{pot}(\rho),$$

$$\int A(\bar{\rho})dx = b^{-3}\int A(a\rho)dx.$$

(a) Pick $a = b^3$ such that $\bar{\rho} \in \Gamma_M$ as well. We claim that

$$E(\bar{\rho}) < 0, \quad \text{for sufficiently small } b > 0.$$

Letting $0 < b < 1$, by (10), we have

$$\begin{aligned} E(\bar{\rho}) &= \int A(\bar{\rho}(x))dx + \int \bar{\rho}(x)J(r(x))dx - \frac{1}{2}\int \bar{\rho}(x)B\bar{\rho}(x)dx \\ &= \int A(\bar{\rho}(x))dx + b^3 \int_G \rho(bx)J(r(x))dx \\ &\qquad - b\Big(\int \rho(x)J(r(x))dx + 3\int P(\rho(x))dx\Big) \\ &= \int A(\bar{\rho}(x))dx - 3b\int P(\rho(x))dx + b\int_G \Big(b^2\rho(bx) - \rho(x)\Big)J(r(x))dx. \end{aligned}$$

Condition (5) implies that

$$\begin{aligned} \int A(\bar{\rho}(x))dx &= \frac{1}{b^3}\int A(b^3\rho(x))dx \\ &\leq mc_0 b^{3/m}\int_G \rho^{1+1/m}(x)dx \\ &= o(b), \quad \text{as} \quad b \to 0. \end{aligned}$$

By $(P1)$, we know that $\int P(\rho)dx > 0$. Let

$$f(b) = \int_G \Big(b^2\rho(bx) - \rho(x) \Big) J(r(x))dx,$$

then

$$f(0) = -\int_G \rho(x)J(r(x))dx.$$

If $f(0) = 0$, then $\mathbf{v}(x) \equiv 0$. This implies that $f(b) \equiv 0$. Otherwise, $f(b) < 0$ for b sufficiently small. So that when b is small enough, our claim is proven. This means our assertion is established.

(b) Under $(\omega 1)$ and $(\omega 3)$, for $b > 1$, we have

$$\begin{aligned} \int \bar\rho J(r(x))dx &= a \int \rho(bx) \int_{r(x)}^{\infty} \omega^2(s(x))sdsdx \\ &\geq ab^{-2\alpha} \int \rho(bx) \int_{r(bx)}^{\infty} \omega^2(s(bx))sdsdx \\ &= ab^{-(5+2\alpha)} \int \rho(x)J(r(x))dx. \end{aligned}$$

Therefore, we choose $a = 1$ and $b = (M/\bar M)^{1/3} > 1$, for $\bar M \leq M < M_c$ it follows that

$$\begin{aligned} E(\bar\rho) &= \int A(\bar\rho(x))dx + \int \bar\rho(x)J(r(x))dx + E_{pot}(\bar\rho) \\ &\geq b^{-3} \int A(\rho(x))dx + b^{-(5+2\alpha)} \int \rho(x)J(r(x))dx + b^{-5}E_{pot}(\rho) \\ &\geq b^{-(5+2\alpha)} \Big(\int A(\rho(x))dx + \int \rho(x)J(r(x))dx + E_{pot}(\rho) \Big) \\ &= (\bar M/M)^{(5+2\alpha)/3} E(\rho). \end{aligned}$$

Since the map $\rho \to \bar\rho$ is one-to-one and onto between Γ_M and $\Gamma_{\bar M}$, thus we prove Part (b). □

Remark 3. We notice that $(\omega 1)$ and $(\omega 3)$ is weaker than $(\omega 1)$ and $(\omega 2)$. Consider $w(r) = r^{-l}$ for $l > 1$ and $r > r_0 > 0$. By a simple calculation, we see that in order to make $\omega(r)$ satisfy the the condition $(\omega 2)$,

$$rJ(r) = r\int_r^{\infty} sw^2(s)ds = r\int_r^{\infty} \frac{ds}{s^{2l-1}} = \frac{r^{3-2l}}{2l-2} \to 0, \text{ as } r \to \infty,$$

it has to require

$$l > \frac{3}{2},$$

while $l > 1$ suffices to make our assumption $(\omega 1)$ and $(\omega 3)$ hold.

Actually, the monotonicity of angular velocity is common and natural in astrophysics and is related to some stability criteria.

