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RESEARCH ARTICLE

Asymptotic estimates for slowly rotating Newtonian stars

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Abstract This work is mainly concerned with the rotating Newtonian stars with prescribed angular velocity law. For general compressible fluids, the existence of rotating star solutions was proved by using concentration-compactness principle. In this paper, we establish the asymptotic estimates on the diameters of the stars with small rotation. The novelty of this paper is that a direct and concise definition of *slowly rotating stars* is given, different from the case with given angular momentum law, and the most general fluids are considered.

Keywords Slowly rotating star, asymptotic estimate, axi-symmetry
MSC 35Q35, 76U05

1 Backgrounds

In this paper, we study the relative equilibrium figures of rotating self-gravitating fluids. This model is of primary interest in astrophysics, where it is taken as a model of stars. Since Newton's time, many distinguished mathematicians and physicists such as Maclaurin, Clairaut, Liouville, Lyapunov, and Poincaré have made great contributions. In 1902, Jeans first supposed the universe to be filled with a non-relativistic fluids, governed by the Euler-Poisson equations (see [20, p. 562]):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \\ \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\dfrac{1}{\rho}\nabla P + \mathbf{g}, \\ \nabla\times\mathbf{g} = 0, \\ \nabla\cdot\mathbf{g} = -4\pi\rho. \end{cases} \tag{1}$$

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Here, $t \in \mathbb{R}^+$, $x \in \mathbb{R}^3$, ρ denotes the density, $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity, P is the pressure of the fluid, $\mathbf{g}$ is the gravitational field, and ∇ denotes the spatial gradient.

It is a classical problem to investigate the stability of fluids in equilibrium. The fact shows that the stability of a fluid depends mainly on two elements: its pressure P and its rotating velocity $\mathbf{v}$. For the equilibrium of compressible fluids with general pressure equation, we find the optimal conditions on the angular velocity in [12]. We now want to establish some quantitative properties for the fluids in equilibrium.

The well-known fact from quantum statistics (see [4, Chap. 10]) shows that the pressure $P(\rho)$ of the white dwarf obeys the asymptotic relations:

$$P(\rho) = \begin{cases} c_1\rho^{5/3} - c_2\rho^{7/3} + O(\rho^3), & \rho \to 0, \\ d_1\rho^{4/3} - d_2\rho^{2/3} + \cdots, & \rho \to \infty, \end{cases} \tag{2}$$

for the envelope and the core, respectively, where c_1, c_2, d_1, and d_2 are positive constants. In the last century, Milne and Chandrasekhar produced an impressive amount of work on slowly rotating stars. The interested readers can see [4,5,19] for more details.

The existence of stationary non-rotating star solution is classical, which shows that the configuration of the star is exactly a ball. However, when a star rotates with a fixed axis, it will be no longer radially symmetric. Therefore, it is challenging and significant, in both astrophysics and mathematics, to study the rotating star. In 1971, a rigorous mathematical theory for rotating compressible fluids was initiated by Auchmuty and Beals [3]. So far, there have been many results, e.g. [1–3,7,11,13–15,18], on the aspect of existence. As far as the property of the star, Friedman and Turkington [8,9] obtained the asymptotic estimates on the diameter of the fluids with prescribed angular momentum. For fluids with prescribed angular velocity, Chanillo and Li [6] gave an *a priori* bound on diameters and the number of connected components of white dwarfs, where they consider the uniformly rotating stars, that is, the angular velocity is a small constant. However, when the angular velocity is not constant, it is very interesting to determine the decay rate at infinity. In this paper, under a nearly optimal angular velocity condition, we obtain the asymptotic estimates on the radius of the rotating stars.

In our previous papers [11,12], we establish the existence of the equilibrium state of general compressible fluids rotating with prescribed angular velocity. The purpose of the present work is to study the quantitative properties of the stars with small angular velocity, which are called *slowly rotating stars*. When the angular velocity is large, the stars will not archive equilibrium states. We show that the diameters of slowly rotating stars depend only on their mass and rotating angular velocity.

We remark that the existence and stability of rotating star solution with given angular momentum was discussed in [16,17] recently.

