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Asymptotic expansion at infinity of solutions of Monge–Ampère type equations[☆]

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ABSTRACT

We obtain a quantitative expansion at infinity of solutions for a kind of Monge–Ampère type equations that origin from mean curvature equations of Lagrangian graph $(x, Du(x))$ and refine the previous study on zero mean curvature equations and the Monge–Ampère equations.

1. Introduction

In 2018, Wang–Huang–Bao [31] studied the second boundary value problem of Lagrangian mean curvature equation of gradient graph $(x, Du(x))$ in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$, where Du denotes the gradient of scalar function u and

$$g_\tau = \sin\tau\delta_0 + \cos\tau g_0, \quad \tau \in \left[0, \frac{\pi}{2}\right]$$

is the linearly combined metric of standard Euclidean metric

$$\delta_0 = \sum_{i=1}^n dx_i \otimes dx_i + \sum_{j=1}^n dy_j \otimes dy_j,$$

with the pseudo-Euclidean metric

$$g_0 = \sum_{i=1}^n dx_i \otimes dy_i + \sum_{j=1}^n dy_j \otimes dx_j.$$

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They proved that for domain $\Omega \subset \mathbb{R}^n$, if $u \in C^2(\Omega)$ is a solution of

$$F_\tau\left(\lambda\left(D^2 u\right)\right) = f(x), \quad x \in \Omega, \tag{1}$$

then $Df(x)$ is the mean curvature of gradient graph $(x, Du(x))$ in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$. Previously, Warren [32] proved that when $f(x) \equiv C_0$ for some constants C_0, the mean curvature of $(x, Du(x))$ is zero. In (1), $f(x)$ is a scalar function with sufficient regularity, $\lambda\left(D^2 u\right) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are n eigenvalues of Hessian matrix $D^2 u$ and

$$F_\tau(\lambda) := \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^n \ln \lambda_i, & \tau = 0, \\ \dfrac{\sqrt{a^2+1}}{2b}\displaystyle\sum_{i=1}^n \ln \dfrac{\lambda_i + a - b}{\lambda_i + a + b}, & 0 < \tau < \frac{\pi}{4}, \\ -\sqrt{2}\displaystyle\sum_{i=1}^n \dfrac{1}{1+\lambda_i}, & \tau = \frac{\pi}{4}, \\ \dfrac{\sqrt{a^2+1}}{b}\displaystyle\sum_{i=1}^n \arctan \dfrac{\lambda_i + a - b}{\lambda_i + a + b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\ \displaystyle\sum_{i=1}^n \arctan \lambda_i, & \tau = \frac{\pi}{2}, \end{cases}$$

$a = \cot\tau, b = \sqrt{\left|\cot^2\tau - 1\right|}$.

If $\tau = 0$, then (1) becomes the Monge–Ampère type equation

$$\det D^2 u = e^{nf(x)} \quad \text{in } \mathbb{R}^n. \tag{2}$$

For $f(x)$ being a constant C_0, there are Bernstein-type results by Jörgens [21], Calabi [8] and Pogorelov [29], which state that any convex classical solution of (2) must be a quadratic polynomial. See Cheng–Yau [9], Caffarelli [3], Jost–Xin [22] and Li–Xu–Simon–Jia [26] for different proofs and extensions. For $f(x) - C_0$ having compact support, there are exterior Bernstein-type results by Ferrer–Martínez–Milán [13] for $n = 2$ and Caffarelli–Li [6], which state that any convex solution must be asymptotic to quadratic polynomials at infinity (for $n = 2$ we need additional ln-term). For $f(x) - C_0$ vanishing at infinity, there are similar asymptotic results by Bao–Li–Zhang [2]. For $f(x) - C_0$ being a periodic function or asymptotically periodic function, there are classification results by Caffarelli–Li [7], Teixeira–Zhang [30] etc.

If $\tau = \frac{\pi}{2}$, then (1) becomes the Lagrangian mean curvature equation

$$\sum_{i=1}^n \arctan \lambda_i\left(D^2 u\right) = f(x) \quad \text{in } \mathbb{R}^n. \tag{3}$$

For $f(x)$ being a constant C_0, there are Bernstein-type results by Yuan [33,34], which state that any classical solution of (3) and

$$D^2 u \geq \begin{cases} -KI, & n \leq 4, \\ -(\frac{1}{\sqrt{3}} + \epsilon(n))I, & n \geq 5, \end{cases} \quad \text{or} \quad C_0 > \frac{n-2}{2}\pi, \tag{4}$$

must be a quadratic polynomial, where I denote the unit $n \times n$ matrix, K is a constant and $\epsilon(n)$ is a small dimensional constant. For $f(x) - C_0$ having compact support, there is an exterior Bernstein-type result by Li–Li–Yuan [25], which states that any classical solution of (3) with (4) must be asymptotic to quadratic polynomials at infinity (for $n = 2$ we need additional ln-term).

For general $\tau \in [0, \frac{\pi}{2}]$, for $f(x)$ being a constant C_0, there are Bernstein-type results under suitable semi-convex conditions by Warren [32], which is based on the results of Jörgens [21]–Calabi [8]–Pogorelov [29], Flanders [14] and Yuan [33,34]. For $f(x) - C_0$ having compact support, there are exterior Bernstein-type results when $n \geq 3$ in our earlier work [27], which state that any classical solution of (1) with suitable semi-convex conditions must be asymptotic to quadratic polynomial at infinity. There are also higher order expansions at infinity, which give the precise gap between exterior maximal/minimal gradient graph and the entire case. Such higher order expansions problem was considered for the Yamabe equation and σ_k-Yamabe equation by Han–Li–Li [17], which refines the study by Caffarelli–Gidas–Spruck [5], Korevaar–Mazzeo–Pacard–Schoen [23], Han–Li–Teixeira [18] etc.

In this paper, we obtain asymptotic expansion at infinity of classical solutions of

$$F_\tau(\lambda(D^2u)) = f(x) \quad \text{in } \mathbb{R}^n, \tag{5}$$

where $n \geq 3$, $\tau \in [0, \frac{\pi}{4}]$ and $f(x)$ is a perturbation of $f(\infty) := \lim_{x\to\infty} f(x)$ at infinity. This partially refines previous study [2,6,19,25,27] etc.

Our first result considers asymptotic behavior and higher order expansions of general classical solution of (5). Hereinafter, we let $\varphi = O_m(|x|^{-k_1}(\ln|x|)^{k_2})$ with $m \in \mathbb{N}, k_1, k_2 \geq 0$ denote

$$|D^k\varphi| = O(|x|^{-k_1-k}(\ln|x|)^{k_2}) \quad \text{as } |x| \to +\infty$$

for all $0 \leq k \leq m$. Let x^T denote the transpose of vector $x \in \mathbb{R}^n$, $\mathtt{Sym}(n)$ denote the set of symmetric $n\times n$ matrix, $\mathcal{H}_k^n$ denote the k-order spherical harmonic function space in $\mathbb{R}^n$, $DF_\tau(\lambda(A))$ denote the matrix with elements being value of partial derivative of $F_\tau(\lambda(M))$ w.r.t M_{ij} variable at matrix A and $[k]$ denote the largest natural number no larger than k.