Then recalling Steiner symmetrization [13], we denote $\check\rho$ as the symmetric rearrangement of ρ with respect to z. By properties of the rearrangement, the integrals

$$\int A(\rho(x))dx, \qquad \int \rho(x)J(r(x))dx$$

do not change, while the potential term

$$\iint \frac{\rho(x)\rho(y)}{|x-y|}dxdy$$

increase. Thus,

$$E(\check{\rho}) \leq E(\rho).$$

We next consider the minimizing sequence $\{\rho_i\} \subset \Gamma_M$ of the energy functional $E(\rho)$ and assume they have been rearranged, so that every ρ_i is even with respect to $z = 0$ and decreasing in $z \geq 0$.

We here list some Lemmas in the following. Their proof can be found in [15] [23] and references therein.

Lemma 4.4. *([15]) Under ($\omega 2$), any minimizing sequence of $E(\rho)$ in Γ_M ($M < M_c$) is bounded in $L^\gamma(\mathbf{R^3})$ ($\gamma \geq 4/3$) and therefore has a subsequence which converges weakly in $L^\gamma(\mathbf{R^3})$.*

Lemma 4.5. *([23]) Let $\rho \in \Gamma_M$ ($M < M_c$). Then for $R > 1$,*

$$\sup_{a\in\mathbf{R^3}} \int_{a+B_R} \rho(x)dx \geq \frac{1}{RM}\left(-2E_{pot}(\rho) - \frac{M^2}{R} - \frac{C\|\rho\|_\gamma^2}{R^{5-6/\gamma}}\right),$$

where $B_R = \{y \in \mathbf{R^3} : |y| < R\}$.

Lemma 4.6. *([15]) Under ($\omega 2$), let $\{\rho_i\} \subset \Gamma_M$ be a minimizing sequence of $E(\rho)$ in Γ_M ($M < M_c$). Then there exist positive constants $\delta_0 = \delta_0(M)$, $R_0 = R_0(M)$, and $i_0 \in \mathbf{N}$ such that*

$$\int_{B_R} \rho_i(x)dx \geq \delta_0, \quad \text{for } i > i_0, \ \ R > R_0.$$

Lemma 4.7. *([23]) Let $\{\rho_i\} \subset L^\gamma(\mathbf{R^3})$ ($\gamma \geq 4/3$) be bounded and*

$$\rho_i \rightharpoonup \tilde{\rho} \text{ weakly in } L^\gamma(\mathbf{R^3}).$$

(a) For any domain $\Omega \subset \mathbf{R^3}$,

$$\nabla B(\chi_\Omega \rho_i) \to \nabla B(\chi_\Omega \tilde{\rho}) \text{ strongly in } L^2(\mathbf{R^3}),$$

where χ_Ω is the indicator function on set Ω.

(b) If in addition $\{\rho_i\}$ is bounded in $L^1(\mathbf{R^3})$, $\tilde{\rho} \in L^1(\mathbf{R^3})$, and for any $\epsilon > 0$ there exist $R > 0$ and $i_0 \in \mathbf{N}$ such that

$$\int_{|x|\geq R} \rho_i(x)dx < \epsilon, \qquad i \geq i_0,$$

then

$$\nabla B\rho_i \to \nabla B\tilde{\rho} \text{ strongly in } L^2(\mathbf{R^3}).$$

Next, for the completeness of this paper, we sketch out the proof of Theorem 2.1.

Proof of Theorem 2.1. Fix $0 < M < M_c$. By Lemma 4.2 and 4.3 (a), we have proved Theorem 2.1 (a). It is sufficient to show Theorem 2.1 (b).

Split $\rho \in \Gamma_M$ into three different parts as in [23],

$$\rho = \rho\chi_{_{B^{(1)}}} + \rho\chi_{_{B^{(2)}}} + \rho\chi_{_{B^{(3)}}} =: \rho^{(1)} + \rho^{(2)} + \rho^{(3)},$$

where $\chi_{_{B^{(j)}}}$ is the indicator function on sets $B^{(j)} (j = 1, 2, 3)$,

$$B^{(1)} = \{x \in \mathbf{R^3} : |x| \leq R_1\}, \quad B^{(2)} = \{x \in \mathbf{R^3} : R_1 < |x| \leq R_2\},$$

$$B^{(3)} = \{x \in \mathbf{R^3} : |x| > R_2\},$$

for $0 < R_1 < R_2$ (to be determined later). Thus we have

$$E(\rho) = E(\rho^{(1)}) + E(\rho^{(2)}) + E(\rho^{(3)}) - P_{12} - P_{13} - P_{23},$$

where

$$P_{kl} = \int\int \frac{\rho^{(k)}(x)\rho^{(l)}(y)}{|x-y|}dxdy, \qquad 1 \leq k < l \leq 3.$$