2 Reformulation and main result

For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we denote

$$r(x) = \sqrt{x_1^2 + x_2^2}, \quad z(x) = x_3.$$

A rotating star solution $(\rho, \mathbf{v})(r, z)$ is an axi-symmetric time-independent solution of system (1), which models a star rotating about z-axis [17]. If the angular velocity $\omega(r)$ $(\geqslant 0)$ is prescribed, then the velocity field

$$\mathbf{v}(x) = (-x_2\omega(r), x_1\omega(r), 0),$$

and

$$|\mathbf{v}(x)| = \omega(r)r.$$

The Euler-Poisson equations (1) can be written as

$$\nabla P(\rho) = \rho\{\nabla B\rho + \omega^2(r)r\mathbf{e}_r\}, \tag{3}$$

where the gravitational field

$$\mathbf{g} = \nabla B\rho,$$

and

$$B\rho(x) = \int \frac{\rho(y)}{|x-y|}\,\mathrm{d}y, \quad \mathbf{e}_r = \Big(\frac{x_1}{r}, \frac{x_2}{r}, 0\Big),$$

the function P is prescribed.

For convenience, we will use $\int$ to denote $\int_{\mathbb{R}^3}$ throughout this paper. As far as the existence, we consider the problem of minimizing the energy functional, which corresponds to (3) (see Theorem 3 (c) below):

$$\begin{aligned} E(\rho) &= \frac{1}{2}\int (2A(\rho(x)) + \rho(x)J(r(x)) - \rho(x)\cdot B\rho(x))\mathrm{d}x \\ &=: E_{\mathrm{int}}(\rho) + E_{\mathrm{kin}}(\rho) + E_{\mathrm{pot}}(\rho) \end{aligned} \tag{4}$$

over the set

$$\Gamma_M = \Big\{\rho \in L^1(\mathbb{R}^3)\colon \rho \text{ is axisymmetric about } z\text{-axis}, \\ \rho \geqslant 0, \int \rho\mathrm{d}x = M, \int \Big(A(\rho) + \frac{1}{2}\rho J(r)\Big)\mathrm{d}x < \infty\Big\}.$$

Here, E_{int}, E_{kin}, and E_{pot} represent the internal energy, the rotating kinetic energy, and the gravitational potential energy of the fluids, respectively. $A(\rho)$ is determined by the pressure function $P(\rho)$, which is continuous, with the relationship

$$A(s) = s\int_0^s P(t)t^{-2}\mathrm{d}t.$$

And

$$J(r) = \int_r^\infty s w^s \mathrm{d}s.$$

Suppose that the equation of state $P(\rho)$ satisfies

($P1$) $P(0) = 0$, and $P'(\rho) > 0$ for $\rho > 0$,

($P2$) $\rho^{(n+1)/n} \leqslant c^0 P(\rho)$ for $\rho > \rho^0$,

($P3$) $P(\rho) \leqslant c_0 \rho^{(m+1)/m}$ for $\rho < \rho_0$,

where c_0, c^0, $\rho_0 < \rho^0$ are positive constants and $0 < m, n \leqslant 3$. By concentration-compactness principle, due to Lions [14], we established the existence of a rotating star with the following conditions on angular velocity:

($\omega 1$) $r\omega^2(r) \in L^1([0, \infty))$,

($\omega 2$) $rJ(r) \to 0$ as $r \to \infty$.

Our result in [12] is as follows.

Theorem 1 *Suppose that* ($\omega 1$) *and* ($\omega 2$) *hold. Then there exists* $M_c > 0$, *depending only on* n *and* c_0, (*if* $n = 3$ *then* $M_c < +\infty$; *if* $n < 3$, *then* $M_c = +\infty$), *such that if* $M < M_c$, *then there exists* $\widetilde{\rho} \in \Gamma_{M,S}$, *which is non-increasing in* $z \geqslant 0$ *and with compact support, such that*

$$E(\widetilde{\rho}) = \min_{\rho \in \Gamma_M} E(\rho) = \min_{\rho \in \Gamma_{M,S}} E(\rho),$$

where $\Gamma_{M,S} = \Gamma_M \cap \Gamma_S$,

$$\Gamma_S = \{\rho \in L^1(\mathbb{R}^3)\colon \rho(x_1, x_2, x_3) = \rho(x_1, x_2, -x_3)\}.$$

Remark 1 When $n = 3$, M_c is a critical mass, which is also called "Chandrasekhar limit" (see [4]).

In order to characterize the compactness of such solution, we first prove that $\widetilde{\rho}(x) \leqslant \rho_*$, the upper bound ρ_*—depending only on M, ρ^0, c^0, and n—is specified in the next section. Then a comparison radius a_* can be defined by

$$\frac{4\pi}{3} \rho_* a_*^3 = M. \tag{5}$$

We say that the fluid is slowly rotating, if

$$J(0) = \int_0^\infty s w^2(s) \mathrm{d}s$$

is sufficiently small. At this moment, we can obtain the estimate on the diameter of the star in asymptotic sense.