Theorem 1.1. *Let $u \in C^2(\mathbb{R}^n)$ be a classical solution of (5), where $f \in C^0(\mathbb{R}^n)$ is C^m outside a compact subset of $\mathbb{R}^n$ and satisfies*

$$\limsup_{|x|\to\infty} |x|^{\zeta+k}|D^k(f(x) - f(\infty))| < \infty, \quad \forall\ k = 0, 1, 2, \ldots, m \tag{6}$$

for some $\zeta > 2$ and $m \geq 2$. Suppose either of the following holds

(1) *$D^2u > 0$ for $\tau = 0$;*

(2)
$$u(x) \leq C(1+|x|^2) \quad \textit{and} \quad D^2u > (-a+b)I, \quad \forall\ x \in \mathbb{R}^n \tag{7}$$
for some constant C, for $\tau \in (0, \frac{\pi}{4})$;

(3)
$$u(x) \leq C(1+|x|^2) \quad \textit{and} \quad D^2u > -I, \quad \forall\ x \in \mathbb{R}^n \tag{8}$$
for some constant C, for $\tau = \frac{\pi}{4}$.

Then there exist $c \in \mathbb{R}, b \in \mathbb{R}^n$ and $A \in \mathtt{Sym}(n)$ with $F_\tau(\lambda(A)) = f(\infty)$ such that

$$u(x) - \left(\frac{1}{2}x^TAx + bx + c\right) = \begin{cases} O_{m+1}(|x|^{2-\min\{n,\zeta\}}), & \zeta \neq n, \\ O_{m+1}(|x|^{2-n}(\ln|x|)), & \zeta = n, \end{cases} \tag{9}$$

as $|x| \to +\infty$.

Remark 1.2. The matrix A in Theorem 1.1 also satisfies $A > 0$ in case (1), $A > (-a+b)I$ in case (2) and $A > -I$ in case (3) respectively.

Remark 1.3. Notice that in condition (6), we only require $m \geq 2$, which is an improvement to the results for $m \geq 3$ by Bao–Li–Zhang [2]. It would be interesting to determine sharp lower bounds for m in Theorem 1.1. There has been an example in [2] that shows the decay rate assumption $\zeta > 2$ in (6) is optimal.

We also have the following higher order expansions for $\zeta > n$, which gives a finer characteristic of the error term in (9).

Theorem 1.4. *Under conditions of Theorem 1.1, there exist $c_0 \in \mathbb{R}$, $c_k(\theta) \in \mathcal{H}_k^n$ with $k = 1, 2, \ldots, n - [2n-\zeta] - 1$ such that*

$$\begin{aligned}
& u(x) - \left(\frac{1}{2}x^T Ax + bx + c\right) \\
- \; & c_0(x^T(DF_\tau(\lambda(A)))^{-1}x)^{\frac{2-n}{2}} - \sum_{k=1}^{n-[2n-\zeta]-1} c_k(\theta)\left(x^T(DF_\tau(\lambda(A)))^{-1}x\right)^{\frac{2-n-k}{2}} \\
= \; & \begin{cases} O_m(|x|^{2-\min\{2n,\zeta\}}), & \min\{2n,\zeta\} - n \notin \mathbb{N}, \\ O_m(|x|^{2-\min\{2n,\zeta\}}(\ln|x|)), & \min\{2n,\zeta\} - n \in \mathbb{N}, \end{cases}
\end{aligned} \tag{10}$$

as $|x| \to +\infty$, where

$$\theta = \frac{(DF_\tau(\lambda(A)))^{-\frac{1}{2}}x}{(x^T(DF_\tau(\lambda(A)))^{-1}x)^{\frac{1}{2}}}.$$

Remark 1.5. By computing $F_\tau(\lambda(D^2u))$ of radially symmetric u of form $\frac{C_1}{2}|x|^2 + C_2|x|^{-k}$, we find expansions (9) and (10) are optimal for all $\zeta > 2$ in the sense that the series of k does not exist or cannot be taken up to $n - [2n-\zeta]$ when $2 < \zeta \le n$ or $\zeta > n$ respectively since $c_{n-[2n-\zeta]}$ does not belong to space $\mathcal{H}_n^{n-[2n-\zeta]}$ in general.

The paper is organized as follows. In Section 2 we prove that the Hessian matrix D^2u converges to some constant matrix $A \in \mathtt{Sym}(n)$ at infinity, in order to make preparation for proving Theorem 1.1. In the next two sections we give the proofs of Theorems 1.1 and 1.4 respectively based on the detailed analysis of the solutions of non-homogeneous linearized equations.

Hereinafter, we let $B_r(x)$ denote a ball centered at $x \in \mathbb{R}^n$ with radius r. Especially for $x = 0$, we let $B_r := B_r(0)$. For any open subset $\Omega \subset \mathbb{R}^n$, we let $\overline{\Omega}$ denote the closure of Ω and Ω^c denote the complement of Ω in $\mathbb{R}^n$.

2. Convergence of Hessian at infinity

In this section, we study the asymptotic behavior at infinity of Hessian matrix of classical solutions of (5). We prove a weaker convergence than (9) in Theorem 1.1 and D^2u has bounded C^α norm for some $0 < \alpha < 1$ under a weaker assumption on f. By interior regularity as Lemma 17.16 of [16] and extension theorem as Theorem 6.10 of [12], we may change the value of u, f on a compact subset of $\mathbb{R}^n$ and prove only for $u \in C^{2,\alpha}(\mathbb{R}^n)$ and $f \in C^\alpha(\mathbb{R}^n)$.

Theorem 2.1. *Let u be as in Theorem 1.1, $f \in C^\alpha(\mathbb{R}^n)$ for some $0 < \alpha < 1$ and satisfy*

$$\limsup_{|x|\to\infty}\left(|x|^\zeta|f(x) - f(\infty)| + |x|^{\alpha+\zeta'}[f]_{C^\alpha(\overline{B_{\frac{|x|}{2}}(x)})}\right) < \infty \tag{11}$$

(1) *with some $\zeta > 1, \zeta' > 0$ for $\tau = 0$;*
(2) *with some $\zeta > 1, \zeta' > 0$ for $\tau \in (0, \frac{\pi}{4})$;*
(3) *with some $\zeta > 0, \zeta' > 0$ for $\tau = \frac{\pi}{4}$.*

Then there exist $\epsilon > 0, A \in \mathtt{Sym}(n)$ *with* $F_\tau(\lambda(A)) = f(\infty)$ *and* $C > 0$ *such that*

$$\|D^2u\|_{C^\alpha(\mathbb{R}^n)} \le C, \quad \text{and} \quad \left|D^2u(x) - A\right| \le \frac{C}{|x|^\epsilon}, \quad \forall\ |x| \ge 1.$$

The proof is separated into three subsections according to three different range of τ.

2.1. $\tau = 0$ case

In $\tau = 0$ case, (5) becomes the Monge–Ampère equation (2).

Theorem 2.2. *Let* $u \in C^0(\mathbb{R}^n)$ *be a convex viscosity solution of*

$$\det D^2u = \psi(x) \quad \text{in } \mathbb{R}^n \tag{12}$$

with $u(0) = \min_{\mathbb{R}^n} u = 0$, *where* $0 < \psi \in C^0(\mathbb{R}^n)$ *and*

$$\psi^{\frac{1}{n}} - 1 \in L^n(\mathbb{R}^n).$$

Then there exists a linear transform T *satisfying* $\det T = 1$ *such that* $v := u \circ T$ *satisfies*

$$\left|v - \frac{1}{2}|x|^2\right| \le C|x|^{2-\varepsilon}, \quad \forall\ |x| \ge 1.$$

for some $C > 0$ *and* $\varepsilon > 0$.

Theorem 2.2 can be found in the proof of Theorem 1.2 in [2], which is based on the level set method by Caffarelli–Li [6].