If we choose $R_2 > 2R_1$, then (see [23])

$$P_{13} \leq \frac{C}{R_2}, \qquad P_{12} + P_{23} \leq C\|\rho\|_\gamma^{\gamma/6(\gamma-1)}\|\nabla B\rho^{(2)}\|_2.$$

Denote

$$M^{(j)} = \int \rho^{(j)}(x)dx, \qquad j = 1, 2, 3.$$

Noticing that $e_M < 0$ and using Lemma 4.3 (b), we find

$$\begin{aligned}
& e_M - E(\rho^{(1)}) - E(\rho^{(2)}) - E(\rho^{(3)}) \\
& \leq e_M \left(1 - \left(\frac{M^{(1)}}{M}\right)^{\frac{5+2\alpha}{3}} - \left(\frac{M^{(2)}}{M}\right)^{\frac{5+2\alpha}{3}} - \left(\frac{M^{(3)}}{M}\right)^{\frac{5+2\alpha}{3}}\right) \\
& \leq e_M \left(1 - \left(\frac{M^{(1)}}{M}\right)^{\frac{5}{3}} - \left(\frac{M^{(2)}}{M}\right)^{\frac{5}{3}} - \left(\frac{M^{(3)}}{M}\right)^{\frac{5}{3}}\right) \\
& \leq \frac{10e_M}{9M^2}\left(M^{(1)}M^{(3)} + M^{(1)}M^{(2)} + M^{(2)}M^{(3)}\right) \\
& \leq \frac{10e_M}{9M^2} \cdot M^{(1)}M^{(3)}.
\end{aligned}$$

Here we have used the elementary formula

$$1 - \left(a^{5/3} + b^{5/3} + c^{5/3}\right) \geq \frac{10}{9}(ab + bc + ac)$$

for $0 < a, b, c < 1$, $a + b + c = 1$. From these estimates above, it follows

$$\begin{aligned}
e_M - E(\rho) &= e_M - E(\rho^{(1)}) - E(\rho^{(2)}) - E(\rho^{(3)}) + P_{12} + P_{13} + P_{23} \\
&\leq Ce_M M^{(1)}M^{(3)} + C\left(R_2^{-1} + \|\rho\|_\gamma^{\gamma/6(\gamma-1)}\|\nabla B\rho^{(2)}\|_2\right). \qquad (11)
\end{aligned}$$

Let $\{\rho_i\} \subset \Gamma_M$ be a minimizing sequence of $E(\rho)$. By Lemma 4.4, $\{\rho_i\}$ is bounded in $L^\gamma(\mathbf{R^3})$. So there exists a subsequence, still denoted by $\{\rho_i\}$, such that

$$\rho_i \rightharpoonup \tilde{\rho}, \quad \text{weakly in } L^\gamma(\mathbf{R^3}).$$

We now choose $R_1 > R_0$, by Lemma 4.6, such that $M_i^{(j)} \geq \delta_0$ for i large. By (11),

$$\begin{aligned}
-Ce_M\delta_0 M_i^{(3)} &\leq \frac{C}{R_2} + C\|\nabla B\rho_i^{(2)}\|_2 + E(\rho_i) - e_M \\
&\leq \frac{C}{R_2} + C\|\nabla B\tilde{\rho}^{(2)}\|_2 + C\|\nabla B\rho_i^{(2)} - \nabla B\tilde{\rho}^{(2)}\|_2 + E(\rho_i) - e_M. \quad (12)
\end{aligned}$$

where

$$\rho_i^{(j)} = \rho_i\chi_{B^{(j)}} \quad M_i^{(j)} = \int \rho_i^{(j)}(x)dx, \quad i = 1, 2, \cdots, j = 1, 2, 3,$$

referring to the splitting.

Given any $\epsilon > 0$, by the same argument as [23], we can increase $R_1 > R_0$ such that the mass of $\tilde{\rho}$ mainly concentrates in the ball $B^{(1)}$, so the potential energy generated by $\tilde{\rho}_2$, the second term on the right-hand side of (12), can be sufficiently small, say less than $\epsilon/4$. Next, we take $R_2 > 2R_1$ such that the first term of (12) is also small and less than $\epsilon/4$. Now, R_1 and R_2 are fixed. By Lemma 4.7 (a), the third term in (12) converges to zero as $i \to \infty$. And the remainder $|E(\rho_i) - e_M|$ can be small if i is large. Therefore, for i sufficiently large, we can make

$$M_i^{(3)} = \int_{B^{(3)}} \rho_i(x)dx < \epsilon,$$

then

$$\int_{B_{R_2}} \rho_i(x)dx = M - M_i^{(3)} \geq M - \epsilon.$$

In virtue of Lemma 4.7 (b), we have

$$\|\nabla B\rho_i - \nabla B\tilde{\rho}\|_2 \to 0, \quad \text{as} \quad i \to \infty.$$