Theorem 2 *For slowly rotating stars, if* $\widetilde{\rho}$ *is a minimizer of* $E(\rho)$ *in* Γ_M *for* $M < M_c$ (*determined in Theorem* 1), *then there exists a positive constant* K, *independent of* M, *such that*

$$G \subseteq \{x\colon |x| \leqslant K a_*\},$$

where $G = \{x\colon \widetilde{\rho}(x) > 0\}$.

Remark 2 Actually, if $n = 3$, then Chandrasekhar's critical mass

$$M_c = \frac{1}{(3/4)^{2/3}\pi^{1/3}c^0}.$$

And at this moment, condition $(P2)$ becomes

$$\frac{1}{c^0}\,\rho^{4/3} \leqslant P(\rho) \quad (\rho > \rho^0).$$

Recalling (2), we know d_1 $(= 1/c^0)$ is a large constant. In astrophysics, we know that the density ρ will archive its supremum at the center of the star. Therefore, this shows that the smaller c^0 is, the more particles the star can bind.

3 Asymptotic estimate for slowly rotating stars

By an argument as in [3], it is not difficult to prove the following theorem.

Theorem 3 *Under assumption* $(\omega 1)$, *let* $\widetilde{\rho}$ *is a minimizer of the energy functional* $E(\rho)$ *in* Γ_M, *and let*

$$G = \{x \in \mathbb{R}^3 \mid \widetilde{\rho}(x) > 0\}.$$

Then

(a) $\widetilde{\rho} \in C(\mathbb{R}^3) \cap C^1(G)$;

(b) *there exists a constant* $\mu < 0$ *such that*

$$\begin{cases} A'(\widetilde{\rho}(x)) + J(r(x)) - B\widetilde{\rho}(x) = \mu, & x \in G, \\ J(r(x)) - B\widetilde{\rho}(x) \geqslant \mu, & x \in \mathbb{R}^3 \setminus G; \end{cases} \tag{6}$$

(c) $\widetilde{\rho}$ *is a solution of* (3).

Recalling the Steiner symmetrization [10], we denote $\hat{\rho}$ as the symmetric rearrangement of ρ with respect to z. By the properties of the rearrangement, the integrals

$$\int A(\rho(x))\mathrm{d}x, \quad \int \rho(x)|\mathbf{v}(x)|^2\mathrm{d}x$$

are not changed, while the potential term

$$\iint \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\mathrm{d}y$$

is increased. Thus,

$$E(\hat{\rho}) \leqslant E(\rho).$$

Therefore, if $\widetilde{\rho}$ is a minimizer of $E(\rho)$, then $\widetilde{\rho}$ is even with respect to $z=0$ and decreasing in $z \geqslant 0$.

The next lemma shows that $\widetilde{\rho}$ can be actually bounded by a constant which depends only on the pressure P and the star's mass M. It will be crucial for the sequent asymptotic analysis of the solution.

Lemma 1 *For $n<3$, if density function $\widetilde{\rho}$ is a minimizer of $E(\rho)$ in Γ_M, then*

$$\overline{\rho} = \sup_{x\in\mathbb{R}^3} \widetilde{\rho}(x) \leqslant \rho_*,$$

where the constant ρ_* depends only on M, ρ^0, c^0, and n.

Proof By the definition of $J(r)$, we have

$$J(r) \to 0 \quad (r\to\infty).$$

It follows from (6) that $\mu<0$, and

$$A'(\widetilde{\rho}(x)) = B\widetilde{\rho}(x) - J(r(x)) + \mu \leqslant B\widetilde{\rho}(x), \quad x\in G.$$

Recalling that $\widetilde{\rho}\in\Gamma_{M,S}$ has compact support and $\widetilde{\rho}$ is symmetric with z-axis and the plane $z=0$, we know, by Theorem 3 (a), that $\widetilde{\rho}$ is bounded. Letting

$$\overline{\rho} = \widetilde{\rho}(x_0), \quad x_0\in G,$$

it is easy to see

$$A'(\overline{\rho}) = A'(\widetilde{\rho}(x_0)) \leqslant B\widetilde{\rho}(x_0).$$