Corollary 2.3. *Let* $u \in C^0(\mathbb{R}^n)$ *be a convex viscosity solution of* (5) *with* $f \in C^0(\mathbb{R}^n)$ *satisfies*

$$\limsup_{|x|\to\infty} |x|^\zeta |f(x) - f(\infty)| < \infty$$

for some $\zeta > 1$. *Then there exists a linear transform* T *satisfying* $\det T = 1$ *such that* $v := u \circ T$ *satisfies*

$$\left|v - \frac{\exp(f(\infty))}{2}|x|^2\right| \le C|x|^{2-\varepsilon}, \quad \forall\ |x| \ge 1 \tag{13}$$

for some $C > 0$ *and* $\varepsilon > 0$.

Proof. By a direct computation,

$$\widetilde{u}(x) := \frac{1}{\exp(f(\infty))}\left(u(x) - Du(0)x - u(0)\right)$$

is a convex viscosity solution of

$$\det D^2\widetilde{u} = e^{n(f(x)-f(\infty))} =: \widetilde{f}(x) \quad \text{in } \mathbb{R}^n.$$

By a direct computation, $|\widetilde{f}(x) - 1| \le C|x|^{-\zeta}$ for some $C > 0$ and

$$\int_{\mathbb{R}^n\setminus B_1} \left|(\widetilde{f}(x))^{\frac{1}{n}} - 1\right|^n dx \le C\int_{\mathbb{R}^n\setminus B_1} \left|\widetilde{f}(x) - 1\right|^n dx \le C\int_{\mathbb{R}^n\setminus B_1} |x|^{-\zeta n} dx < \infty.$$

The result follows immediately by applying Theorem 2.2 to $\widetilde{u}$. □

As a consequence, we have the following convergence of Hessian matrix for solutions of (12). The proof is similar to the one in Bao–Li–Zhang [2] and in Caffarelli–Li [6]. Since there are some differences from their proof, we provide the details here for reading simplicity.

Theorem 2.4. *Let $u \in C^0(\mathbb{R}^n)$ be a convex viscosity solution of (5), $f \in C^\alpha(\mathbb{R}^n)$ satisfy (11) for some $0 < \alpha < 1$, $\zeta > 1$ and $\zeta' > 0$. Then $u \in C^{2,\alpha}(\mathbb{R}^n)$,*

$$\|D^2u\|_{C^\alpha(\mathbb{R}^n)} \le C, \tag{14}$$

and

$$u - \left(\frac{1}{2}x^TAx + bx + c\right) = O_2(|x|^{2-\epsilon}) \tag{15}$$

as $|x| \to \infty$, where $\epsilon := \min\{\varepsilon, \zeta, \zeta'\}$, ε is the positive constant from Theorem 2.2, $A \in \mathtt{Sym}(n)$ with $\det A = \exp(nf(\infty))$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $C > 0$.

Proof. By Corollary 2.3, there exist a linear transform T, $\varepsilon > 0$ and $C > 0$ such that $v := u \circ T$ satisfies (13).

Step 1: We prove C^α boundedness of Hessian (14). Let

$$v_R(y) = \left(\frac{4}{R}\right)^2 v\left(x + \frac{R}{4}y\right), \quad |y| \le 2$$

for $|x| = R > 2$. By (13),

$$\|v_R\|_{C^0(\overline{B_2})} \le C$$

for some $C > 0$ for all $R \ge 2$. Then v_R satisfies

$$\det\left(D^2v_R(y)\right) = \exp\left(nf\left(x + \frac{R}{4}y\right)\right) =: f_R(y) \quad \text{in } B_2. \tag{16}$$

By a direct computation, there exists $C > 0$ uniform to x such that

$$\|f_R - \exp(nf(\infty))\|_{C^0(\overline{B_2})} \le CR^{-\zeta}$$

and for all $y_1, y_2 \in B_2$,

$$\frac{|f_R(y_1) - f_R(y_2)|}{|y_1 - y_2|^\alpha} = \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} \cdot (\frac{R}{4})^\alpha \le CR^{-\zeta'},$$

where $z_i := x + \frac{R}{4}y_i \in B_{\frac{|x|}{2}}(x)$. Applying the interior estimate by Caffarelli [3], Jian–Wang [20] on B_2, we have

$$\left\|D^2v_R\right\|_{C^\alpha(\overline{B_1})} \le C \tag{17}$$

and hence

$$\frac{1}{C}I \le D^2v_R \le CI \quad \text{in } B_1 \tag{18}$$

for some C independent of R. For any $|x| = R \ge 2$, we have

$$|D^2v(x)| = |D^2v_R(0)| \le \|D^2v_R\|_{C^0(\overline{B_1})} \le C. \tag{19}$$

For any $x_1, x_2 \in B_2^c$ with $0 < |x_2 - x_1| \le \frac{1}{4}|x_1|$, let $R := |x_1| > 2$, by (17),

$$\begin{aligned}\frac{\left|D^2v(x_1) - D^2v(x_2)\right|}{|x_1 - x_2|^\alpha} &= \frac{\left|D^2v_R(0) - D^2v_R\left(\frac{4(x_2-x_1)}{|x_1|}\right)\right|}{|x_1 - x_2|^\alpha}\\ &\le [D^2v_R]_{C^\alpha(\overline{B_1})} \cdot \left(\frac{4}{|x_1|}\right)^\alpha\\ &\le CR^{-\alpha}.\end{aligned}$$

For any $x_1, x_2 \in B_2^c$ with $|x_2 - x_1| \geq \frac{1}{4}|x_1|$, by (19),

$$\frac{|D^2 v(x_1) - D^2 v(x_2)|}{|x_1 - x_2|^{\alpha}} \leq 2^{\alpha} \cdot 2\|D^2 v\|_{C^0(\mathbb{R}^n)} \leq C.$$

Since the linear transform T from Theorem 2.2 is invertible, (14) follows immediately.

Step 2: We prove convergence speed at infinity (15). Let

$$w(x) := v(x) - \frac{\exp(f(\infty))}{2}|x|^2 \quad \text{and} \quad w_R(y) := \left(\frac{4}{R}\right)^2 w\left(x + \frac{R}{4}y\right), \quad |y| \leq 2$$

for $|x| = R \geq 2$. By (13) in Theorem 2.2,

$$\|w_R\|_{C^0(\overline{B_2})} \leq CR^{-\varepsilon}.$$

Applying Newton–Leibnitz formula between (16) and $\det(\exp(f(\infty))I) = \exp(nf(\infty))$,

$$\widetilde{a_{ij}}(y) D_{ij} w_R = f_R(y) - \exp(nf(\infty)) \quad \text{in } B_2,$$

where $\widetilde{a_{ij}}(y) = \int_0^1 D_{M_{ij}}(\det\left(I + tD^2 w_R(y)\right))dt$.

By (17) and (18), there exists constant C independent of $|x| = R > 2$ such that

$$\frac{I}{C} \leq \widetilde{a_{ij}} \leq CI \quad \text{in } B_1, \quad \|\widetilde{a_{ij}}\|_{C^{\alpha}(\overline{B_1})} \leq C.$$

By interior Schauder estimates, see for instance Theorem 6.2 of [16],

$$\begin{aligned} \|w_R\|_{C^{2,\alpha}\left(\overline{B_{\frac{1}{2}}}\right)} &\leq C\left(\|w_R\|_{C^0(\overline{B_1})} + \|f_R - \exp(nf(\infty))\|_{C^{\alpha}(\overline{B_1})}\right) \\ &\leq CR^{-\min\{\varepsilon,\zeta,\zeta'\}}. \end{aligned} \tag{20}$$

The result (15) follows immediately by scaling back. □

Remark 2.5. In the proof of Theorem 2.4, the interior Schauder estimates used in (20) can be replaced by the $W^{2,\infty}$ type estimates (see for instance Remark 1.3 of [11]),

$$\|w_R\|_{W^{2,\infty}(\overline{B_{\frac{1}{2}}})} \leq C\left(\|w_R\|_{C^0(\overline{B_1})} + \|f_R - \exp(nf(\infty))\|_{C^{\alpha}(\overline{B_1})}\right) \leq CR^{-\min\{\varepsilon,\zeta,\zeta'\}}.$$