Obviously, $\tilde{\rho} \geq 0$ a.e. By the weak convergence we have that for any $\epsilon > 0$,

$$M \geq \int_{B_R} \tilde{\rho}(x)dx \geq M - \epsilon, \quad \text{if } R \text{ large enough},$$

which in particular implies that $\tilde{\rho} \in L^1(\mathbf{R^3})$ with $\int \tilde{\rho}dx = M$. Since $\rho \mapsto \int A(\rho)dx$ is convex, by a standard argument, we have

$$E(\tilde{\rho}) = e_M.$$

Therefore, $\tilde{\rho}$ is a minimizer of $E(\rho)$. □

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REFERENCES

[1] J. F. G. Auchmuty, *Existence of equilibrium figures*, Arch. Rational Mech. Anal., **65** (1977), 249–261.

[2] J. F. G. Auchmuty, *The global branching of rotating stars*, Arch. Rational Mech. Anal., **114** (1991), 179–194.

[3] J. F. G. Auchmuty and R. Beals, *Variation solutions of some non-linear free boundary problems*, Arch. Rational Mech. Anal., **43** (1971), 255–271.

[4] J. F. G. Auchmuty and R. Beals, *Models of rotating stars*, Astrophysical J., **165** (1971), 79–82.

[5] L. A. Caffarelli and A. Friedman, *The shape of axisymmetric rotating fluid*, J. Funct. Anal., **35** (1980), 109–142.

[6] S. Chandrasekhar, "Introduction to the Stellar Structure," University of Chicago Press, 1939

[7] S. Chandrasekhar, "Ellipsoidal Figures of Equilibrium," Dover Publication Inc., New York, 1987

[8] S. Chanillo and Y. Y. Li, *On diameters of uniformly rotating stars*, Comm. Math. Phys., **166** (1994), 417–430.

[9] A. Friedman and B. Turkington, *Asymptotic estimates for an axisymmetric rotating fluid*, J. Fun. Anal., **37** (1980), 136–163.

[10] A. Friedman and B. Turkington, *The oblateness of an axisymmetric rotating fluid*, Indiana Univ. Math. J., **29** (1980), 777–792.

[11] A. Friedman and B. Turkington, *Existence and dimensions of a rotating white dwarf*, J. Diff. Equations, **42** (1981), 414–437.
[12] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," $2^n d$ edition, Springer, 1983.
[13] G. H. Hardy, J. E. Littlewood and G. Polya, "Inequality," Cambridge Univ. Press, Cambridge, 1934.
[14] J. Jang, *Nonlinear instability in gravitational Euler-Poisson system for* $\gamma = \frac{6}{5}$, Arch. Rational Mech. Anal., **188** (2008), 265–307.
[15] H. G. Li and J. G. Bao, *Existence of the rotating stars with prescribed angular velocity law*, to appear in Houston J. Math.
[16] Y. Y. Li, *On uniformly rotating stars*, Arch. Rational Mech. Anal., **115** (1991), 367–393.
[17] P. L. Lions, *The concentration-compactness principle in the calculus of variation, The locally case, part I*, Ann. I. H. Anal. Nonli., **1** (1984), 109–145.
[18] P. L. Lions, *Minimization problems in* $L^1(\mathbf{R}^3)$, J. Funct. Anal., **41** (1981), 236–275.
[19] T. Luo and J. Smoller, *Rotating fluids with self-gravitation in bounded domains*, Arch. Rational Mech. Anal., **173** (2004), 345–377.
[20] T. Luo and J. Smoller, *Nonlinear dynamical stability of Newtonian rotating white dwarfs and supermassive stars*, Comm. Math. Physics, **284** (2008), 425–457.
[21] T. Luo and J. Smoller, *Existence and nonlinear stability of rotating star solutions of the compressible Euler-Poisson equations*, Arch. Rational Mech. Anal., **191** (2009), 447–496.
[22] R. J. McCann, *Stable rotating binary stars and fluid in a tube*, Houston J. Math., **32** (2006), 603–632.
[23] G. Rein, *Reduction and a concentration-compactness principle for energy-Casimir functionals*, SIAM J. Math. Anal., **33** (2001), 896–912.
[24] S. H. Shapiro and S. A. Teukolsky, "Black Holes, White Dwarfs, and Neutron Stars," WILEY-VCH, New York, 1983.
[25] J. L. Tassoul, "Theory of Rotating Stars," Princeton Univ. Press, New Jersey, 1978.
[26] S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity," John Wiley and Sons, Inc. New York, 1972.

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