And by the rearrangement (see [8, p. 142]), we have

$$B\widetilde{\rho}(x_0) \leqslant C_0\overline{\rho}^{1/3}M^{2/3},$$

where

$$C_0 = \pi\Big(\frac{3}{4\pi}\Big)^{2/3}.$$

Since

$$A'(s) = \frac{A(s)}{s} + \frac{p(s)}{s}, \quad p(s)>0,$$

it follows that

$$P(\overline{\rho}) \leqslant \overline{\rho}A'(\overline{\rho}) \leqslant C_0\overline{\rho}^{4/3}M^{2/3}.$$

Using assumption $(P2)$, we obtain that

$$\overline{\rho}^{(n+1)/n} < c^0P(\overline{\rho}) \leqslant c^0C_0\overline{\rho}^{4/3}M^{2/3} \quad (\overline{\rho}>\rho^0). \tag{7}$$

We get the conclusion by $n<3$. □

Now, we give a specific definition of slowly rotating stars.

Definition 1 We say that a star or fluid is *slowly rotating*, if

$$\|J(r)\|_{L^\infty} \leqslant \frac{\delta M}{a_*},$$

where M is the mass of the star, a_* is defined by (5), and δ is a small positive constant to be determined later.

Remark 3 This definition may look more concise and simple than that given by Friedman and Turkington [8] for the case with given angular momentum per unit mass. It is worth pointing out that this bound of J, depends only on the mass M and the comparison radius a_*, or say, the central density ρ_* of the star, but is independent of the support of the star.

In order to prove $\widetilde{\rho}$ has a compact support, we need the following lemma due to Friedman and Turkington (see [8, Lemma 3.6]).

Lemma 2 [8, Lemma 3.6] *If density function $\widetilde{\rho}$ is a minimizer of $E(\rho)$ in Γ_M, then the reduced gravitational potential $B\widetilde{\rho}(x)$ satisfies*

$$B\widetilde{\rho}(x) \leqslant \frac{CM}{a_*} \cdot \frac{\log K}{K} \quad (|x| > Ka_*)$$

for any $K > 2$, where C is a positive constant independent of M and A.

Proof of Theorem 2 For $0 < n < 3$, pick $x = x' \notin G$ in the set

$$D := \left\{x \in \mathbb{R}^3 \,\Big|\, |x| < K_1 a_*,\, r(x) \geqslant \frac{1}{2} K_1 a_*\right\}$$

such that

$$\int_D \widetilde{\rho}(x)\mathrm{d}x = \frac{M}{100},$$

and then fix K_1.

Now, choose $x'' \in \partial G,\ |x''| > K_2 a_*$. We shall derive a contradiction if K_2 is sufficiently large.

By Lemma 2, we have

$$B\widetilde{\rho}(x') = \int \frac{\widetilde{\rho}(y)}{|x'-y|}\mathrm{d}y > \frac{1}{2K_1 a_*}\int\limits_D \widetilde{\rho}(y)\mathrm{d}y = \frac{M}{200K_1 a_*},$$

$$B\widetilde{\rho}(x'') \leqslant \frac{CM}{a_*} \cdot \frac{\log K_2}{K_2}.$$

Since $x' \notin G$ and $x'' \in \partial G$, it follows from (6) and $A'(\rho(x'')) = 0$ that

$$B\widetilde{\rho}(x') - J(r(x')) + \mu \leqslant 0,$$
$$B\widetilde{\rho}(x'') - J(r(x'')) + \mu = 0.$$

This implies that

$$\frac{M}{200K_1a_*} \leqslant B\widetilde{\rho}(x') \leqslant B\widetilde{\rho}(x'') + \int_{r(x')}^{\infty} s\omega^2(s)\mathrm{d}s \leqslant B\widetilde{\rho}(x'') + \frac{M}{400K_1a_*}.$$

Here, we make use of the definition of slowly rotating and take

$$\delta = \frac{1}{400K_1}.$$

Thus, we have

$$\frac{M}{400K_1a_*} \leqslant \frac{CM}{a_*} \cdot \frac{\log K_2}{K_2}.$$

If K_2 is sufficiently large, the contradiction is yielded.

For $n = 3$, we need only to prove the boundness of $\widetilde{\rho}$ similar as Lemma 1. If, to the contrary, $\overline{\rho} \leqslant \rho_*$ were not true, then we assume $\overline{\rho} > \rho^0$. Similar as (7), we have

$$\overline{\rho}^{4/3} < c^0C_0\overline{\rho}^{4/3}M^{2/3} \quad (\overline{\rho} > \rho^0).$$

This clearly contradicts with $M < M_c$. The remaining argument is the same. □

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