Remark 2.6. The condition (11) in Theorem 2.4 holds if for some $C > 0$,

$$|x|^{\zeta}|f(x) - f(\infty)| + |x|^{1+\zeta'}|Df(x)| \leq C, \quad \forall\ |x| > 2. \tag{21}$$

Even if $f(x)$ is C^1, condition (11) is weaker than (21). For example, we consider $f(x) := e^{-|x|}\sin(e^{|x|})$. On the one hand, $Df(x)$ does not admit a limit at infinity, hence f does not satisfy condition (21). On the other hand, for any $|x| = R > 1$ and $z_1, z_2 \in B_{\frac{|x|}{2}}(x)$,

$$\begin{aligned} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{\alpha}} &\leq e^{-|z_2|}\frac{\left|\sin(e^{|z_1|}) - \sin(e^{|z_2|})\right|}{|z_1 - z_2|^{\alpha}} + \sin(e^{|z_1|})\frac{\left|e^{-|z_1|} - e^{-|z_2|}\right|}{|z_1 - z_2|^{\alpha}} \\ &\leq Ce^{-\frac{R}{2}} \cdot \frac{|z_1 - z_2|}{|z_1 - z_2|^{\alpha}} \\ &\leq Ce^{-\frac{R}{2}} \cdot R^{1-\alpha} \end{aligned}$$

for constant C independent of R. Hence f satisfies condition (11) for all $\alpha \in (0,1)$ and any $\zeta, \zeta' > 0$.

This finishes the proof of Theorem 2.1 for $\tau = 0$ case.

2.2. $\tau \in (0, \frac{\pi}{4})$ case

In this subsection, we deal with $\tau \in (0, \frac{\pi}{4})$ case by Legendre transform and the results in previous subsection.

Let $f \in C^{\alpha}(\mathbb{R}^n)$ satisfy (11) for some $0 < \alpha < 1, \zeta > 1, \zeta' > 0$ and $u \in C^{2,\alpha}(\mathbb{R}^n)$ be a classical solution of (5) satisfying (2). Let

$$\overline{u}(x) := u(x) + \frac{a+b}{2}|x|^2,$$

then

$$D^2\overline{u} = D^2u + (a+b)I > 2bI \quad \text{in } \mathbb{R}^n. \tag{22}$$

Let $(\widetilde{x}, v)$ be the Legendre transform of $(x, \overline{u})$, i.e.,

$$\begin{cases} \widetilde{x} := D\overline{u}(x), \\ Dv(\widetilde{x}) := x, \end{cases} \tag{23}$$

and we have

$$D^2v(\widetilde{x}) = \left(D^2\overline{u}(x)\right)^{-1} = (D^2u(x) + (a+b)I)^{-1} < \frac{1}{2b}I.$$

Let

$$\bar{v}(\widetilde{x}) := \frac{1}{2}|\widetilde{x}|^2 - 2bv(\widetilde{x}). \tag{24}$$

By a direct computation, $D\bar{u}(\mathbb{R}^n) = \mathbb{R}^n$ and

$$\widetilde{\lambda}_i\left(D^2\bar{v}\right) = 1 - 2b \cdot \frac{1}{\lambda_i + a + b} = \frac{\lambda_i + a - b}{\lambda_i + a + b} \in (0,1). \tag{25}$$

Thus $\bar{v}(\widetilde{x})$ satisfies the following Monge–Ampère type equation

$$\det D^2\bar{v} = \exp\left\{\frac{2b}{\sqrt{a^2+1}} f\left(\frac{1}{2b}(\widetilde{x} - D\bar{v}(\widetilde{x}))\right)\right\} =: g(\widetilde{x}) \quad \text{in } \mathbb{R}^n. \tag{26}$$

Step 1: There exists $C_0 > 1$ such that

$$\frac{1}{C_0}|x| \le |\widetilde{x}| \le C_0|x|, \quad \forall\ |x| > 1. \tag{27}$$

We prove the two inequalities in (27) separately.

By the definition of $\widetilde{x} = D\bar{u}(x)$ and (22),

$$|\widetilde{x} - \widetilde{0}| = |D\bar{u}(x) - D\bar{u}(0)| > 2b|x|.$$

Hence by triangle inequality,

$$|\widetilde{x}| \ge -|\widetilde{0}| + |\widetilde{x} - \widetilde{0}| > -|\widetilde{0}| + 2b|x|, \tag{28}$$

and the first inequality of (27) follows immediately.

By the quadratic growth condition in (7), we prove the linear growth result of $Du(x)$. In fact, for any $|x| \ge 1$, let $e := \frac{Du(x)}{|Du(x)|} \in \partial B_1$. By Newton–Leibnitz formula and (7)

$$\begin{aligned} u(x + |x|e) &= u(x) + \int_0^{|x|} e \cdot Du(x+se)\mathtt{d}s \\ &= u(x) + \int_0^{|x|}\int_0^s e \cdot D^2u(x+te)\cdot e\mathtt{d}t\mathtt{d}s + \int_0^{|x|} e \cdot Du(x)\mathtt{d}s \\ &\ge u(x) + \frac{(-a+b)}{2}|x|^2 + |Du(x)| \cdot |x|. \end{aligned} \tag{29}$$

Furthermore by (7), there exists $C>0$ independent of $|x|\geq 1$ such that

$$|Du(x)|\leq \frac{1}{|x|}\left(C(1+|(x+|x|e)|^2)+C(1+|x|^2)+\frac{a-b}{2}|x|^2\right)\leq C(1+|x|).$$

Hence there exists $C>0$ such that

$$|Du(x)|\leq C(1+|x|),\quad \forall\, x\in\mathbb{R}^n. \tag{30}$$

By (30), there exists $C>0$ such that

$$|\widetilde{x}|=|Du(x)+(a+b)x|\leq |Du(x)|+(a+b)|x|\leq C(|x|+1).$$

The second inequality of (27) follows immediately.

Now we study Eq. (26) by applying Theorem 2.4 and Remark 2.6, which require a knowledge on the asymptotic behavior of $g(\widetilde{x})$.

Step 2: $g(\widetilde{x})$ satisfies condition (11). By the equivalence (27),

$$\lim_{\widetilde{x}\to\infty} g(\widetilde{x})=\exp\left\{\frac{2b}{\sqrt{a^2+1}}f(\infty)\right\}=: g(\infty)\in(0,1].$$

By a direct computation,

$$\begin{aligned}
&|\widetilde{x}|^{\zeta}|g(\widetilde{x})-g(\infty)|\\
=\;& e^{\frac{2b}{\sqrt{a^2+1}}f(\infty)}\frac{|\widetilde{x}|^{\zeta}}{\left|\frac{\widetilde{x}-D\bar{v}(\widetilde{x})}{2b}\right|^{\zeta}}\cdot\left|\frac{\widetilde{x}-D\bar{v}(\widetilde{x})}{2b}\right|^{\zeta}\cdot\left|e^{\frac{2b}{\sqrt{a^2+1}}(f(\frac{\widetilde{x}-D\bar{v}(\widetilde{x})}{2b})-f(\infty))}-1\right|\\
\leq\;& C|x|^{\zeta}\left|e^{\frac{2b}{\sqrt{a^2+1}}(f(x)-f(\infty))}-1\right|\\
\leq\;& C|x|^{\zeta}\,|f(x)-f(\infty)|<C.
\end{aligned}$$

For any $\widetilde{y},\widetilde{z}\in B_{\frac{|\widetilde{x}|}{2}\cdot 2b}(\widetilde{x}),\widetilde{y}\neq\widetilde{z}$ with $|\widetilde{x}|>C_0$, by (22) we have

$$y,z\in B_{\frac{|x|}{2}}(x),\quad |\widetilde{y}-\widetilde{z}|\geq 2b|y-z|>0\quad\text{and}\quad y\neq z.$$

Thus by condition (11),

$$\frac{|g(\widetilde{y})-g(\widetilde{z})|}{|\widetilde{y}-\widetilde{z}|^{\alpha}}\leq (2b)^{-\alpha}\frac{\exp\{\frac{2b}{\sqrt{a^2+1}}f(y)\}-\exp\{\frac{2b}{\sqrt{a^2+1}}f(z)\}}{|y-z|^{\alpha}}\leq C[f]_{C^{\alpha}(\overline{B_{\frac{|x|}{2}}(x)})}. \tag{31}$$

Thus $g(\widetilde{x})$ satisfies (11) for $0<\alpha<1,\zeta>1$ and $\zeta'>0$ as given.

By Theorem 2.4, we have

$$\|D^2\bar{v}\|_{C^{\alpha}(\mathbb{R}^n)}\leq C$$

and

$$\bar{v}-\left(\frac{1}{2}\widetilde{x}^T\widetilde{A}\widetilde{x}+\widetilde{b}\cdot\widetilde{x}+\widetilde{c}\right)=O_2(|\widetilde{x}|^{2-\epsilon}) \tag{32}$$

for some $0<\widetilde{A}\in\mathtt{Sym}(n)$ satisfying $\det\widetilde{A}=g(\infty)$, $\widetilde{b}\in\mathbb{R}^n,\widetilde{c}\in\mathbb{R}$ and $C,\epsilon>0$.

Step 3: We finish the proof of Theorem 2.1 (2). By strip argument as in [25,27] etc, we prove that $I-\widetilde{A}$ is invertible. In fact, by (25), $\widetilde{A}\leq I$ and it remains to prove $\lambda_i(\widetilde{A})<1$ for all $i=1,2,\ldots,n$. Arguing by contradiction and rotating the $\widetilde{x}$-space to make $\widetilde{A}$ diagonal, we may assume that $\widetilde{A}_{11}=1$. By (32) with the definition of Legendre transform (24) and (28), there exists $\widetilde{b_1}$ such that

$$x_1=D_1v(\widetilde{x})=\widetilde{b_1}+O(|\widetilde{x}|^{1-\epsilon})\quad\text{as } |\widetilde{x}|\to\infty. \tag{33}$$

This becomes a contradiction to (27).

Let

$$A := 2b\left(I-\widetilde{A}\right)^{-1}-(a+b)I.$$

By a direct computation, $F_\tau(\lambda(A))=f(\infty)$ and

$$\begin{aligned}\left|D^2u(x)-A\right| &= 2b\left|\left(I-D^2\bar{v}(\widetilde{x})\right)^{-1}-\left(I-\widetilde{A}\right)^{-1}\right|\\ &\le C|D^2\bar{v}(\widetilde{x})-\widetilde{A}|\\ &\le \frac{C}{|\widetilde{x}|^\epsilon}\quad \forall\ |x|\ge 1.\end{aligned}$$

By the equivalence (27), we have

$$\left|D^2u(x)-A\right|\le \frac{C}{|x|^\epsilon},\quad \forall\ |x|\ge 1. \tag{34}$$

Furthermore, by (24), for any $x,y\in\mathbb{R}^n$,

$$\left|D^2u(x)-D^2u(y)\right| = 2b\left|\left(I-D^2\bar{v}(\widetilde{x})\right)^{-1}-\left(I-D^2\bar{v}(\widetilde{y})\right)^{-1}\right|.$$

By (34), $D^2\bar{v}(\widetilde{x})$ is bounded away from 0 and I, it follows that $\exists\ C>0$ such that

$$\left|D^2u(x)-D^2u(y)\right|\le 2bC\left|D^2\bar{v}(\widetilde{x})-D^2\bar{v}(\widetilde{y})\right| \tag{35}$$

Combining (35) and the equivalence (27), D^2u has bounded C^α norm.

So far, we finished the proof of Theorem 2.1 for $\tau\in(0,\frac{\pi}{4})$ case.

2.3. $\tau=\frac{\pi}{4}$ case

In this subsection, we deal with $\tau=\frac{\pi}{4}$ case by Legendre transform and analysis on the Poisson equations.

Let $f\in C^\alpha(\mathbb{R}^n)$ satisfy (11) for some $0<\alpha<1,\zeta,\zeta'>0$ and $u\in C^{2,\alpha}(\mathbb{R}^n)$ be a classical solution of (5) satisfying (3). Let

$$\overline{u}(x):=u(x)+\frac{1}{2}|x|^2,$$

then $D^2\overline{u}>0$ in $\mathbb{R}^n$. By Eq. (5), for all $i=1,2,\ldots,n$,

$$-\frac{1}{\lambda_i(D^2\overline{u})}\ge-\sum_{j=1}^n\frac{1}{\lambda_j(D^2\overline{u})}\ge\frac{\sqrt{2}}{2}\inf_{\mathbb{R}^n}f.$$

Thus there exists $\delta>0$ such that

$$D^2\overline{u}(x)>\delta I,\quad \forall\ x\in\mathbb{R}^n.$$

Let $(\widetilde{x},v)$ be the Legendre transform of $(x,\overline{u})$ as in (23) and we have

$$0<D^2v(\widetilde{x})=(D^2\overline{u}(x))^{-1}<\frac{1}{\delta}I.$$

By a direct computation, $D\bar{u}(\mathbb{R}^n)=\mathbb{R}^n$ and $v(\widetilde{x})$ satisfies the following Poisson equation

$$\Delta v=-\frac{\sqrt{2}}{2}f(Dv(\widetilde{x}))=:g(\widetilde{x})\quad \text{in }\mathbb{R}^n. \tag{36}$$

Step 1: There exists $C_0>1$ such that (27) holds. The proof is separated into two parts similarly.

By the definition of Legendre transform in (23),

$$|\widetilde{x}-\widetilde{0}|=|D\bar{u}(x)-D\bar{u}(0)|>\delta|x|.$$

Hence by triangle inequality,

$$|\widetilde{x}| \geq -|\widetilde{0}| + |\widetilde{x} - \widetilde{0}| > -|\widetilde{0}| + \delta|x|$$

and the first inequality of (27) follows immediately. The second inequality of (27) follows similarly by (8) and (29).

Step 2: Asymptotic behavior of $g(\widetilde{x})$ at infinity. By the equivalence (27),

$$g(\widetilde{x}) = -\frac{\sqrt{2}}{2}f(x) \to -\frac{\sqrt{2}}{2}f(\infty) =: g(\infty)$$

as $|\widetilde{x}| \to +\infty$. Similar to the proof of (31), we have

$$\limsup_{|\widetilde{x}|\to+\infty}\left(|\widetilde{x}|^{\zeta}|g(\widetilde{x}) - g(\infty)| + |\widetilde{x}|^{\alpha+\zeta'}[g]_{C^{\alpha}(\overline{B_{\frac{|\widetilde{x}|}{2}}(\widetilde{x})})}\right) < \infty$$

for the give $0 < \alpha < 1$, $\zeta, \zeta' > 0$.

Step 3: Asymptotic behavior of $v(\widetilde{x})$ at infinity.

Since (11) remains when $\zeta > 0$ becomes smaller, we only need to prove for $0 < \zeta < 2$ case for reading simplicity. By a direct computation, $\Delta|x|^{2-\zeta} = c_{n,\zeta}|x|^{-\zeta}$ in B_1^c. Thus there exist subsolution $\underline{v}$ and supersolution $\overline{v}$ of Poisson equation

$$\Delta\widetilde{v} = g(\widetilde{x}) - g(\infty) \quad \text{in } \mathbb{R}^n \tag{37}$$

with $\underline{v}, \overline{v} = O(|\widetilde{x}|^{2-\zeta})$ as $|x| \to \infty$. By Perron's method (see for instance [2,10,24]) and interior regularity, we have a classical solution $\widetilde{v} \in C^{2,\alpha}(\mathbb{R}^n)$ of (37) with $\widetilde{v} = O(|\widetilde{x}|^{2-\zeta})$ as $|\widetilde{x}| \to \infty$.

For any $|\widetilde{x}| = R \geq 1$, let

$$\widetilde{v}_R(y) := \left(\frac{2}{R}\right)^2 \widetilde{v}(\widetilde{x} + \frac{R}{2}y), \quad y \in B_1.$$

Then $\widetilde{v}_R$ satisfies

$$\Delta\widetilde{v}_R = g(\widetilde{x} + \frac{R}{2}y) - g(\infty) =: g_R(y) \quad \text{in } B_1.$$

By a direct computation,

$$\|g_R\|_{C^{\alpha}(\overline{B_1})} \leq CR^{-\min\{\zeta,\zeta'\}} \quad \text{and} \quad \|\widetilde{v}_R\|_{C^0(\overline{B_1})} \leq CR^{-\zeta}.$$

By interior Schauder estimates, we have

$$\|\widetilde{v}_R\|_{C^{2,\alpha}(\overline{B_{1/2}})} \leq CR^{-\min\{\zeta,\zeta'\}}$$

and then

$$\widetilde{v}(\widetilde{x}) = O_2(|\widetilde{x}|^{2-\min\{\zeta,\zeta'\}})$$

as $|\widetilde{x}| \to \infty$. Then

$$\Delta(v - \widetilde{v}) = g(\infty) \quad \text{in } \mathbb{R}^n$$

and $D^2(v - \widetilde{v})$ is bounded. By Liouville type theorem, $v - \widetilde{v}$ is a quadratic function and hence

$$v - \left(\frac{1}{2}\widetilde{x}^T\widetilde{A}\widetilde{x} + \widetilde{b}\widetilde{x} + \widetilde{c}\right) = O_2(|\widetilde{x}|^{2-\min\{\zeta,\zeta'\}})$$

for some $\widetilde{A} \in \mathtt{Sym}(n)$ with $\mathtt{trace}\widetilde{A} = g(\infty)$, $\widetilde{b} \in \mathbb{R}^n$ and $\widetilde{c} \in \mathbb{R}$. Similarly we have (33) and $\widetilde{A}$ is invertible. Taking $A := \widetilde{A}^{-1} - I$ and the result follows similar to $\tau \in (0, \frac{\pi}{4})$ case.

3. Asymptotics of solutions of (5)

In this section, we prove Theorem 1.1. As an integral part of the preparation, we analyze the linearized equation of (5) and obtain the asymptotic behavior at infinity. The major difficulty is that the linearized equation is not homogeneous.

3.1. Asymptotics of solutions of nonhomogeneous linear elliptic equations

Consider the linear elliptic equation

$$Lu := a_{ij}(x)D_{ij}u(x) = f(x) \quad \text{in } \mathbb{R}^n, \tag{38}$$

where the coefficients are uniformly elliptic, satisfying

$$\|a_{ij}\|_{C^\alpha(\mathbb{R}^n)} < \infty, \tag{39}$$

for some $0 < \alpha < 1$ and

$$|a_{ij}(x) - a_{ij}(\infty)| \le C|x|^{-\varepsilon}, \tag{40}$$

for some $0 < (a_{ij}(\infty)) \in \mathtt{Sym}(n)$ and $\varepsilon, C > 0$.

Theorem 3.1. *Let v be a classical solution of (38) that bounded from at least one side, the coefficients satisfy (39) and (40) and $f \in C^0(\mathbb{R}^n)$ satisfy*

$$\limsup_{|x|\to+\infty} |x|^\zeta |f(x)| < \infty \tag{41}$$

for some $\zeta > 2$. Then there exists a constant v_∞ such that

$$v(x) = v_\infty + \begin{cases} O\left(|x|^{2-\min\{n,\zeta\}}\right), & \zeta \ne n, \\ O\left(|x|^{2-n}(\ln|x|)\right), & \zeta = n, \end{cases} \tag{42}$$

as $|x| \to \infty$.

The homogeneous version of Theorem 3.1 has been proved earlier, see for instance Gilbarg–Serrin [15] and Li–Li–Yuan [25]. Hence we start with constructing a special solution of (38) and translate the question into homogeneous case.

By the criterion in [28], the Green's function of operator L is equivalent to the Green's function of Laplacian under conditions (39) and (40). More precisely, let $G_L(x,y)$ be the Green's function centered at y, there exists constant C such that

$$\begin{array}{lll} C^{-1}|x-y|^{2-n} \le G_L(x,y) \le C|x-y|^{2-n}, & & \forall\, x \ne y, \\ \left|D_{x_i}G_L(x,y)\right| \le C|x-y|^{1-n}, & i = 1,\ldots,n, & \forall\, x \ne y, \\ \left|D_{x_i}D_{x_j}G_L(x,y)\right| \le C|x-y|^{-n}, & i,j = 1,\ldots,n, & \forall\, x \ne y. \end{array} \tag{43}$$

By an elementary estimate as in Bao–Li–Zhang [2], we construct a solution that vanishes at infinity. More rigorously, we introduce the following result.

Lemma 3.2. *There exists a bounded strong solution $u \in W^{2,p}_{loc}(\mathbb{R}^n)$ with $p > n$ of (38) satisfying*

$$u(x) = \begin{cases} O(|x|^{2-\min\{n,\zeta\}}), & \zeta \ne n, \\ O(|x|^{2-n}(\ln|x|)), & \zeta = n, \end{cases}$$

as $|x| \to \infty$.

Proof. By (43) and Calderón–Zygmund inequality,

$$w(x) := \int_{\mathbb{R}^n} G_L(x,y)f(y)\mathtt{d}y$$

belongs to $W^{2,p}_{loc}(\mathbb{R}^n)$ for $p > n$ and is a strong solution of (38) (see for instance [1,35]). It remains to compute the vanishing speed at infinity. Let

$$\begin{aligned} E_1 &:= \{y \in \mathbb{R}^n, \quad |y| \le |x|/2\}, \\ E_2 &:= \{y \in \mathbb{R}^n, \quad |y-x| \le |x|/2\}, \\ E_3 &:= \mathbb{R}^n \setminus (E_1 \cup E_2). \end{aligned}$$

By a direct computation,

$$\int_{E_1} \frac{1}{|x-y|^{n-2}} f(y)\mathtt{d}y \le C \int_{B_{\frac{|x|}{2}}} f(y)\mathtt{d}y \cdot |x|^{2-n} \le \begin{cases} C|x|^{2-\min\{n,\zeta\}}, & \zeta \ne n, \\ C|x|^{2-n}(\ln|x|), & \zeta = n. \end{cases}$$

Similarly, we have $\frac{|x|}{2} \le |y|$ in E_2 and hence

$$\int_{E_2} \frac{1}{|x-y|^{n-2}} f(y)\mathtt{d}y \le C \int_{|x-y|\le\frac{|x|}{2}} \frac{1}{|x-y|^{n-2}}\mathtt{d}y \cdot \frac{1}{|x|^{\zeta}} \le C|x|^{2-\zeta}.$$

Now we separate E_3 into two parts

$$E_3^+ := \{y \in E_3 : |x-y| \ge |y|\}, \quad E_3^- := E_3 \setminus E_3^+.$$

Then

$$\int_{E_3^+} \frac{1}{|x-y|^{n-2} \cdot |y|^{\zeta}}\mathtt{d}y \le \int_{|y|\ge\frac{|x|}{2}} \frac{1}{|y|^{n+\zeta-2}}\mathtt{d}y \le C|x|^{2-\zeta}$$

and

$$\int_{E_3^-} \frac{1}{|x-y|^{n-2} \cdot |y|^{\zeta}}\mathtt{d}y \le \int_{|y-x|\ge\frac{|x|}{2}} \frac{1}{|y-x|^{n+\zeta-2}}\mathtt{d}y \le C|x|^{2-\zeta}.$$

Hence there exists $C > 0$ such that

$$|w(x)| \le C \left| \int_{E_1\cup E_2\cup E_3} \frac{1}{|x-y|^{n-2}} f(y)\mathtt{d}y \right| \le \begin{cases} C|x|^{2-\min\{n,\zeta\}}, & \zeta \ne n, \\ C|x|^{2-n}(\ln|x|), & \zeta = n. \end{cases} \quad \square$$

Proof of Theorem 3.1. We may assume without loss of generality that v is bounded from below, otherwise consider $-v$ instead. Let $w(x)$ be the bounded strong solution of (38) from Lemma 3.2, then

$$\widetilde{v} := v - w - \inf_{\mathbb{R}^n}(v-w) \ge 0$$

is a strong solution of (38) with $f \equiv 0$. By interior regularity, $\widetilde{v}$ is a positive classical solution. By Theorem 2.2 in [25],

$$\widetilde{v}(x) = \widetilde{v}_\infty + O\left(|x|^{2-n}\right) \quad \text{as } |x| \to \infty,$$

for some constant $\widetilde{v}_\infty$. Then the result follows immediately from Lemma 3.2. $\square$

Remark 3.3. If v is a classical solution of (38) with $|Dv(x)| = O(|x|^{-1})$ as $|x| \to \infty$ and $f \in C^0(\mathbb{R}^n)$ satisfy (41), then v is bounded from at least one side. The proof is similar to $f \equiv 0$ case, which can be found in Corollary 2.1 of [25].

3.2. Proof of Theorem 1.1

Let $u \in C^2(\mathbb{R}^n)$ be a classical solution of (5), where f satisfies (6) for some $\zeta > 2, m \geq 2$ and either of cases (1)–(3) holds. By extension and interior estimates, we may assume that $u \in W^{4,p}_{loc}(\mathbb{R}^n)$ for some $p > n$. By Theorem 2.1, Hessian matrix D^2u have finite C^α norm on $\mathbb{R}^n$ and converge to some $A \in \mathtt{Sym}(n)$ at a Hölder speed as in (34).

Let $v := u(x) - \frac{1}{2}x^TAx$. Applying Newton–Leibnitz formula between

$$F_\tau\left(\lambda\left(D^2v + A\right)\right) = f(x) \quad \text{and} \quad F_\tau(\lambda(A)) = f(\infty),$$

we have

$$\overline{a_{ij}}(x)D_{ij}v := \int_0^1 D_{M_{ij}}F_\tau\left(\lambda(tD^2v + A)\right)\mathrm{d}t \cdot D_{ij}v = f(x) - f(\infty) =: \overline{f}(x) \tag{44}$$

For any $e \in \partial B_1$, by the concavity of operator F, the partial derivatives $v_e := D_ev$ and $v_{ee} := D_e^2v$ are strong solutions of

$$\widehat{a_{ij}}(x)D_{ij}v_e := D_{M_{ij}}F_\tau\left(\lambda(D^2v + A)\right)D_{ij}v_e = f_e(x), \tag{45}$$

and

$$\widehat{a_{ij}}(x)D_{ij}v_{ee} \geq f_{ee}(x). \tag{46}$$

By Theorem 2.1, there exist $\epsilon > 0$ and $C > 0$ such that

$$\left|\overline{a_{ij}}(x) - D_{M_{ij}}F_\tau(\lambda(A))\right| + \left|\widehat{a_{ij}}(x) - D_{M_{ij}}F_\tau(\lambda(A))\right| \leq \frac{C}{|x|^\epsilon}.$$

By condition (6) and constructing barrier functions for (46), there exists $C > 0$ such that for all $x \in \mathbb{R}^n$,

$$v_{ee}(x) \leq \begin{cases} C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

By the arbitrariness of e,

$$\lambda_{\max}\left(D^2v\right)(x) \leq \begin{cases} C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

By (6) and the ellipticity of Eq. (44),

$$\lambda_{\min}\left(D^2v\right)(x) \geq -C\lambda_{\max}\left(D^2v\right) - C|\overline{f}(x)| \geq \begin{cases} -C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ -C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

Hence

$$\left|D^2v(x)\right| \leq \begin{cases} C|x|^{2-\min\{n,\zeta+2\}}, & \zeta \neq n-2, \\ C|x|^{2-n}(\ln|x|), & \zeta = n-2. \end{cases}$$

By Theorem 2.1, the coefficients $\overline{a_{ij}}$, $\widehat{a_{ij}}$ have bounded C^α norm on exterior domain. Since $\zeta > 2$, applying Remark 3.3 to Eq. (45), for any $e \in \partial B_1$, $v_e(x)$ is bounded from one side and there exists $b_e \in \mathbb{R}$ such that

$$v_e(x) = b_e + \begin{cases} O\left(|x|^{2-\min\{n,\zeta+1\}}\right), & \zeta \neq n-1, \\ O\left(|x|^{2-n}(\ln|x|)\right), & \zeta = n-1, \end{cases} \quad \text{as } |x| \to \infty. \tag{47}$$

Picking e as n unit coordinate vectors of $\mathbb{R}^n$, we found $b \in \mathbb{R}^n$ from (47) and let

$$\overline{v}(x) := v(x) - bx = u(x) - \left(\frac{1}{2}x^TAx + bx\right).$$

By (47),

$$|D\overline{v}(x)| = |(\partial_{x_1} v - b_1, \ldots, \partial_{x_n} v - b_n)| = \begin{cases} O\left(|x|^{2-\min\{n,\zeta+1\}}\right), & \zeta \neq n-1, \\ O\left(|x|^{2-n}(\ln|x|)\right), & \zeta = n-1, \end{cases}$$

as $|x| \to \infty$. By (44),

$$\overline{a_{ij}}(x) D_{ij}\overline{v} = \overline{a_{ij}}(x) D_{ij} v = \overline{f}(x).$$

By the arguments above again, there exists $c \in \mathbb{R}$ such that

$$\overline{v}(x) = c + \begin{cases} O(|x|^{2-\min\{n,\zeta\}}), & \zeta \neq n, \\ O\left(|x|^{2-n}(\ln|x|)\right), & \zeta = n, \end{cases} \quad \text{as } |x| \to \infty.$$

Notice that here we used $\zeta > 2$ for $|D\overline{v}| = O(|x|^{-1})$ and $\overline{f} = O(|x|^{-\zeta})$ at infinity. Let $Q(x) := \frac{1}{2}x^T Ax + bx + c$. Then

$$|u - Q| = |\overline{v} - c| = \begin{cases} O(|x|^{2-\min\{n,\zeta\}}), & \zeta \neq n, \\ O(|x|^{2-n}(\ln|x|)), & \zeta = n, \end{cases} \quad \text{as } |x| \to \infty.$$

Finally, we give the estimates of derivatives of u. For $|x| \geq 1$, let

$$E(y) = \left(\frac{2}{|x|}\right)^2 (u - Q)\left(x + \frac{|x|}{2}y\right).$$

Then by Newton–Leibnitz formula,

$$\underline{a^{ij}}(y) D_{ij}E(y) = F_\tau\left(\lambda(A + D^2E(y))\right) - F_\tau(\lambda(A)) = f(x + \frac{|x|}{2}y) - f(\infty) =: \underline{f}(y) \quad \text{in } B_1,$$

where

$$\underline{a^{ij}}(y) = \int_0^1 D_{M_{ij}} F_\tau\left(\lambda(A + tD^2E(y))\right) \mathrm{d}t.$$

By the Evans–Krylov estimate and interior Schauder estimate (see for instance Chap.8 of [4] and Chap.6 of [16]), for all $0 < \alpha < 1$, we have

$$\begin{aligned} \|E\|_{C^{2,\alpha}(\overline{B_{\frac{1}{2}}})} &\leq C(\|E\|_{C^0(\overline{B_1})} + \|\underline{f}\|_{C^\alpha(\overline{B_2})}) \\ &\leq C(\|E\|_{C^0(\overline{B_1})} + \|\underline{f}\|_{C^1(\overline{B_2})}) \\ &= \begin{cases} O(|x|^{-\min\{n,\zeta\}}), & \zeta \neq n, \\ O(|x|^{-n}(\ln|x|)), & \zeta = n, \end{cases} \quad \text{as } |x| \to \infty. \end{aligned}$$

By taking further derivatives and iterate, we have for all $k \leq m+1$,

$$\begin{aligned} \left(\tfrac{|x|}{2}\right)^{k-2} \left|D^k(u - Q)(x)\right| &= |D^kE(0)| \\ &\leq C_k(\|E\|_{C^0(\overline{B_1})} + \|\underline{f}\|_{C^{k-2,\alpha}(\overline{B_1})}) \\ &\leq C_k(\|E\|_{C^0(\overline{B_1})} + \|\underline{f}\|_{C^{k-1}(\overline{B_1})}) \\ &= \begin{cases} O(|x|^{-\min\{n,\zeta\}}), & \zeta \neq n, \\ O(|x|^{-n}(\ln|x|)), & \zeta = n, \end{cases} \quad \text{as } |x| \to \infty. \end{aligned}$$

This finishes the proof of Theorem 1.1.

4. Proof of Theorem 1.4

In this section, we consider asymptotic expansion at infinity for classical solutions of (5). Assume that u, f are as in Theorem 1.1. Let $\overline{a_{ij}}, \overline{f}$ and v be as in (44) and Section 3.2 respectively.

In the following, we only need to focus on $\zeta > n$ case as explained in Remark 1.5. It follows from (9) in Theorem 1.1,

$$\left|\overline{a_{ij}}(x) - D_{M_{ij}}F_\tau(\lambda(A))\right| \le C\left|D^2v(x)\right| = O_{m-1}\left(|x|^{-n}\right)$$

and hence

$$\begin{aligned} D_{M_{ij}}F_\tau(\lambda(A))D_{ij}v &= \overline{f} - (\overline{a_{ij}}(x) - D_{M_{ij}}F_\tau(\lambda(A)))D_{ij}v =: g(x) \\ &= O_m(|x|^{-\zeta}) + O_{m-1}\left(|x|^{-2n}\right) \\ &= O_{m-1}(|x|^{-\min\{2n,\zeta\}}) \end{aligned}$$

by (6) as $|x| \to \infty$.

Let

$$Q := [D_{M_{ij}}F_\tau(\lambda(A))]^{\frac{1}{2}} \quad \text{and} \quad \widetilde{v}(x) := v(Qx).$$

Then

$$\Delta\widetilde{v}(x) = g(Qx) =: \widetilde{g}(x) \quad \text{in } \mathbb{R}^n. \tag{48}$$

By a direct computation,

$$\widetilde{v} = O_{m+1}(|x|^{2-n}) \quad \text{and} \quad \widetilde{g} = O_{m-1}\left(|x|^{-\min\{2n,\zeta\}}\right).$$

Let $\Delta_{\mathbb{S}^{n-1}}$ be the Laplace–Beltrami operator on unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and

$$\Lambda_0 = 0,\ \Lambda_1 = n-1,\ \Lambda_2 = 2n, \ldots,\ \Lambda_k = k(k+n-2), \ldots,$$

be the sequence of eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ with eigenfunctions

$$Y_1^{(0)} = 1,\ Y_1^{(1)}(\theta),\ Y_2^{(1)}(\theta), \ldots,\ Y_n^{(1)}(\theta), \ldots,\ Y_1^{(k)}(\theta), \ldots,\ Y_{m_k}^{(k)}(\theta), \ldots$$

i.e.,

$$-\Delta_{\mathbb{S}^{n-1}}Y_m^{(k)}(\theta) = \Lambda_k Y_m^{(k)}(\theta), \quad \forall\ m = 1, 2, \ldots, m_k.$$

By Lemmas 3.1 and 3.2 of [27], there exists a solution $\widetilde{v}_{\widetilde{g}}$ of $\Delta\widetilde{v}_{\widetilde{g}} = \widetilde{g}$ in $\mathbb{R}^n \setminus \overline{B_1}$ with

$$\widetilde{v}_{\widetilde{g}} = \begin{cases} O_m\left(|x|^{2-\min\{2n,\zeta\}}\right), & \min\{2n,\zeta\} - n \notin \mathbb{N}, \\ O_m\left(|x|^{2-\min\{2n,\zeta\}}(\ln|x|)\right), & \min\{2n,\zeta\} - n \in \mathbb{N}. \end{cases}$$

Thus $\overline{v}(x) := \widetilde{v} - \widetilde{v}_{\widetilde{g}}$ is harmonic on $\mathbb{R}^n \setminus \overline{B_1}$ with $\overline{v} = O(|x|^{2-n})$ as $|x| \to \infty$. By spherical harmonic expansions, there exist constants $C_{k,m}^{(1)}, C_{k,m}^{(2)}$ such that

$$\overline{v} = \sum_{k=0}^{\infty}\sum_{m=1}^{m_k} C_{k,m}^{(1)}Y_m^{(k)}(\theta)|x|^k + \sum_{k=0}^{\infty}\sum_{m=1}^{m_k} C_{k,m}^{(2)}Y_m^{(k)}(\theta)|x|^{2-n-k}.$$

By the vanishing speed of $\overline{v}$, we have $C_{k,m}^{(1)} = 0$ for all k, m. Thus similar to the proof of Lemma 3.3 in [27], there exist constants $c_{k,m}$ with $k \in \mathbb{N}$, $m = 1, \ldots, m_k$ such that

$$\widetilde{v} = \begin{cases} \displaystyle\sum_{k=0}^{[\zeta]-n}\sum_{m=1}^{m_k} c_{k,m}Y_m^{(k)}(\theta)|x|^{2-n-k} + O_m\left(|x|^{2-\zeta}\right), & n < \zeta < 2n,\ \zeta \notin \mathbb{N}, \\ \displaystyle\sum_{k=0}^{\zeta-n-1}\sum_{m=1}^{m_k} c_{k,m}Y_m^{(k)}(\theta)|x|^{2-n-k} + O_m\left(|x|^{2-\zeta}(\ln|x|)\right), & n < \zeta < 2n,\ \zeta \in \mathbb{N}, \\ \displaystyle\sum_{k=0}^{n-1}\sum_{m=1}^{m_k} c_{k,m}Y_m^{(k)}(\theta)|x|^{2-n-k} + O_m\left(|x|^{2-2n}(\ln|x|)\right), & 2n \le \zeta. \end{cases}$$

By rotating backwards by Q^{-1}, the results in Theorem 1.4 follow immediately.

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